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Backstepping boundary control of Burgers' equation with actuator dynamics ☆

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Abstract

In this paper, we propose a backstepping boundary control law for Burgers' equation with actuator dynamics. While the control law without actuator dynamics depends only on the signals $u(0,t)$ and $u(1,t)$, the backstepping control also depends on $u_x(0,t)$, $u_x(1,t)$, $u_{xx}(0,t)$ and $u_{xx}(1,t)$, making the regularity of the control inputs the key technical issue of the paper. With elaborate Lyapunov analysis, we prove that all these signals are sufficiently regular and the closed-loop system, including the boundary dynamics, is globally H^3 stable and well posed.

Keywords: Burgers' equation; Backstepping; Boundary control; Stabilization; Distributed parameter systems

1. Introduction

In this paper, we develop a backstepping boundary feedback control for Burgers' equation

$$u_t - \varepsilon u_{xx} + uu_x = 0, \quad 0 < x < 1, \quad t > 0, \tag{1.1}$$

$$u_{xt}|_{x=0} = \varphi_0, \quad t > 0, \tag{1.2}$$

$$u_{xt}|_{x=1} = \varphi_1, \quad t > 0, \tag{1.3}$$

$$u|_{t=0} = u^0, \quad 0 < x < 1, \tag{1.4}$$

where $\varepsilon > 0$ is a viscosity parameter. In ODEs (1.2) and (1.3), φ_0 and φ_1 are control inputs. The function $u^0 = u^0(x)$ is an initial state in an appropriate function space. Since the above system is composed of a PDE

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(the viscous Burgers equation) and two ODEs, it is often referred to as a hybrid system in the literature (see, e.g., [16]).

The integrators separating φ_0 from $u_x|_{x=0}$ and φ_1 from $u_x|_{x=1}$ can be regarded as a part of actuator dynamics which prevent direct actuation via boundary values of u_x. The focus on Burgers' equation is much less for its physical relevance than for its structural properties that allow to address nontrivial nonlinear issues in a notationally simple setting, as a start towards future more practical nonlinear PDE stabilization problems.

To see that control is indeed needed for system (1.1)–(1.4), consider the uncontrolled system (i.e., $\varphi_0 = \varphi_1 = 0$)

$$u_t - \varepsilon u_{xx} + u u_x = 0, \quad 0 < x < 1, \quad t > 0, \tag{1.5}$$

$$u_{xt}|_{x=0} = 0, \quad t > 0, \tag{1.6}$$

$$u_{xt}|_{x=1} = 0, \quad t > 0, \tag{1.7}$$

$$u|_{t=0} = u^0, \quad 0 < x < 1 \tag{1.8}$$

with special initial conditions

$$u_x|_{x=0,t=0} = u_x|_{x=1,t=0} = \frac{2\varepsilon}{\cos^2(1/2)}. \tag{1.9}$$

To find the equilibrium points of (1.5)–(1.9), let us solve the steady equation of (1.5)–(1.9)

$$-\varepsilon v_{xx} + v v_x = 0, \quad 0 < x < 1, \tag{1.10}$$

$$v_x(0) = v_x(1) = \frac{2\varepsilon}{\cos^2(1/2)}. \tag{1.11}$$

Eqs. (1.10) and (1.11) can be written as

$$(\varepsilon v_x - \tfrac{1}{2} v^2)_x = 0$$

and then

$$\varepsilon v_x - \tfrac{1}{2} v^2 = C,$$

where C is a constant. Taking $C = 4\varepsilon^2$, we obtain one of the solutions of (1.10) and (1.11)

$$v(x) = 2\varepsilon \tan\left(x + \frac{\pi - 1}{2}\right), \tag{1.12}$$

which blows up at $x = \frac{1}{2}$.

The problem of control of Burgers' equation has received extensive attention recently [2–5,7,8,10,15]. In the present paper, we propose a backstepping control building upon the design

$$u_x(0,t) = k[u(0,t)^3 + u(0,t)], \tag{1.13}$$

$$u_x(1,t) = -k[u(1,t)^3 + u(1,t)], \tag{1.14}$$

in [10] where global boundary feedback stabilization was achieved for the positive constant k large enough. While controls (1.13) and (1.14) use only $u(0,t)$ and $u(1,t)$ for feedback, the backstepping controls applied through integrators (1.2) and (1.3) will employ also $u_x(0,t)$, $u_x(1,t)$, $u_{xx}(0,t)$ and $u_{xx}(1,t)$. To establish the regularity of those variables, which is the main subject of the analysis in this paper, we will show $H^3(0,1)$ regularity of u.

To our knowledge, Coron and d'Andrea-Novel [6] were the first to apply backstepping to a PDE system. Common features our work has with theirs are that

- both handle actuator dynamics with backstepping,
- both show regularity of control inputs.

The distinguishing features of our work are:

- while [6] deals with a hyperbolic PDE, our paper is the first dealing with a parabolic one,
- while both [6] and our design employ scalar control inputs, theirs is applied in a distributed fashion, whereas ours acts only from the boundary. [2]

The rest of the paper is organized as follows. We design a backstepping boundary control and present our main results in Section 2. By using the Lyapunov method, we prove our main results in Section 3.

