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PRESCRIBED-TIME MEAN-NONOVERSHOOTING CONTROL UNDER FINITE-TIME VANISHING NOISE*

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Abstract. We develop a prescribed-time mean-nonovershooting stabilizing feedback law for stochastic nonlinear systems with noise that vanishes in finite time. The prescribed time of stabilization must be strictly after the noise vanishes, but it may occur as early as the user desires after the noise vanishes. In fact, more generally, the feedback is stabilizing and mean-overshoot free for any noise that, given the prescribed time of stabilization, satisfies certain decay rate properties which require, in particular, that no noise component vanish slower than linearly in the "time to go" until the prescribed time. In contrast to the existing stochastic prescribed-time designs where only multiplicative noise is allowed, our design can deal with multiplicative and additive noise simultaneously. A new controller is designed to guarantee that the mean of the system output prescribed time tracks a given trajectory without overshooting, that the fourth moment of the tracking error between states and derivatives of the reference trajectory converges to zero in prescribed time, and that the controller and all of the states are mean-square bounded. Finally, a simulation example is given to illustrate the prescribed-time mean-nonovershooting design.

Key words. prescribed-time, mean-nonovershooting, multiplicative and additive noise

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1. Introduction.

Safe control of nonlinear systems is challenging due to the tight coupling between potentially conflicting control objectives and safety constraints. References [1, 2, 3, 19, 21, 27, 29] use control barrier functions (CBFs) to characterize the long-term safety of dynamical systems. A CBF certifies whether a control scheme achieves forward invariance of a safe set by checking whether the system trajectory remains away from the boundary of this set.

As demonstrated by [4, 20], overshoot is undesirable in many practical safety control problems. For example, the optimum setpoint may be close to an economic or safety constraint in process control problems (e.g., product quality constraints, metallurgical limits). Consequently, overshoot of a setpoint could lead to violation of a constraint and endanger process operation. For nonlinear systems, [11] is the first paper to consider the nonovershooting safety control where a modified backstepping method is employed to guarantee a nonovershooting response for strict-feedback nonlinear systems. The nonovershooting design possesses all of the attributes of a safety design with a CBF of a uniform and high relative degree [1].

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Recently, [12] generalized the results in [11] to the stochastic setting. The safe control scheme in [12] guarantees that the mean of the output asymptotically tracks a given trajectory without any overshooting while keeping all of the states mean-square bounded. In many real-world applications, the control tasks require the given trajectory to be prescribed-time tracked, rather than asymptotically tracked, by the mean of the output without any overshooting. Thus, it is important to study the prescribed-time mean-nonovershooting control for stochastic nonlinear systems.

Prescribed-time control is popular in applications where there exists a short, finite amount of time remaining to achieve the control objective. The appeal of such control is that it allows the user to prescribe the convergence time a priori and irrespective of initial conditions. Many prescribed-time control results have recently been developed for systems with different structures. For deterministic systems, [22, 23] provide prescribed-time designs for robust regulation of normal-form nonlinear systems; [26] investigates the prescribed finite time consensus control for multiagent systems; [5, 6] address the prescribed-time observer design and output-feedback design for linear systems; [8, 9, 10] focus on the prescribed-time stabilization for uncertain nonlinear strict-feedback-like systems; and [24, 25] consider prescribed-time designs for the Schrödinger equation and reaction-diffusion equations, respectively. When it comes to stochastic systems, [15] is the first paper on prescribed-time control to solve stochastic nonlinear prescribed-time stabilization and inverse optimality problems; [13] presents a different design to reduce the control effort; and [14] introduces prescribed-time output-feedback designs for stochastic nonlinear systems without and with sensor uncertainty. We stress that the stochastic prescribed-time designs in [13, 14, 15] only focus on stochastic prescribed-time stabilization, without considering the prescribed-time mean-nonovershooting and safety. Although [1] develops a prescribed-time safety design for a chain of integrators, the system considered is deterministic and disturbance free. There exist no analogous results for prescribed-time mean-nonovershooting control for stochastic nonlinear systems.

Motivated by the above observations, we develop prescribed-time mean- nonovershooting designs for stochastic nonlinear systems where the matched noise should be at least linearly vanishing. The contributions of this paper are threefold:

- We propose a new mean-nonovershooting design framework for stochastic nonlinear systems. The design in this paper is completely different from the mean-nonovershooting design in [12] in the following two aspects: First, our controller is characterized by a time-varying blow-up function that grows unbounded towards the terminal time, which makes the control design and safety analysis in this paper much more difficult than those in [12]. Second, unlike [12] where the mean of the output can track a given trajectory asymptotically without overshooting, our design has the clear advantage of achieving mean-nonovershooting tracking in prescribed-time, and not merely asymptotically.
- In contrast to the stochastic prescribed-time designs in [13, 14, 15], we consider more general models and present a new design scheme. On the one hand, the control schemes in [13, 14, 15] are only effective for systems with multiplicative noise, but the design in this paper can deal with multiplicative and additive noise simultaneously. On the other hand, different from the stochastic prescribed-time designs in [13, 14, 15] for stabilization, the control objective of this paper is prescribed-time mean-nonovershooting tracking of a time-varying reference trajectory. In our prescribed-time mean-nonovershooting design, many time-varying nonlinear terms produced by the time-varying

reference trajectory and the time-varying blow-up function are absorbed into the virtual controllers, which makes the design in this paper much more complex than the designs in [13, 14, 15].

- In contrast to [1, 2, 3, 19, 21, 27, 29] where control barrier functions, "nominal" control input, and "safety filter" are designed to achieve safety control objective, we propose a different safety control scheme for stochastic systems. In our scheme, there is no need for a "safety filter" redesign, and only a nominal controller is designed to ensure that the mean of the system output can track a given trajectory "from below" in prescribed time without overshooting while keeping all of the states mean-square bounded. In other words, although our design is different from those in [1, 2, 3, 19, 21, 27, 29], our design is not inferior in terms of safety.

The remainder of this paper is organized as follows. Section 2 presents problem formulation. Section 3 is devoted to the prescribed-time mean-nonovershooting control design. Section 4 focuses on prescribed-time safety analysis. Section 5 gives an example illustrating the theoretical results. Section 6 includes concluding remarks. Appendices A–G introduces useful tools and the proofs of Proposition 1, Lemmas 1–4 and Theorem 1.

2. Problem formulation. Consider a class of stochastic nonlinear systems described by

$$dx_i = x_{i+1}dt + \varphi_i^T(t,x)d\omega, i = 1,\ldots,n-1, \tag{2.1}$$

$$dx_n = (u + \phi(t,x))dt + \varphi_n^T(t,x)d\omega, \tag{2.2}$$

$$y = x_1, \tag{2.3}$$

where $x = (x_1,\ldots,x_n)^T \in R^n$, $u \in R$, and $y \in R$ are the system state, control input, and output, respectively. The functions $\phi : R^+ \times R^n \to R$ and $\varphi_i : R^+ \times R^n \to R^m$ are continuous of their arguments and locally Lipschitz in x, and $i = 1,\ldots,n$. ω is an m-dimensional independent standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathcal{F}_0$ contains all P-null sets).

We introduce the following two functions:

$$\mu(t) = \left(\frac{T}{t_0 + T - t}\right)^2, \quad t \in [t_0, t_0 + T), \tag{2.4}$$

$$\nu(t) = 1 - \frac{t - t_0}{T}, \quad t \in [t_0, t_0 + T), \tag{2.5}$$

where $T > 0$ is the freely prescribed time.

Obviously, the blow-up function $\mu(t)$ is a monotonically increasing function on $[t_0, t_0+T)$ with $\mu(t_0) = 1$ and $\lim_{t\to t_0+T} \mu(t) = +\infty$; $\nu(t)$ is a monotonically decreasing function on $[t_0, t_0+T)$ with $\nu(t_0) = 1$ and $\lim_{t\to t_0+T} \nu(t) = 0$ (In this paper, $\lim_{t\to t_0+T}$ means t approaches $t_0 + T$ "from the left" or "from below").

For system (2.1)–(2.3), we make the following assumptions.

Assumption 1. The given trajectory $y_r(t) \in R$ and its derivatives $y_r^{(1)}(t),\ldots, y_r^{(n-1)}(t)$ exist. Moreover, there is a known positive constant M such that

$$|y_r^{(i)}(t)| \leq M,\ i = 0,\ldots,n-1,\ \forall t \in [t_0, +\infty), \tag{2.6}$$

where $y_r^{(0)}(t) = y_r(t)$.

Assumption 2. There exist nonnegative constants $c_{0,1}$, $c_{0,2}$, $c_{i,1}$, $c_{i,2}$, and σ_i $(i=1,\dots,n)$ such that

$$|\phi(t,x)| \leq c_{0,1}(|x_1|+\cdots+|x_n|)+c_{0,2} \quad \forall t\in[t_0,t_0+T), \tag{2.7}$$

$$|\varphi_i(t,x)| \leq (c_{i,1}(|x_1|+\cdots+|x_i|)+c_{i,2})\gamma_i(t) \quad \forall t\in[t_0,t_0+T), \tag{2.8}$$

where $\gamma_i(t)$ satisfies

$$0 \leq \gamma_i(t) \leq \nu^{\sigma_i}(t) \tag{2.9}$$

with

$$\sigma_i \geq 2n-2i+1. \tag{2.10}$$

Remark 1. From Assumption 2, the noise in system (2.1)–(2.3) is allowed to be additive and multiplicative simultaneously. In the existing results on stochastic prescribed-time control [13, 14, 15], the noise is only allowed to be multiplicative. From this aspect, we consider more general systems for stochastic prescribed-time design in this paper. In contrast to the stochastic prescribed-time stabilization problems in [13, 14, 15] and due to the effect of the time-varying reference trajectory $y_r(t)$ and the additive noise, in order to solve the prescribed-time mean-nonovershooting control problem, we require the noise to be vanishing in prescribed time T. More specifically, it can be observed from Assumption 2 that $\sigma_n \geq 1$, which means, along with (2.5), that the most slowly vanishing noise in the system (i.e., the matched noise) should vanish at a rate that is no slower than linear in the "time to go" given by t_0+T-t.

The objective of this paper is, with Assumptions 1–2, to design a prescribed-time mean-nonovershooting controller for system (2.1)–(2.3), such that the closed-loop system has an almost surely unique solution on $[t_0,t_0+T)$, the mean of the system output can prescribed-time track a given trajectory without overshooting, the fourth moment of the tracking error between states and derivatives of the reference trajectory converges to zero in prescribed time, and the control input and all of the states are mean-square bounded.

Remark 2. How can the requirement that the noise vanish no later than the time t_0+T (Assumption 2) be motivated? In real-world applications, an important scenario is where the noise is vanishing by some time $t_0+T_{\max}$, the time $T_{\max}>0$ is known, and we are able to pick any time T for prescribed stabilization, with T strictly greater than $T_{\max}$. In this case, Assumption 2 is modified as follows.

