Backstepping boundary control of Burgers’ equation with actuator dynamics

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Abstract

In this paper, we propose a backstepping boundary control law for Burgers’ equation with actuator dynamics. While the control law without actuator dynamics depends only on the signals \( u(0,t) \) and \( u(1,t) \), the backstepping control also depends on \( u_x(0,t) \), \( u_x(1,t) \), \( u_{xx}(0,t) \) and \( u_{xx}(1,t) \), making the regularity of the control inputs the key technical issue of the paper. With elaborate Lyapunov analysis, we prove that all these signals are sufficiently regular and the closed-loop system, including the boundary dynamics, is globally \( H^3 \) stable and well posed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we develop a backstepping boundary feedback control for Burgers’ equation

\begin{align}
    u_t - u u_x + u u_x &= 0, \quad 0 < x < 1, \quad t > 0, \\
    u_x |_{x=0} &= \varphi_0, \quad t > 0, \\
    u_x |_{x=1} &= \varphi_1, \quad t > 0, \\
    u |_{t=0} &= u^0, \quad 0 < x < 1,
\end{align}

where \( \varepsilon > 0 \) is a viscosity parameter. In ODEs (1.2) and (1.3), \( \varphi_0 \) and \( \varphi_1 \) are control inputs. The function \( u^0 = u^0(x) \) is an initial state in an appropriate function space. Since the above system is composed of a PDE

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\end{itemize}
(the viscous Burgers equation) and two ODEs, it is often referred to as a hybrid system in the literature (see, e.g., [16]).

The integrators separating $\varphi_0$ from $u_x|_{x=0}$ and $\varphi_1$ from $u_x|_{x=1}$ can be regarded as a part of actuator dynamics which prevent direct actuation via boundary values of $u_x$. The focus on Burgers’ equation is much less for its physical relevance than for its structural properties that allow to address nontrivial nonlinear issues in a notationally simple setting, as a start towards future more practical nonlinear PDE stabilization problems.

To see that control is indeed needed for system (1.1)−(1.4), consider the uncontrolled system (i.e., $\varphi_0 = \varphi_1 = 0$)

$$u_t - uu_x + uux = 0, \quad 0 < x < 1, \quad t > 0,$$
(1.5)

$$u_x|_{x=0} = 0, \quad t > 0,$$
(1.6)

$$u_x|_{x=1} = 0, \quad t > 0,$$
(1.7)

$$u|_{t=0} = u^0, \quad 0 < x < 1$$
(1.8)

with special initial conditions

$$u_x|_{x=0, t=0} = u_x|_{x=1, t=0} = \frac{2\varepsilon}{\cos^2(1/2)}.$$  
(1.9)

To find the equilibrium points of (1.5)−(1.9), let us solve the steady equation of (1.5)−(1.9)

$$-\varepsilon v_x + \varphi_x = 0, \quad 0 < x < 1,$$
(1.10)

$$\varphi_x(0) = \varphi_x(1) = \frac{2\varepsilon}{\cos^2(1/2)},$$  
(1.11)

Eqs. (1.10) and (1.11) can be written as

$$(\varepsilon v_x - \frac{1}{2}v^2)_x = 0$$

and then

$$\varepsilon v_x - \frac{1}{2}v^2 = C,$$

where $C$ is a constant. Taking $C = 4\varepsilon^2$, we obtain one of the solutions of (1.10) and (1.11)

$$\varphi(x) = 2\varepsilon \tan (x + \frac{\pi - 1}{2}),$$  
(1.12)

which blows up at $x = \frac{1}{2}$.

The problem of control of Burgers’ equation has received extensive attention recently [2–5,7,8,10,15]. In the present paper, we propose a backstepping control building upon the design

$$u_t(0, t) = k[u(0, t)^3 + u(0, t)],$$
(1.13)

$$u_x(1, t) = -k[u(1, t)^3 + u(1, t)]$$
(1.14)

in [10] where global boundary feedback stabilization was achieved for the positive constant $k$ large enough. While controls (1.13) and (1.14) use only $u(0, t)$ and $u(1, t)$ for feedback, the backstepping controls applied through integrators (1.2) and (1.3) will employ also $u_t(0, t)$, $u_x(1, t)$, $u_{xx}(0, t)$ and $u_{xx}(1, t)$. To establish the regularity of those variables, which is the main subject of the analysis in this paper, we will show $H^3(0,1)$ regularity of $u$. 

To our knowledge, Coron and d’Andrea-Novel [6] were the first to apply backstepping to a PDE system. Common features our work has with theirs are that

- both handle actuator dynamics with backstepping,
- both show regularity of control inputs.

The distinguishing features of our work are:

- while [6] deals with a hyperbolic PDE, our paper is the first dealing with a parabolic one,
- while both [6] and our design employ scalar control inputs, theirs is applied in a distributed fashion, whereas ours acts only from the boundary.\(^2\)

The rest of the paper is organized as follows. We design a backstepping boundary control and present our main results in Section 2. By using the Lyapunov method, we prove our main results in Section 3.

