Brief paper

Stabilization of chains of linear parabolic PDE–ODE cascades

Xiang Xu a, Lu Liu b,∗, Miroslav Krstic c, Gang Feng b

a Department of Electrical and Electronic Engineering, Southern University of Science and Technology, Shenzhen, Guangdong Province, China
b Department of Biomedical Engineering, City University of Hong Kong, Kowloon, Hong Kong Special Administrative Region
c Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA

ARTICLE INFO

Article history:
Received 4 January 2022
Received in revised form 27 June 2022
Accepted 18 October 2022
Available online 2 December 2022

ABSTRACT

Over the past decade, stabilization problems have been solved for various cascade and “sandwich” configurations involving linear ODEs and PDEs of both hyperbolic and parabolic types. In this paper, we consider systems in which the output of the (i + 1)th ODE subsystem is the control input of the ith PDE subsystem, and in which the state of the ith PDE subsystem enters as control into the (i + 1)th ODE subsystems. We present some results, among which a representative case is for the case where the ODEs in the chain are scalar and the PDEs are pure delays, in two major directions. First, we allow for the virtual inputs to be affected by PDE dynamics different from pure delays: we allow the PDEs to include diffusion, i.e. to be parabolic, and to even have counter-convection, and, in addition, for the PDE dynamics to enter the ODEs not only with the PDE’s boundary value but also in a spatially-distributed (integrated) fashion. Second, we allow the ODEs in the chain to be not just scalar ODEs in a strict-feedback configuration but general LTI systems. We develop an n-step backstepping procedure and prove that the resulting closed-loop system is exponentially stable. A simulation example is provided to illustrate the effectiveness of our controllers.

© 2022 Elsevier Ltd. All rights reserved.

1. Introduction

Diffusion equations, as classical parabolic partial differential equations (PDEs), can be used to model a variety of phenomena in different fields (see, e.g., Beauchard and Cannarsa (2017), Evans (1998), Koga, Camacho-Solorio, and Krstic (2021), Kolmogoroff, Petrovsky, and Piscounoff (1938), Miró, Ginestar, Verdú, and Hennig (2002), Rasheed et al. (2021), Wettlaufer (1991) and Zalba, Marín, Cabeza, and Mehling (2003)). In past decades, boundary control of diffusion equations has been extensively studied (see, e.g., Baccoli, Pisano, and Orlov (2015), Bekiaris-Liberis and Krstic (2010a), Boskovic, Krstic, and Liu (2001), Espitia, Karafyllis, and Krstic (2021), Hashimoto and Krstic (2016), Karafyllis, Espitia, and Krstic (2021), Karafyllis, Smyshlyaev, and Chrysafi (2019), Krstic (2009a, 2009b), Krstic and Smyshlyaev (2008a), Smyshlyaev and Krstic (2007a, 2007b), Susto and Krstic (2010) and Wang and Krstic (2019)). In particular, the authors in Bekiaris-Liberis and Krstic (2010a), Krstic (2009a) and Susto and Krstic (2010) considered boundary control problems of an ordinary differential equation (ODE) cascaded with a second-order diffusion PDE. In Hashimoto and Krstic (2016) and Krstic (2009b), the authors discussed control of diffusion PDEs with time delays. Adaptive boundary control was developed for diffusion PDEs in Karafyllis et al. (2019), Krstic and Smyshlyaev (2008a) and Smyshlyaev and Krstic (2007a, 2007b). In Wang and Krstic (2019), the authors investigated a system consisting of a diffusion PDE sandwiched between two ODEs. More recently, in Espitia et al. (2021) and Karafyllis et al. (2021), event-triggered boundary control problems of diffusion PDEs were studied.

In the study of boundary control for diffusion PDEs, the backstepping approach is frequently used. The backstepping technique was initially developed in the 1990s for designing stabilizing controls for dynamic systems with a triangular structure (Kokotovic, 1992; Krstic, Kokotovic, & Kanellakopoulos, 1995). It was further successfully applied to designing feedback controllers for linear systems with bounded delays in a series of works (Bekiaris-Liberis & Krstic, 2010b, 2011; Krstic, 2008, 2010; Krstic & Smyshlyaev, 2008b). In our recent work (Xu, Liu, Krstic, & Feng, 2022), a predictor feedback controller was developed based on the backstepping approach for linear systems with distributed infinite delays. This technique was first applied to deal with linear systems with actuator and sensor dynamics modelled by diffusion equations rather than pure delays in Krstic (2009a).
The results obtained in Krstic (2009a) were then extended to the case of input dynamics governed by diffusion with counter-convection in Susto and Krstic (2010). In Bekiaris-Liberis and Krstic (2010a), the authors further developed a backstepping-forwarding approach to handle linear systems with distributed actuator dynamics modelled by diffusion equations with counter-convection. It is noted that all of these works only considered a single PDE–ODE cascade system.

In this paper, we consider a more general PDE–ODE cascade system, that is, a chain of subsystems as shown in Fig. 1. This system consists of n subsystems, where each subsystem is composed of an ODE cascaded by a second-order diffusion PDE. The output of the (i + 1)th subsystem is the control input of the ith subsystem. Such a chain of subsystems is to be controlled by one control signal $U(t)$. Many practical systems have a chain structure, such as vehicle platoon systems (Liang, Martensson, & Johansson, 2015; Floeg, Van De Wouw, & Nijmeijer, 2013), mobile robots systems for formation control (Cruz-Morales, Velasco-Villa, & Rodriguez-Angelos, 2018; Dai, He, Lin, & Wang, 2017), and density flow systems (Bastin & Coron, 2013). Moreover, there exist some practical systems described by diffusion equations with chain or even more complex structures, see, for example, Deutscher (2021), Kondor and Lafferty (2002), Yang et al. (2018) and Zhou et al. (2022). More recently, control of PDE systems with a chain structure has also been considered (Auriol, 2020; Auriol, Bribiesca-Argomedo, Niculescu, & Redaud, 2021; Redaud, Auriol, & Niculescu, 2021). Inspired by these observations, this work considers control of a chain of PDE–ODE cascade subsystems, and a new controller is proposed based on the backstepping technique. Distributed diffusion–counterconvection is considered in this work. The main contributions of this work, compared with relevant existing works, can be summarized as follows.

First, a more general PDE–ODE cascade system is considered. In most existing works, the authors only consider a system consisting of a single PDE cascaded with a single ODE (Bekiaris-Liberis & Krstic, 2010a; Krstic, 2009a; Susto & Krstic, 2010) or a single PDE sandwiched between two ODEs (Wang & Krstic, 2019). In contrast, our work considers a more general case where multiple PDEs are cascaded with multiple ODEs. The system considered in our work includes those of Bekiaris-Liberis and Krstic (2010a), Krstic (2009a) and Susto and Krstic (2010) as its special cases.

Second, a new controller is proposed based on the backstepping technique. In particular, we adopt an n-step induction procedure to construct the controller via the backstepping technique. At each step, the backstepping-forwarding transformation proposed in Bekiaris-Liberis and Krstic (2010a) is performed. It is proven that under the proposed new controller, the system under consideration is globally exponentially stabilized.

The rest of this paper is organized as follows. In Section 2, our problem formulation is given. In Section 3, we propose a novel backstepping controller for a chain of PDE–ODE cascades and provide rigorous stability analysis. A simulation example is given in Section 4, and conclusions are drawn in Section 5.