Assumption 2′. Let $T>T_{\max}>0$. There exist nonnegative constants $c_{0,1}$, $c_{0,2}$, $c_{i,1}$, and $c_{i,2}$ such that

$$|\phi(t,x)| \leq c_{0,1}(|x_1|+\cdots+|x_n|)+c_{0,2} \quad \forall t\in[t_0,t_0+T), \tag{2.11}$$

$$|\varphi_i(t,x)| \leq (c_{i,1}(|x_1|+\cdots+|x_i|)+c_{i,2})\gamma_i(t) \quad \forall t\in[t_0,t_0+T), \tag{2.12}$$

where

$$\gamma_i(t) = \begin{cases} 1, & t_0 \leq t < t_0+T_{\max}, \\ 0, & t_0+T_{\max} \leq t < t_0+T. \end{cases} \tag{2.13}$$

Next, we prove that Assumption 2′ is a special case of Assumption 2.

Choosing

$$m_i = \frac{T^{\sigma_i}}{(T-T_{\max})^{\sigma_i}}, \tag{2.14}$$

we have

$$\gamma_i(t) \leq m_i \nu^{\sigma_i}(t) \quad \forall t \in [t_0, t_0+T). \tag{2.15}$$

By (2.15), (2.12) can be rewritten as

$$|\varphi_i(t,x)| \leq (c_{i,1} m_i(|x_1| + \cdots + |x_i|) + c_{i,2} m_i)\nu^{\sigma_i}(t), \tag{2.16}$$

which shows that Assumption 2 can be satisfied easily by satisfying the simple and practically reasonable, albeit more conservative, Assumption 2′.

3. Prescribed-time mean-nonovershooting control design. In this section, we design the prescribed-time mean-nonovershooting controller step by step.

Step 1. In this step, we aim to design the virtual controller x_2^*.

Defining

$$\xi_1 = x_1 - y_r, \tag{3.1}$$

$$V_1 = \frac{1}{4}\xi_1^4, \tag{3.2}$$

by (2.1) and (3.1)–(3.2) we get

$$\mathcal{L}V_1 = \xi_1^3(x_2 - \dot{y}_r) + \frac{3}{2}\xi_1^2|\varphi_1|^2. \tag{3.3}$$

It follows from (2.5)–(3.1) and Lemma A.1 that

$$\begin{aligned}
\frac{3}{2}\xi_1^2|\varphi_1|^2 &\leq \frac{3}{2}\nu^{2\sigma_1}\xi_1^2(c_{1,1}|x_1| + c_{1,2})^2 \\
&\leq \frac{3}{2}\nu^{2\sigma_1}\xi_1^2(c_{1,1}|\xi_1| + c_{1,1}|y_r| + c_{1,2})^2 \\
&\leq 3\nu^{2\sigma_1}\xi_1^2\left(c_{1,1}^2\xi_1^2 + (c_{1,1}M + c_{1,2})^2\right) \\
&\leq \left(3c_{1,1}^2 + \frac{9}{4\beta_1}(c_{1,1}M + c_{1,2})^4\mu\right)\xi_1^4 + \beta_1\nu^{4\sigma_1+2} \\
&\leq \left(3c_{1,1}^2 + \frac{9}{4\beta_1}(c_{1,1}M + c_{1,2})^4\right)\mu\xi_1^4 + \beta_1\nu^{4\sigma_1+2},
\end{aligned} \tag{3.4}$$

where β_1 is an arbitrary positive constant.

Substituting (3.4) into (3.3) yields

$$\begin{aligned}
\mathcal{L}V_1 \leq\ & \xi_1^3(x_2 - x_2^*) + \xi_1^3(x_2^* - \dot{y}_r) + \left(3c_{1,1}^2 + \frac{9}{4\beta_1}(c_{1,1}M + c_{1,2})^4\right)\mu\xi_1^4 \\
& + \beta_1\nu^{4\sigma_1+2}.
\end{aligned} \tag{3.5}$$

If we choose the virtual controller as

$$\alpha_1 = c_1 + 3c_{1,1}^2 + \frac{9}{4\beta_1}(c_{1,1}M + c_{1,2})^4, \tag{3.6}$$

$$x_2^* = \dot{y}_r - \mu\alpha_1\xi_1, \tag{3.7}$$

and define

$$\xi_2 = x_2 - x_2^*, \tag{3.8}$$

then (3.5) can be rewritten as

$$\mathcal{L}V_1 \leq -c_1\mu\xi_1^4 + \xi_1^3\xi_2 + \beta_1\nu^{4\sigma_1+2}, \tag{3.9}$$

where $c_1 > 0$ is a design parameter.

From (2.1), (3.7), and (3.8) we get

$$d\xi_1 = \big(-\alpha_1\mu\xi_1 + \xi_2\big)dt + \varphi_1^T d\omega. \tag{3.10}$$

Step 2. In this step, we aim to design the virtual controller x_3^*.

From (3.7)–(3.8) we get

$$\xi_2 = x_2 - \dot{y}_r + \mu\alpha_1\xi_1. \tag{3.11}$$

It can be inferred from (2.1) and (3.11) that

$$d\xi_2 = \Big(x_3 - \ddot{y}_r + \Big(\frac{2}{T}\mu^{3/2}\alpha_1 - \alpha_1^2\mu^2\Big)\xi_1 + \alpha_1\mu\xi_2\Big)dt + \Big(\varphi_2^T + \alpha_1\mu\varphi_1^T\Big)d\omega. \tag{3.12}$$

Choose the new scaled Lyapunov function

$$V_2 = V_1 + \frac{1}{4\mu^4}\xi_2^4. \tag{3.13}$$

By (3.9), (3.12), and (3.13) we get

$$\begin{aligned}\mathcal{L}V_2 \leq& -c_1\mu\xi_1^4 + \xi_1^3\xi_2 - \frac{2}{T}\mu^{-7/2}\xi_2^4 + \frac{3}{2\mu^4}\xi_2^2|\varphi_2 + \mu\alpha_1\varphi_1|^2 + \beta_1\nu^{4\sigma_1+2}\\ &+ \frac{1}{\mu^4}\xi_2^3\Big(x_3 - \ddot{y}_r + \Big(\frac{2}{T}\mu^{3/2}\alpha_1 - \alpha_1^2\mu^2\Big)\xi_1 + \alpha_1\mu\xi_2\Big).\end{aligned} \tag{3.14}$$

By Lemma A.1 we obtain

$$\xi_1^3\xi_2 \leq \varepsilon_{2,1,1}\mu\xi_1^4 + \frac{27}{256\varepsilon_{2,1,1}^3\mu^3}\xi_2^4, \tag{3.15}$$

where $\varepsilon_{2,1,1}$ is an arbitrary positive constant.

From (3.1) and (3.11) we have

$$|x_1| \leq |\xi_1| + M, \tag{3.16}$$

$$|x_2| \leq \alpha_1\mu|\xi_1| + |\xi_2| + M. \tag{3.17}$$

It follows from (2.8), (2.9), and (3.16)–(3.17) that

$$\begin{aligned}|\varphi_2 + \mu\alpha_1\varphi_1|^2 &\leq 2\nu^{2\sigma_2}(c_{2,1}(|x_1| + |x_2|) + c_{2,2})^2 + 2\alpha_1^2\nu^{2\sigma_1-4}(c_{1,1}|x_1| + c_{1,2})^2\\ &\leq \bar{a}_{2,1}\nu^{2\sigma_2}(\mu^2\xi_1^2 + \xi_2^2 + M^2 + 1) + \bar{a}_{2,2}\nu^{2\sigma_1-4}(\xi_1^2 + M^2 + 1),\end{aligned} \tag{3.18}$$

where $\bar{a}_{2,1}$ and $\bar{a}_{2,2}$ are positive constants.

From Lemma A.1 we have

$$\begin{aligned}\frac{3}{2\mu^4}(\bar{a}_{2,1}\nu^{2\sigma_2}\mu^2 + \bar{a}_{2,2}\nu^{2\sigma_1-4})\xi_1^2\xi_2^2 &\leq \frac{3}{2\mu^2}(\bar{a}_{2,1} + \bar{a}_{2,2})\xi_1^2\xi_2^2\\ &\leq \varepsilon_{2,1,2}\mu\xi_1^4 + \frac{9}{16\varepsilon_{2,1,2}\mu^3}(\bar{a}_{2,1} + \bar{a}_{2,2})^2\xi_2^4,\end{aligned} \tag{3.19}$$

$$\frac{3\bar{a}_{2,2}}{2\mu^4}(M^2+1)\nu^{2\sigma_1-4}\xi_2^2 \le \beta_{2,1}\nu^{4\sigma_1+2} + \frac{9\bar{a}_{2,2}^2}{16\beta_{2,1}\mu^3}(M^2+1)^2\xi_2^4, \tag{3.20}$$

$$\frac{3\bar{a}_{2,1}}{2\mu^4}(M^2+1)\nu^{2\sigma_2}\xi_2^2 \le \beta_{2,2}\nu^{4\sigma_2+10} + \frac{9\bar{a}_{2,1}^2}{16\beta_{2,2}\mu^3}(M^2+1)^2\xi_2^4, \tag{3.21}$$

where $\varepsilon_{2,1,2}$, $\beta_{2,1}$, and $\beta_{2,2}$ are arbitrary positive constants.

