We solve the output-feedback stabilization problem for a tank with a liquid modeled by the viscous Saint-Venant PDE system. The control input is the acceleration of the tank and a Control Lyapunov Functional methodology is used. The measurements are the tank position and the liquid level at the tank walls. The control scheme is a combination of a state feedback law with functional observers for the tank velocity and the liquid momentum. Four different types of output feedback stabilizers are proposed. A full-order observer and a reduced-order observer are used in order to estimate the tank velocity while the unmeasured liquid momentum is either estimated by using an appropriate scalar filter or is ignored. The reduced order observer differs from the full order observer because it omits the estimation of the measured tank position. Exponential convergence of the closed-loop system to the desired equilibrium point is achieved in each case. An algorithm is provided that guarantees that a robotic arm can move a glass of water to a pre-specified position no matter how full the glass is, without spilling water out of the glass, without residual end point sloshing and without measuring the water momentum and the glass velocity. Finally, the efficiency of the proposed output feedback laws is validated by numerical examples, obtained by using a simple finite-difference numerical scheme. The properties of the proposed, explicit, finite-difference scheme are determined.

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Bing, Jordan, & Warn-Varnas, 2011; Gerbeau & Perthame, 2001; Lannes, 2013; Mascia & Rousset, 2006; Sundby, 1996). Feedback stabilization problems involving various variations of the Saint-Venant model have attracted the attention of many researchers during the last decades (see Bastin & Coron, 2016; Bastin, Coron, & Hayat, 2021; Bastin, Coron, & d’Andréa Novel, 2009; Coron, 2002, 2007; Coron, d’Andréa Novel, & Bastin, 2007; de Halleux & Bastin, 2002; Diagne, Diagne, Tang, & Krstic, 2017; Diagne, Tang, Diagne, and Krstic, 2017; Karafyllis & Krstic, 2022b; Litrico & Fromion, 2006; Petit & Rouchon, 2002; Prieur & de Halleux, 2004; Smyshlyaev & Krstic, 2005). Most works study the inviscid Saint-Venant model (i.e., the model that ignores viscous stresses and surface tension and takes into account gravity and friction forces), which provides a system of first-order hyperbolic Partial Differential Equations (PDEs). Many papers study stabilization problems for the linearization of this system around an equilibrium point, employing either the backstepping methodology (see Diagne, Diagne et al., 2017; Diagne, Tang et al., 2017) or the Control Lyapunov Functional (CLF) methodology (providing local stabilization results for the original nonlinear system in many cases; see Bastin & Coron, 2016; Bastin et al., 2021, 2009; Coron, 2007; Coron et al., 2007; de Halleux & Bastin, 2002; Litrico & Fromion, 2006). In Karafyllis and Krstic (2022b) a state feedback stabilizer for the viscous, nonlinear, Saint-Venant system has been proposed. The feedback law proposed in Karafyllis and Krstic (2022b) requires measurement of the tank position and velocity, the liquid level at the tank walls and the total liquid momentum. The latter quantity is very difficult to be measured directly.

In this paper we utilize the state feedback law proposed in Karafyllis and Krstic (2022b) in order to solve the output feedback stabilization problem of the motion of a tank containing a viscous, incompressible, Newtonian liquid. We assume that we measure the tank position and the liquid level at the tank walls and we construct functional observers for the tank velocity and the liquid momentum. The problem is studied by means of the CLF methodology initially presented in Krstic (1999) for nonlinear parabolic PDEs and studied also in Coron (2007), Karafyllis (2021) and Karafyllis and Krstic (2019, 2022a, 2022b). We construct two different observers for the tank velocity: a full-order observer and a reduced-order observer. We also construct two different estimators for the liquid momentum (strictly speaking they are not functional observers of the liquid momentum). Therefore, we construct four different types of output feedback stabilizers. Exponential convergence of the closed-loop system to the desired equilibrium point is achieved in each case. As in Karafyllis and Krstic (2022b), the constructed CLFs are also size functionals (see Sonntag, 2022 for the notion of the size function) as they provide positive upper and lower bounds of the liquid level. As far as we know, this is the first paper in the literature that achieves output feedback stabilization of the nonlinear viscous Saint-Venant system.