**Notation:** \(H^s(0,1)\) denotes the usual Sobolev space (see [1,13]) for any \(s \in \mathbb{R}\). For \(s \geq 0\), \(H^s_0(0,1)\) denotes the completion of \(C_0^\infty(0,1)\) in \(H^s(0,1)\), where \(C_0^\infty(0,1)\) denotes the space of all infinitely differentiable functions on \((0,1)\) with compact support in \((0,1)\). We use the following \(H^1\) norm of \(H^1(0,1)\):

\[
\|u\|_{H^1} = \left( u(0)^2 + \int_0^1 u_x^2 \, dx \right)^{1/2}, \quad u \in H^1(0,1),
\]

which is equivalent to the usual one. The norm on \(L^2(0,1)\) is denoted by \(\| \cdot \| \). It is easy to see that

\[
\|u\|^2 \leq 2\|u\|_{H^1}, \quad (1.15)
\]

Let \(X\) be a Banach space and \(T > 0\). We denote by \(C^n([0,T];X)\) the space of \(n\) times continuously differentiable functions defined on \([0,T]\) with values in \(X\), and write \(C([0,T];X)\) for \(C^0([0,T];X)\). In what follows, for simplicity, we omit the indication of the varying range of \(x\) and \(t\) in equations and we understand that \(x\) varies from 0 to 1 and \(t\) from 0 to \(\infty\).

### 2. Main result

For notational convenience, in what follows, we denote

\[
w_0 = u|_{x=0}, \quad w_1 = u|_{x=1}. \tag{2.1}
\]

In order to use the backstepping method, we introduce the errors \(z_0, z_1\) of control (1.13) and (1.14) as follows:

\[
z_0 = u_\epsilon|_{x=0} - k(w_0 + w_0^3), \tag{2.2}
\]

\[
z_1 = u_\epsilon|_{x=1} + k(w_1 + w_1^3), \tag{2.3}
\]

where \(k\) is a positive constant. With (1.1)–(1.3), we have

\[
z_0 = \phi_0 - k(1 + 3w_0^2)(aw_{xx} - uu_x)|_{x=0}, \tag{2.4}
\]

\[
z_1 = \phi_1 + k(1 + 3w_1^2)(aw_{xx} - uu_x)|_{x=1}. \tag{2.5}
\]

To achieve that the errors \(z_0, z_1\) decay exponentially, that is,

\[
z_0 = -\alpha z_0, \tag{2.6}
\]

\[
z_1 = -\alpha z_1, \tag{2.7}
\]

\(^2\)Additionally, by nature of the two problems, [6] shows stability only in terms of basic energy, whereas we show stability in higher-order norms.
(where $\alpha > 0$), we choose the controls

$$
\varphi_0 = -2z_0 + k(1 + 3w_0^3)(u_{xx} - uu_x)|_{x=0} = -2(u_x|_{x=0} - k(w_0 + w_0^3) + k(1 + 3w_0^2)(u_{xx} - uu_x)|_{x=0},
$$

$$
\varphi_1 = -2z_1 - k(1 + 3w_1^3)(u_{xx} - uu_x)|_{x=1} = -2(u_x|_{x=1} + k(w_1 + w_1^3)) - k(1 + 3w_1^2)(u_{xx} - uu_x)|_{x=1},
$$

(2.8)

Now the closed-loop system (1.1), (2.2)–(2.3) and (2.6)–(2.9) can be summarized as

$$
u_t - \alpha u_{xx} + uu_x = 0,
$$

$$
u_x|_{x=0} = k(w_0 + w_0^3) + z_0,
$$

$$
u_x|_{x=1} = -k(w_1 + w_1^3) + z_1,
$$

$$z_0 = -\alpha z_0,
$$

$$z_1 = -\alpha z_1
$$

(2.10)

which can be viewed as system (1.1), (1.13) and (1.14) perturbed by the exponentially decaying $z_0$ and $z_1$. Hence, the closed-loop system differs from those previously analyzed in [2,14] in a very minor way. The novelty here is that the control

$$
\varphi_0 = -2z_0 + k(1 + 3w_0^3)(u_{xx} - uu_x)|_{x=0} - w_0(k(w_0 + w_0^3) + z_0)),
$$

$$
\varphi_1 = -2z_1 - k(1 + 3w_1^3)(u_{xx} - uu_x)|_{x=1} - w_1(-k(w_1 + w_1^3) + z_1)),
$$

(2.15)

besides employing the signals $w_0, w_1, z_0, z_1$, also employs $u_{xx}|_{x=0}$ and $u_{xx}|_{x=1}$. To show that the latter signals, and hence the control, are sufficiently regular, we will require $u^0$ to be in $H^3(0,1)$ and will need to perform substantial additional analysis.

Note that, in general, exponentially decaying perturbations can cause finite escape time phenomena in nonlinear systems [11], the prevention of which is ensured by employing the backstepping approach which forces error variables like $z_0$ and $z_1$ to decay faster than exponential, if needed. As we shall see, in the present problem, exponential decay of $z_0$ and $z_1$ will be sufficient to maintain global stability, which is a consequence of a specific way they enter Burgers’ equation.