Notation: $H^s(0,1)$ denotes the usual Sobolev space (see [1,13]) for any $s \in \mathbb{R}$. For $s \geqslant 0$, $H_0^s(0,1)$ denotes the completion of $C_0^\infty(0,1)$ in $H^s(0,1)$, where $C_0^\infty(0,1)$ denotes the space of all infinitely differentiable functions on $(0,1)$ with compact support in $(0,1)$. We use the following H^1 norm of $H^1(0,1)$:

$$\|u\|_{H^1} = \left(u(0)^2 + \int_0^1 u_x^2 \, \mathrm{d}x \right)^{1/2}, \quad u \in H^1(0,1),$$

which is equivalent to the usual one. The norm on $L^2(0,1)$ is denoted by $\|\cdot\|$. It is easy to see that

$$\|u\|^2 \leqslant 2\|u\|_{H^1}^2. \tag{1.15}$$

Let X be a Banach space and $T > 0$. We denote by $C^n([0,T];X)$ the space of n times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T];X)$ for $C^0([0,T];X)$. In what follows, for simplicity, we omit the indication of the varying range of x and t in equations and we understand that x varies from 0 to 1 and t from 0 to ∞.

2. Main result

For notational convenience, in what follows, we denote

$$w_0 = u|_{x=0}, \quad w_1 = u|_{x=1}. \tag{2.1}$$

In order to use the backstepping method, we introduce the errors z_0, z_1 of control (1.13) and (1.14) as follows:

$$z_0 = u_x|_{x=0} - k(w_0 + w_0^3), \tag{2.2}$$

$$z_1 = u_x|_{x=1} + k(w_1 + w_1^3), \tag{2.3}$$

where k is a positive constant. With (1.1)–(1.3), we have

$$\dot{z}_0 = \varphi_0 - k(1 + 3w_0^2)(\varepsilon u_{xx} - uu_x)|_{x=0}, \tag{2.4}$$

$$\dot{z}_1 = \varphi_1 + k(1 + 3w_1^2)(\varepsilon u_{xx} - uu_x)|_{x=1}. \tag{2.5}$$

To achieve that the errors z_0 and z_1 decay exponentially, that is,

$$\dot{z}_0 = -\alpha z_0, \tag{2.6}$$

$$\dot{z}_1 = -\alpha z_1 \tag{2.7}$$

[2] Additionally, by nature of the two problems, [6] shows stability only in terms of basic energy, whereas we show stability in higher-order norms.

(where $\alpha > 0$), we choose the controls

$$\varphi_0 = -\alpha z_0 + k(1+3w_0^2)(\varepsilon u_{xx} - uu_x)|_{x=0}$$
$$= -\alpha(u_x|_{x=0} - k(w_0 + w_0^3)) + k(1+3w_0^2)(\varepsilon u_{xx} - uu_x)|_{x=0}, \tag{2.8}$$

$$\varphi_1 = -\alpha z_1 - k(1+3w_1^2)(\varepsilon u_{xx} - uu_x)|_{x=1}$$
$$= -\alpha(u_x|_{x=1} + k(w_1 + w_1^3)) - k(1+3w_1^2)(\varepsilon u_{xx} - uu_x)|_{x=1}. \tag{2.9}$$

Now the closed-loop system (1.1), (2.2)–(2.3) and (2.6)–(2.9) can be summarized as

$$u_t - \varepsilon u_{xx} + uu_x = 0, \tag{2.10}$$

$$u_x|_{x=0} = k(w_0 + w_0^3) + z_0, \tag{2.11}$$

$$u_x|_{x=1} = -k(w_1 + w_1^3) + z_1, \tag{2.12}$$

$$\dot{z}_0 = -\alpha z_0, \tag{2.13}$$

$$\dot{z}_1 = -\alpha z_1 \tag{2.14}$$

which can be viewed as system (1.1), (1.13) and (1.14) perturbed by the exponentially decaying z_0 and z_1. Hence, the *closed-loop* system differs from those previously analyzed in [2,14] in a very minor way. The novelty here is that the *control*

$$\varphi_0 = -\alpha z_0 + k(1+3w_0^2)(\varepsilon u_{xx}|_{x=0} - w_0(k(w_0 + w_0^3) + z_0)), \tag{2.15}$$

$$\varphi_1 = -\alpha z_1 - k(1+3w_1^2)(\varepsilon u_{xx}|_{x=1} - w_1(-k(w_1 + w_1^3) + z_1)), \tag{2.16}$$

besides employing the signals w_0, w_1, z_0, z_1, also employs $u_{xx}|_{x=0}$ and $u_{xx}|_{x=1}$. To show that the latter signals, and hence the control, are sufficiently regular, we will require u^0 to be in $H^3(0,1)$ and will need to perform substantial additional analysis.

Note that, in general, exponentially decaying perturbations can cause finite escape time phenomena in nonlinear systems [11], the prevention of which is ensured by employing the backstepping approach which forces error variables like z_0 and z_1 to decay faster than exponential, if needed. As we shall see, in the present problem, exponential decay of z_0 and z_1 will be sufficient to maintain global stability, which is a consequence of a specific way they enter Burgers' equation.

To show that control (2.8) and (2.9) indeed exponentially stabilizes system (1.1)–(1.4) at least in L^2 norm, we define the Lyapunov function

$$V = E + \frac{6\varepsilon^2}{\alpha}(z_0^2 + z_1^2), \tag{2.17}$$

where the energy function E is defined by

$$E = \int_0^1 u^2 \, dx. \tag{2.18}$$

Using (1.1)–(1.3) and integrating by parts, we obtain

$$\dot{E} = 2\int_0^1 u(\varepsilon u_{xx} - uu_x)\, dx$$
$$= 2\varepsilon w_1[z_1 - k(w_1 + w_1^3)] - 2\varepsilon w_0[z_0 + k(w_0 + w_0^3)] - 2\varepsilon \int_0^1 u_x^2\, dx - \frac{2}{3}(w_1^3 - w_0^3)$$
$$\leqslant \left(\frac{1}{3} + \frac{1}{6} - 2\varepsilon k\right)(w_0^2 + w_0^4 + w_1^2 + w_1^4) + 6\varepsilon^2(z_0^2 + z_1^2) - 2\varepsilon \int_0^1 u_x^2\, dx. \tag{2.19}$$