Notation: Throughout this paper, the following notation is used. The notation $| \cdot |$ represents the absolute value of real numbers, the module of complex numbers, the p norm of vectors, or the induced 2-norm of matrices. The symbols I and 0 represent identity matrix and zero matrix with compatible dimensions, respectively. For any square matrix $A$, $\lambda(A)$ represents any of its eigenvalue. Let $u(x, t) : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a differentiable function with respect to $(x, t)$. Then $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ denote the first- and second-order partial derivatives of $u$ with respect to $x$, respectively, and $\frac{\partial u}{\partial t}$ is the first-order partial derivative of $u$ with respect to $t$.

### 2. Problem formulation

In this paper, we consider the following system consisting of $n$ subsystems, as shown in Fig. 1, where each subsystem is a diffusion PDE cascaded with an ODE,

\[
\dot{X}_i = A_i X_i + \int_0^{D_i} B_i(x) u_i(x, t) \, dx,
\]

\[
\frac{\partial u_i}{\partial t} = \frac{\partial^2 u_i}{\partial x^2} - b_i \frac{\partial u_i}{\partial x},
\]

where $X_i \in \mathbb{R}^{h_i}$ and $u_i : [0, D_i] \times [0, +\infty) \rightarrow \mathbb{R}$ are the plant state and the actuator state of the $i$th subsystem, respectively, $U(t)$ is the control signal. Moreover, for $i = 1, 2, \ldots, n$, $A_i \in \mathbb{R}^{h_i \times h_i}$, $b_i \geq 0$, $B_i : [0, D_i] \rightarrow \mathbb{R}^{h_i \times 1}$ is a vector function satisfying $\int_0^{D_i} |B_i(x)| \, dx < +\infty$ and for $i = 2, \ldots, n$, $C_i \in \mathbb{R}^{1 \times h_i}$.

When $b_i > 0$, the term $-b_i \frac{\partial u_i}{\partial x}$ has the effect of counter-convection which opposes the propagation of the control signal (Bekiaris-Liberis & Krstic, 2010a; Susto & Krstic, 2010).

### 3. Main results

In this section, we construct a controller by the backstepping technique and perform the backstepping-forwarding transformation in each step. We construct the controller in the following steps.

**Step 1:** The first subsystem is written as follows:

\[
\dot{X}_1 = A_1 X_1 + \int_0^{D_1} B_1(x) u_1(x, t) \, dx,
\]

\[
\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2} - b_1 \frac{\partial u_1}{\partial x},
\]

\[
\frac{\partial u_1}{\partial t}(0, t) = 0,
\]

\[
u_1(D_1, t) = U_1(t),
\]

where $U_1 = C_2 X_2$ is the virtual input of the first subsystem. We apply the backstepping-forwarding transformation proposed...
in Bekiaris-Liberis and Krstic (2010a) for system (3) and define $Z_i$ as follows:

$$Z_i = X_i + \int_0^{D_i} g_i(x) u_i(x, t) \, dx,$$

(4)

where $g_i$ is the solution to the following boundary value problem.

$$g_i''(x) + b_1 g_i'(x) + B_i(x) = A_1 g_i(x),$$

(5a)

$$b_1 g_i(0) + g_i(0) = 0,$$

(5b)

$$g_i(D_i) = 0,$$

(5c)

with $g_i'(x)$ and $g_i''(x)$ being the first and second order derivatives of $g_i$ with respect to $x$, respectively. Then we have

$$\dot{Z}_i = \dot{A}_i Z_i + \dot{B}_i U_i,$$

(6)

where $\dot{A}_i = A_i$ and $\dot{B}_i = -g_i'(D_i)$. It follows from Bekiaris-Liberis and Krstic (2010a) that if we can choose $U_i = K_i Z_i$ such that $\dot{A}_i + \dot{B}_i K_i$ is Hurwitz, then system (3) is globally exponentially stabilized. However, based on our problem formulation, the virtual input $U_i$ equals $C_i X_i$ which is related to the second subsystem. Thus we need to consider the next subsystem in the next step.

**Step 2 ($2 \leq i \leq n - 1$):** Assume we have already constructed $Z_{i-1}$ in Step $i-1$ such that

$$\dot{Z}_{i-1} = \dot{A}_{i-1} Z_{i-1} + \dot{B}_{i-1} C_i X_i,$$

(7)

Then we have

$$\begin{bmatrix} \dot{Z}_{i-1} \\ \dot{X}_i \end{bmatrix} = \begin{bmatrix} \hat{A}_{i-1} & \hat{B}_{i-1} C_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} Z_{i-1} \\ X_i \end{bmatrix} + \int_0^{D_i} \begin{bmatrix} 0 \\ \hat{B}_i(x) \end{bmatrix} u_i(x, t) \, dx,$$

(8a)

$$\frac{\partial u_i}{\partial t} = \frac{\partial^2 u_i}{\partial x^2} - b_1 \frac{\partial u_i}{\partial x},$$

(8b)

$$\frac{\partial u_i}{\partial x}(0, t) = 0,$$

(8c)

$$u_i(D_i, t) = U_i,$$

(8d)

where $U_i = C_i X_{i+1}$ is the virtual input of the $i$th subsystem. Then we can perform the backstepping-forwarding transformation in Bekiaris-Liberis and Krstic (2010a) for system (8) and define $Z_i$ as follows:

$$Z_i = \begin{bmatrix} Z_{i-1} \\ X_i \end{bmatrix} + \int_0^{D_i} g_i(x) u_i(x, t) \, dx,$$

(9)

where $g_i$ is the solution to the following boundary value problem.

$$g_i''(x) + b_1 g_i'(x) + \begin{bmatrix} 0 \\ \hat{B}_i(x) \end{bmatrix} g_i(x) = \begin{bmatrix} \hat{A}_{i-1} \\ 0 \end{bmatrix} g_i(x),$$

(10a)

$$b_1 g_i(0) + g_i(0) = 0,$$

(10b)

$$g_i(D_i) = 0.$$  

(10c)

Then we have

$$\dot{Z}_i = \dot{A}_i Z_i + \dot{B}_i U_i,$$

(11)

where

$$\dot{A}_i = \begin{bmatrix} \hat{A}_{i-1} \\ 0 \end{bmatrix},$$

(12)

$$\dot{B}_i = -g_i'(D_i).$$

(13)

It follows from Bekiaris-Liberis and Krstic (2010a) that if we can choose $U_i = K_i Z_i$ such that $\dot{A}_i + \dot{B}_i K_i$ is Hurwitz, then system (8) is globally exponentially stabilized. Similarly, the virtual input $U_i$ equals $C_{i+1} X_{i+1}$ which is related to the next subsystem. We continue this process until $i = n - 1$.