To show that control (2.8) and (2.9) indeed exponentially stabilizes system (1.1)–(1.4) at least in $L^2$ norm, we define the Lyapunov function

$$V = E + \frac{6\alpha^2}{\alpha} (z_0^2 + z_1^2),
$$

(2.17)

where the energy function $E$ is defined by

$$E = \int_0^1 u^2 \, dx.
$$

(2.18)

Using (1.1)–(1.3) and integrating by parts, we obtain

$$\dot{E} = 2 \int_0^1 u(u_{xx} - uu_x) \, dx
$$

$$= 2\alpha w_1[z_1 - k(w_1 + w_1^3)] - 2\alpha w_0[z_0 + k(w_0 + w_0^3)] - 2\alpha \int_0^1 u_x^2 \, dx - \frac{2}{3} (v_1^3 - w_0^3)
$$

$$\leq \left( \frac{1}{3} + \frac{1}{6} - 2\alpha k \right) (w_0^2 + w_0^4 + w_1^2 + w_1^4) + 6\alpha^2 (z_0^2 + z_1^2) - 2\alpha \int_0^1 u_x^2 \, dx.
$$

(2.19)
Taking \( k > 1/4\epsilon \) and denoting
\[
E_1 = \int_0^1 u_t^2 \, dx + \frac{k}{2} (w_0^2 + 2w_0^4 + w_1^2 + 2w_1^4),
\]
we obtain
\[
\dot{E} \leq \left( \frac{1}{2} - 2\epsilon k \right) (w_0^2 + w_0^4 + w_1^2 + w_1^4) + 6\epsilon^2 (z_0^2 + z_1^2) - 2\epsilon \int_0^1 u_t^2 \, dx
\]
\[
\leq 6\epsilon^2 (z_0^2 + z_1^2) - \min \{ 2\epsilon k - 1/2, 2\epsilon \} \left( \int_0^1 u_t^2 \, dx + w_0^2 + w_0^4 + w_1^2 + w_1^4 \right)
\]
\[
= 6\epsilon^2 (z_0^2 + z_1^2) - \min \{ 2\epsilon k - 1/2, 2\epsilon \} \left( \int_0^1 u_t^2 \, dx + \frac{k}{2k} (2w_0^2 + w_0^4 + 2w_1^2 + w_1^4) \right)
\]
\[
\leq 6\epsilon^2 (z_0^2 + z_1^2) - \min \{ 2\epsilon k - 1/2, 2\epsilon \} \left( \int_0^1 u_t^2 \, dx + \frac{k}{2} (2w_0^2 + w_0^4 + 2w_1^2 + w_1^4) \right)
\]
(\text{note that } 1 + k > 1 \text{ and } 1 + k < \frac{1}{k})
\]
\[
\leq 6\epsilon^2 (z_0^2 + z_1^2) - \frac{1}{1 + k} \min \{ 2\epsilon k - 1/2, 2\epsilon \} \left( \int_0^1 u_t^2 \, dx + \frac{k}{2} (2w_0^2 + w_0^4 + 2w_1^2 + w_1^4) \right)
\]
\[
= 6\epsilon^2 (z_0^2 + z_1^2) - \beta E_1,
\]
where
\[
\beta = \frac{1}{1 + k} \min \{ 2\epsilon k - 1/2, 2\epsilon \} > 0.
\]
It therefore follows from (2.6) and (2.7) that
\[
\dot{\gamma} = \dot{E} + \frac{12\epsilon^2}{\alpha} (z_0 \hat{z}_0 + z_1 \hat{z}_1)
\]
\[
\leq 6\epsilon^2 (z_0^2 + z_1^2) - \beta E_1 - 12\epsilon^2 (z_0^2 + z_1^2)
\]
\[
\leq -\gamma (E_1 + z_0^2 + z_1^2),
\]
(2.23)
where
\[
\gamma = \min \{ \beta, 6\epsilon^2 \}.
\]
Inequality (2.23) shows the \( L^2 \) exponential stability. Further, the closed-loop system
\[
u_t - \alpha u_{xx} + uu_x = 0,
\]
(2.25)
\[
u_{xt} \big|_{t=0} = \left[ -\alpha (u_t - k(u + u_x^3)) + k(1 + 3u^2)(uu_{xx} - uu_x) \right] \big|_{t=0},
\]
(2.26)
\[
u_{xt} \big|_{t=1} = \left[ -\alpha (u_t + k(u + u_x^3)) - k(1 + 3u^2)(uu_{xx} - uu_x) \right] \big|_{t=1},
\]
(2.27)
\[
u \big|_{t=0} = u^0
\]
(2.28)
is \( H^3 \) stable and well posed, as stated in the following theorem.