The paper is structured as follows. In Section 2 we present the output feedback control problem of the motion of a tank. In Section 3 we present the main ideas for the construction of dynamic output feedback laws. Section 4 contains the statements of the main results. In Section 5 we present an algorithm which guarantees that a robotic arm will move a glass of water to a pre-specified position. We show that no matter how full the glass is, the robot can transfer the glass without spilling water out of the glass, without residual end point sloshing and without measuring the water momentum and the glass velocity. Section 6 presents illustrative numerical examples which show the efficiency of the proposed dynamic output feedback laws. While there are many numerical schemes devoted to the numerical approximation of the inviscid Saint-Venant model, there are only few papers in the literature that study the numerical approximation of the viscous Saint-Venant system with no control (see Delestre & Marche, 2011; Gunawan, 2014; Katsaounis & Simeoni, 2002) and there is no paper that studies the numerical approximation of the viscous Saint-Venant model with control. Here, we used a simple finite-difference scheme and in Section 6 we state the properties and the accuracy of the scheme. The proofs of all results are provided in Section 7. Finally, the concluding remarks are given in Section 8.

Notation. Throughout this paper, we adopt the following notation.

- \( \mathbb{R}_+ = [0, +\infty) \) denotes the set of non-negative real numbers.
- Let \( S \subseteq \mathbb{R}^n \) be an open set and let \( A \subseteq \mathbb{R}^n \) be a set that satisfies \( S \subseteq A \subseteq \text{cl}(S) \). By \( C^0(A; \Omega) \), we denote the class of continuous functions on \( A \), which take values in \( \Omega \subseteq \mathbb{R}^m \).
- By \( C^k_A; (A; \Omega) \), where \( k \geq 1 \) is an integer, we denote the class of functions on \( A \subseteq \mathbb{R}^n \), which takes values in \( \Omega \subseteq \mathbb{R}^m \) and has continuous derivatives of order \( k \). In other words, the functions of class \( C^k_A; (A; \Omega) \) are the functions which have continuous derivatives of order \( k \) in \( S = \text{int}(A) \) that can be continued continuously to all points in \( \partial S \cap A \). When \( \Omega = \mathbb{R}^m \) then we write \( C^k(A; \Omega) \) or \( C^k(A) \). When \( I \subseteq \mathbb{R} \) is an interval and \( G \in C^1(I) \) is a function of a single variable, \( G(h) \) denotes the derivative with respect to \( h \) in \( I \).
- Let \( I \subseteq \mathbb{R} \) be an interval, let \( a < b \) be given constants and let \( u : I \times [a, b] \to \mathbb{R} \) be a given function. We use the notation \( u[t] \) to denote the profile at certain \( t \in I \), i.e., \( u[t](x) = u(t, x) \) for all \( x \in [a, b] \). When \( u(t, x) \) is (twice) differentiable with respect to \( x \in [a, b] \), we use the notation \( u_x(t, x) \) (\( u_{xx}(t, x) \)) for the (second) derivative of \( u \) with respect to \( x \in [a, b] \), i.e., \( u_x(t, x) = \frac{\partial u(t, x)}{\partial x} = \frac{\partial u}{\partial x}(t, x) \) (\( u_{xx}(t, x) = \frac{\partial^2 u(t, x)}{\partial x^2} \)). When \( u(t, x) \) is differentiable with respect to \( t \), we use the notation \( u_t(t, x) \) for the derivative of \( u \) with respect to \( t \), i.e., \( u_t(t, x) = \frac{\partial u}{\partial t}(t, x) \).
- Given a set \( U \subseteq \mathbb{R}^n \), \( x_U \) denotes the characteristic function of \( U \), i.e., the function defined by \( x_U(x) := 1 \) for all \( x \in U \) and \( x_U(x) := 0 \) for all \( x \notin U \). The sign function \( \text{sgn} : \mathbb{R} \to \mathbb{R} \) is the function defined by the relations \( \text{sgn}(x) = 1 \) for \( x > 0 \), \( \text{sgn}(0) = 0 \) and \( \text{sgn}(x) = -1 \) for \( x < 0 \).
- Let \( a < b \) be given constants. For \( p \in [1, +\infty) \), \( L^p(a, b) \) is the set of equivalence classes of Lebesgue measurable functions \( u : (a, b) \to \mathbb{R} \) with
- \( \|u\|_p := \left( \int_a^b |u(x)|^p \ dx \right)^{1/p} \to +\infty \)
- \( L^\infty(a, b) \) is the set of equivalence classes of Lebesgue measurable functions \( u : (a, b) \to \mathbb{R} \) with
- \( \|u\|_\infty := \text{ess sup}_{x \in (a,b)} (|u(x)|) < +\infty \).
- For an integer \( k \geq 1 \), \( H^k(a, b) \) denotes the Sobolev space of functions in \( L^2(a, b) \) with all its weak derivatives up to order \( k \geq 1 \) in \( L^2(a, b) \).