**Step n:** We have already constructed $Z_{n-1}$ in previous steps such that

$$\dot{Z}_{n-1} = \dot{A}_{n-1} Z_{n-1} + \dot{B}_{n-1} C_n X_n,$$

(14)

Then we have

$$\begin{bmatrix} \dot{Z}_{n-1} \\ X_n \end{bmatrix} = \begin{bmatrix} \hat{A}_{n-1} & \hat{B}_{n-1} C_n \\ 0 & A_n \end{bmatrix} \begin{bmatrix} Z_{n-1} \\ X_n \end{bmatrix} + \int_0^{D_n} \begin{bmatrix} 0 \\ \hat{B}_n(x) \end{bmatrix} u_n(x, t) \, dx,$$

(15a)

$$\frac{\partial u_n}{\partial t} = \frac{\partial^2 u_n}{\partial x^2} - b_n \frac{\partial u_n}{\partial x},$$

(15b)

$$\frac{\partial u_n}{\partial x}(0, t) = 0,$$

(15c)

$$u_n(D_n, t) = U.$$

(15d)

Define $Z_n$ as follows:

$$Z_n = \begin{bmatrix} Z_{n-1} \\ X_n \end{bmatrix} + \int_0^{D_n} g_n(x) u_n(x, t) \, dx,$$

(16)

where $g_n$ is the solution to the following boundary value problem,

$$g_n''(x) + b_n g_n'(x) + \begin{bmatrix} 0 \\ \hat{B}_n(x) \end{bmatrix} g_n(x) = \begin{bmatrix} \hat{A}_{n-1} \\ 0 \end{bmatrix} g_n(x),$$

(17a)

$$b_n g_n(0) + g_n(0) = 0,$$

(17b)

$$g_n(D_n) = 0.$$  

(17c)

Then we have

$$\dot{Z}_n = \dot{A}_n Z_n + \dot{B}_n U(t),$$

(18)

where

$$\dot{A}_n = \begin{bmatrix} \hat{A}_{n-1} & \hat{B}_{n-1} C_n \\ 0 & A_n \end{bmatrix},$$

(19)

$$\dot{B}_n = -g_n'(D_n).$$

(20)

Therefore, inspired by Bekiaris-Liberis and Krstic (2010a), we can construct the following controller,

$$U(t) = K_n Z_n,$$

(21)

where $K_n$ is chosen such that $\dot{A}_n + \dot{B}_n K_n$ is Hurwitz.

Our control strategy can be illustrated Fig. 2. It can be seen that in each Step $i (1 \leq i \leq n - 1)$, we start from a subsystem as described in (3) and (8) with a virtual input and perform the backstepping-forwarding transformation for it. However, since the virtual input is determined by the $(i+1)$th subsystem, we have to go to Step $i+1$. In the final step $n$, we obtain a subsystem (15) with the real control input and compensate it via the backstepping-forwarding transformation. Moreover, since in the final step, we transform the system into the form of the closed-loop system considered in Bekiaris-Liberis and Krstic (2010a), the well-posedness of our closed-loop system is equivalent to that in Bekiaris-Liberis and Krstic (2010a).

Furthermore, to ensure the existence of controller (21), the following two assumptions are needed.

**Assumption 3.1.** The following matrices are invertible:

$$E_i = \begin{bmatrix} I \\ 0 \end{bmatrix} \, e^{\hat{A}_i t} \begin{bmatrix} 0 \\ -b_i I \end{bmatrix} \, d_i, \ i = 1, 2, \ldots, n.$$  

(22)
Remark 3.2. The matrix pair \((\hat{A}_n, \hat{B}_n)\) is stabilizable.

Assumption 3.2. The matrix pair \((\hat{A}_n, \hat{B}_n)\) is stabilizable.

Remark 3.1. Assumption 3.1 is sufficient and necessary for existence of \(g_i(x)\) and Assumption 3.2 is sufficient and necessary for existence of \(K_i\). In fact, it can be calculated from (6) and (10) that

\[
g_i(x) = \left[ \begin{array}{cc} I & 0 \\ -b_i & I \end{array} \right] M_i + \int_0^x e^{\hat{A}_i (x-y)} \left[ \begin{array}{cc} 0 & I \\ -b_i & 0 \end{array} \right] \left[ \begin{array}{cc} \hat{A}_i & 0 \\ 0 & \hat{B}_i \end{array} \right] dy,
\]

where \(M_i = E_i^{-1} [I 0] \int_0^{D_i} e^{\hat{A}_i (0-y)} \left[ \begin{array}{cc} 0 & 1 \\ -b_i & 0 \end{array} \right] \left[ \begin{array}{cc} \hat{A}_i & 0 \\ 0 & \hat{B}_i \end{array} \right] dy.\)

Eq. (24) shows exactly why Assumption 3.1 is needed.

Remark 3.2. When \(b_i = 0, i = 1, 2, \ldots, n\), Assumption 3.1 holds if and only if

\[
\lambda(A_i) \notin \left\{ -\frac{\pi^2 (2j + 1)^2}{4D_i^2}, j = 1, 2, \ldots, n \right\},
\]

for \(i = 1, 2, \ldots, n\). Moreover, if \(b_i = 0, i = 1, 2, \ldots, n\) and \(B_i(x) = B_i \delta(x)\) where \(\delta\) represents the Dirac function, then Assumption 3.2 holds if and only if the matrix pair \((A_i, B_i)\) is stabilizable and for \(i = 2, 3, \ldots, n\),

\[
\text{rank} \left[ \begin{array}{cc} A_i - \lambda I & B_i \\ \lambda C_i & 0 \end{array} \right] = k_i + 1, \quad \forall \lambda \in \mathbb{C}^+.\]

Then we are ready to present our main result, which is given in the following theorem.

Theorem 3.1. Under Assumptions 3.1 and 3.2, the system (1)–(2) with controller (21) is globally exponentially stable in the sense that there exist \(\eta, \kappa > 0\) such that

\[
\Omega(t) \leq \eta e^{-\kappa t} \Omega(0), \forall t \geq 0,
\]

where

\[
\Omega(t) = \sum_{i=1}^n \left( |X_i|^2 + \int_0^{D_i} |u_i(x, t)|^2 dx \right).
\]

Proof. We only provide a sketch of the proof here. Detailed calculations can be found in the Appendix. The theorem can be proven by induction.

Step 1: Define

\[
\omega_i(x, t) = u_i(x, t) - C_i X_i .
\]

Then we have

\[
\frac{\partial \omega_i(x, t)}{\partial t} = \frac{\partial^2 \omega_i(x, t)}{\partial x^2} - b_i \frac{\partial \omega_i(x, t)}{\partial x} - C_i \tilde{X}_i ,
\]

\[
\frac{\partial \omega_i(x, t)}{\partial x}(0, t) = 0,
\]

\[
\omega_i(D_i, t) = 0.
\]

We define the following function,

\[
V_i = \frac{1}{2} \int_0^{D_i} e^{-h_i \eta} \omega_i^2(x, t) dx + |Z_i|^2 .
\]

It can be proven that there exist positive constants \(\lambda_1, \alpha_1, \beta_1, \gamma_1, M_i, \bar{M}_i, \xi\) such that

\[
V_i \leq -\lambda_i V_i + \alpha_1 |X_i|^2 + \beta_1 |Z_i|^2 + \gamma_1 \sup_{x \in [0, D_i]} |u_i(x, t)|^2 ,
\]

where

\[
\Omega_i = \left( |X_i|^2 + \int_0^{D_i} |u_i(x, t)|^2 dx \right).
\]

Step 2: \(i \leq n - 1\): Assume that for the \((i-1)\)th subsystem, there exist a function \(V_{i-1}\) and positive constants \(\lambda_{i-1}, \alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}, M_{i-1}, \bar{M}_{i-1}, \xi_{i-1}\) such that

\[
V_{i-1} \leq -\lambda_{i-1} V_{i-1} + \alpha_{i-1} |X_{i-1}|^2 + \beta_{i-1} |Z_{i-1}|^2 + \gamma_{i-1} \sup_{x \in [0, D_{i-1}]} |u_{i-1}(x, t)|^2 ,
\]

where

\[
\Omega_{i-1} = \sum_{j=1}^{i-1} \left( |X_j|^2 + \int_0^{D_j} |u_j(x, t)|^2 dx \right).
\]