**Theorem 2.1.** Suppose that the initial data \( u^0 \in H^3(0, 1) \) and \( k > 1/(4\epsilon) \). Let \( z_0 \) and \( z_1 \) be defined by (2.2) and (2.3), respectively. Then problem (2.25)–(2.28) has a unique global classical solution \( u \) with
\[
u \in C([0, \infty); H^3(0, 1)) \cap C^1([0, \infty); H^1(0, 1)).
Moreover, the solution satisfies the following $L^2$, $H^1$ and $H^3$ stability estimates

\[
\|u(t)\|^2 + \frac{6\gamma}{\alpha} [z_0(t)^2 + z_1(t)^2] \leq \left( \|u(t)\|^2 + \frac{6\gamma}{\alpha} [z_0(0)^2 + z_1(0)^2] \right) e^{-2\omega_0 t},
\]

\[
\|u(t)\|^2_{H^1} + u(t,0)^2 + u(t,1)^2 \leq C(\|u(0)\|^2_{H^1} + u(0)^2 + u(1)^2) \times \exp(C(\|u(0)\|^2_{H^3} + u(0)^2 + u(1)^2) e^{-\omega_1 t}),
\]

\[
\|u(t)\|^2_{H^3} \leq C(\|u(0)\|^6_{H^3}) \exp(C(\|u(0)\|^6_{H^3}) e^{-\omega_2 t}),
\]

for $t \geq 0$, where

\[
0 < \omega_0 = \frac{k\gamma}{4(1 + k) \max\{1, \frac{6\gamma}{\alpha}\}} < \varepsilon/2,
\]

\[
0 < \omega_1 < \min \left\{ \frac{\omega_0}{(3k + 8)(4\alpha(1 + k) + 6\varepsilon^2)}, \omega_2 = \min\{\omega_1, 2\varepsilon/5, c\varepsilon/5\} \right\},
\]

and $\gamma$ is defined by (2.24) and $C = C(\alpha, \varepsilon, k)$ is a positive constant.

**Remark 2.1.** In the case where $u^0 \in H^3(0,1)$, it can be seen from (2.15), (2.16) and (2.31) that the control inputs $\phi_0$ and $\phi_1$ are bounded and converge to zero exponentially as $t \to \infty$.

### 3. Proof

In this section we prove our main result. By applying Theorem 7.4 of [12, p. 491], we show that problem (2.10)–(2.14), which is equivalent to (2.25)–(2.28), is well posed. The stability estimates (2.30) and (2.31) are proved with the Lyapunov method.

**Proof of Theorem 2.1.** Step 1: *Well-posedness.* Solving ordinary differential equations (2.26) and (2.27), we obtain

\[
u_{c,0} = z_{0,0} e^{-\gamma t} + k(w_0 + w_0^3),
\]

\[
u_{c,1} = z_{1,0} e^{-\gamma t} - k(w_1 + w_1^3),
\]

where

\[
z_{0,0} = u^0(0) - k[u^0(0) + u^0(0)^3],
\]

\[
z_{1,0} = u^0(1) + k[u^0(1) + u^0(1)^3].
\]

Consequently, in the class of classical solutions, problem (2.25)–(2.28) is equivalent to

\[
u_t - \alpha u_{xx} + uu_x = 0,
\]

\[
u_{c,0} = z_{0,0} e^{-\gamma t} + k(w_0 + w_0^3),
\]

\[
u_{c,1} = z_{1,0} e^{-\gamma t} - k(w_1 + w_1^3),
\]

\[
u|_{t=0} = u^0.
\]

Set

\[
a(x, t) = x(x - 1)^2 z_{0,0} e^{-\gamma t} + x^3(x - 1)^3 z_{1,0} e^{-\gamma t}.
\]
\( \nu(x,t) = u(x,t) - a(x,t) \). \hspace{1cm} (3.10)

Then we have

\[
\begin{align*}
\nu_x|_{x=0} &= u_x|_{x=0} - a_x|_{x=0} \\
&= z_0^0 e^{-\nu t} + k(w_0 + w_3^0) - z_0^0 e^{-\nu t} \\
&= k(\nu(0,t) + \nu(0,t)^3)
\end{align*}
\]

and likewise,

\[
\nu_x|_{x=1} = -k(\nu(1,t) + \nu(1,t)^3).
\]

Hence, problem (3.5)–(3.8) becomes

\[
\begin{align*}
\nu_t - \nu_{xx} + (v + a)(\nu_x + a_x) + a_t - \nu_{xx} &= 0, \\
\nu_x|_{x=0} &= k(\nu(0,t) + \nu(0,t)^3), \\
\nu_x|_{x=1} &= -(\nu(1,t) + \nu(1,t)^3), \\
\nu|_{t=0} &= u^0 - a(x,0).
\end{align*}
\]

We can readily verify that problem (3.13)–(3.16) satisfies all conditions of Theorem 7.4 of [12, p. 491] with any positive constant \( \beta \) there (especially note that condition (7.36) on p. 487 of [12] is satisfied). Therefore we conclude from this theorem that problem (3.13)–(3.16) has a unique classical solution \( \nu \) in the class \( H^{2+\beta,1+\beta/2}(\Omega \times [0, T]) \) for suitable \( T > 0 \). Consequently, problem (3.5)–(3.8) has a unique classical solution \( u = v + a \) in the class \( H^{2+\beta,1+\beta/2}(\Omega \times [0, T]) \). Global solutions are obtained by establishing a priori estimates below.