2. Description of the problem

We study a 1-D model for the motion of a tank that contains a viscous, Newtonian, incompressible liquid. We assume that a force, that can be manipulated, acts on the tank. Assuming that the liquid pressure is hydrostatic, the liquid is modeled by the 1-D viscous Saint-Venant equations and the tank obeys Newton’s second law.

The control objective is to drive asymptotically the tank to a specified position without liquid spilling out and having both the tank and the liquid within the tank at rest.
Let the position of the left wall of the tank at time $t \geq 0$ be $a(t)$ and let the length of the tank be $L > 0$ (a constant). The following equations describe the motion of the liquid within the tank:

$$H_t + (Hu)_z = 0, \quad \text{for } t > 0, \quad Z \in [a(t), a(t) + L]$$

(1)

$$H_u + \left( \frac{1}{2} g H^2 \right)_z = \mu (Hu)_z, \quad \text{for } t > 0,$$

$$Z \in (a(t), a(t) + L)$$

(2)

where $H(t, Z) > 0$, $u(t, Z) \in \mathbb{R}$ are the liquid level and the liquid velocity, respectively, at time $t \geq 0$ and position $Z \in [a(t), a(t) + L]$. While $g$, $\mu > 0$ (constants) are the acceleration of gravity and the kinematic viscosity of the liquid, respectively. The liquid velocities at the walls of the tank coincide with the tank velocity, i.e., we have:

$$u(t, a(t)) = u(t, a(t) + L) = u(t), \quad \text{for } t \geq 0$$

(3)

where $u(t) = \dot{a}(t)$ is the velocity of the tank at time $t \geq 0$. For the tank we have

$$\ddot{a}(t) = -f(t), \quad \text{for } t > 0$$

(4)

where $-f(t)$ is the control input to the problem, is the acceleration of the tank at time $t \geq 0$, which by virtue of Newton’s 2nd law is equal to the force acting on the tank divided by the mass of the tank. The conditions for avoiding the liquid spilling out of the tank are:

$$H(t, a(t)) < H_{\text{max}}$$

$$H(t, a(t) + L) < H_{\text{max}}$$

(5)

where $H_{\text{max}} > 0$ is the height of the tank walls. Applying the transformation

$$v(t, x) = u(t, a(t) + x) - u(t)$$

$$h(t, x) = H(t, a(t) + x)$$

(6)

$$\xi(t) = a(t) - a^*$$

(7)

where $a^* \in \mathbb{R}$ is the specified position (a constant) in which we want the left wall of the tank to be placed, we obtain the model

$$\dot{\xi} = w, \quad \dot{w} = -f \quad \text{for } t \geq 0$$

(7)

$$h_t + (hv)_x = 0, \quad \text{for } t > 0, \quad x \in [0, L]$$

(8)

$$\left( hv \right)_x + \left( \frac{h^2 + \frac{1}{2} g h^2}{2} \right)_x = \mu (hv)_x + hf, \quad \text{for } t > 0, \quad x \in (0, L)$$