Taking $k > 1/4\varepsilon$ and denoting

$$E_1 = \int_0^1 u_x^2 \,\mathrm{d}x + \frac{k}{2}\,(w_0^4 + 2w_0^2 + w_1^4 + 2w_1^2), \tag{2.20}$$

we obtain

$$\begin{aligned}
\dot{E} &\leqslant \left(\frac{1}{2} - 2\varepsilon k\right)(w_0^2 + w_0^4 + w_1^2 + w_1^4) + 6\varepsilon^2(z_0^2 + z_1^2) - 2\varepsilon \int_0^1 u_x^2 \,\mathrm{d}x \\
&\leqslant 6\varepsilon^2(z_0^2 + z_1^2) - \min\{2\varepsilon k - 1/2,\ 2\varepsilon\}\left(\int_0^1 u_x^2 \,\mathrm{d}x + w_0^2 + w_0^4 + w_1^2 + w_1^4\right) \\
&= 6\varepsilon^2(z_0^2 + z_1^2) - \min\{2\varepsilon k - 1/2,\ 2\varepsilon\}\left(\int_0^1 u_x^2 \,\mathrm{d}x + \frac{k}{2k}(2w_0^2 + 2w_0^4 + 2w_1^2 + 2w_1^4)\right) \\
&\leqslant 6\varepsilon^2(z_0^2 + z_1^2) - \min\{2\varepsilon k - 1/2,\ 2\varepsilon\}\left(\int_0^1 u_x^2 \,\mathrm{d}x + \frac{k}{2k}(2w_0^2 + w_0^4 + 2w_1^2 + w_1^4)\right) \\
&\qquad \left(\text{note that } \frac{1}{1+k} < 1 \text{ and } \frac{1}{1+k} < \tfrac{1}{k}\right) \\
&\leqslant 6\varepsilon^2(z_0^2 + z_1^2) - \frac{1}{1+k}\min\{2\varepsilon k - 1/2,\ 2\varepsilon\}\left(\int_0^1 u_x^2 \,\mathrm{d}x + \frac{k}{2}(2w_0^2 + w_0^4 + 2w_1^2 + w_1^4)\right) \\
&= 6\varepsilon^2(z_0^2 + z_1^2) - \beta E_1,
\end{aligned} \tag{2.21}$$

where

$$\beta = \frac{1}{1+k}\min\{2\varepsilon k - 1/2,\ 2\varepsilon\} > 0. \tag{2.22}$$

It therefore follows from (2.6) and (2.7) that

$$\begin{aligned}
\dot{V} &= \dot{E} + \frac{12\varepsilon^2}{\alpha}\,(z_0\dot{z}_0 + z_1\dot{z}_1) \\
&\leqslant 6\varepsilon^2(z_0^2 + z_1^2) - \beta E_1 - 12\varepsilon^2(z_0^2 + z_1^2) \\
&\leqslant -\gamma(E_1 + z_0^2 + z_1^2),
\end{aligned} \tag{2.23}$$

where

$$\gamma = \min\{\beta,\ 6\varepsilon^2\}. \tag{2.24}$$

Inequality (2.23) shows the L^2 exponential stability. Further, the closed-loop system

$$u_t - \varepsilon u_{xx} + u u_x = 0, \tag{2.25}$$

$$u_{xt}|_{x=0} = [-\alpha(u_x - k(u + u^3)) + k(1 + 3u^2)(\varepsilon u_{xx} - u u_x)]|_{x=0}, \tag{2.26}$$

$$u_{xt}|_{x=1} = [-\alpha(u_x + k(u + u^3)) - k(1 + 3u^2)(\varepsilon u_{xx} - u u_x)]|_{x=1}, \tag{2.27}$$

$$u|_{t=0} = u^0 \tag{2.28}$$

is H^3 stable and well posed, as stated in the following theorem.

Theorem 2.1. *Suppose that the initial data $u^0 \in H^3(0,1)$ and $k > 1/(4\varepsilon)$. Let z_0 and z_1 be defined by* (2.2) *and* (2.3), *respectively. Then problem* (2.25)–(2.28) *has a unique global classical solution u with*

$$u \in C([0,\infty); H^3(0,1)) \cap C^1([0,\infty); H^1(0,1)).$$

Moreover, the solution satisfies the following L^2, H^1 and H^3 stability estimates

$$\|u(t)\|^2 + \frac{6\varepsilon^2}{\alpha}[z_0(t)^2 + z_1(t)^2] \leqslant \left(\|u^0\|^2 + \frac{6\varepsilon^2}{\alpha}[z_0(0)^2 + z_1(0)^2]\right) e^{-2\omega_0 t}, \tag{2.29}$$

$$\|u(t)\|_{H^1}^2 + u_x(0,t)^2 + u_x(1,t)^2 \leqslant C(\|u^0\|_{H^1}^2 + u_x^0(0)^2 + u_x^0(1)^2) \times \exp(C(\|u^0\|_{H^1}^2 + u_x^0(0)^2 + u_x^0(1)^2)^3) e^{-\omega_1 t}, \tag{2.30}$$

$$\|u(t)\|_{H^3}^2 \leqslant C(\|u^0\|_{H^3}^2)\exp(C\|u^0\|_{H^3}^6) e^{-\omega_2 t} \tag{2.31}$$

for $t \geqslant 0$, where

$$0 < \omega_0 = \frac{k\gamma}{4(1+k)\max\{1,\ 6\varepsilon^2/\alpha\}} < \varepsilon/2, \tag{2.32}$$

$$0 < \omega_1 < \min\left\{\omega_0, \frac{k^2\alpha\gamma}{(3k+8)(4\alpha(1+k)+6k\varepsilon^2)}\right\}, \tag{2.33}$$

$$\omega_2 = \min\{\omega_1, 2\varepsilon/5, \varepsilon k/5\} \tag{2.34}$$

and γ is defined by (2.24) and $C = C(\varepsilon, \alpha, k)$ is a positive constant.