**Step 2:** \( L^2 \)-estimate. In what follows, we denote by \( C = C(\epsilon, \sigma, k) \) a generic positive constant that may vary from line to line.

Since

\[
V \leq \frac{2(1 + k)}{k} \max \{1, 6\epsilon^2/\sigma\} (E_1 + \sigma_0^2 + \sigma_1^2),
\]

we deduce from (2.23) that

\[
\dot{V} \leq -\frac{k\gamma}{2(1 + k) \max \{1, 6\epsilon^2/\sigma\}} V,
\]

which implies that

\[
V \leq V(0)e^{-2\nu_{out}}, \hspace{1cm} t \geq 0,
\]

where \( \gamma \) and \( \nu_{out} \) are defined by (2.24) and (2.32), respectively. This proves (2.29).

**Step 3:** \( H^1 \)-estimate. On the other hand, integrating by parts and using Eqs. (3.5)–(3.7), we obtain

\[
\dot{E}_1 = 2 \int_0^1 u_x u_{xt} \, dx + 2k[(w_1^3 + w_1)\dot{w}_1 + (w_0^3 + w_0)\dot{w}_0] \\
= 2\dot{w}_1 z_1 - 2\dot{w}_0 z_0 - 2 \int_0^1 u_x(u_{xx} - uu_x) \, dx \\
\times \left( \text{note that } \max \{0 \leq x \leq 1\} u(x)^2 \leq \frac{2(1 + k)}{k} E_1 \text{ and } \int_0^1 u_{xx} u_{xt} \, dx \leq \varepsilon \int_0^1 u_{xx}^2 + \frac{1}{\sigma} \max \{0 \leq x \leq 1\} u(x)^2 \int_0^1 u_x^2 \, dx \right) \\
\leq 2\dot{w}_1 z_1 - 2\dot{w}_0 z_0 - \varepsilon \int_0^1 u_{xx}^2 \, dx + \frac{2(1 + k)}{k} E_1^2 \\
\leq 2 \frac{d}{dt}(w_1 z_1) + 2\dot{x}_1 w_1 - 2 \frac{d}{dt}(w_0 z_0) - 2\dot{x}_0 w_0 + \frac{2(1 + k)}{k} E_1^2. \hspace{1cm} (3.20)
\]
Adding $d/dt(\frac{2}{k}z_0^2)$ and $d/dt(\frac{2}{k}z_1^2)$ to both sides, we obtain

\[
\frac{d}{dt}\left(\frac{1}{2}(E_1 - \frac{k}{2}(w_0^2 + w_1^2) + \frac{2}{k}\left(\frac{k}{2}w_0 + z_0\right)^2 + \frac{2}{k}\left(\frac{k}{2}w_1 - z_1\right)^2\right) \leq 2xz_1w_1 - 2xz_0w_0 - \frac{4x}{k}(z_0^2 + z_1^2) + \frac{2(1 + k)}{\epsilon k}E_1^2 \leq C(w_0^2 + w_1^2) + \frac{2(1 + k)}{\epsilon k}E_1^2. \tag{3.21}
\]

Denoting

\[
J = E_1 - \frac{k}{2}(w_0^2 + w_1^2) + \frac{2}{k}\left(\frac{k}{2}w_0 + z_0\right)^2 + \frac{2}{k}\left(\frac{k}{2}w_1 - z_1\right)^2 = \int_0^1 (u_0^2 + \frac{k}{2}(w_0^2 + w_0^2 + w_1^2 + w_1^2) + \frac{2}{k}\left(\frac{k}{2}w_0 - kw_0\right)^2 + \left(u_1|_{x=0} - \frac{k}{2}w_0 - kw_0\right)^2 + \left(u_1|_{x=0} + \frac{k}{2}w_1 + kw_1\right)^2).
\tag{3.22}
\]

we get

\[
J \leq C(w_0^2 + w_1^2) + \frac{2(1 + k)}{\epsilon k}E_1^2. \tag{3.23}
\]

It is clear that

\[
\frac{1}{2}E_1 \leq J \leq \frac{3}{2}E_1 + \frac{4}{k}(z_0^2 + z_1^2) \leq \frac{3}{2k}(E_1 + z_0^2 + z_1^2). \tag{3.24}
\]