(9)

$$v(t, 0) = v(t, L) = 0, \quad \text{for } t \geq 0$$

(10)

where the control input $f$ appears additively in the second equation of (7) and multiplicatively in (9). Due to (10) there is no need for a boundary condition for the liquid level $h$. Moreover, the conditions (5) for avoiding the liquid spilling out of the tank become:

$$\max \left( h(t, 0), h(t, L) \right) < H_{\text{max}}, \quad \text{for } t \geq 0$$

(11)

We consider classical solutions for the PDE–ODE system (7)–(10), i.e., we consider functions

$$\xi \in C^1(\mathbb{R}_+) \cap C(\{0, +\infty\}), \quad w \in C^0(\mathbb{R}_+) \cap C^1(\{0, +\infty\}),$$

$$v \in C^0([0, +\infty) \times [0, L]) \cap C^1(\{0, +\infty\} \times [0, L]),$$

$$h \in C^1([0, +\infty) \times [0, L]; \{0, +\infty\} \times C(\{0, +\infty\} \times [0, L]),$$

with $v(t) \in C^2([0, L])$ for each $t > 0$ that satisfy Eqs. (7)–(10) for a given input $f \in C^0(\mathbb{R}_+)$. Using (8) and (10), we can prove that for every solution of (7)–(10) it holds that $\frac{1}{2} \int_0^L h(t, x) \, dx = 0$ for all $t > 0$. Hence, the total mass of the liquid $m \geq 0$ is constant. Therefore, without loss of generality, we assume that every solution of (7)–(10) satisfies the equation

$$\int_0^L h(t, x) \, dx = m.$$  

(12)

The open-loop system (7)–(10), (12), i.e., system (7)–(10), (12) with $f(t) \equiv 0$, allows a continuum of equilibrium points, namely the points

$$h(x) \equiv h^*, v(x) \equiv 0, \quad v(x) \equiv 0, \quad x \in [0, L]$$

(13)

$$\xi \in \mathbb{R}, \quad w = 0$$

(14)

where $h^* = m/L$. We assume that the equilibrium points satisfy the condition for no spilling out (11), i.e., $h^* < H_{\text{max}}$. Due to the existence of a continuum of equilibrium points for the open-loop system given by (13), (14), the desired equilibrium point is not asymptotically stable for the open-loop system. In addition to that, experience shows that it is possible to find smooth initial conditions $(\xi(0), w(0), h(0), v(0)) \in \mathbb{R} \times \mathbb{R} \times C^1([0, L]; \{0, +\infty\}) \times C^1([0, L])$ with $(\xi(0)) = (v(0))(0) = 0$ for which we cannot avoid liquid spilling out of the tank—no matter what the applied input $f$ is. Thus, the described control problem is far from trivial.

In contrast with Karafyllis and Krstic (2022b), here we assume that we do not measure the whole state vector $(\xi(t), w(t), h(t), v(t))$ but we measure only the tank position $\xi(t)$ and the liquid level at the tank walls $h(t, 0)$ and $h(t, L)$. In other words, the measured output is given by the equation:

$$y(t) = \left[ \xi(t), h(t, 0), h(t, L) \right] \in \mathbb{R}^3.$$  

(15)

Our objective is to design a finite-dimensional, dynamic, output feedback law of the form

$$\dot{\xi}(t) = F_1(s(t), y(t)), \quad s(t) \in \mathbb{R}^k, \quad f(t) = F_2(s(t), y(t)),$$

(16)

that leads exponentially the solution of the closed-loop system (7)–(10) with (16) to the equilibrium point with $\xi = 0$. Moreover, we additionally require that the “spill-free condition” (11) holds for every $t \geq 0$.

Finally, it should be noted that in this paper we do not study existence/uniqueeness of solutions for the closed-loop system. However the obtained stability estimates combined with ideas in the literature (see Sundbye, 1996) can be used for a future study of the existence/uniqueeness issue.