Define

\[
\omega_i(x, t) = u_i(x, t) - C_{i+1} X_{i+1}
\]

and one has that

\[
\frac{\partial \omega_i(x, t)}{\partial t} = \frac{\partial^2 \omega_i(x, t)}{\partial x^2} - b_i \frac{\partial \omega_i(x, t)}{\partial x} - C_{i+1} \tilde{X}_{i+1} ,
\]

\[
\frac{\partial \omega_i(x, t)}{\partial x}(0, t) = 0,
\]

\[
\omega_i(D_i, t) = 0.
\]
Define the following function,
\[ V_i = V_{i-1} + k_i \dot{V}_i, \]  
where
\[ \dot{V}_i = \frac{1}{2} \int_0^{\theta_i} e^{-b_\tau \omega_i^2(x, t)dx} + |Z_i|^2. \]
(40)
Then by choosing sufficiently large \( k_i > 0 \), one has that for the \( i \)th subsystem, there exist a function \( V_i \) and positive constants \( \lambda_i, \alpha_i, \beta_i, \gamma_i, M_i, M_{i-1} \) such that
\[ \dot{V}_i \leq -\lambda_i V_i + \alpha_i |X_{i+1}|^2 + \beta_i |Z_i|^2 + \gamma_i \sup_{x \in [0, \theta_i+1]} |u_{i+1}(x, t)|^2. \]
(42)
\[ M_i \Omega_i - l_i |X_{i+1}|^2 \leq V_i \leq M_{i-1} \Omega_i + |X_{i+1}|^2, \]
(43)
where
\[ \Omega_i = \sum_{j=1}^{i} \left( |X_j|^2 + \int_0^{\theta_j} |u_j(x, t)|^2 dx \right). \]
(44)

**Step n:** It follows from the previous steps that there exist a function \( V_{n-1} \) and positive constants \( \lambda_{n-1}, \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}, M_{n-1}, M_{n-1}, l_{n-1} \) such that
\[ \dot{V}_{n-1} \leq -\lambda_{n-1} V_{n-1} + \alpha_{n-1} |X_{n}|^2 + \beta_{n-1} |Z_{n-1}|^2 + \gamma_{n-1} \sup_{x \in [0, \theta_{n-1}]} |u_n(x, t)|^2, \]
(45)
\[ M_{n-1} \Omega_{n-1} - l_{n-1} |X_n|^2 \leq V_{n-1} \leq M_{n-1} \Omega_{n-1} + |X_n|^2, \]
(46)
where
\[ \Omega_{n-1} = \sum_{j=1}^{n-1} \left( |X_j|^2 + \int_0^{\theta_j} |u_j(x, t)|^2 dx \right). \]
(47)
Define \( \omega_n(x, t) \) as follows:
\[ \omega_n(x, t) = u_n(x, t) - K_n Z_n. \]
(48)
Then after the \( n \)-step backstepping procedure, we can transform the system (1)–(2) with controller (21) into the following “target” system
\[ \dot{Z}_n = (\hat{A}_n + \hat{B}_n K_n)Z_n, \]
(49a)
\[ \frac{\partial \omega_n(x, t)}{\partial t} = \frac{\partial^2 \omega_n(x, t)}{\partial x^2} - b_n \omega_n(x, t) - K_n(\hat{A}_n + \hat{B}_n K_n)Z_n, \]
(49b)
\[ \frac{\partial \omega_n(0, t)}{\partial x} = 0, \]
(49c)
\[ \omega_n(D_n, t) = 0. \]
(49d)

Define the following function,
\[ \dot{V}_n = \frac{1}{2} \int_0^{\theta_n} e^{-b_\tau \omega_n^2(x, t)dx} + cZ_n^2 P_n Z_n, \]
(50)
where \( c > 0 \) is a positive constant and \( P_n \) is a positive definite symmetric matrix such that
\[ P_n(\hat{A}_n + \hat{B}_n K_n) + (\hat{A}_n + \hat{B}_n K_n)^T P_n = -I. \]
(51)
Consider the following Lyapunov function candidate,
\[ V = V_{n-1} + k_n \dot{V}_n. \]
(52)
It can be proven that by choosing sufficiently large \( k_n > 0 \), there exist positive constants \( \lambda, M \) and \( \bar{M} \) such that
\[ \dot{V} \leq -\lambda V, \]
(53)
\[ M \Omega \leq V \leq \bar{M} \Omega, \]
(54)
The theorem can be thus proven. □

**Remark 3.3.** In Step \( n \) of this proof, inspired by Bekiaris-Liberis and Krstic (2010a), there is another choice of \( \omega_n(x, t) \) that is given as follows:
\[ \omega_n(x, t) = u_n(x, t) - r(x)Z_n, \]
(55)
where \( r(x) \) is the solution to the following boundary value problem,
\[ r''(x) = b_n r'(x) + r(x)(\hat{A}_n + \hat{B}_n K_n), \]
(56a)
\[ r'(0) = 0, \]
(56b)
\[ r(D_n) = K_n, \]
(56c)
The “target” system in this case can thus be written as:
\[ \dot{Z}_n = (\hat{A}_n + \hat{B}_n K_n)Z_n, \]
(57a)
\[ \frac{\partial \omega_n(x, t)}{\partial t} = \frac{\partial^2 \omega_n(x, t)}{\partial x^2} - b_n \frac{\partial \omega_n(x, t)}{\partial x}, \]
(57b)
\[ \frac{\partial \omega_n(0, t)}{\partial x} = 0, \]
(57c)
\[ \omega_n(D_n, t) = 0, \]
(57d)
which is simpler compared with (49). However, it follows from Remark 3.1 that \( r(x) \) exists if and only if \( K_n \) is chosen such that the following matrix is invertible
\[ \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \left[ \begin{array}{cc} \hat{A}_n + \hat{B}_n K_n & -b_n I \\ -b_n I & \end{array} \right]^{-1} \begin{bmatrix} I \\ -b_n I \end{bmatrix}. \]
(58)
In contrast, by choosing a different \( \omega_n(x, t) = u_n(x, t) - K_n Z_n \), and constructing a new Lyapunov function, we avoid this restriction in our proof. This is an additional contribution of this work.

4. A simulation example

In this section, we provide an illustrative example to show the effectiveness of the proposed controller. We consider the following system with distributed diffusion–counterconvection
\[ \dot{X}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} X_1 + \int_0^1 x u_1(x, t)dx, \]
(59a)
\[ \frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial u_1}{\partial x}, \]
(59b)
\[ \frac{\partial u_1(0, t)}{\partial x} = 0, \]
(59c)
\[ u_1(1, t) = 2X_1 + \int_0^1 x u_2(x, t)dx, \]
(59d)
\[ \dot{X}_2 = 2X_2 + \int_0^1 x u_2(x, t)dx, \]
(59e)
\[ \frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2} - 2 \frac{\partial u_2}{\partial x}, \]
(59f)
\[ \frac{\partial u_2(0, t)}{\partial x} = 0, \]
(59g)
\[ u_2(1, t) = X_3, \]
(59h)
\[ \dot{X}_3 = u_3(0, t) + u_3(1, t), \]
(59i)
\[ \frac{\partial u_3}{\partial t} = \frac{\partial^2 u_3}{\partial x^2} - \frac{\partial u_3}{\partial x}, \]
(59j)
\[ \frac{\partial u_3(0, t)}{\partial x} = 0, \]
(59k)
\[ u_3(2, t) = U(t). \]
(59l)
The initial conditions are given as follows:
\[ X_1(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \]
(60)
It can be calculated that Assumption 3.1 is satisfied if the following three matrices are all invertible,

\[
E_1 = \begin{bmatrix} 1 & 0.7839 \\ 0 & 1.7839 \end{bmatrix},
\]

\[
E_2 = \begin{bmatrix} 1 & 1.2015 \\ 0 & 2.2015 \end{bmatrix},
\]

\[
E_3 = \begin{bmatrix} 1 & 6.2399 \\ 0 & 7.2399 \end{bmatrix}.
\]

They are indeed invertible and Assumption 3.1 is thus satisfied. Furthermore, we can calculate that

\[
\hat{A}_3 = \begin{bmatrix} 0 & -0.2411 \\ 0 & 0.7589 \end{bmatrix},
\]

\[
\hat{B}_3 = \begin{bmatrix} -1.6644 \\ 0.2232 \end{bmatrix}.
\]

The stabilizability of \((\hat{A}_3, \hat{B}_3)\) can thus be verified, which implies that Assumption 3.2 is satisfied. Therefore, we can design controller (21) for system (59). Fig. 3 shows the closed-loop response for system (59) under controller (21). It can be seen that our controller (21) is effective.