For $M > 0$ yet to be determined, we define the Lyapunov function $G$ as

\[
G = MV + \ln\left(1 + J + \frac{6\epsilon^2}{\alpha}(z_0^2 + z_1^2)\right), \tag{3.25}
\]

which, in view of (3.22) and (3.24), is positive definite and radially unbounded in the argument of $\|u\|_{l_1}$, $u_0(0)$ and $u_0(1)$. It is also easy to see that

\[
G \leq MV + J + \frac{6\epsilon^2}{\alpha}(z_0^2 + z_1^2)
\leq 2M(1 + k)E_1 + \frac{6Mc^2}{\alpha}(z_0^2 + z_1^2) + J + \frac{6\epsilon^2}{\alpha}(z_0^2 + z_1^2)
\leq \frac{4M(1 + k)}{k}J + \frac{6Mc^2}{\alpha}(z_0^2 + z_1^2) + J + \frac{6\epsilon^2}{\alpha}(z_0^2 + z_1^2)
\leq \frac{4M(1 + k) + k}{k}J + \frac{6\epsilon^2(M + 1)}{\alpha}(z_0^2 + z_1^2)
\leq \frac{2(4M(1 + k) + k) + 6\epsilon^2(M + 1)}{k\alpha}(J + z_0^2 + z_1^2). \tag{3.26}
\]
It then follows from (2.23), (3.23) and (3.24) that for $M > 16(1+k)(3k+8)/2\xi k^2$
\[
\dot{G} = MV + \frac{J + (12\xi^2/\alpha)(z_0\dot{z}_0 + z_1\dot{z}_1)}{1 + J + (6\xi^2/\alpha)(z_0^2 + z_1^2)}
\]
(use (3.23) and note that $z_0\dot{z}_0 + z_1\dot{z}_1 = -\alpha(z_0^2 + z_1^2)$ is negative)
\[
\leq MV + \frac{C(w_0^2 + w_1^2) + [2(1+k)/\varepsilon k]E_1^2}{1 + J + (6\xi^2/\alpha)(z_0^2 + z_1^2)}
\]
(use (2.23) and the first inequality of (3.24))
\[
\leq -M\gamma(E_1 + z_0^2 + z_1^2) + \frac{8(1+k)}{\varepsilon k} J + C(w_0^2 + w_1^2)
\]
\[
\leq -\frac{M\gamma}{2} (E_1 + z_0^2 + z_1^2) + \frac{M\gamma}{2} (z_0^2 + z_1^2) + \frac{8(1+k)}{\varepsilon k} J + \left( C - \frac{M\gamma}{2} \right) (w_0^2 + w_1^2)
\]
\[
\leq -\frac{2kM\gamma}{2(3k+8)} J - \frac{M\gamma}{2} (z_0^2 + z_1^2) + \frac{8(1+k)}{\varepsilon k} J + \left( C - \frac{M\gamma}{2} \right) (w_0^2 + w_1^2)
\]
\[
\leq -\left( \frac{2kM\gamma}{2(3k+8)} - \frac{8(1+k)}{\varepsilon k} \right) (J + z_0^2 + z_1^2) + \left( C - \frac{M\gamma}{2} \right) (w_0^2 + w_1^2)
\]
\[
\leq -\frac{k\xi}{\alpha(4M(1+k) + k)} \left( \frac{2kM\gamma}{2(3k+8)} - \frac{8(1+k)}{\varepsilon k} \right) G
\]
\[
+ \left( C - \frac{M\gamma}{2} \right) (w_0^2 + w_1^2).
\]
(3.27)

Since the function
\[
f(M) = \frac{k\xi}{\alpha(4M(1+k) + k)} \left( \frac{2kM\gamma}{2(3k+8)} - \frac{8(1+k)}{\varepsilon k} \right)
\]
is increasing and
\[
\lim_{M \to \infty} f(M) = \frac{k^2\xi\gamma}{(3k+8)(4\xi(1+k) + 6\xi^2)},
\]
we can find an $M_0$ sufficiently large such that $\omega_1 < f(M_0)$ and $C - M_0k\gamma/2 < 0$, where $\omega_1$ is defined by (2.33). Consequently, we obtain
\[
\dot{G} \leq -\omega_1 G.
\]
(3.28)

Hence, we have
\[
G \leq G(0)e^{-\omega_1 t}, \quad t \geq 0.
\]
(3.29)