3. Construction of the dynamic output feedback law

3.1. A nonlinear state feedback law

The static nonlinear state feedback law

$$f(t) = -\sigma \left( \int_0^L h(t, x) v(t, x) \, dx + \mu \left( h(t, L) - h(t, 0) \right) - q \left( u(t) + k\xi(t) \right) \right)$$  

(17)

was proposed in Karafyllis and Krstic (2022b) for the stabilization of the equilibrium point with $\xi = 0$. Here $\sigma, k, q > 0$ are constants that must satisfy the conditions

$$k < \frac{q \theta^2}{b + G^{-1}(-cr)}$$

(18)

where

$$c := \frac{1}{\mu \sqrt{g}}, \quad \theta := \frac{\sigma g}{g + \mu \sigma L}, \quad b := \frac{4mL^2H_{\text{max}}}{\mu \pi^2}$$

(19)
and \( G^{-1} : \left( -\frac{3}{2} h^* \sqrt{h^*}, +\infty \right) \to (0, +\infty) \) is the inverse function of the increasing \( C \) function \( G : (0, +\infty) \to \left( -\frac{3}{2} h^* \sqrt{h^*}, +\infty \right) \) defined by means of the formula
\[
G(h) := \frac{2}{3} \text{sgn} (h - h^*) \left( h \sqrt{h} - 3h^* \sqrt{h} + 2h^* \sqrt{h^*} \right)
\] (20)
and \( r \) is a constant that satisfies
\[
0 < r < R := \frac{2\mu \sqrt{\frac{3}{2}}}{\frac{1}{2} \left( h^* \sqrt{h^*} + \sqrt{H_{\max} \min (H_{\max} - 3h^*, 0)} \right)}
\] (21)

The design of the feedback law (17) was based on the Control Lyapunov Functional (CLF) \( V : S \to \mathbb{R}_+ \), defined by
\[
V(\xi, w, h, v) := W(h, v) + E(h, v) + \frac{1}{2} \xi^2 + \frac{q}{2} (w + k\xi)^2
\] (22)

where
\[
E(h, v) := \frac{1}{2} \int_0^t \left( h(x)v^2(x)dx + \frac{1}{2} \xi^2 \left( h(x) - h^* \right)^2 dx \right.
\] (23)
\[
W(h, v) := \frac{1}{2} \int_0^t \left. \left( h^{-1}(x)(h(x)v(x) + \mu h_c(x))^2 dx \right) \right. + \frac{1}{2} g \int_0^t \left. \left( h(x) - h^* \right)^2 dx \right.
\] (24)

and \( S \) is the set defined by
\[
(\xi, w, h, v) \in S \iff \begin{cases} h \in C^0 ([0, L], (0, +\infty)) \cap H^1(0, L) \\ v \in C^0 ([0, L]) \\ \int_0^L \hat{h}(x)dx = m \\ (\xi, w, v) \in \mathbb{R}^2, v(0) = v(L) = 0 \end{cases}
\] (25)

The feedback law (17) guarantees (Theorem 1 in Karafyllis & Krstic, 2022b) that for each \( r \in [0, R] \) there exist constants \( M \geq 1, \beta > 0 \) with the following property:

**Q** Every classical solution of the PDE–ODE system (7)–(10), (12) and (17) with \( V(\xi(t), w(t), h(t), v(t)) \leq r \), satisfies \( \xi(t), w(t), h(t), v(t) \in X \) and the following estimate for all \( t \geq 0 \):
\[
\|\xi(t), w(t), h(t) - h^* \chi_{(0, L)}(t)\|_X 
\leq M \exp (-\beta t) \|\xi(0), w(0), h(0) - h^* \chi_{(0, L)}(0), v(0)\|_X
\] (26)

Here \( X \subset \mathbb{R}^2 \times H^1(0, L) \times L^2(0, L) \) is the metric space
\[
X := \{ (\xi, w, h, v) \in S : V(\xi, w, h, v) < R \}
\] (27)

with metric induced by the norm of the underlying linear space \( \mathbb{R}^2 \times \mathbb{H}^1(0, L) \times \mathbb{L}^2(0, L) \), i.e., we have for all \( (\xi, w, h, v) \in X \)
\[
\|\xi, w, h, v\|_X = \left( \xi^2 + w^2 + \|h\|^2 + \|h^*\|^2 + \|v\|^2 + \|v\|^2 \right)^{1/2}
\] (28)