From (3.18)–(3.21) we have

$$\begin{aligned}\frac{3}{2\mu^4}\xi_2^2|\varphi_2+\mu\alpha_1\varphi_1|^2 \le{}& \varepsilon_{2,1,2}\mu\xi_1^4 + \Big(\frac{3}{2}\bar{a}_{2,1} + \frac{9}{16\varepsilon_{2,1,2}}(\bar{a}_{2,1}+\bar{a}_{2,2})^2 \\ &+ \frac{9\bar{a}_{2,2}^2}{16\beta_{2,1}}(M^2+1)^2 + \frac{9\bar{a}_{2,1}^2}{16\beta_{2,2}}(M^2+1)^2\Big)\frac{1}{\mu^3}\xi_2^4 \\ &+ \beta_{2,1}\nu^{4\sigma_1+2} + \beta_{2,2}\nu^{4\sigma_2+10}.\end{aligned} \tag{3.22}$$

Substituting (3.15) and (3.22) into (3.14) yields

$$\begin{aligned}\mathcal{L}V_2 \le{}& -(c_1-\varepsilon_{2,1})\mu\xi_1^4 + \Big(\frac{3}{2}\bar{a}_{2,1} + \frac{27}{256\varepsilon_{2,1,1}^3} + \frac{9}{16\varepsilon_{2,1,2}}(\bar{a}_{2,1}+\bar{a}_{2,2})^2 \\ &+ \frac{9\bar{a}_{2,2}^2}{16\beta_{2,1}}(M^2+1)^2 + \frac{9\bar{a}_{2,1}^2}{16\beta_{2,2}}(M^2+1)^2\Big)\frac{1}{\mu^3}\xi_2^4 + \frac{1}{\mu^4}\xi_2^3\Big(x_3 - x_3^* \\ &+ x_3^* - \ddot{y}_r + \alpha_1\mu\xi_2 + \Big(\frac{2}{T}\mu^{3/2}\alpha_1 - \alpha_1^2\mu^2\Big)\xi_1\Big) + (\beta_1+\beta_{2,1})\nu^{4\sigma_1+2} \\ &+ \beta_{2,2}\nu^{4\sigma_2+10},\end{aligned} \tag{3.23}$$

where

$$\varepsilon_{2,1} = \varepsilon_{2,1,1} + \varepsilon_{2,1,2}. \tag{3.24}$$

If we choose

$$\begin{aligned}\alpha_2 ={}& c_2 + \frac{3}{2}\bar{a}_{2,1} + \frac{27}{256\varepsilon_{2,1,1}^3} + \frac{9}{16\varepsilon_{2,1,2}}(\bar{a}_{2,1}+\bar{a}_{2,2})^2 + \frac{9\bar{a}_{2,2}^2}{16\beta_{2,1}}(M^2+1)^2 \\ &+ \frac{9\bar{a}_{2,1}^2}{16\beta_{2,2}}(M^2+1)^2,\end{aligned} \tag{3.25}$$

$$x_3^* = \ddot{y}_r - \Big(\frac{2}{T}\mu^{3/2}\alpha_1 - \alpha_1^2\mu^2\Big)\xi_1 - (\alpha_1+\alpha_2)\mu\xi_2, \tag{3.26}$$

and define

$$\xi_3 = x_3 - x_3^*, \tag{3.27}$$

then we have

$$\mathcal{L}V_2 \le -(c_1-\varepsilon_{2,1})\mu\xi_1^4 - \frac{c_2}{\mu^3}\xi_2^4 + \frac{1}{\mu^4}\xi_2^3\xi_3 + \beta_2(\nu^{4\sigma_1+2} + \nu^{4\sigma_2+10}), \tag{3.28}$$

where $c_2 > 0$ is a design parameter and

$$\beta_2 = \max\{\beta_{11}+\beta_{2,1}, \beta_{2,2}\}. \tag{3.29}$$

By (3.12) and (3.26) we obtain

$$d\xi_2 = \big(-\alpha_2\mu\xi_2 + \xi_3\big)dt + \Big(\varphi_2^T + \alpha_1\mu\varphi_1^T\Big)d\omega. \tag{3.30}$$

Deductive step. In this step, we aim to design the virtual controller x_{k+2}^*. Assume that at step k there are a set of virtual controllers defined by

$$x_i^* = y_r^{(i-1)} + \frac{2}{T}\frac{\partial x_{i-1}^*}{\partial\mu}\mu^{3/2} + \sum_{j=1}^{i-2}\frac{\partial x_{i-1}^*}{\partial\xi_j}(-\alpha_j\mu\xi_j + \xi_{j+1})$$
$$- \alpha_{i-1}\mu\xi_{i-1}, \quad 3 \le i \le k+1, \tag{3.31}$$

$$\xi_i = x_i - x_i^*, \quad 2 \le i \le k+1, \tag{3.32}$$

such that

$$d\xi_i = \big(-\alpha_i\mu\xi_i + \xi_{i+1}\big)dt + \Big(\varphi_i^T - \sum_{j=1}^{i-1}\frac{\partial x_i^*}{\partial x_j}\varphi_j^T\Big)d\omega, \ 2 \le i \le k, \tag{3.33}$$

and

$$\mathcal{L}V_k \le -\sum_{i=1}^{k}\frac{c_i - \varepsilon_{k,i}}{\mu^{4i-5}}\xi_i^4 + \frac{1}{\mu^{4(k-1)}}\xi_k^3\xi_{k+1} + \beta_k\sum_{j=1}^{k}\nu^{4\sigma_j+8j-6}, \tag{3.34}$$

where $\alpha_2, \ldots, \alpha_k$ are positive constants, $c_1, \ldots, c_k$ are design parameters, $\varepsilon_{k,1}, \ldots, \varepsilon_{k,k-1}$ and $\beta_1, \ldots, \beta_k$ are arbitrary positive constants, $\varepsilon_{k,k} = 0$ and $V_k(\bar{\xi}_k) = \sum_{i=1}^{k}\frac{1}{4\mu^{4(i-1)}}\xi_i^4$, and $\bar{\xi}_k = (\xi_1, \ldots, \xi_k)^T$.

To complete the induction, at the $(k+1)$th step $(2 \le k \le n-2)$, we consider the ξ_{k+1}-system.

It follows from (2.1), (3.31), and (3.32) that

$$d\xi_{k+1} = \left(x_{k+2} - y_r^{(k+1)} - \frac{2}{T}\frac{\partial x_{k+1}^*}{\partial\mu}\mu^{3/2} - \sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial\xi_j}(-\alpha_j\mu\xi_j + \xi_{j+1})\right)dt$$
$$+ \left(\varphi_{k+1}^T - \sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_j}\varphi_j^T\right)d\omega. \tag{3.35}$$

We choose the Lyapunov function

$$V_{k+1}(\bar{\xi}_{k+1}) = V_k(\bar{\xi}_k) + \frac{1}{4\mu^{4k}}\xi_{k+1}^4. \tag{3.36}$$

By (3.34)–(3.36) and Itô's formula we get

$$\mathcal{L}V_{k+1} \le -\sum_{i=1}^{k}\frac{c_i - \varepsilon_{k,i}}{\mu^{4i-5}}\xi_i^4 + \frac{1}{\mu^{4(k-1)}}\xi_k^3\xi_{k+1} + \frac{3}{2\mu^{4k}}\xi_{k+1}^2\left|\varphi_{k+1} - \sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_j}\varphi_j\right|^2$$
$$- \frac{2k}{T\mu^{4k-1/2}}\xi_{k+1}^4 + \frac{1}{\mu^{4k}}\xi_{k+1}^3\left(x_{k+2} - y_r^{(k+1)} - \frac{2}{T}\frac{\partial x_{k+1}^*}{\partial\mu}\mu^{3/2} - \sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial\xi_j}\right.$$
$$\left.\cdot(-\alpha_j\mu\xi_j + \xi_{j+1})\right) + \beta_k\sum_{j=1}^{k}\nu^{4\sigma_j+8j-6}. \tag{3.37}$$

By Lemma A.1 we obtain

$$\frac{1}{\mu^{4(k-1)}}\xi_k^3\xi_{k+1} \le \varepsilon_{k+1,k,1}\frac{1}{\mu^{4k-5}}\xi_k^4 + \frac{1}{4}\left(\frac{4}{3}\varepsilon_{k,k-1,1}\right)^{-3}\frac{1}{\mu^{4k-1}}\xi_{k+1}^4, \tag{3.38}$$

where $\varepsilon_{k+1,k,1}$ is an arbitrary positive constant.

To proceed further, we need some estimates. The following proposition supplies these estimates, whose proof is given in Appendix B.

PROPOSITION 1. *For $k \ge 2$, there exist positive constants b_k, $b_{k,0}, b_{k,1}, \dots, b_{k,k-1}$ such that*

$$|x_k^*| \le b_k\Big(\sum_{j=1}^{k-1}\mu^{k-j}|\xi_j| + M\Big), \tag{3.39}$$

$$\Big|\frac{\partial x_k^*}{\partial \mu}\Big| \le b_{k,0}\sum_{j=1}^{k-1}\mu^{k-j-1}|\xi_j|, \tag{3.40}$$

$$\Big|\frac{\partial x_k^*}{\partial \xi_j}\Big| \le b_k\mu^{k-j},\ 1 \le j \le k-1, \tag{3.41}$$

$$\Big|\frac{\partial x_k^*}{\partial x_j}\Big| \le b_{k,j}\mu^{k-j},\ 1 \le j \le k-1. \tag{3.42}$$

It follows from Assumption 1, (3.1), (3.32), *and* (3.39) *that*

$$|x_1| \le |\xi_1| + M, \tag{3.43}$$

$$|x_j| \le |\xi_j| + b_j\Big(\sum_{s=1}^{j-1}\mu^{j-s}|\xi_s| + M\Big),\ 2 \le j \le n. \tag{3.44}$$

From (2.8), (2.9), (3.42), (3.43), *and* (3.44) *we get*

$$\begin{aligned}\Big|\varphi_{k+1} - \sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_j}\varphi_j\Big|^2 &\le 2\nu^{2\sigma_{k+1}}\Big(c_{k+1,1}\sum_{s=1}^{k+1}|x_s| + c_{k+1,2}\Big)^2 \\ &\quad + 2\Big(\sum_{j=1}^{k}b_{k+1,j}\nu^{\sigma_j}\mu^{k+1-j}\Big(c_{j,1}\sum_{s=1}^{j}|x_s| + c_{j,2}\Big)\Big)^2 \\ &\le \tilde{b}_{k+1,1}\sum_{i=1}^{k+1}\mu^{2(k+1-i)}\xi_i^2 + \tilde{b}_{k+1,1}(M+1)^2\mu^{2k}\Big(\sum_{i=1}^{k+1}\nu^{2\sigma_i+4i-4}\Big),\end{aligned} \tag{3.45}$$

where $\tilde{b}_{k+1,1}$ is a positive constant.

By (3.45) we obtain

$$\begin{aligned}\frac{3}{2\mu^{4k}}\xi_{k+1}^2\Big|\varphi_{k+1} - \sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_j}\varphi_j\Big|^2 &\le \frac{3\tilde{b}_{k+1,1}}{2\mu^{4k}}\xi_{k+1}^4 + \sum_{i=1}^{k}\frac{3\tilde{b}_{k+1,1}}{2\mu^{2(k+i-1)}}\xi_i^2\xi_{k+1}^2 \\ &\quad + \frac{3\tilde{b}_{k+1,1}}{2\mu^{2k}}(M+1)^2\sum_{i=1}^{k+1}\nu^{2\sigma_i+4i-4}\xi_{k+1}^2.\end{aligned} \tag{3.46}$$

By Lemma A.1 we have

$$\frac{3\tilde{b}_{k+1,1}}{2\mu^{2(k+i-1)}}\xi_i^2\xi_{k+1}^2 \le \frac{1}{\mu^{4i-5}}\varepsilon_{k+1,i,2}\xi_i^4 + \frac{9}{16\varepsilon_{k+1,i,2}}\tilde{b}_{k+1,1}^2\frac{1}{\mu^{4k-1}}\xi_{k+1}^4 \tag{3.47}$$

and

$$\begin{aligned}&\frac{3\tilde{b}_{k+1,1}}{2\mu^{2k}}(M+1)^2\nu^{2\sigma_i+4i-4}\xi_{k+1}^2\\ &\le \beta_{k+1,i}\nu^{4\sigma_i+8i-6}+\frac{9\tilde{b}_{k+1,1}^2}{16\beta_{k+1,i}\mu^{4k-1}}(M+1)^4\xi_{k+1}^4,\end{aligned}\tag{3.48}$$

where $\varepsilon_{k+1,i,2}$ and $\beta_{k+1,i}$ are arbitrary positive constants.