It therefore follows that
\[
1 + J + \frac{6\xi^2}{\alpha} (z_0^2 + z_1^2) \leq \exp(G(0)e^{-\omega_1 t}),
\]
and then
\[
J + \frac{6\xi^2}{\alpha} (z_0^2 + z_1^2) \leq \exp(G(0)e^{-\omega_1 t}) - 1
\]
\[
\leq CG(0) \exp(G(0))e^{-\omega_1 t}
\]
\[
\leq C(||u^0||_{H^1}^2 + ||u^0||_{H^1}^6 + u^0(0)^2 + u^0(1)^2)
\]
\[
\times \exp(C(||u^0||_{H^1}^2 + ||u^0||_{H^1}^6 + u^0(0)^2 + u^0(1)^2))e^{-\omega_1 t}, \quad t \geq 0,
\]
(3.30)
which implies (2.30) since
\[ \|u_t^0\|_{H^1}^2 + \|u_t^0\|_{H^1}^6 + u_t^0(0)^2 + u_t^0(1)^2 \leq \|u_t^0\|_{H^1}^2 + u_t^0(0)^2 + (\|u_t^0\|_{H^1}^2 + u_t^0(1)^2)^3 \leq C(\|u_t^0\|_{H^1}^2 + u_t^0(0)^2 + u_t^0(1)^2) \exp(\|u_t^0\|_{H^1}^2 + u_t^0(0)^2 + u_t^0(1)^2) \leq eC(\|u_t^0\|_{H^1}^2 + u_t^0(0)^2 + u_t^0(1)^2) \exp(\|u_t^0\|_{H^1}^2 + u_t^0(0)^2 + u_t^0(1)^2) \leq eC(\|u_t^0\|_{H^1}^2 + u_t^0(0)^2 + u_t^0(1)^2) \exp(\|u_t^0\|_{H^1}^2 + u_t^0(0)^2 + u_t^0(1)^2) \] (3.31)
and
\[ \exp(C(\|u_t^0\|_{H^1}^2 + \|u_t^0\|_{H^1}^6 + u_t^0(0)^2 + u_t^0(1)^2)) \leq \exp(C(1 + (\|u_t^0\|_{H^1}^2 + u_t^0(0)^2 + u_t^0(1)^2))) \leq e^C \exp(C(\|u_t^0\|_{H^1}^2 + u_t^0(0)^2 + u_t^0(1)^2)^3). \] (3.32)

**Step 4: \( H^2 \)-estimate.** In order to obtain the \( H^2 \)-estimate, we estimate the \( L^2 \) norm of \( u_t \). Integrating by parts and using Eqs. (3.5)–(3.7), we obtain
\[
\frac{d}{dt} \int_0^1 u_t^2 \, dx = 2 \int_0^1 u_t(u_{ttx} - u_t u_x - uu_{xx}) \, dx \\
\leq -2\alpha k[w_0^2(3w_0^2 + 1) + \bar{w}_1^2(3w_1^2 + 1)] + 2\alpha \bar{w}_1 \bar{z}_1 - 2\alpha \bar{w}_0 \bar{z}_0 - 2\varepsilon \|u_t\|^2 + C\|u_t\|_{H^1} \|u_t\|_{H^1} \\
= -2\alpha k[w_0^2(3w_0^2 + 1) + \bar{w}_1^2(3w_1^2 + 1)] \\
-2\alpha \bar{w}_1 \bar{z}_1 + 2\alpha \bar{w}_0 \bar{z}_0 \quad \text{(use (2.6) and (2.7))} \\
-2\varepsilon \|u_t\|^2 + C\|u_t\|_{H^1} \|u_t\|_{H^1} \\
(\text{note definition (2.34) of } \omega_2) \\
\leq -\frac{1}{2} \alpha k[w_0^2(3w_0^2 + 1) + \bar{w}_1^2(3w_1^2 + 1)] + C(z_1^2 + z_0^2) - 5\omega_2 \|u_t\|_{H^1}^2 + C\|u_t\|_{H^1} \|u_t\|_{H^1} \\
\leq -\frac{1}{2} \alpha k[w_0^2(3w_0^2 + 1) + \bar{w}_1^2(3w_1^2 + 1)] + C(z_1^2 + z_0^2) - 4\omega_2 \|u_t\|_{H^1}^2 + C\|u_t\|_{H^1} \|u_t\|_{H^1} \\
\leq -\frac{1}{2} \alpha k[w_0^2(3w_0^2 + 1) + \bar{w}_1^2(3w_1^2 + 1)](z_1^2 + z_0^2) - 2\omega_2 \|u_t\|^2 + C\|u_t\|_{H^1} \|u_t\|_{H^1}. \] (3.33)

Multiplying (3.33) by \( e^{2\alpha z_t} \) gives
\[
\frac{d}{dt}(\|u_t\|^2 e^{2\alpha z_t}) \leq C(z_1^2 + z_0^2) e^{2\alpha z_t} + C\|u_t\|^2 \|u_t\|^2 e^{2\alpha z_t} \\
\leq C(z_1^2 + z_0^2) e^{2\alpha z_t} + C\|u_t\|^2 \|u_t\|^2 e^{2\alpha z_t}. \] (3.34)

It, therefore, follows from (2.29) that
\[
\|u_t\|^2 e^{2\alpha z_t} \leq \|u_t(0)\|^2 + C(\|u_t^0\|_{H^2}^2 + \|u_t^0\|_{H^2}^6) + C \int_0^t \|u_t\|^2 \|u_t\|^2 e^{2\alpha z_s} \, ds, \]
and then by (2.29) and Gronwalls inequality we obtain
\[
\|u_t\|^2 \leq C(\|u_t^0\|_{H^2}^2 + \|u_t^0\|_{H^2}^6) \exp(C(\|u_t^0\|_{H^2}^2 + \|u_t^0\|_{H^2}^6)) e^{-2\alpha z_t}, \quad t \geq 0. \] (3.35)