Remark 2.1. In the case where $u^0 \in H^3(0,1)$, it can be seen from (2.15), (2.16) and (2.31) that the control inputs φ_0 and φ_1 are bounded and converge to zero exponentially as $t \to \infty$.

3. Proof

In this section we prove our main result. By applying Theorem 7.4 of [12, p. 491], we show that problem (2.10)–(2.14), which is equivalent to (2.25)–(2.28), is well posed. The stability estimates (2.30) and (2.31) are proved with the Lyapunov method.

Proof of Theorem 2.1. *Step* 1: *Well-posedness.* Solving ordinary differential equations (2.26) and (2.27), we obtain

$$u_x|_{x=0} = z_0^0 e^{-\alpha t} + k(w_0 + w_0^3), \tag{3.1}$$

$$u_x|_{x=1} = z_1^0 e^{-\alpha t} - k(w_1 + w_1^3), \tag{3.2}$$

where

$$z_0^0 = u_x^0(0) - k[u^0(0) + u^0(0)^3], \tag{3.3}$$

$$z_1^0 = u_x^0(1) + k[u^0(1) + u^0(1)^3]. \tag{3.4}$$

Consequently, in the class of classical solutions, problem (2.25)–(2.28) is equivalent to

$$u_t - \varepsilon u_{xx} + uu_x = 0, \tag{3.5}$$

$$u_x|_{x=0} = z_0^0 e^{-\alpha t} + k(w_0 + w_0^3), \tag{3.6}$$

$$u_x|_{x=1} = z_1^0 e^{-\alpha t} - k(w_1 + w_1^3), \tag{3.7}$$

$$u|_{t=0} = u^0. \tag{3.8}$$

Set

$$a(x,t) = x(x-1)^2 z_0^0 e^{-\alpha t} + x^2(x-1) z_1^0 e^{-\alpha t}, \tag{3.9}$$

$$v(x,t)=u(x,t)-a(x,t). \tag{3.10}$$

Then we have

$$\begin{aligned} v_x|_{x=0} &= u_x|_{x=0}-a_x|_{x=0} \\ &= z_0^0 e^{-\alpha t}+k(w_0+w_0^3)-z_0^0e^{-\alpha t} \\ &\quad \text{(note that } v(0,t)=u(0,t)=w_0) \\ &= k(v(0,t)+v(0,t)^3) \end{aligned} \tag{3.11}$$

and likewise,

$$v_x|_{x=1}=-k(v(1,t)+v(1,t)^3). \tag{3.12}$$

Hence, problem (3.5)–(3.8) becomes

$$v_t-\varepsilon v_{xx}+(v+a)(v_x+a_x)+a_t-\varepsilon a_{xx}=0, \tag{3.13}$$

$$v_x|_{x=0}=k(v(0,t)+v(0,t)^3), \tag{3.14}$$

$$v_x|_{x=1}=-(v(1,t)+v(1,t)^3), \tag{3.15}$$

$$v|_{t=0}=u^0-a(x,0). \tag{3.16}$$

We can readily verify that problem (3.13)–(3.16) satisfies all conditions of Theorem 7.4 of [12, p. 491] with any positive constant β there (especially note that condition (7.36) on p. 487 of [12] is satisfied). Therefore we conclude from this theorem that problem (3.13)–(3.16) has a unique classical solution v in the class $H^{2+\beta,1+\beta/2}(\bar{\Omega}\times[0,T])$ for suitable $T>0$. Consequently, problem (3.5)–(3.8) has a unique classical solution $u=v+a$ in the class $H^{2+\beta,1+\beta/2}(\bar{\Omega}\times[0,T])$. Global solutions are obtained by establishing a priori estimates below.

Step 2: *L^2-estimate.* In what follows, we denote by $C=C(\varepsilon,\alpha,k)$ a generic positive constant that may vary from line to line.

Since

$$V\leqslant\frac{2(1+k)}{k}\max\{1,\ 6\varepsilon^2/\alpha\}(E_1+z_0^2+z_1^2), \tag{3.17}$$

we deduce from (2.23) that

$$\dot V\leqslant-\frac{k\gamma}{2(1+k)\max\{1,\ 6\varepsilon^2/\alpha\}}\,V, \tag{3.18}$$

which implies that

$$V\leqslant V(0)e^{-2\omega_0 t},\quad t\geqslant 0, \tag{3.19}$$

where γ and ω_0 are defined by (2.24) and (2.32), respectively. This proves (2.29).

Step 3: *H^1-estimate.* On the other hand, integrating by parts and using Eqs. (3.5)–(3.7), we obtain

$$\begin{aligned} \dot E_1 &= 2\int_0^1 u_xu_{xt}\,dx+2k[(w_1^3+w_1)\dot w_1+(w_0^3+w_0)\dot w_0] \\ &= 2\dot w_1z_1-2\dot w_0z_0-2\int_0^1 u_{xx}(\varepsilon u_{xx}-uu_x)\,dx \\ &\quad\times\left(\text{note that }\max_{0\leqslant x\leqslant 1}u(x)^2\leqslant\frac{2(1+k)}{k}E_1\text{ and }2\int_0^1 u_{xx}uu_x\,dx\leqslant\varepsilon\int_0^1 u_{xx}^2+\frac{1}{\varepsilon}\max_{0\leqslant x\leqslant 1}u(x)^2\int_0^1 u_x^2\,dx\right) \\ &\leqslant 2z_1\dot w_1-2z_0\dot w_0-\varepsilon\int_0^1 u_{xx}^2\,dx+\frac{2(1+k)}{\varepsilon k}E_1^2 \\ &\leqslant 2\frac{d}{dt}(w_1z_1)+2\alpha z_1w_1-2\frac{d}{dt}(w_0z_0)-2\alpha z_0w_0+\frac{2(1+k)}{\varepsilon k}E_1^2. \end{aligned} \tag{3.20}$$