With (3.47) and (3.48), (3.46) can be rewritten as

$$\begin{aligned}&\frac{3}{2\mu^{4k}}\xi_{k+1}^2\left|\varphi_{k+1}-\sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial x_j}\varphi_j\right|^2\\ &\le\sum_{i=1}^{k}\left(\varepsilon_{k+1,i,2}\frac{1}{\mu^{4i-5}}\right)\xi_i^4+\Big(\sum_{i=1}^{k}\frac{9}{16\varepsilon_{k+1,i,2}}\tilde{b}_{k+1,1}^2+\frac{3}{2}\tilde{b}_{k+1,1}\\ &\quad+\sum_{i=1}^{k+1}\frac{9\tilde{b}_{k+1,1}^2}{16\beta_{k+1,i}}(M+1)^4\Big)\frac{1}{\mu^{4k-1}}\xi_{k+1}^4+\sum_{i=1}^{k+1}\beta_{k+1,i}\nu^{4\sigma_i+8i-6}.\end{aligned}\tag{3.49}$$

Substituting (3.38) and (3.49) into (3.37) yields

$$\begin{aligned}\mathcal{L}V_{k+1}\le&-\sum_{i=1}^{k}\frac{c_i-\varepsilon_{k+1,i}}{\mu^{4i-5}}\xi_i^4+\frac{1}{\mu^{4k}}\xi_{k+1}^3\Bigg(x_{k+2}-x_{k+2}^*+x_{k+2}^*-y_r^{(k+1)}-\frac{2}{T}\frac{\partial x_{k+1}^*}{\partial\mu}\mu^{3/2}\\ &-\sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial\xi_j}(-\alpha_j\mu\xi_j+\xi_{j+1})\Bigg)+\left(\frac{3}{2}\tilde{b}_{k+1,1}+\frac{1}{4}\left(\frac{4}{3}\varepsilon_{k,k-1,1}\right)^{-3}\right.\\ &\left.+\sum_{i=1}^{k}\frac{9}{16\varepsilon_{k+1,i,2}}\tilde{b}_{k+1,1}^2+\sum_{i=1}^{k+1}\frac{9\tilde{b}_{k+1,1}^2}{16\beta_{k+1,i}}(M+1)^4\right)\frac{1}{\mu^{4k-1}}\xi_{k+1}^4\\ &+\beta_{k+1}\sum_{i=1}^{k+1}\nu^{4\sigma_i+8i-6},\end{aligned}\tag{3.50}$$

where

$$\varepsilon_{k+1,i}=\varepsilon_{k,i}+\varepsilon_{k+1,i,2},\ \ 1\le i\le k-1,\tag{3.51}$$

$$\varepsilon_{k+1,k}=\varepsilon_{k+1,k,1}+\varepsilon_{k+1,k,2},\tag{3.52}$$

$$\beta_{k+1}=\max\Big\{\max_{1\le i\le k}\{\beta_{k+1,i}\}+\beta_k,\beta_{k+1,k+1}\Big\}.\tag{3.53}$$

If we choose

$$\begin{aligned}\alpha_{k+1}=&\,c_{k+1}+\frac{3}{2}\tilde{b}_{k+1,1}+\frac{1}{4}\left(\frac{4}{3}\varepsilon_{k,k-1,1}\right)^{-3}+\sum_{i=1}^{k}\frac{9}{16\varepsilon_{k+1,i,2}}\tilde{b}_{k+1,1}^2\\ &+\sum_{i=1}^{k+1}\frac{9\tilde{b}_{k+1,1}^2}{16\beta_{k+1,i}}(M+1)^4,\end{aligned}\tag{3.54}$$

$$x_{k+2}^*=y_r^{(k+1)}+\frac{2}{T}\frac{\partial x_{k+1}^*}{\partial\mu}\mu^{3/2}-\alpha_{k+1}\mu\xi_{k+1}+\sum_{j=1}^{k}\frac{\partial x_{k+1}^*}{\partial\xi_j}(-\alpha_j\mu\xi_j+\xi_{j+1}),\tag{3.55}$$

and define

$$\xi_{k+2} = x_{k+2} - x_{k+2}^*, \tag{3.56}$$

then we have

$$\mathcal{L}V_{k+1} \leq -\sum_{i=1}^{k+1} \frac{c_i - \varepsilon_{k+1,i}}{\mu^{4i-5}} \xi_i^4 + \frac{1}{\mu^{4k}} \xi_{k+1}^3 \xi_{k+2} + \beta_{k+1} \sum_{i=1}^{k+1} \nu^{4\sigma_i + 8i - 6} \tag{3.57}$$

and

$$d\xi_{k+1} = (-\alpha_{k+1}\mu\xi_{k+1} + \xi_{k+2})dt + \left(\varphi_{k+1}^T - \sum_{j=1}^{k} \frac{\partial x_{k+1}^*}{\partial x_j} \varphi_j^T\right) d\omega, \tag{3.58}$$

where $c_{k+1} > 0$ is a design parameter and $\varepsilon_{k+1,k+1} = 0$.

Step n. In this step, we aim to design the actual controller u.

Similar to (3.55), we choose the actual control law as

$$u = y_r^{(n)} + \frac{2}{T} \frac{\partial x_n^*}{\partial \mu} \mu^{3/2} + \sum_{j=1}^{n-1} \frac{\partial x_n^*}{\partial \xi_j} (-\alpha_j \mu \xi_j + \xi_{j+1}) - \alpha_n \mu \xi_n - \phi(t, x), \tag{3.59}$$

and then we get

$$\mathcal{L}V_n(\bar{\xi}_n) \leq -\sum_{i=1}^{n} \frac{c_i - \varepsilon_{n,i}}{\mu^{4i-5}} \xi_i^4 + \beta_n \sum_{i=1}^{n} \nu^{4\sigma_i + 8i - 6} \tag{3.60}$$

and

$$d\xi_n = -\alpha_n \mu \xi_n dt + \left(\varphi_n^T - \sum_{j=1}^{n-1} \frac{\partial x_n^*}{\partial x_j} \varphi_j^T\right) d\omega, \tag{3.61}$$

where $c_n > 0$ is a design parameter, α_n is a positive constant, $\varepsilon_{n,1}, \ldots, \varepsilon_{n,n-1}$ and β_n are arbitrary positive constants, $\varepsilon_{n,n} = 0$, $\xi_n = x_n - x_n^*$, $\bar{\xi}_n = (\xi_1, \ldots, \xi_n)^T$, and

$$V_n(\bar{\xi}_n) = \sum_{i=1}^{n} \frac{1}{4\mu^{4(i-1)}} \xi_i^4. \tag{3.62}$$

Remark 3. In this section, we propose a new stochastic prescribed-time control design, which is essentially different from those in [13, 14, 15]. Comparing this design with those in [13, 14, 15], we observe that the virtual controllers x_2^* in (3.7), x_3^* in (3.26), x_{k+2}^* in (3.55), and the actual controller u in (3.59) are much more complex than their corresponding parts in [13, 14, 15]. This is mainly because [13, 14, 15] aim for stochastic prescribed-time stabilization, while the objective of this paper is to solve the prescribed-time mean-nonovershooting control problem. In [13, 14, 15], the drift terms of $d\xi_{k+1}$ in (3.35), i.e., $-y_r^{(k+1)} - \frac{m}{T} \frac{\partial x_{k+1}^*}{\partial \mu} \mu^{1+1/m} - \sum_{j=1}^{k} \frac{\partial x_{k+1}^*}{\partial \xi_j} (-\alpha_j \mu \xi_j + \xi_{j+1})$, are damped by some negative μ-scaling terms, which yields much simpler virtual controllers and a much simpler actual controller. However, in order to achieve prescribed-time mean-nonovershooting control in this paper, we need to ensure that these drift terms are directly absorbed into the virtual controller x_{k+2}^* in (3.55), which heightens the challenge of the design of the virtual controllers $x_{k+3}^*, \ldots, x_n^*$ and the actual controller u.

4. Prescribed-time safety analysis. From (3.10), (3.30), (3.58), and (3.61) we get

$$\xi_k = e^{-\int_{t_0}^t \alpha_k\mu(s)ds}\xi_k(t_0) + \int_{t_0}^t e^{-\int_s^t \alpha_k\mu(\tau)d\tau}\xi_{k+1}(s)ds \\ + M_k(t),\ 1\le k\le n-1, \tag{4.1}$$

$$\xi_n = e^{-\int_{t_0}^t \alpha_n\mu(s)ds}\xi_n(t_0) + M_n(t), \tag{4.2}$$

where

$$M_1(t) = \int_{t_0}^t e^{-\int_s^t \alpha_1\mu(s)ds}\varphi_1^T d\omega(s) \tag{4.3}$$

and

$$M_k(t) = \int_{t_0}^t e^{-\int_s^t \alpha_k\mu(s)ds}\left(\varphi_k^T - \sum_{j=1}^{k-1}\frac{\partial x_k^*}{\partial x_j}\varphi_j^T\right)d\omega(s) \tag{4.4}$$

for $2\le k\le n$.

We choose the design parameters $c_1,\dots,c_n$ as

$$c_k > \max\{\check{c}_k, \varepsilon_{n,k}\},\quad 1\le k\le n-1, \tag{4.5}$$

$$c_n > 0, \tag{4.6}$$

where

$$\check{c}_1 = \frac{1}{x_1(t_0)-y_r(t_0)}(-x_2(t_0)+\dot{y}_r(t_0)), \tag{4.7}$$

$$\check{c}_k = \frac{1}{\xi_k(t_0)}\Bigg(- x_{k+1}(t_0) + y_r^{(k)}(t_0) + \frac{2}{T}\frac{\partial x_k^*}{\partial \mu}|_{t=t_0} + \sum_{j=1}^{k-1}\frac{\partial x_k^*}{\partial \xi_j}|_{t=t_0}(-\alpha_j\xi_j(t_0) \\ + \xi_{j+1}(t_0))\Bigg),\ 2\le k\le n-1, \tag{4.8}$$

and

$$\xi_k(t_0) = x_k(t_0) - y_r^{(k-1)}(t_0) - \frac{2}{T}\frac{\partial x_{k-1}^*}{\partial\mu}|_{t=t_0} - \sum_{j=1}^{k-2}\frac{\partial x_{k-1}^*}{\partial\xi_j}|_{t=t_0}(-\alpha_j\xi_j(t_0)+\xi_{j+1}(t_0)) \\ + \alpha_{k-1}\xi_{k-1}(t_0),\ 2\le k\le n-1. \tag{4.9}$$

With (3.62), (4.5), and (4.6), (3.60) can be rewritten as

$$\mathcal{L}V_n(\bar{\xi}_n) \le -c_0\mu V_n + \beta_n\sum_{j=1}^n \nu^{4\sigma_j+8j-6}, \tag{4.10}$$

where

$$c_0 = 4\min_{1\le i\le n}\{c_i - \varepsilon_{n,i}\}. \tag{4.11}$$

From (4.5) and (4.6) we obtain that c_0 is a positive constant.