Since
\[
\|u_{xx}\|^2 \leq C(\|u_t\|^2 + \|u_t\|^4), \] (3.36)
it follows from (3.30), (3.35) and (3.36) that
\[
\|u_{xx}\|^2 \leq C(\|u_t^0\|_{H^2}^2 + \|u_t^0\|_{H^2}^6) \exp(C(\|u_t^0\|_{H^2}^2 + \|u_t^0\|_{H^2}^6)) e^{-2\alpha z_t}, \quad t \geq 0. \] (3.37)
Adding d

We now estimate

Thus, we deduce from (3.40) that

It, therefore, follows that

\[ \int \frac{d}{dt} (\sum_{i=0}^{1} \xi_{i} w_{i}^{2}) = C \| u \|^{4} + C \| u \|^{2} \| u_{\sigma} \|^{2}. \]

Thus, we deduce from (3.40) that

\[ \hat{J} \leq C \sum_{i=0}^{1} \xi_{i} (w_{i}^{2} + 1) + C \| u \|^{4} + C(\| u \|^{2} + \| u \|_{H^{1}}^{2}) \hat{J}. \]
On the other hand, multiplying (3.33) by $e^{2t}$, one obtains
\[
\frac{d}{dt} (\|u_t\| e^{2t}) + \frac{1}{2} k \left[ \dot{w}_0^2(3w_0^2 + 1) + \dot{w}_1^2(3w_1^2 + 1) \right] e^{2t} \\
\leq C(\varepsilon_1^2 + \varepsilon_0^2) e^{2t} + C \|u\| H^1_0 \|u_t\| e^{2t} \\
\leq C(\varepsilon_1^2 + \varepsilon_0^2) e^{2t} + C \|u\| H^1_0 \|u_t\|^2 e^{2t}.
\]
(3.45)

Integrating from 0 to $\infty$ and using (3.30) and (3.35), one obtains
\[
\int_0^\infty \sum_{i=0}^1 \dot{w}_i^2(w_i^2 + 1)e^{2t} \, dt \leq C(\|u_0\|_{H^1}^2 + \|u^0\|_{H^2}^6).
\]
(3.46)

It, therefore, follows from Gronwall’s inequality (see, e.g., [9, p. 63]) that
\[
J_t \leq C(\|u_0\|_{H^1}^2 + \|u^0\|_{H^2}^{12}) \exp(C(\|u_0\|_{H^1}^2 + \|u^0\|_{H^2}^6)) e^{-\varepsilon_0 t}, \quad t \geq 0.
\]
(3.47)

Since
\[
\|u_{xx}\|^2 \leq C(\|u_t\|^2 + \|u^0\|_{H^2}^4)
\]
(3.48)

(2.31) follows from (2.30), (3.37) and (3.47) with the use of (3.31) and (3.32). $\square$

4. Conclusion

In summary, we propose here a backstepping boundary control law for Burgers’ equation with actuator dynamics, which depends not only on the signals $u(0,t)$ and $u(1,t)$, but also on $u_t(0,t)$, $u_t(1,t)$, $u_{xx}(0,t)$ and $u_{xx}(1,t)$. With elaborate Lyapunov analysis, and applying a result from the classical theory of nonlinear partial differential equations of parabolic type, we prove that the controlled closed-loop system is globally $H^3$ stable and has a unique global classical solution.

As we pointed out in the introduction, system (1.1)–(1.3) is only an example of a more general class of boundary control problems (yet to be investigated) tractable by backstepping. To give a preview of possible future extensions, we offer the system
\[
u_t - \varepsilon u_{xx} + uu_x = 0,
\]
(4.1)

\[
u_{x|1} = f_{01} \left( u_{|x=0}, u_x_{|x=0}, u_{xx|1} = 0, \int_0^1 u \, dx \right) + \tilde{\xi}_0,
\]
(4.2)

\[
u_{x|1} = f_{11} \left( u_{|x=1}, u_x_{|x=1}, u_{xx|1} = 1, \int_0^1 u \, dx \right) + \tilde{\xi}_1,
\]
(4.3)

\[
\tilde{\xi}_0 = f_{02} \left( u_{|x=0}, u_x_{|x=0}, u_{xx|0} = 0, \int_0^1 u \, dx, \tilde{\xi}_0 \right) + \varphi_0,
\]
(4.4)

\[
\tilde{\xi}_1 = f_{12} \left( u_{|x=1}, u_x_{|x=1}, u_{xx|1} = 1, \int_0^1 u \, dx, \tilde{\xi}_1 \right) + \varphi_1,
\]
(4.5)

where $f_{ij}$ are smooth functions and $\varphi_0, \varphi_1$ are controls. The variables $\tilde{\xi}_0, \tilde{\xi}_1$ serve as “virtual controls” to the $u|_{x=0}, u|_{x=1}$ equations, effectively separating the inputs $\varphi_0, \varphi_1$ from the Burgers equation but not one but two cascaded integrators (on each end of the $[0,1]$ interval). After designing the control laws in two steps of backstepping, one would need $H^3(0,1)$ estimates on $u$ to show regularity of the control signals. Adding more integrators would be standard insofar as the backstepping design task is concerned, however, regularity analysis would require estimates of increasingly high order.
References