Adding $\mathrm{d}/\mathrm{d}t(\frac{2}{k}z_0^2)$ and $\mathrm{d}/\mathrm{d}t(\frac{2}{k}z_1^2)$ to both sides, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(E_1-\frac{k}{2}(w_0^2+w_1^2)+\frac{2}{k}\left(\frac{k}{2}w_0+z_0\right)^2+\frac{2}{k}\left(\frac{k}{2}w_1-z_1\right)^2\right)\\&\leqslant 2\alpha z_1w_1-2\alpha z_0w_0-\frac{4\alpha}{k}(z_0^2+z_1^2)+\frac{2(1+k)}{\varepsilon k}E_1^2\\&\leqslant C(w_0^2+w_1^2)+\frac{2(1+k)}{\varepsilon k}E_1^2.\end{aligned}\tag{3.21}$$

Denoting

$$\begin{aligned}J&=E_1-\frac{k}{2}(w_0^2+w_1^2)+\frac{2}{k}\left(\frac{k}{2}w_0+z_0\right)^2+\frac{2}{k}\left(\frac{k}{2}w_1-z_1\right)^2\\&=\int_0^1 u_x^2+\frac{k}{2}(w_0^2+w_0^4+w_1^2+w_1^4)\\&\quad+\frac{2}{k}\left[\left(u_x|_{x=0}-\frac{k}{2}w_0-kw_0^3\right)^2+\left(u_x|_{x=1}+\frac{k}{2}w_1+kw_1^3\right)^2\right],\end{aligned}\tag{3.22}$$

we get

$$\dot{J}\leqslant C(w_0^2+w_1^2)+\frac{2(1+k)}{\varepsilon k}E_1^2.\tag{3.23}$$

It is clear that

$$\frac{1}{2}E_1\leqslant J\leqslant\frac{3}{2}E_1+\frac{4}{k}(z_0^2+z_1^2)\leqslant\frac{3k+8}{2k}(E_1+z_0^2+z_1^2).\tag{3.24}$$

For $M>0$ yet to be determined, we define the Lyapunov function G as

$$G=MV+\ln\left(1+J+\frac{6\varepsilon^2}{\alpha}(z_0^2+z_1^2)\right),\tag{3.25}$$

which, in view of (3.22) and (3.24), is positive definite and radially unbounded in the argument of $\|u\|_{H^1}$, $u_x(0)$ and $u_x(1)$. It is also easy to see that

$$\begin{aligned}G&\leqslant MV+J+\frac{6\varepsilon^2}{\alpha}(z_0^2+z_1^2)\\&\leqslant\frac{2M(1+k)}{k}E_1+\frac{6M\varepsilon^2}{\alpha}(z_0^2+z_1^2)+J+\frac{6\varepsilon^2}{\alpha}(z_0^2+z_1^2)\\&\leqslant\frac{4M(1+k)}{k}J+\frac{6M\varepsilon^2}{\alpha}(z_0^2+z_1^2)+J+\frac{6\varepsilon^2}{\alpha}(z_0^2+z_1^2)\\&\leqslant\frac{4M(1+k)+k}{k}J+\frac{6\varepsilon^2(M+1)}{\alpha}(z_0^2+z_1^2)\\&\leqslant\frac{\alpha(4M(1+k)+k)+6k\varepsilon^2(M+1)}{k\alpha}(J+z_0^2+z_1^2).\end{aligned}\tag{3.26}$$

It then follows from (2.23), (3.23) and (3.24) that for $M > 16(1+k)(3k+8)/2\varepsilon\gamma k^2$

$$\begin{aligned}
\dot{G} &= M\dot{V} + \frac{\dot{J} + (12\varepsilon^2/\alpha)(z_0\dot{z}_0 + z_1\dot{z}_1)}{1 + J + (6\varepsilon^2/\alpha)(z_0^2 + z_1^2)} \\
&\quad \text{(use (3.23) and note that } z_0\dot{z}_0 + z_1\dot{z}_1 = -\alpha(z_0^2 + z_1^2) \text{ is negative)} \\
&\leqslant M\dot{V} + \frac{C(w_0^2 + w_1^2) + [2(1+k)/\varepsilon k]E_1^2}{1 + J + (6\varepsilon^2/\alpha)(z_0^2 + z_1^2)} \\
&\quad \text{(use (2.23) and the first inequality of (3.24))} \\
&\leqslant -M\gamma(E_1 + z_0^2 + z_1^2) + \frac{8(1+k)}{\varepsilon k} J + C(w_0^2 + w_1^2) \\
&\leqslant -\frac{M\gamma}{2}(E_1 + z_0^2 + z_1^2) - \frac{M\gamma}{2}(z_0^2 + z_1^2) + \frac{8(1+k)}{\varepsilon k} J + \left(C - \frac{Mk\gamma}{2}\right)(w_0^2 + w_1^2) \\
&\leqslant -\frac{2kM\gamma}{2(3k+8)} J - \frac{M\gamma}{2}(z_0^2 + z_1^2) + \frac{8(1+k)}{\varepsilon k} J + \left(C - \frac{Mk\gamma}{2}\right)(w_0^2 + w_1^2) \\
&\leqslant -\left(\frac{2kM\gamma}{2(3k+8)} - \frac{8(1+k)}{\varepsilon k}\right)(J + z_0^2 + z_1^2) + \left(C - \frac{Mk\gamma}{2}\right)(w_0^2 + w_1^2) \\
&\leqslant -\frac{k\alpha}{\alpha(4M(1+k)+k) + 6k\varepsilon^2(M+1)}\left(\frac{2kM\gamma}{2(3k+8)} - \frac{8(1+k)}{\varepsilon k}\right) G \\
&\quad + \left(C - \frac{Mk\gamma}{2}\right)(w_0^2 + w_1^2).
\end{aligned} \tag{3.27}$$