Before presenting the main safety results, we first give four technical lemmas, which are important in the stability analysis.

The first lemma shows that $\xi_i(t_0) < 0$ can be ensured if we choose $y(t_0) < y_r(t_0)$. The proof of this lemma is given in Appendix C.

LEMMA 1. *Consider the plant* (2.1)–(2.3), (3.59), *and* (4.5)–(4.8). *If* $y(t_0) < y_r(t_0)$, *then we have* $\xi_i(t_0) < 0$, $i = 1, \ldots, n$.

Based on (4.10), the second lemma provides an estimate for EV_n. The proof of this lemma is given in Appendix D.

LEMMA 2. *Consider the plant* (2.1)–(2.3), (3.59), *and* (4.5)–(4.8). *If Assumptions* 1–2 *hold, then the following conclusions hold:*

(1) *The plant has an almost surely unique strong solution on* $[t_0, t_0 + T)$ *for any* $x_0 \in R^n$.

(2) *The function* V_n *defined in* (3.62) *satisfies*

$$EV_n \le f(t), \quad \forall t \in [t_0, t_0 + T), \tag{4.12}$$

where

$$f(t) = e^{-c_0 \int_{t_0}^{t} \mu(s) ds} \Big(V_n(t_0) + \beta_n \int_{t_0}^{t} \sum_{j=1}^{n} \nu^{4\sigma_j + 8j - 6}(\tau) e^{c_0 \int_{t_0}^{\tau} \mu(s) ds} d\tau \Big) \tag{4.13}$$

satisfies

$$f(t) \sim \frac{\beta_n}{c_0} \sum_{j=1}^{n} \nu^{4\sigma_j + 8j - 4} \qquad \text{as } t \to t_0 + T. \tag{4.14}$$

In other words, $f(t)$ *is the equivalent infinitesimal of* $\frac{\beta_n}{c_0} \sum_{j=1}^{n} \nu^{4\sigma_j + 8j - 4}$ *as* $t \to t_0 + T$.

Lemma 3 provides estimates for the Riemann integrals, whose integrands are stochastic processes. The proof of this lemma is given in Appendix E.

LEMMA 3. *Consider the plant* (2.1)–(2.3), (3.59), *and* (4.5)–(4.8). *If Assumptions* 1–2 *hold, then, for* $k = 1, \ldots, n-1$ *we have*

$$E\left\{ \int_{t_0}^{t} e^{-\int_s^t \alpha_k \mu(\tau) d\tau} \xi_{k+1}(s) ds \right\} = \int_{t_0}^{t} e^{-\int_s^t \alpha_k \mu(\tau) d\tau} E\{\xi_{k+1}(s)\} ds. \tag{4.15}$$

Lemma 4 provides an estimate for $M_k(t)$. The proof of this lemma is given in Appendix F.

LEMMA 4. *If Assumptions* 1–2 *hold for the plant* (2.1)–(2.3), (3.59), *and* (4.5)–(4.8), *then we have*

$$E\{M_k(t)\} = 0, \ k = 1, \ldots, n. \tag{4.16}$$

In the following theorem, we state the main safety results on system (2.1)–(2.3). The proof of this theorem is given in Appendix G.

THEOREM 1. *Consider the plant consisting of* (2.1)–(2.3), (3.59), *and* (4.5)–(4.8). *If Assumptions* 1–2 *hold, then the following conclusions hold:*

(1) *The plant has an almost surely unique strong solution on* $[t_0, t_0 + T)$ *for any* $x_0 \in R^n$.

(2) *The mean-square of the state and the controller is bounded.*

(3) *The state* x_k *converges to* $y_r^{(k-1)}$ *in* L^4 *as* $t \to t_0 + T$. *Specifically,*

$$\lim_{t \to t_0 + T} E|x_k - y_r^{(k-1)}|^4 = 0, \ 1 \le i \le n, \tag{4.17}$$

where L^4 *denotes the family of quartic integrable random variables.*

(4) *In addition, if* $y(0) < y_r(0)$*, we have*

$$E\{y(t)\} \leq y_r(t) \quad \forall t \in [t_0, t_0 + T), \tag{4.18}$$

$$\lim_{t \to t_0 + T} E\{y(t) - y_r(t)\} = 0. \tag{4.19}$$

Remark 4. From the proofs of Lemmas 2–4 and Theorem 1, it can be observed that the requirement that the noise be vanishing (Assumption 2) is crucial for achieving the safety control objectives. If we remove this requirement with $\gamma_i(t) = 1$, $i = 1, \ldots, n$, by repeating the proof of Lemma 2 we can still prove that $\lim_{t \to t_0+T} E|x_1 - y_r|^4 = 0$ holds. However, we will also obtain that $\lim_{t \to t_0+T} Ex_i^2 = +\infty$ $(2 \leq i \leq n)$ and $\lim_{t \to t_0+T} Eu^2 = +\infty$, which makes the control scheme infeasible. The main reasons are as follows: with the effect of the additive noise and the time-varying reference trajectory, and without the vanishing assumption, the convergence rate of the Lyapunov function $V_n(\bar{\xi}_n)$ is only polynomial, rather than exponential, which results in $\lim_{t \to t_0+T} Eu^2 = +\infty$.

Remark 5. From the proof of Theorem 1, we obtain that the convergence rates of states $x_2, \ldots, x_n$ are determined by the noise vanishing rates $\delta_1, \ldots, \delta_n$. The larger the noise vanishing rates, the faster the convergence rate of states. The convergence rate of $y(t)$ is heavily relied on $\alpha_1, \ldots, \alpha_n$, which are determined by the design parameters $c_1, \ldots, c_n$. In other words, a larger control effort yields a faster convergence rate of $y(t)$.

Remark 6. In contrast to [1, 2, 3, 19, 21, 27, 29], where control barrier functions, a "nominal" control input, and a "safety filter" are designed to achieve the safe control objective, we propose a different safety design for stochastic system (2.1)–(2.3). Specifically, our "nominal" controller (3.59) ensures that the mean of the system output can track a given trajectory "from below" in prescribed time without overshooting, while keeping all of the states mean-square bounded. In other words, we achieve safety by the nominal controller (3.59) without the need for a "safety filter" redesign. In fact, our control barrier function is $h(t) = E\{y(t)\} - y_r(t)$. Thus, although we develop a different design from those in [1, 2, 3, 19, 21, 27, 29], we maintain safety in the same CBF-based sense as in those articles.

5. A simulation example. In this section, we use a stochastic mass-spring-damper system to show the effectiveness of the prescribed-time mean-nonovershooting control schemes developed in this paper.

The mass-spring-damper system is shown in Figure 1, where a mass m is attached to a ceiling through a spring-damper combination. The mass is driven by an external force which serves as a control variable. Let y be the displacement from a reference position; k is the spring parameter and a is the damper parameter. We assume that the displacement is relative small, with the restoring force modeled as ky. By Newton's law of motion, the system is described as

$$mg + u - ky - a\dot{y} = m\ddot{y}. \tag{5.1}$$

The damper parameter $a(t)$ has a nominal value a_0 and $a(t) \in (a_0 - 0.2, a_0 + 0.2)$. Let $\Delta(t) = a(t) - a_0$. $\Delta(t)$ is the Gaussian white noise process with zero mean and $E(\Delta(t))^2 = (1 - \frac{t}{2})^4$.

To obtain a state-space model for the mass-spring-damper system, take the state variables to be $x_1 = y$ and $x_2 = \dot{y}$. Then, from (5.1) we get the state-space form

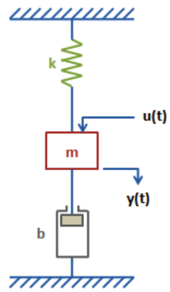

FIG. 1. *Mass-spring-damper system.*

$$dx_1 = x_2 dt, \tag{5.2}$$

$$dx_2 = \left(\frac{u}{m} + g - \frac{k}{m}x_1 - \frac{a_0}{m}x_2\right) dt - \frac{1}{4m}(2-t)^2 x_2 d\omega, \tag{5.3}$$

$$y = x_1. \tag{5.4}$$

Choosing $m=1$ and $a_0 = k = 0.1$, (5.2)–(5.4) can be rewritten as

$$dx_1 = x_2 dt, \tag{5.5}$$

$$dx_2 = (u + g - 0.1x_1 - 0.1x_2)\, dt - \frac{1}{4}(2-t)^2 x_2 d\omega, \tag{5.6}$$

$$y = x_1. \tag{5.7}$$

We choose the reference trajectory as $y_r(t) = \sin t$. It is obvious that Assumption 1 is satisfied.

Letting $t_0 = 0$ and $T = 2$, (2.4) and (2.5) can be rewritten as

$$\mu(t) = \left(\frac{2}{2-t}\right)^2, \quad t \in [0, 2), \tag{5.8}$$

$$\nu(t) = 1 - \frac{t}{2}, \quad t \in [0, 2). \tag{5.9}$$

By (5.6) we obtain

$$|g - 0.1x_1 - 0.1x_2| \leq 0.1(|x_1| + |x_2|) + g, \tag{5.10}$$

$$\frac{1}{4}|(2-t)^2 x_2| = \nu^2 |x_2| \leq \nu |x_2|, \tag{5.11}$$

which shows that Assumption 2 is satisfied with $c_{01} = 0.1$, $c_{02} = g$, $c_{11} = 0$, $c_{12} = 0$, $\gamma_1 = 0$, $c_{21} = 1$, $c_{22} = 0$, $\gamma_2(t) = \nu^2(t)$, and $\sigma_2 = 1$.

By following the design procedure developed in section 3, we get the controller

$$\begin{aligned} u = &- g + 0.1x_1 + 0.1x_2 - \sin t - c_1\mu^{3/2}(x_1 - \sin t) - c_1\mu(x_2 - \cos t) \\ &- \left(c_2 + 5 + \frac{27}{16}c_1^2\right)\mu(x_2 - \cos t + c_1\mu(x_1 - \sin t)). \end{aligned} \tag{5.12}$$

In the simulation, we randomly set the initial conditions as $x_1(0) = -1$, $x_2(0) = 2$. From (4.5) and (4.6) we get

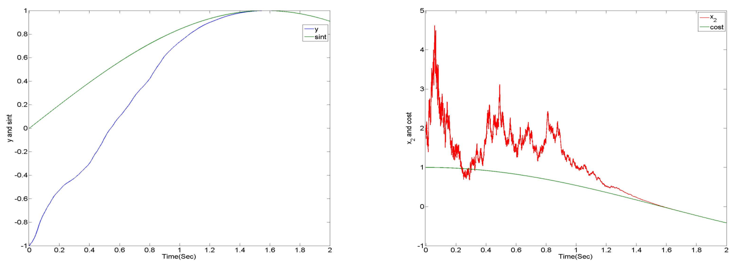

FIG. 2. *The response of states of the closed-loop system* (5.5)–(5.9) *and* (5.12).