5. Conclusions

In this paper, we have proposed a novel controller based on the backstepping technique for a system consisting of multiple subsystems. Each subsystem is an ODE cascaded by a second-order diffusion PDE. We have adopted an n-step backstepping procedure to construct the controller. It has been proven that under the proposed new controller, the cascade system is globally exponentially stabilized. Distributed diffusion–counterconvection is also considered. A simulation example has been given to illustrate the effectiveness of our controller. In our future research, we will consider ODEs cascaded with different kinds of PDEs, such as high-dimensional wave PDEs and Navier–Stokes PDEs.

Appendix

In the appendix, we provide detailed calculations for the proof of Theorem 3.1, which is given as follows.

Detailed Calculations for the Proof of Theorem 3.1.

Step 1: We first prove (32). The derivative of \(V_1\) can be calculated as follows:

\[
\dot{V}_1 = \int_0^{\eta_1} e^{-b_1 \int_0^t \omega_1(x,t) dt} \int_0^{\eta_1} \omega_1(x,t) \frac{\partial \omega_1(x,t)}{\partial t} dx + Z_1^2 \dot{Z}_1,
\]

\[
= \int_0^{\eta_1} e^{-b_1 \int_0^t \omega_1(x,t) dt} \left( \frac{\partial^2 \omega_1(x,t)}{\partial x^2} - b_1 \frac{\partial \omega_1(x,t)}{\partial x} - C_2 \dot{X}_2 \right) dx
\]

\[
+ Z_1^2 (\dot{A}_1 Z_1 + \dot{B}_1 C_2 X_2) 
\]

\[
= e^{-b_1 \int_0^t \omega_1(x,t) dt} \left( \frac{\partial \omega_1(x,t)}{\partial x} \right) \bigg|_{\omega_1=0} - \int_0^{\eta_1} e^{-b_1 \int_0^t \omega_1(x,t) dt} \left( \frac{\partial \omega_1(x,t)}{\partial x} \right)^2 dx
\]

\[
- \int_0^{\eta_1} e^{-b_1 \int_0^t \omega_1(x,t) dt} \omega_1(x,t) dx \dot{Z}_1 + \dot{A}_1 ||Z_1||^2 + \dot{B}_1 C_2 X_2
\]

\[
\leq e^{-b_1 \int_0^t \omega_1(x,t) dt} \left( \frac{\partial \omega_1(x,t)}{\partial x} \right)^2 dx + \left( \dot{A}_1 + \frac{1}{2} \right) ||Z_1||^2
\]

\[
+ \frac{1}{2} \left( \dot{B}_1 C_2 X_2 \right)^2 - \int_0^{\eta_1} e^{-b_1 \int_0^t \omega_1(x,t) dt} \omega_1(x,t) dx \dot{C}_2 X_2. 
\]

It follows from Poincaré’s inequality that

\[
\int_0^{\eta_1} e^{-b_1 \int_0^t \omega_1(x,t) dt} \left( \frac{\partial \omega_1(x,t)}{\partial x} \right)^2 dx \geq e^{-b_1 \eta_1} \int_0^{\eta_1} \left( \frac{\partial \omega_1(x,t)}{\partial x} \right)^2 dx
\]

\[
\geq \frac{e^{-b_1 \eta_1}}{4D_1^2} \int_0^{\eta_1} \omega_1^2(x,t) dx \geq \frac{e^{-b_1 \eta_1}}{4D_1^2} \int_0^{\eta_1} e^{-b_1 \omega_1^2(x,t) dx} dx.
\]

Moreover, since \(\dot{Z}_2 = A_2 Z_2 + B_2(x) u_2(x,t) dx\), one has

\[
\dot{V}_1 = \leq - \frac{b_1}{2} \dot{B}_1 C_2^2 |X_2|^2 + e^{-b_1 \int_0^t \omega_1^2(x,t) dx} \left( \dot{A}_1 + \frac{3}{2} |Z_1|^2 \right) + \frac{1}{2} \dot{B}_1 C_2^2 |X_2|^2 + \frac{e^{b_1 \int_0^t \omega_1^2(x,t) dx}}{8D_1^2} \int_0^{\eta_1} e^{-b_1 \omega_1^2(x,t) dx} dx
\]

\[
+ \frac{1}{2} \frac{e^{b_1 \int_0^t \omega_1^2(x,t) dx}}{2} \int_0^{\eta_1} B_2(x) u_2(x,t) dx \]

\[
\leq - \lambda_1 V_1 + \alpha_1 |X_2|^2 + \beta_1 |Z_1|^2 + \gamma_1 \sup_{x \in [0, \eta_2]} \eta_1 \left( u_2(x,t) \right)^2, 
\]

where

\[
\lambda_1 = \min \left( \frac{e^{-b_1 \eta_1}}{4D_1^2}, 1 \right),
\]

\[
\alpha_1 = \frac{1}{2} |\dot{B}_1 C_2|^2 + 4e^{b_1 \int_0^t \omega_1^2(x,t) dx} |A_2|^2,
\]

\[
\beta_1 = |A_1| + \frac{3}{2} |\dot{A}_1|,
\]

\[
\gamma_1 = 4e^{b_1 \int_0^t \omega_1^2(x,t) dx} \left( \frac{\int_0^{\eta_1} B_2(x) dx \right)^2. 
\]

We further prove (33). The right hand side of (33) can be proven directly by the definitions of \(\omega_1\) and \(Z_1\), and

\[
\mathcal{M}_1 = 3 + 2 \int_0^{\eta_1} |g_i(x)|^2 dx + D_1 |C_2|^2.
\]

For the left hand side of (33), it is noted that

\[
X_1 = Z_1 - \int_0^{\eta_1} g_i(x) \omega_1(x,t) dx - \int_0^{\eta_1} g_i(x) dx C_2 X_2.
\]

\[
u_i(x,t) = \omega_1(x,t) + C_2 X_2.
\]

Then by applying Cauchy–Schwarz inequality, one has

\[
|X_1|^2 \leq 3 |Z_1|^2 + 3e^{b_1 \int_0^t \omega_1^2(x,t) dx} \int_0^{\eta_1} |g_i(x)|^2 dx \int_0^{\eta_1} e^{-b_1 \omega_1^2(x,t) dx} dx
\]

\[
+ 3D_1 |C_2|^2 \int_0^{\eta_1} |g_i(x)|^2 dx |X_2|^2,
\]

\[
\int_0^{\eta_1} |\nu_i(x,t)|^2 dx \leq 2e^{b_1 \int_0^t \omega_1^2(x,t) dx} \int_0^{\eta_1} e^{-b_1 \omega_1^2(x,t) dx} dx + 2D_1 |C_2|^2 |X_2|^2,
\]

which imply

\[
\mathcal{M}_1 \Omega_1 - \mathcal{L}_1 |X_2|^2 \leq V_1,
\]
Fig. 3. Responses of system (59) under controller (21).