Since the function

$$f(M) = \frac{k\alpha}{\alpha(4M(1+k)+k) + 6k\varepsilon^2(M+1)}\left(\frac{2kM\gamma}{2(3k+8)} - \frac{8(1+k)}{\varepsilon k}\right)$$

is increasing and

$$\lim_{M\to\infty} f(M) = \frac{k^2\alpha\gamma}{(3k+8)(4\alpha(1+k) + 6k\varepsilon^2)},$$

we can find an M_0 sufficiently large such that $\omega_1 < f(M_0)$ and $C - M_0 k\gamma/2 < 0$, where ω_1 is defined by (2.33). Consequently, we obtain

$$\dot{G} \leqslant -\omega_1 G. \tag{3.28}$$

Hence, we have

$$G \leqslant G(0)e^{-\omega_1 t}, \quad t \geqslant 0. \tag{3.29}$$

It therefore follows that

$$1 + J + \frac{6\varepsilon^2}{\alpha}(z_0^2 + z_1^2) \leqslant \exp(G(0)e^{-\omega_1 t}),$$

and then

$$\begin{aligned}
J + \frac{6\varepsilon^2}{\alpha}(z_0^2 + z_1^2) &\leqslant \exp(G(0)e^{-\omega_1 t}) - 1 \\
&\leqslant CG(0)\exp(G(0))e^{-\omega_1 t} \\
&\leqslant C(\|u^0\|_{H^1}^2 + \|u^0\|_{H^1}^6 + u_x^0(0)^2 + u_x^0(1)^2) \\
&\quad \times \exp(C(\|u^0\|_{H^1}^2 + \|u^0\|_{H^1}^6 + u_x^0(0)^2 + u_x^0(1)^2))e^{-\omega_1 t}, \quad t \geqslant 0,
\end{aligned} \tag{3.30}$$

which implies (2.30) since

$$\begin{aligned}
&\|u^0\|_{H^1}^2 + \|u^0\|_{H^1}^6 + u_x^0(0)^2 + u_x^0(1)^2 \\
&\quad \leqslant \|u^0\|_{H^1}^2 + u_x^0(0)^2 + u_x^0(1)^2 + (\|u^0\|_{H^1}^2 + u_x^0(0)^2 + u_x^0(1)^2)^3 \\
&\quad \leqslant C(\|u^0\|_{H^1}^2 + u_x^0(0)^2 + u_x^0(1)^2)\exp(\|u^0\|_{H^1}^2 + u_x^0(0)^2 + u_x^0(1)^2) \\
&\quad \leqslant eC(\|u^0\|_{H^1}^2 + u_x^0(0)^2 + u_x^0(1)^2)\exp((\|u^0\|_{H^1}^2 + u_x^0(0)^2 + u_x^0(1)^2)^3)
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
&\exp(C(\|u^0\|_{H^1}^2 + \|u^0\|_{H^1}^6 + u_x^0(0)^2 + u_x^0(1)^2)) \\
&\quad \leqslant \exp(C(1 + (\|u^0\|_{H^1}^2 + u_x^0(0)^2 + u_x^0(1)^2)^3)) \\
&\quad = \mathrm{e}^C \exp(C(\|u^0\|_{H^1}^2 + u_x^0(0)^2 + u_x^0(1)^2)^3).
\end{aligned} \tag{3.32}$$

Step 4: *H^2-estimate.* In order to obtain the H^2-estimate, we estimate the L^2 norm of u_t. Integrating by parts and using Eqs. (3.5)–(3.7), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2\,\mathrm{d}x &= 2\int_0^1 u_t(\varepsilon u_{txx} - u_t u_x - u u_{tx})\,\mathrm{d}x \\
&\leqslant -2\varepsilon k[\dot{w}_0^2(3w_0^2+1) + \dot{w}_1^2(3w_1^2+1)] + 2\varepsilon\dot{w}_1\dot{z}_1 - 2\varepsilon\dot{w}_0\dot{z}_0 - 2\varepsilon\|u_{xt}\|^2 + C\|u_t\|\,\|u_t\|_{H^1}\|u\|_{H^1} \\
&= -2\varepsilon k[\dot{w}_0^2(3w_0^2+1) + \dot{w}_1^2(3w_1^2+1)] \\
&\qquad -2\varepsilon\alpha\dot{w}_1 z_1 + 2\varepsilon\alpha\dot{w}_0 z_0 \quad \text{(use (2.6) and (2.7))} \\
&\qquad -2\varepsilon\|u_{xt}\|^2 + C\|u_t\|\|u_t\|_{H^1}\|u\|_{H^1} \\
&\qquad \text{(note definition (2.34) of } \omega_2) \\
&\leqslant -\tfrac{1}{2}\varepsilon k[\dot{w}_0^2(3w_0^2+1) + \dot{w}_1^2(3w_1^2+1)] + C(z_1^2+z_0^2) - 5\omega_2\|u_t\|_{H^1}^2 + C\|u_t\|\|u_t\|_{H^1}\|u\|_{H^1} \\
&\leqslant -\tfrac{1}{2}\varepsilon k[\dot{w}_0^2(3w_0^2+1) + \dot{w}_1^2(3w_1^2+1)] + C(z_1^2+z_0^2) - 4\omega_2\|u_t\|_{H^1}^2 + C\|u_t\|^2\|u\|_{H^1}^2 \\
&\leqslant -\tfrac{1}{2}\varepsilon k[\dot{w}_0^2(3w_0^2+1) + \dot{w}_1^2(3w_1^2+1)]C(z_1^2+z_0^2) - 2\omega_2\|u_t\|^2 + C\|u_t\|^2\|u\|_{H^1}^2.
\end{aligned} \tag{3.33}$$