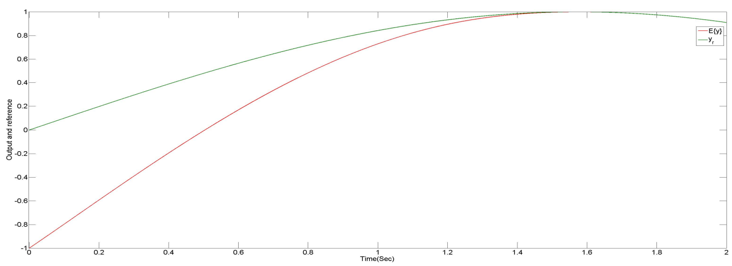

FIG. 3. *The response of* $E\{y\}$ *and* y_r *of the closed-loop system* (5.5)–(5.9) *and* (5.12).

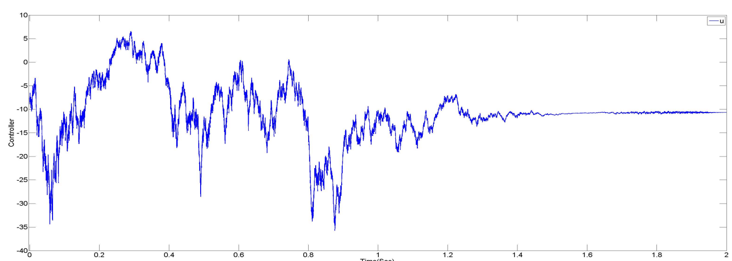

FIG. 4. *The response of the controller of the closed-loop system* (5.5)–(5.9) *and* (5.12).

$$c_1 > \max\left\{\frac{1 - x_2(0)}{x_1(0)}, 1\right\} = 1, \tag{5.13}$$

$$c_2 > 0. \tag{5.14}$$

By (5.13) and (5.14), we choose the controller gains $c_1 = 1.1$ and $c_2 = 1$. We employ gain clipping due to the finite-precision arithmetic in the computation of the control signal. Figure 2 gives the response of states, which shows that $\lim_{t\to 2} |x_1 - y_r|^4 = 0$ and $\lim_{t\to 2} |x_2 - \dot{y}_r|^4 = 0$. Figure 3 describes the response of $E\{y(t)\}$ and $y_r(t)$, which shows that $y_r(t)$ is prescribed-time tracked by $E\{y(t)\}$ from "below" with $\lim_{t\to 2} E\{y(t) - y_r(t)\} = 0$. Figure 4 gives the response of the controller, from which we can see that the mean-square of the controller is bounded. Therefore, the effectiveness of the controller design developed in section 3 is demonstrated.

Remark 7. Although the mean-nonovershooting control problem of the above stochastic mass-spring-damper system has been studied in [12], this reference only considers the mean-nonovershooting safety control, where the reference trajectories can be tracked "from below" in the asymptotic case, without considering the prescribed-time mean-nonovershooting control problem. Compared with the design in [12], our controller (5.12) is characterized by a time-varying blow-up function (5.8) that grows unbounded towards the terminal time $T=2$, which yields $\lim_{t\to 2} E\{y(t)-y_r(t)\}=0$, a better and more practical performance than $\lim_{t\to\infty} E\{y(t)-y_r(t)\}=0$ in [12].

6. Concluding remarks. We have developed a prescribed-time mean-nonovershooting design for stochastic nonlinear systems where the matched noise should be vanishing at a rate that is no slower than linear in the "time to go" until the prescribed time of convergence. This design is applicable to any noise that vanishes strictly before the prescribed time. In comparison to existing designs, our design has two advantages. First, we can achieve mean-nonovershooting tracking in prescribed time rather than asymptotically. Second, our design can deal with multiplicative and additive noise simultaneously. A new controller is designed to ensure that the closed-loop system has an almost surely unique solution on $[t_0, t_0+T)$, that the mean of the system output can prescribe-time track a given trajectory without overshooting, that the fourth moment of the tracking error between states and derivatives of the reference trajectory converges to zero in prescribed time, and that all of the states are mean-square bounded.

For the prescribed-time mean-nonovershooting control of stochastic nonlinear systems, many important issues are still open and worth investigating. For example, in Assumption 2, $c_{0,1}$, $c_{0,2}$, $c_{i,1}$, and $c_{i,2}$ are required to be constants. Such a requirement is a bit conservative since there exist some real-world applications where $\phi(t,x)$ and $\varphi_i(t,x)$ don't satisfy the linear growth condition. Therefore, an important future work is generalizing Assumption 2 to a more general form. In addition, it would be interesting to apply the results in this paper to stochastic benchmark systems [16].

Appendix A. Useful tools. In this appendix, we collect four lemmas which are useful in the controller design and safety analysis.

LEMMA A.1 ([17]). *Let x, y be real variables; then for any positive real numbers a, m, and n, we have*

$$ax^m y^n \le b|x|^{m+n} + \frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-\frac{m}{n}} a^{\frac{m+n}{n}} b^{-\frac{m}{n}} |y|^{m+n}, \tag{A.1}$$

where $b>0$ is any real number.

LEMMA A.2 (Fubini's theorem [7]). *Let $X(t)$ be a stochastic process $0\le t\le T$ (for all t, $X(t)$ is a stochastic variable), with regular sample paths (for all ω at any point t, $X(t)$ has left and right limits). Then*

$$\int_0^T E|X(t)|dt = E\left(\int_0^T |X(t)|dt\right). \tag{A.2}$$

Furthermore if this quantity is finite, then

$$\int_0^T EX(t)dt = E\left(\int_0^T X(t)dt\right). \tag{A.3}$$

LEMMA A.3 (Dynkin's formula [18]). *Let $V \in C^{2,1}(R^n \times R_+; R_+)$, and let τ_1, τ_2 be bounded stopping times such that $0 \le \tau_1 \le \tau_2$ almost surely. If $V(x,t)$ and $\mathcal{L}V(x,t)$ are bounded on $t \in [\tau_1,\ \tau_2]$ almost surely, then*

$$E[V(x,\tau_2) - V(x,\tau_1)] = E\left\{\int_{\tau_1}^{\tau_2} \mathcal{L}V(x,t)dt\right\}. \tag{A.4}$$

LEMMA A.4 ([28]). *Let $x_1, \dots, x_n, p$ be positive real numbers; then*

$$(x_1 + \cdots + x_n)^p \le \max\{n^{p-1}, 1\}(x_1^p + \cdots + x_n^p). \tag{A.5}$$

Appendix B. Proof of Proposition 1. From (2.6), (3.6), and (3.7) we have

$$|x_2^*| \le b_2\Big(\mu|\xi_1| + M\Big), \tag{B.1}$$

$$\Big|\frac{\partial x_2^*}{\partial \mu}\Big| \le b_{2,0}|\xi_1|, \tag{B.2}$$

$$\Big|\frac{\partial x_2^*}{\partial \xi_1}\Big| \le b_2\mu, \tag{B.3}$$

$$\Big|\frac{\partial x_2^*}{\partial x_1}\Big| \le b_{2,1}\mu, \tag{B.4}$$

where $b_{2,0}, b_{2,1}$, and b_2 are positive constants.

By (3.6), (3.25), and (3.26) we get

$$|x_3^*| \le b_3\Big(\mu^2|\xi_1| + \mu|\xi_2| + M\Big), \tag{B.5}$$

$$\Big|\frac{\partial x_3^*}{\partial \mu}\Big| \le b_{3,0}\Big(\mu|\xi_1| + |\xi_2|\Big), \tag{B.6}$$

$$\Big|\frac{\partial x_3^*}{\partial \xi_i}\Big| \le b_3\mu^{3-i},\ i = 1, 2, \tag{B.7}$$

$$\Big|\frac{\partial x_3^*}{\partial x_i}\Big| \le b_{3,i}\mu^{3-i},\ i = 1, 2, \tag{B.8}$$

where $b_{3,0}, b_{3,1}, b_{3,2}$, and b_3 are positive constants.

With the observation of (B.1)–(B.8), for x_{k-1}^*, we assume that

$$|x_{k-1}^*| \le b_{k-1}\left(\sum_{j=1}^{k-2} \mu^{k-j-1}|\xi_j| + M\right), \tag{B.9}$$

$$\Big|\frac{\partial x_{k-1}^*}{\partial \mu}\Big| \le b_{k-1,0}\sum_{j=1}^{k-2} \mu^{k-j-2}|\xi_j|, \tag{B.10}$$

$$\Big|\frac{\partial x_{k-1}^*}{\partial \xi_j}\Big| \le b_{k-1}\mu^{k-j-1},\ 1 \le j \le k-2, \tag{B.11}$$

$$\Big|\frac{\partial x_{k-1}^*}{\partial x_j}\Big| \le b_{k-1,j}\mu^{k-j-1},\ 1 \le j \le k-2, \tag{B.12}$$

where $b_{k-1,0}, b_{k-1,1}, \dots, b_{k-1,k-2}$, and b_{k-1} are positive constants.

Next, we prove that (3.39)–(3.42) hold by induction.

By (3.31) we obtain

$$x_k^* = y_r^{(k-1)} + \frac{2}{T}\frac{\partial x_{k-1}^*}{\partial \mu}\mu^{3/2} + \sum_{j=1}^{k-2}\frac{\partial x_{k-1}^*}{\partial \xi_j}(-\alpha_j\mu\xi_j + \xi_{j+1}) - \mu\alpha_{k-1}\xi_{k-1}. \tag{B.13}$$

It can be inferred from (B.9)–(B.13) that (3.39)–(3.42) hold.