where

\[
M_1 = \frac{1}{3 + 4e^{b_1D_1} + 6e^{b_1D_1} \int_0^{D_1} |g_1(x)|^2 \, dx},
\]

\[
l_1 = \frac{\left(3D_1|C_2|^2 \int_0^{D_1} |g_1(x)|^2 \, dx + 2D_1|C_2|^2\right)}{\left(3 + 4e^{b_1D_1} + 6e^{b_1D_1} \int_0^{D_1} |g_1(x)|^2 \, dx\right)}.
\]

The first step is thus completed.
\textbf{Step 2 (}2 \leq i \leq n - 1\textbf{)): The derivative of }\hat{V}_i\text{ can be calculated as follows.}

\[
\dot{V}_i = -\int_0^{D_i} e^{-b_i x} \left( \frac{\partial \omega(x, t)}{\partial x} \right)^2 \, dx - \int_0^{D_i} e^{-b_i x} \frac{\partial \omega(x, t)}{\partial x} C_{i+1} \dot{X}_{i+1} + Z_i^2 (\hat{A}_i Z_i + \hat{B}_i C_{i+1} X_{i+1}) \\
\leq - \frac{1}{2} \int_0^{D_i} e^{-b_i x} \left( \frac{\partial \omega(x, t)}{\partial x} \right)^2 \, dx - \int_0^{D_i} e^{-b_i x} \frac{\partial \omega(x, t)}{\partial x} (x) \, dx \\
= - \dot{\hat{A}}_i |Z_i|^2 + \hat{\beta}_i |Z_i|^2 + \hat{\gamma}_i \sup_{x \in [0, D_{i+1}]} |u_{i+1}(x, t)|^2 \\
\leq - \dot{\hat{A}}_i |Z_i|^2 + \hat{\beta}_i |Z_i|^2 + \hat{\gamma}_i \sup_{x \in [0, D_{i+1}]} |u_{i+1}(x, t)|^2, \quad (86)
\]

where

\[
\dot{\hat{A}}_i = \min \left( \frac{e^{-b_i D_i}}{8 D_i^2}, 1 \right), \quad (87)
\]

\[
\hat{\alpha}_i = 8 e^{-b_i D_i} \left| C_{i+1} \right|^2 |A_{i+1}|^2 + \frac{1}{2} \left| \dot{\hat{B}}_i C_{i+1} \right|^2, \quad (88)
\]

\[
\hat{\beta}_i = |\hat{A}_i| + \frac{3}{2}, \quad (89)
\]

\[
\hat{\gamma}_i = 8 e^{-b_i D_i} \left| C_{i+1} \right|^2 \left( \int_0^{D_{i+1}} B_{i+1}(x) \, dx \right)^2. \quad (90)
\]

Moreover, with the similar argument as in Step 1, we can also obtain that there exist positive constants \( \hat{M}_i \), \( \hat{M}_0 \), and \( \hat{l}_i \) such that

\[
\hat{M}_i \dot{\hat{\Omega}}_i - \hat{l}_i |X_{i+1}|^2 \leq \dot{V}_i \leq \hat{M}_i (\dot{\hat{\Omega}}_i + |X_{i+1}|^2), \quad (91)
\]

where

\[
\dot{\hat{\Omega}}_i = |Z_{i-1}|^2 + |X_i|^2 + \int_0^{D_i} |u(x, t)|^2 \, dx. \quad (92)
\]

Next, we recall the following function,

\[
V_i = V_{i-1} + k_i \dot{V}_i, \quad (93)
\]

where \( k_i \) is to be determined. It is further noted that

\[
|u(x, t)|^2 \leq 2 \omega(x, t)^2 + 2 |C_{i+1} X_{i+1}|^2 \\
= 2 \left( \int_0^{D_i} \frac{\partial \omega(y, t)}{\partial y} \, dy \right)^2 + 2 |C_{i+1} X_{i+1}|^2 \\
\leq 2 D_i \left( \int_0^{D_i} \frac{\partial \omega(y, t)}{\partial y} \, dy \right)^2 + 2 |C_{i+1}|^2 |X_{i+1}|^2. \quad (94)
\]

which yields that

\[
\sup_{x \in [0, D_i]} |u(x, t)|^2 \leq 2 D_i e^{b_i D_i} \left( \int_0^{D_i} e^{-b_i x} \left( \frac{\partial \omega(x, t)}{\partial x} \right)^2 \, dx + 2 |C_{i+1}|^2 |X_{i+1}|^2 \right). \quad (95)
\]

It can thus be calculated that

\[
\dot{V}_i = V_{i-1} + k_i \dot{V}_i \\
\leq - \lambda_i \dot{V}_{i-1} + \alpha_i |X_i|^2 + \beta_i |Z_{i-1}|^2 \\
+ \gamma_i \sup_{x \in [0, D_i]} |u(x, t)|^2 - k_i \dot{\hat{A}}_i \dot{V}_i \\
- \frac{k_i}{2} \int_0^{D_i} e^{-b_i x} \left( \frac{\partial \omega(x, t)}{\partial x} \right)^2 \, dx \\
+ k_i \hat{\alpha}_i |X_{i+1}|^2 + k_i \hat{\beta}_i |Z_i|^2 + k_i \hat{\gamma}_i \sup_{x \in [0, D_{i+1}]} |u_{i+1}(x, t)|^2 \\
\leq \lambda_i \dot{V}_{i-1} + \min(\alpha_i, \beta_i, \gamma_i) \dot{V}_i + k_i \hat{\alpha}_i |X_{i+1}|^2 + k_i \hat{\beta}_i |Z_i|^2 + k_i \hat{\gamma}_i \sup_{x \in [0, D_{i+1}]} |u_{i+1}(x, t)|^2 \\
+ \left( 2 \gamma_i - D_i e^{b_i D_i} - \frac{k_i}{2} \right) \int_0^{D_i} e^{-b_i x} \left( \frac{\partial \omega(x, t)}{\partial x} \right)^2 \, dx \\
+ (k_i \hat{\alpha}_i + 2 \gamma_i |C_{i+1}|^2) |X_{i+1}|^2 \\
+ k_i \hat{\beta}_i |Z_i|^2 + k_i \hat{\gamma}_i \sup_{x \in [0, D_{i+1}]} |u_{i+1}(x, t)|^2. \quad (96)
\]

Since \( \dot{V}_i \geq \hat{M}_i \dot{\hat{\Omega}}_i - \hat{l}_i |X_{i+1}|^2 \), then by choosing \( k_i \) such that

\[
k_i \leq \frac{\hat{M}_i \dot{\hat{\Omega}}_i - \hat{l}_i |X_{i+1}|^2}{\hat{l}_i |X_{i+1}|^2}, \quad (97)
\]

one can obtain

\[
V_i \leq - \lambda_i V_i + \alpha_i |X_i|^2 + \beta_i |Z_i|^2 + \gamma_i \sup_{x \in [0, D_{i+1}]} |u_{i+1}(x, t)|^2, \quad (99)
\]

where

\[
\lambda_i = \min \left( \lambda_i - \frac{\hat{l}_i}{2} \right), \quad (100)
\]

\[
\alpha_i = k_i \hat{\alpha}_i + 2 \gamma_i |C_{i+1}|^2 + \frac{k_i}{2} \hat{\beta}_i \hat{\gamma}_i, \quad (101)
\]

\[
\beta_i = k_i \hat{\beta}_i, \quad (102)
\]

\[
\gamma_i = k_i \hat{\gamma}_i, \quad (103)
\]