Multiplying (3.33) by $\mathrm{e}^{2\omega_2 t}$ gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\|u_t\|^2\mathrm{e}^{2\omega_2 t}) &\leqslant C(z_1^2+z_0^2)\mathrm{e}^{2\omega_2 t} + C\|u\|_{H^1}^2\|u_t\|^2\mathrm{e}^{2\omega_2 t} \\
&\leqslant C(z_1^2+z_0^2)\mathrm{e}^{2\omega_2 t} + C\|u\|_{H^1}^2\|u_t\|^2\mathrm{e}^{2\omega_2 t}.
\end{aligned} \tag{3.34}$$

It, therefore, follows from (2.29) that

$$\|u_t\|^2\mathrm{e}^{2\omega_2 t} \leqslant \|u_t(0)\|^2 + C(\|u^0\|_{H^2}^2 + \|u^0\|_{H^2}^6) + C\int_0^t \|u\|_{H^1}^2\|u_t\|^2\mathrm{e}^{2\omega_2 s}\,\mathrm{d}s,$$

and then by (2.29) and Gronwall's inequality we obtain

$$\|u_t\|^2 \leqslant C(\|u^0\|_{H^2}^2 + \|u^0\|_{H^2}^6)\exp(C(\|u^0\|_{H^2}^2 + \|u^0\|_{H^2}^6))\mathrm{e}^{-2\omega_2 t}, \quad t \geqslant 0. \tag{3.35}$$

Since

$$\|u_{xx}\|^2 \leqslant C(\|u_t\|^2 + \|u\|_{H^1}^4), \tag{3.36}$$

it follows from (3.30), (3.35) and (3.36) that

$$\|u_{xx}\|^2 \leqslant C(\|u^0\|_{H^2}^2 + \|u^0\|_{H^2}^{12})\exp(C(\|u^0\|_{H^2}^2 + \|u^0\|_{H^2}^6))\mathrm{e}^{-2\omega_2 t}, \quad t \geqslant 0. \tag{3.37}$$

Step 5: H^3-*estimate*. To prove that (2.31), we estimate u_{xt}. Integrating by parts and using (3.5)–(3.7), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_{xt}^2\,\mathrm{d}x &= 2u_{xt}u_{tt}\Big|_0^1 - 2\int_0^1 u_{xxt}(\varepsilon u_{txx} - u_t u_x - u u_{xt})\,\mathrm{d}x \\ &\leqslant -k\left[\frac{\mathrm{d}}{\mathrm{d}t}(\dot{w}_0^2)(3w_0^2+1) + \frac{\mathrm{d}}{\mathrm{d}t}(\dot{w}_1^2)(3w_1^2+1)\right] \\ &\quad +2\ddot{w}_1\dot{z}_1 - 2\ddot{w}_0\dot{z}_0 - 2\varepsilon\|u_{xxt}\|^2 + 8\|u_{xxt}\|\|u_t\|_{H^1}\|u\|_{H^1} \\ &\leqslant -k\frac{\mathrm{d}}{\mathrm{d}t}[\dot{w}_0^2(3w_0^2+1) + \dot{w}_1^2(3w_1^2+1)] + 6k(\dot{w}_0^3 w_0 + \dot{w}_1^3 w_1) \\ &\quad -2\alpha z_1\ddot{w}_1 + 2\alpha z_0\ddot{w}_0 \quad \text{(use (2.6) and (2.7))} \\ &\quad +C\|u_t\|_{H^1}^2\|u\|_{H^1}^2 \\ &= -k\frac{\mathrm{d}}{\mathrm{d}t}[\dot{w}_0^2(3w_0^2+1) + \dot{w}_1^2(3w_1^2+1)] + 6k(\dot{w}_0^3 w_0 + \dot{w}_1^3 w_1) \\ &\quad -2\alpha\frac{\mathrm{d}}{\mathrm{d}t}(\dot{w}_1 z_1) + 2\alpha\frac{\mathrm{d}}{\mathrm{d}t}(\dot{w}_0 z_0) - 2z_1\alpha^2\dot{w}_1 + 2z_0\alpha^2\dot{w}_0 + C\|u_t\|_{H^1}^2\|u\|_{H^1}^2. \end{aligned} \tag{3.38}$$

Adding $\mathrm{d}/\mathrm{d}t(\alpha^2 z_0^2/k)$ and $\mathrm{d}/\mathrm{d}t(\alpha^2 z_1^2/k)$ to both sides and setting

$$J_1 = \int_0^1 u_{xt}^2\,\mathrm{d}x + 3k(\dot{w}_0^2 w_0^2 + \dot{w}_1^2 w_1^2) + (\sqrt{k}\dot{w}_1 + z_1/\sqrt{k})^2 + (\sqrt{k}\dot{w}_0 - z_0/\sqrt{k})^2, \tag{3.39}$$

we obtain

$$\begin{aligned}\dot{J}_1 &\leqslant 6k(\dot{w}_0^3 w_0 + \dot{w}_1^3 w_1) + 2|z_1|\alpha^2|\dot{w}_1| + 2|z_0|\alpha^2|\dot{w}_0| - \frac{2\alpha^3}{k}(z_0^2 + z_1^2) + C\|u_t\|_{H^1}^2\|u\|_{H^1}^2 \\ &\leqslant \underbrace{6k(\dot{w}_0^3 w_0 + \dot{w}_1^3 w_1)}_{=I} + C(\dot{w}_0^2 + \dot{w}_1^2) + C\|u\|_{H^1}^2 J_1. \end{aligned} \tag{3.40}$$