This completes the proof of Proposition 1. ∎

Appendix C. Proof of Lemma 1. Since $y(t_0) < y_r(t_0)$, from (2.3) and (3.1) we have

$$\xi_1(t_0) = y(t_0) - y_r(t_0) < 0. \tag{C.1}$$

By (3.6), (3.11), (4.7), and (C.1) we have

$$\begin{aligned}\xi_2(t_0) &= x_2(t_0) - \dot{y}_r(t_0) + \alpha_1 \xi_1(t_0)\\ &\le x_2(t_0) - \dot{y}_r(t_0) + c_1 \xi_1(t_0)\\ &< x_2(t_0) - \dot{y}_r(t_0) + \check{c}_1 \xi_1(t_0)\\ &= 0.\end{aligned} \tag{C.2}$$

By (3.54), (4.5), and (4.8)–(4.9) and using an induction argument, we obtain

$$\begin{aligned}\xi_{k+1}(t_0) &= x_{k+1}(t_0) - y_r^{(k)}(t_0) - \frac{2}{T}\frac{\partial x_k^*}{\partial \mu}|_{t=t_0} + \alpha_k \xi_k(t_0)\\ &\quad - \sum_{j=1}^{k-1} \frac{\partial x_k^*}{\partial \xi_j}|_{t=t_0}(-\alpha_j \xi_j(t_0) + \xi_{j+1}(t_0))\\ &\le x_{k+1}(t_0) - y_r^{(k)}(t_0) - \frac{2}{T}\frac{\partial x_k^*}{\partial \mu}|_{t=t_0} + c_k \xi_k(t_0)\\ &\quad - \sum_{j=1}^{k-1} \frac{\partial x_k^*}{\partial \xi_j}|_{t=t_0}(-\alpha_j \xi_j(t_0) + \xi_{j+1}(t_0))\\ &< x_{k+1}(t_0) - y_r^{(k)}(t_0) - \frac{2}{T}\frac{\partial x_k^*}{\partial \mu}|_{t=t_0} + \check{c}_k \xi_k(t_0)\\ &\quad - \sum_{j=1}^{k-1} \frac{\partial x_k^*}{\partial \xi_j}|_{t=t_0}(-\alpha_j \xi_j(t_0) + \xi_{j+1}(t_0))\\ &= 0, \quad k = 2, \dots, n-1.\end{aligned} \tag{C.3}$$

This completes the proof of Lemma 1. ∎

Appendix D. Proof of Lemma 2. *Step* 1. We first prove that the plant has an almost surely unique solution on $[t_0, t_0 + T)$.

From (3.59), for every real number T_1 satisfying $0 < T_1 < T$ and integer $k \ge 1$, there exists a positive constant $K_{T_1,k}$ such that

$$|u(t,x) - u(t,y)| \le K_{T_1,k}|x - y| \tag{D.1}$$

holds for all $t \in [t_0, t_0 + T_1]$ and all $x, y \in R^n$ with $|x| \vee |y| \le k$. Additionally, $\varphi_i(t,x)$ is locally Lipschitz in x. Thus, the plant satisfies the local Lipschitz condition.

By Theorem 3.15 in [18], the plant has an almost surely unique strong solution $x(t)$ on $[t_0, \rho_\infty)$, where $\rho_\infty = (t_0 + T) \wedge \lim_{\tau \to +\infty} \inf\{t \ge t_0 : |x(t)| \ge \tau\}$. Next, we prove $\rho_\infty = t_0 + T$ almost surely. If this is not true, we can find positive constants ε and T_2 $(0 < T_2 < T)$ such that

$$P\{\rho_\infty \le t_0 + T_2\} > 2\varepsilon. \tag{D.2}$$

For each integer $k > 0$, define

$$\rho_k = (t_0 + T) \wedge \inf\{t : t_0 \le t < t_0 + T, |x(t)| \ge k\}. \tag{D.3}$$

Since $\rho_k \to \rho_\infty$ almost surely, there exists a sufficiently large integer k_0 such that

$$P\left\{\rho_k \le t_0 + T_2\right\} > \varepsilon \quad \forall k \ge k_0. \tag{D.4}$$

From (2.5), (2.10), and (4.10) we have

$$\mathcal{L}V_n \le n\beta_n. \tag{D.5}$$

Fix $k \ge k_0$. For any $t_0 \le t \le t_0 + T_2$, by (D.5) we have

$$\begin{aligned} EV_n(t \wedge \rho_k, x(t \wedge \rho_k)) &= V_n(t_0) + E\left\{\int_{t_0}^{t\wedge\rho_k} \mathcal{L}V_n(\tau, x(\tau))d\tau\right\} \\ &\le V_n(t_0) + n\beta_n(t - t_0), \end{aligned} \tag{D.6}$$

where $V_n(t_0, x(t_0))$ is abbreviated as $V_n(t_0)$.

By (D.6) we get

$$EV_n((t_0 + T_2) \wedge \rho_k, x((t_0 + T_2) \wedge \rho_k))) \le V_n(t_0) + n\beta_n T_2, \tag{D.7}$$

which shows that

$$E\chi_{\rho_k \le t_0 + T_2} V_n(\rho_k, x(\rho_k)) \le V_n(t_0) + n\beta_n T_2 < +\infty. \tag{D.8}$$

Define

$$b_k = \inf\Big\{V_n(t, x) : |x| \ge k, t \in [t_0, t_0 + T_2]\Big\}. \tag{D.9}$$

By (2.6), (3.1), (3.11), (3.31), (3.32), and (3.62) we have

$$\lim_{|x|\to+\infty} \inf_{t\in[t_0,T_3]} V_n = +\infty \quad \forall T_3 \in (t_0, t_0 + T), \tag{D.10}$$

from which, together with (D.9), we obtain

$$\lim_{k\to+\infty} b_k = +\infty. \tag{D.11}$$

From (D.4) and (D.8) we obtain

$$V_n(t_0) + n\beta_n T_2 \ge b_k P\left\{\rho_k \le t_0 + T_2\right\} > \varepsilon b_k. \tag{D.12}$$

Letting $k \to +\infty$ in both sides of (D.12), from (D.11) we obtain

$$V_n(t_0) + n\beta_n T_2 = +\infty, \tag{D.13}$$

which is a contradiction with (D.8). Thus, we have $\rho_\infty = t_0 + T$.

Step 2. We then prove (4.12)–(4.14). Choose

$$V = e^{c_0 \int_{t_0}^t \mu(s)ds} V_n. \tag{D.14}$$

From (4.10) and (D.14) we have

$$\mathcal{L}V = e^{c_0 \int_{t_0}^t \mu(s)ds}(\mathcal{L}V_n + c_0\mu V_n) \le \beta_n \sum_{j=1}^n \nu^{4\sigma_j + 8j - 6} e^{c_0 \int_{t_0}^t \mu(s)ds}. \tag{D.15}$$

Let k be a positive integer. Define the stopping time

$$m_k = \inf\{t : t_0 \le t < t_0 + T,\ |x| \ge k\}. \tag{D.16}$$

From Step 1, the plant has an almost surely unique solution on $[t_0, t_0 + T)$. Thus, $m_k \to +\infty$ almost surely as $k \to +\infty$.

Let $t_k = m_k \wedge t$ for any $t \in [t_0, t_0 + T)$. Noting $V(t_0, x(t_0)) = V_n(t_0)$, with (D.15), and using Lemma A.3 on the interval $[t_0, t_k]$, we get

$$\begin{aligned} EV(t_k, x(t_k)) &= V_n(t_0) + E\left\{\int_{t_0}^{t_k} \mathcal{L}V(\tau, x(\tau))d\tau\right\} \\ &\le V_n(t_0) + \beta_n E\bigg\{\int_{t_0}^{t_k} \sum_{j=1}^{n} \nu^{4\sigma_j+8j-6}(\tau) e^{c_0 \int_{t_0}^{\tau} \mu(s)ds} d\tau\bigg\} \\ &\le V_n(t_0) + \beta_n \int_{t_0}^{t} \sum_{j=1}^{n} \nu^{4\sigma_j+8j-6}(\tau) e^{c_0 \int_{t_0}^{\tau} \mu(s)ds} d\tau. \end{aligned} \tag{D.17}$$

Letting $k \to +\infty$, using Fatou's lemma, (D.17) can be rewritten as

$$EV(t, x(t)) \le V_n(t_0) + \beta_n \int_{t_0}^{t} \sum_{j=1}^{n} \nu^{4\sigma_j+8j-6}(\tau) e^{c_0 \int_{t_0}^{\tau} \mu(s)ds} d\tau. \tag{D.18}$$

From (D.14) and (D.18) we have

$$E\{V_n(t, x(t))\} \le f(t) \quad \forall t \in [t_0, t_0 + T), \tag{D.19}$$

where

$$f(t) = e^{-c_0 \int_{t_0}^{t} \mu(s)ds} \left(V_n(t_0) + \beta_n \int_{t_0}^{t} \sum_{j=1}^{n} \nu^{4\sigma_j+8j-6} e^{c_0 \int_{t_0}^{\tau} \mu(s)ds} d\tau\right). \tag{D.20}$$

By L'Hospital's rule we get

$$f(t) \sim \frac{\beta_n}{c_0} \sum_{j=1}^{n} \nu^{4\sigma_j+8j-4} \qquad \text{as } t \to t_0 + T. \tag{D.21}$$

This completes the proof of Lemma 2. ∎

Appendix E. Proof of Lemma 3. By (3.62) and Schwarz's inequality we get

$$\begin{aligned} \int_{t_0}^{t} E\left|e^{-\int_s^t \alpha_k \mu(\tau)d\tau} \xi_{k+1}(s)\right| ds &= \int_{t_0}^{t} e^{-\int_s^t \alpha_k \mu(\tau)d\tau} E|\xi_{k+1}(s)| ds \\ &\le \int_{t_0}^{t} e^{-\int_s^t \alpha_k \mu(\tau)d\tau} (E\xi_{k+1}^4)^{1/4} ds \\ &\le \sqrt{2} \int_{t_0}^{t} \mu^k e^{-\int_s^t \alpha_k \mu(\tau)d\tau} (EV_n)^{1/4} ds. \end{aligned} \tag{E.1}$$

From (2.10) and Lemma 2 we obtain that EV_n is bounded on $[t_0, t_0 + T)$, which, together with (E.1), shows that

$$\int_{t_0}^{t} E\left|e^{-\int_s^t \alpha_k \mu(\tau)d\tau} \xi_{k+1}(s)\right| ds < +\infty \tag{E.2}$$

holds $\forall t \in [t_0, t_0 + T)$, where $k = 1, \ldots, n-1$.

By (E.2) and Fubini's theorem, in Lemma A.2 we get (4.15).

This completes the proof of Lemma 3. ∎

Appendix F. Proof of Lemma 4. The proof includes two steps.

Step 1. We first prove $E\{M_1(t)\}=0$.

By (3.4) we get

$$
\begin{aligned}
E\bigg\{\int_{t_0}^{t} e^{-2\int_s^t \alpha_1\mu(s)ds}|\varphi_1|^2 ds\bigg\} &= \int_{t_0}^{t} e^{-2\int_s^t \alpha_1\mu(s)ds} E|\varphi_1|^2 ds \\
&\le 2c_{1,1}^2\int_{t_0}^{t} e^{-2\int_s^t \alpha_1\mu(s)ds}\nu^{2\sigma_1}E\xi_1^2 ds \\
&\quad + 2(c_{1,1}M+c_{1,2})^2\int_{t_0}^{t} e^{-2\int_s^t \alpha_1\mu(s)ds}\nu^{2\sigma_1}ds \\
&\le 4c_{1,1}^2\int_{t_0}^{t} e^{-2\int_s^t \alpha_1\mu(s)ds}\nu^{2\sigma_1}\sqrt{EV_n} \\
&\quad + 2(c_{1,1}M+c_{1,2})^2\int_{t_0}^{t} e^{-2\int_s^t \alpha_1\mu(s)ds}\nu^{2\sigma_1}ds. \qquad \text{(F.1)}
\end{aligned}
$$

By (2.10) and Lemma 2 we obtain that EV_n is bounded on $[t_0,t_0+T)$, which, together with (F.1), shows that

$$
E\bigg\{\int_{t_0}^{t} e^{-2\int_s^t \alpha_1\mu(s)ds}|\varphi_1|^2 ds\bigg\} < +\infty \qquad \text{(F.2)}
$$

holds $\forall t\in[t_0,t_0+T)$.