Moreover, one has

\[
V_i \geq \bar{M}_i \dot{\bar{\Omega}}_i - l_i |X_{i+1}|^2 \geq \bar{M}_i (\dot{\bar{\Omega}}_i + |X_{i+1}|^2) \\
\geq \bar{M}_i (\dot{\bar{\Omega}}_i + |X_{i+1}|^2) + \left( \frac{k_i}{2} \bar{M}_i - l_i - 1 \right) |X_{i+1}|^2 - k_i |X_{i+1}|^2. \quad (104)
\]

Choose \( k_i \) such that

\[
k_i \leq \frac{2 \bar{M}_i - l_i}{2 \bar{M}_i}, \quad (105)
\]

Then we have

\[
V_i \geq \bar{M}_i \dot{\bar{\Omega}}_i - l_i |X_{i+1}|^2, \quad (106)
\]

where

\[
\bar{M}_i = \min(\bar{M}_{i-1}, \frac{k_i}{2} \bar{M}_{i-1}), \quad (107)
\]

\[
l_i = k_i \hat{l}_i. \quad (108)
\]

It follows from the definition of \( Z_{i-1} \) that there exists \( M_{Z_{i-1}} > 0 \) such that

\[
|Z_{i-1}|^2 \leq M_{Z_{i-1}} \dot{\bar{\Omega}}_{i-1}. \quad (109)
\]

Then we can obtain

\[
V_i \leq \bar{M}_{i-1} (\dot{\bar{\Omega}}_{i-1} + |X_i|^2) + k_i \bar{M}_i (\dot{\bar{\Omega}}_i + |X_{i+1}|^2) \\
\leq \bar{M}_{i-1} (\dot{\bar{\Omega}}_{i-1} + |X_i|^2) + k_i \bar{M}_i \left( M_{Z_{i-1}} \dot{\bar{\Omega}}_{i-1} + |X_{i+1}|^2 \right) \\
+ |X_{i+1}|^2 + \int_0^{D_i} |u(x, t)|^2 \, dx + |X_{i+1}|^2 \leq \bar{M}_i (\dot{\bar{\Omega}}_i + |X_{i+1}|^2) \quad (110)
\]

where

\[
\bar{M}_i = \bar{M}_{i-1} + k_i \bar{M}_i (M_{Z_{i-1}} + 1). \quad (111)
\]
Therefore, \( V_t \) satisfies the conditions (42)-(43) if one chooses \( k_i \) such that
\[
k_i = \max \left\{ \frac{2l_{i-1}}{\hat{M}_i}, \frac{2\max\{\alpha_{i-1}, \beta_{i-1}\}}{\lambda_{\min}(P_i)}, 4\gamma_{i-1}D_i e^{b_i D_i} \right\}. \tag{112}
\]
This step is thus completed.

**Step n:** It can be first calculated that
\[
\hat{V}_n = -c|Z_n|^2 - \int_0^{D_n} e^{-bx} \left( \frac{\partial \omega_n(x, t)}{\partial x} \right)^2 dx \\
+ \int_0^{D_n} e^{-bx} \frac{\partial \omega_n(x, t)}{\partial x} dx K_n \hat{A}_n + \hat{B}_n K_n Z_n \\
\leq -c|Z_n|^2 - \frac{1}{2} \int_0^{D_n} e^{-bx} \left( \frac{\partial \omega_n(x, t)}{\partial x} \right)^2 dx \\
- \frac{e^{-b_i D_n}}{16D_n^2} \int_0^{D_n} e^{-bx} \omega_n^2(x, t) dx \\
+ 4e^{b_i D_n} D_i^2 K_n \hat{A}_n \hat{B}_n K_n^2 |Z_n|^2. \tag{113}
\]
Furthermore, if we choose \( c > 0 \) such that
\[
c \geq 4e^{b_i D_n} D_i^2 K_n \hat{A}_n \hat{B}_n K_n^2 |Z_n|^2,
\]
then we have
\[
\hat{V}_n \leq -\frac{c}{2} |Z_n|^2 - \frac{1}{2} \int_0^{D_n} e^{-bx} \left( \frac{\partial \omega_n(x, t)}{\partial x} \right)^2 dx \\
- \frac{e^{-b_i D_n}}{16D_n^2} \int_0^{D_n} e^{-bx} \omega_n^2(x, t) dx \\
\leq -\frac{1}{2} \int_0^{D_n} e^{-bx} \left( \frac{\partial \omega_n(x, t)}{\partial x} \right)^2 dx - \lambda_n \hat{V}_n, \tag{115}
\]
where
\[
\lambda_n = \min \left\{ \frac{e^{-b_i D_n}}{2D_n^2}, \frac{1}{2\lambda_{\max}(P_n)} \right\}. \tag{116}
\]
Now we are going to prove that there exist \( \hat{M}_n \) and \( \hat{M}_n \) such that
\[
\hat{M}_n \hat{A}_n \leq \hat{V}_n \leq \hat{M}_n \hat{A}_n, \tag{117}
\]
where
\[
\hat{A}_n = |Z_{n-1}|^2 + |X_n|^2 + \int_0^{D_n} |u_n(x, t)|^2 dx. \tag{118}
\]
We recall the following facts:
\[
Z_n = \left[ \begin{array}{c} Z_{n-1} \\ X_n \end{array} \right] + \int_0^{D_n} g_n(x) u_n(x, t) dx \\
= \left[ \begin{array}{c} Z_{n-1} \\ X_n \end{array} \right] + \int_0^{D_n} g_n(x) \omega_n(x, t) dx \\
+ \int_0^{D_n} g_n(x) dx K_n Z_n, \tag{119}
\]
\[
\omega_n(x, t) := u_n(x, t) - K_n Z_n. \tag{120}
\]
By following the similar arguments as in Step 1, and using Cauchy–Schwarz inequality, one can obtain the property (117) with
\[
\hat{M}_n = c\lambda_{\min}(P_n) \left\{ \left( 2 \right) I - \int_0^{D_n} g_n(x) dx K_n \right\}^2 \\
+ 2D_n |K_n|^2 + c\lambda_{\min}(P_n) e^{b_i D_i} D_i \\
\times \left( 4 + 4 \int_0^{D_n} |g_n(x)|^2 dx \right), \tag{121}
\]
\[
\hat{M}_n = \left( c\lambda_{\max}(P_n) + D_n |K_n|^2 \right) \left( 2 + 2 \int_0^{D_n} |g_n(x)|^2 dx \right) + 1. \tag{122}
\]
Next, we prove that \( V \) satisfies (53)-(54). It can be obtained that
\[
\hat{V} \leq -\lambda_n V_{n-1} + \alpha_{n-1} |X_n|^2 + \beta_{n-1} |Z_{n-1}|^2 \\
+ \gamma_{n-1} \sup_{x \in [0, D_n]} |u_n(x, t)|^2 - k_n \int_0^{D_n} e^{-bx} \left( \frac{\partial \omega_n(x, t)}{\partial x} \right)^2 dx \\
- k_n \lambda_n \hat{V}_n, \tag{123}
\]
and
\[
\sup_{x \in [0, D_n]} |u_n(x, t)|^2 \\
\leq 2D_n e^{b_i D_n} \int_0^{D_n} e^{-bx} \left( \frac{\partial \omega_n(x, t)}{\partial x} \right)^2 dx + 2|K_n|^2 |Z_n|^2. \tag{124}
\]
Therefore, one can obtain
\[
\hat{V} \leq -\lambda_n V_{n-1} - k_n \lambda_n \hat{V}_n + (\alpha_{n-1} + \beta_{n-1}) \hat{A}_n \\
+ (2\gamma_{n-1} D_n e^{b_i D_n} - k_n) \int_0^{D_n} e^{-bx} \left( \frac{\partial \omega_n(x, t)}{\partial x} \right)^2 dx \\
+ 2\gamma_{n-1} |K_n|^2 \hat{V}_n \\
\leq -\lambda_n V_{n-1} + \left( -k_n \lambda_n + \frac{(\alpha_{n-1} + \beta_{n-1})}{\hat{M}_n} \right) \hat{V}_n \\
+ 2\gamma_{n-1} |K_n|^2 \hat{V}_n + (2\gamma_{n-1} D_n e^{b_i D_n} - k_n) \int_0^{D_n} e^{-bx} \left( \frac{\partial \omega_n(x, t)}{\partial x} \right)^2 dx, \tag{125}
\]
Choosing \( k_n \) such that
\[
k_n \geq \frac{(\alpha_{n-1} + \beta_{n-1})}{\hat{M}_n} + \frac{2\gamma_{n-1} |K_n|^2}{c\lambda_{\min}(P_n)}, \tag{126}
\]
\[
k_n \geq 2\gamma_{n-1} D_n e^{b_i D_n}, \tag{127}
\]
one can obtain
\[
\hat{V} \leq -\lambda_n V_{n-1} - k_n \lambda_n \hat{V}_n \leq -\lambda \hat{V}, \tag{128}
\]
where
\[
\lambda = \min\{\lambda_{n-1}, \lambda_n \}. \tag{129}
\]
Moreover, choosing \( k_n \) such that
\[
k_n \geq \frac{M_n}{2} \hat{M}_n \geq l_n - 1, \tag{130}
\]
one has
\[
\hat{V} \geq M_n - \Omega n - l_n - 1 |X_n|^2 + k_n \hat{M}_n \hat{A}_n \geq M \hat{V} + \left( \frac{k_n}{2} \hat{M}_n - l_n - 1 \right) |X_n|^2 \tag{131}
\]
where
\[
M = \text{min}\{M_n - \frac{k_n}{2} M_n \}, \tag{132}
\]
\[
\Omega = \sum_{i=1}^{n} \left( |X_i|^2 + \int_0^{D_n} |u_i(x, t)|^2 dx \right). \tag{133}
\]
Furthermore, it follows from the definition of $Z_{n-1}$ that there exists $M_{z_{n-1}} > 0$ such that