We now estimate I. Since for any $x, y \in [0,1]$

$$v(x)^2 = v(y)^2 + \int_y^x (v^2)_\xi\,\mathrm{d}\xi \leqslant v(y)^2 + \|v\|\|v_x\|,$$

we have

$$\max_{0\leqslant x\leqslant 1} v(x)^2 \leqslant \|v\|^2 + \|v\|\|v_x\|. \tag{3.41}$$

It, therefore, follows that

$$\begin{aligned}|\dot{w}_i^3 w_i| &\leqslant |\dot{w}_i w_i|(\|u_t\|^2 + \|u_t\|\|u_{xt}\|) \\ &\leqslant \tfrac{1}{2}(\dot{w}_i^2 w_i^2 + \|u_t\|^4 + \|u_t\|^2\|u_{xt}\|^2), \quad i = 0,1 \end{aligned} \tag{3.42}$$

and then

$$I \leqslant C\sum_{i=0}^{1}\dot{w}_i^2 w_i^2 + C\|u_t\|^4 + C\|u_t\|^2\|u_{xt}\|^2. \tag{3.43}$$

Thus, we deduce from (3.40) that

$$\dot{J}_1 \leqslant C\sum_{i=0}^{1}\dot{w}_i^2(w_i^2 + 1) + C\|u_t\|^4 + C(\|u_t\|^2 + \|u\|_{H^1}^2)J_1. \tag{3.44}$$

On the other hand, multiplying (3.33) by $e^{\omega_2 t}$, one obtains

$$\frac{d}{dt}(\|u_t\|^2 e^{\omega_2 t}) + \tfrac{1}{2}\varepsilon k[\dot{w}_0^2(3w_0^2+1) + \dot{w}_1^2(3w_1^2+1)]e^{\omega_2 t}$$
$$\leqslant C(z_1^2+z_0^2)e^{\omega_2 t} + C\|u\|_{H^1}^2\|u_t\|^2 e^{\omega_2 t}$$
$$\leqslant C(z_1^2+z_0^2)e^{\omega_2 t} + C\|u\|_{H^1}^2\|u_t\|^2 e^{\omega_2 t}. \tag{3.45}$$

Integrating from 0 to ∞ and using (3.30) and (3.35), one obtains

$$\int_0^\infty \sum_{i=0}^1 \dot{w}_i^2(w_i^2+1)e^{\omega_2 t}\,dt \leqslant C(\|u^0\|_{H^2}^2 + \|u^0\|_{H^2}^6). \tag{3.46}$$

It, therefore, follows from Gronwall's inequality (see, e.g., [9, p. 63]) that

$$J_1 \leqslant C(\|u^0\|_{H^3}^2 + \|u^0\|_{H^3}^{12})\exp(C(\|u^0\|_{H^2}^2 + \|u^0\|_{H^2}^6))e^{-\omega_2 t}, \quad t\geqslant 0. \tag{3.47}$$

Since

$$\|u_{xxx}\|^2 \leqslant C(\|u_{xt}\|^2 + \|u\|_{H^2}^4) \tag{3.48}$$

(2.31) follows from (2.30), (3.37) and (3.47) with the use of (3.31) and (3.32). □

4. Conclusion

In summary, we propose here a backstepping boundary control law for Burgers' equation with actuator dynamics, which depends not only on the signals $u(0,t)$ and $u(1,t)$, but also on $u_x(0,t)$, $u_x(1,t)$, $u_{xx}(0,t)$ and $u_{xx}(1,t)$. With elaborate Lyapunov analysis, and applying a result from the the classical theory of nonlinear partial differential equations of parabolic type, we prove that the controlled closed-loop system is globally H^3 stable and has a unique global classical solution.

As we pointed out in the introduction, system (1.1)–(1.3) is only an example of a more general class of boundary control problems (yet to be investigated) tractable by backstepping. To give a preview of possible future extensions, we offer the system

$$u_t - \varepsilon u_{xx} + uu_x = 0, \tag{4.1}$$

$$u_{xt}|_{x=0} = f_{01}\left(u|_{x=0}, u_x|_{x=0}, u_{xx}|_{x=0}, \int_0^1 u\,dx\right) + \xi_0, \tag{4.2}$$

$$u_{xt}|_{x=1} = f_{11}\left(u|_{x=1}, u_x|_{x=1}, u_{xx}|_{x=1}, \int_0^1 u\,dx\right) + \xi_1, \tag{4.3}$$

$$\dot{\xi}_0 = f_{02}\left(u|_{x=0}, u_x|_{x=0}, u_{xx}|_{x=0}, \int_0^1 u\,dx, \xi_0\right) + \varphi_0, \tag{4.4}$$

$$\dot{\xi}_1 = f_{12}\left(u|_{x=1}, u_x|_{x=1}, u_{xx}|_{x=1}, \int_0^1 u\,dx, \xi_1\right) + \varphi_1, \tag{4.5}$$

where f_{ij} are smooth functions and φ_0, φ_1 are controls. The variables ξ_0, ξ_1 serve as "virtual controls" to the $u|_{x=0}, u|_{x=1}$ equations, effectively separating the inputs ϕ_0, ϕ_1 from the Burgers equation but not one but two cascaded integrators (on each end of the $[0,1]$ interval). After designing the control laws in two steps of backstepping, one would need $H^5(0,1)$ estimates on u to show regularity of the control signals. Adding more integrators would be standard insofar as the backstepping design task is concerned, however, regularity analysis would require estimates of increasingly high order.

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