By (4.3) and (F.2) we get $E\{M_1(t)\}=0$.

Step 2. We then prove $E\{M_k(t)\}=0$, $k=2,\dots,n$.

From (3.45) we get

$$
\begin{aligned}
&E\bigg\{\int_{t_0}^{t} e^{-2\int_s^t \alpha_k\mu(s)ds}\Big|\varphi_k^T-\sum_{j=1}^{k-1}\frac{\partial x_k^*}{\partial x_j}\varphi_j^T\Big|^2 ds\bigg\} \\
&= \int_{t_0}^{t} e^{-2\int_s^t \alpha_k\mu(s)ds}E\Big|\varphi_k^T-\sum_{j=1}^{k-1}\frac{\partial x_k^*}{\partial x_j}\varphi_j^T\Big|^2 ds \\
&\le \tilde b_{k,1}\int_{t_0}^{t} e^{-2\int_s^t \alpha_k\mu(s)ds}\sum_{j=1}^{k}\mu^{2(k-j)}E\xi_j^2 ds \\
&\quad + \tilde b_{k,1}(M+1)^2\int_{t_0}^{t} e^{-2\int_s^t \alpha_k\mu(s)ds}\sum_{j=1}^{k}\nu^{2\sigma_j+4j-4k}(s)ds. \qquad \text{(F.3)}
\end{aligned}
$$

From (3.62) and Schwarz's inequality we have

$$
\sum_{j=1}^{k}\mu^{2(k-j)}E\xi_j^2 \le \sum_{j=1}^{k}\mu^{2(k-j)}\sqrt{E\xi_j^4} \le 2k\mu^{2k-2}\sqrt{EV_n}. \qquad \text{(F.4)}
$$

By Lemma 2 we obtain that EV_n is bounded on $[t_0,t_0+T)$, which, together with (2.10) and (F.3)–(F.4), shows that

$$
E\bigg\{\int_{t_0}^{t} e^{-2\int_s^t \alpha_k\mu(s)ds}\Big|\varphi_k^T-\sum_{j=1}^{k-1}\frac{\partial x_k^*}{\partial x_j}\varphi_j^T\Big|^2 ds\bigg\} < +\infty \qquad \text{(F.5)}
$$

holds $\forall t\in[t_0,t_0+T)$.

By (4.4) and (F.5), we get $E\{M_k(t)\}=0$, $k=2,\dots,n$.

This completes the proof of Lemma 4. $\square$

Appendix G. Proof of Theorem 1. By Lemma 2, conclusion (1) holds. Next, we prove conclusions (2)–(4).

Step 1. We first prove conclusion (2).

By (3.1), (3.32), (3.39), and Assumption 1 we obtain

$$x_1^2 \le 2(\xi_1^2 + M^2), \tag{G.1}$$

$$x_k^2 \le 2(\xi_k^2 + x_k^{*2}) \le 2kb_k^2 \sum_{j=1}^{k-1} \mu^{2k-2j}\xi_j^2 + 2\xi_k^2 + 2kb_k^2 M^2, \ k \ge 2. \tag{G.2}$$

From (G.1)–(G.2), (3.62), and Lemma A.4 we have

$$\begin{aligned} |x|^2 &\le \bar{b}\left(\sum_{j=1}^{n} \mu^{2n-2j}\xi_j^2 + M^2\right) \\ &\le \bar{b}\left(\sqrt{n}\left(\sum_{j=1}^{n} \mu^{4n-4j}\xi_j^4\right)^{1/2} + M^2\right) \\ &\le \bar{b}\big(2\sqrt{n}\mu^{2n-2}\sqrt{V_n} + M^2\big), \end{aligned} \tag{G.3}$$

where $\bar{b}$ is a positive constant.

By (G.3) and Lemma 2 we get

$$\begin{aligned} E|x|^2 &\le 2\bar{b}\sqrt{n}\mu^{2n-2}E\sqrt{V_n} + \bar{b}M^2 \\ &\le 2\bar{b}\sqrt{n}\mu^{2n-2}\sqrt{EV_n} + \bar{b}M^2 \\ &\le 2\bar{b}\sqrt{n}\mu^{2n-2}\sqrt{f(t)} + \bar{b}M^2. \end{aligned} \tag{G.4}$$

By (2.10) and Lemma 2 we obtain that $\mu^{2n-2}\sqrt{f(t)}$ is bounded on $[t_0, t_0 + T)$. From (G.4) we obtain that the mean-square of the state is bounded. Similarly, with Assumption 2, Proposition 1, (3.59), and (G.4) we can prove that the mean-square of the controller is bounded. Thus, conclusion (2) holds.

Step 2. We now prove conclusion (3).

When $k = 1$, from (3.1) and Lemma 2 we obtain

$$E|x_1 - y_r|^4 \le E\xi_1^4 \le 4EV_1 \le 4f(t). \tag{G.5}$$

By (2.10), (4.14), and (G.5) we get

$$\lim_{t \to t_0+T} E|x_1 - y_r|^4 = 0. \tag{G.6}$$

When $k \ge 2$, by (3.31)–(3.32) and (3.40)–(3.41) we get

$$\begin{aligned} |x_k - y_r^{(k-1)}| &\le \left|\xi_k + \frac{2}{T}\frac{\partial x_{k-1}^*}{\partial \mu}\mu^{3/2} + \sum_{j=1}^{k-2}\frac{\partial x_{k-1}^*}{\partial \xi_j}(-\alpha_j\mu\xi_j + \xi_{j+1}) - \mu\alpha_{k-1}\xi_{k-1}\right| \\ &\le \tilde{b}_k \sum_{j=1}^{k} \mu^{k-j}|\xi_j|, \end{aligned} \tag{G.7}$$

which, together with Lemma A.4, shows that

$$E|x_k - y_r^{(k-1)}|^4 \le \tilde{b}_k^4 k^3 \sum_{j=1}^{k} \mu^{4k-4j}E\xi_j^4 \le 4\tilde{b}_k^4 k^3 \mu^{4k-4}EV \le 4\tilde{b}_k^4 k^3 \mu^{4k-4} f(t), \tag{G.8}$$

where $\tilde{b}_k$ is a positive constant independent of M.

It follows from (2.10), (4.14), and (G.8) that

$$\begin{aligned}\lim_{t\to t_0+T} E|x_k - y_r^{(k-1)}|^4 &\le 4\tilde{b}_k^4 k^3 \lim_{t\to t_0+T} \mu^{4k-4} f(t)\\ &\le \frac{4\tilde{b}_k^4 k^3 \beta_n}{c_0} \lim_{t\to t_0+T} \sum_{j=1}^{n} \nu^{4\sigma_j+8j+4-8n}\\ &= 0, \end{aligned} \tag{G.9}$$

which shows that

$$\lim_{t\to t_0+T} E|x_i - y_r^{(i-1)}|^4 = 0, \ 1\le i\le n. \tag{G.10}$$

Step 3. We finally prove conclusion (4).

Noting $\alpha_n > 0$ is a constant, from (2.4), (4.2), and Lemma 4 we get

$$E\{\xi_n(t)\} = e^{-\int_{t_0}^t \alpha_n \mu(s)ds}\xi_n(t_0) = e^{-\alpha_n T^2(\frac{1}{t_0+T-t}-\frac{1}{T})}\xi_n(t_0). \tag{G.11}$$

By Lemma 1 we know that $\xi_n(t_0) < 0$, which, along with (G.11), implies that $E\{\xi_n\}$ is a nonpositive function satisfying

$$\lim_{t\to t_0+T} E\{\xi_n(t)\} = 0. \tag{G.12}$$

From (4.1) and Lemma 4 we have

$$E\{\xi_{n-1}(t)\} = e^{-\int_{t_0}^t \alpha_{n-1}\mu(s)ds}\xi_{n-1}(t_0) + E\bigg\{\int_{t_0}^t e^{-\int_s^t \alpha_{n-1}\mu(\tau)d\tau}\xi_n(s)ds\bigg\}, \tag{G.13}$$

which, in view of (2.4), (G.11), and Lemma 3, can be expressed as

$$\begin{aligned}E\{\xi_{n-1}(t)\} &= e^{-\int_{t_0}^t \alpha_{n-1}\mu(s)ds}\xi_{n-1}(t_0) + \int_{t_0}^t e^{-\int_s^t \alpha_{n-1}\mu(\tau)d\tau}E\{\xi_n(s)\}ds\\ &= e^{-\alpha_{n-1}T^2(\frac{1}{t_0+T-t}-\frac{1}{T})}\xi_{n-1}(t_0) + \frac{\int_{t_0}^t e^{\int_{t_0}^s (\alpha_{n-1}-\alpha_n)\mu(\tau)d\tau}\xi_n(t_0)ds}{e^{\int_{t_0}^t \alpha_{n-1}\mu(\tau)d\tau}}. \end{aligned} \tag{G.14}$$

Since α_{n-1} and α_n are positive constants, by using (G.14) we get

$$\begin{aligned}\lim_{t\to t_0+T} E\{\xi_{n-1}(t)\} &= \lim_{t\to t_0+T} \frac{\int_{t_0}^t e^{\int_{t_0}^s (\alpha_{n-1}-\alpha_n)\mu(\tau)d\tau}\xi_n(t_0)ds}{e^{\int_{t_0}^t \alpha_{n-1}\mu(\tau)d\tau}}\\ &= \lim_{t\to t_0+T} \frac{\xi_n(t_0)}{\alpha_{n-1}\mu(t)e^{\int_{t_0}^t \alpha_n\mu(\tau)d\tau}}\\ &= \lim_{t\to t_0+T} \frac{\xi_n(t_0)}{\alpha_{n-1}\mu(t)e^{\alpha_n T^2(\frac{1}{t_0+T-t}-\frac{1}{T})}}\\ &= 0. \end{aligned} \tag{G.15}$$

From Lemma 1 we obtain that $E\{\xi_{n-1}\}$ is a nonpositive function.

By using induction we conclude that $E\{\xi_1(t)\}$ is a nonpositive function with

$$\lim_{t\to t_0+T} E\{\xi_1(t)\} = 0. \tag{G.16}$$

Thus, (4.17) and (4.19) hold.

This completes the proof of Theorem 1. ∎

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