$$\|Z_{n-1}\|^2 \leq M_{z_{n-1}} \Omega_{n-1}.$$  (134)

One can thus obtain

$$V \leq M_{z_{n-1}} \Omega_{n-1} + \|X_n\|^2 + k_{n} \tilde{M}_{n} \tilde{\Omega}_{n} \leq \tilde{M}_{n-1} \Omega_{n-1} + \|X_n\|^2 + k_{n} \tilde{M}_{n} (M_{z_{n-1}} + 1) + \|X_n\|^2 + \int_0^T \|u_n(x, t)\|^2 \, dx \leq \tilde{M} \Omega,$$  (135)

where

$$\tilde{M} = M_{z_{n-1}} + k_{n} \tilde{M}_{n} (M_{z_{n-1}} + 1).$$  (136)

The theorem is thus proven. □

Remark A.1. In the proof, the terms $\int_0^t |B_i(x)|^2 \, dx$, $i = 1, 2, \ldots, n$ are avoided as they could be ill-defined, for example, when $B_i(x) \equiv B_i(x)$ where $\delta$ is the Dirac function. In addition, if $B_i(x)$, $i = 1, 2, \ldots, n$ are bounded on $[0, T]$, then $\int_0^T |B_i(x)|^2 \, dx$, $i = 1, 2, \ldots, n$ are well-defined and a simpler proof could be adopted by applying Cauchy–Schwarz inequality to (73) and (86).

References


Xiang Xu received the Bachelor of Engineering degree from Nanjing University of Science and Technology, China in 2014 and the Ph.D. degree from City University of Hong Kong, Hong Kong in 2018. From 2018 to 2021, he was a postdoctoral fellow in the Department of Biomedical Engineering, City University of Hong Kong. He is now an associate professor in the Department of Electronic and Electrical Engineering, Southern University of Science and Technology, Shenzhen, China. His research interests include multi-agent systems, time delay systems, nonlinear control and boundary control of PDEs.

Lu Liu received the Ph.D. degree from the Department of Mechanical and Automation Engineering, Chinese University of Hong Kong, Hong Kong, in 2008. From 2009 to 2012, she was an Assistant Professor with the University of Tokyo, Japan, and then a Lecturer with the University of Nottingham, U.K. After that, she joined City University of Hong Kong, Hong Kong, where she is currently an Associate Professor. Her current research interests include networked dynamical systems, control theory and applications, and biomedical devices.


Miroslav Krstic is Distinguished Professor of Mechanical and Aerospace Engineering, holds the Alsopch endowed chair, and is the founding director of the Cymer Center for Control Systems and Dynamics at UC San Diego. He also serves as Senior Associate Vice Chancellor for Research at UCSD. As a graduate student, Krstic won the UC Santa Barbara best dissertation award and student best paper awards at CDC and ACC. Krstic has been elected Fellow of seven scientific societies – IEEE, IFAC, ASME, SIAM, AAAS, IET (UK), and AAAA (Assoc. Fellow) – and as a foreign member of the Serbian Academy of Sciences and Arts and of the Academy of Engineering of Serbia. He has received the Richard E. Bellman Control Heritage Award, SIAM Reid Prize, ASME Oldenburger Medal, Nyquist Lecture Prize, Paynter Outstanding Investigator Award, Ragazzini Education Award, IFAC Ruth Curtain Distributed Parameter Systems Award, IFAC Nonlinear Control Systems Award, Chestnut textbook prize, Control Systems Society Distinguished Member Award, the PECASE, NSF Career, and ONR Young Investigator awards, the Schuck ('96 and '19) and Axelby paper prizes, and the first UCSD Research Award given to an engineer. Krstic has also been awarded the Springer Visiting Professorship at UC Berkeley, the Distinguished Visiting Fellowship of the Royal Academy of Engineering, the Invitation Fellowship of the Japan Society for the Promotion of Science, and four honorary professorships outside of the United States. He serves as Editor-in-Chief of Systems & Control Letters and has been serving as Senior Editor in Automatica and IEEE Transactions on Automatic Control, as editor of two Springer book series, and has served as Vice President for Technical Activities of the IEEE Control Systems Society and as chair of the IEEE CSS Fellow Committee. Krstic has coauthored eighteen books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.

Gang Feng received the Ph.D. degree in Electrical Engineering from the University of Melbourne, Australia. He has been with City University of Hong Kong since 2000 after serving as lecturer/senior lecturer at School of Electrical Engineering, University of New South Wales, Australia, 1992–1999. He is now Chair Professor of Mechatronic Engineering. He has been awarded IEEE Computational Intelligence Society Fuzzy Systems Pioneer Award, an Alexander von Humboldt Fellowship, the IEEE Transactions on Fuzzy Systems Outstanding Paper Award, Changjiang chair professorship from Education Ministry of China, and CityU Outstanding Research Award. He is listed as a SCI highly cited researcher by Clarivate Analytics. His current research interests include multi-agent systems and control, intelligent systems and control, and networked systems and control.