The quantity
\[
\|\xi, w, h - h^* \chi_{(0, L)}(t)\|_X = \left( \xi^2 + w^2 + \|h - h^* \chi_{(0, L)}\|^2 + \|h^*\|^2 + \|v\|^2 \right)^{1/2}
\] (29)

i.e., the distance of the state \( (\xi, w, h, v) \) from the equilibrium \( (0, 0, h^* \chi_{(0, L)}, 0 \chi_{(0, L)}) \) is a combined measure of the sloshing of the liquid in the tank (given by the component \( \|h - h^* \chi_{(0, L)}\|^2 + \|h^*\|^2 + \|v\|^2 \)), the tank velocity (given by the component \( v^2 \)), and the deviation of the tank from the desired position (given by the component \( \xi^2 \)). Finally, the “spill-free condition” (11) holds for every classical solution of the PDE–ODE system (7)–(10), (12) with (17) and \( V(\xi(0), w(0), h(0), v(0)) \leq r \), since Lemma 1 in Karafyllis & Krstic (2022b) implies that for all \( (\xi, w, h, v) \in X \) it holds that
\[
0 < G^{-1}(-cV(\xi, w, h, v) \leq h(x) \leq G^{-1}(cV(\xi, w, h, v)) < H_{\max} \text{ for all } x \in (0, L).
\] (29)

The above inequality in conjunction with definitions (27), (19), (21) guarantees the fact that the “spill-free condition” (11) holds for every classical solution of the PDE–ODE system (7)–(10), (12) with (17) and \( V(\xi(0), w(0), h(0), v(0)) \leq r \).

### 3.2. Ideas for the construction of dynamic output feedback law

The main idea for the construction of a dynamic feedback law is the replacement of the quantities \( \int_0^t h(t, x) \times v(t, x)dx \) (the liquid momentum) and \( w(t) \) (the tank velocity) – which are not measured – in the feedback law (17) by appropriate estimates. For the tank velocity \( w(t) \) an estimate \( \hat{w}(t) \) can be obtained either by means of the finite-dimensional full-order observer
\[
\begin{align*}
\frac{d\hat{\xi}}{dt} &= -2\gamma \left( \hat{\xi}(t) - \xi(t) \right) \\
\frac{d\hat{w}}{dt} &= -f(t) - \left( \frac{1}{2} + \gamma^2 \right) \left( \hat{t}(t) - \xi(t) \right)
\end{align*}
\] (30)

where \( \gamma > 0 \) is a constant or by means of the reduced-order observer
\[
\begin{align*}
\hat{\xi}(t) &= -\gamma \xi(t) - \gamma^2 \xi(t) - f(t) \\
\hat{\xi}(t) &= -\gamma \xi(t) + \gamma \xi(t) - w(t)
\end{align*}
\] (31)

where \( \gamma > 0 \) is a constant and the velocity estimate is given by
\[
\hat{w}(t) = \xi(t) + \gamma \xi(t)
\] (32)

The variable \( \xi \) in the reduced-order observer (31) is the observer state. The reduced-order observer is designed based on the linear finite-dimensional system (7) (a double integrator) with output equal to \( \xi \). Both observers (30) and (31) guarantee global exponential estimation of the tank velocity \( w(t) \) with rate of convergence equal to \( \gamma \) since by using (7), (30) and (31) we get:
\[
\begin{align*}
\frac{d}{dt} \left( \hat{\xi}(t) - \xi(t) \right) &= \hat{w}(t) - w(t) - 2\gamma \left( \hat{t}(t) - \xi(t) \right) \\
\frac{d}{dt} \left( \hat{w}(t) - w(t) \right) &= -f(t) - \left( \frac{1}{2} + \gamma^2 \right) \left( \hat{t}(t) - \xi(t) \right) \\
\frac{d}{dt} \left( \xi(t) + \gamma \xi(t) - w(t) \right) &= -\gamma \xi(t) + \gamma \xi(t) - w(t)
\end{align*}
\] (33)

Moreover, exponential estimation of \( v \) is guaranteed independently of the input (i.e., the observers (30) and (31) can also work in open-loop). For the liquid momentum we need to follow a different approach. If we define
\[
\tilde{v}(t) := \int_0^t h(t, x)v(t, x)dx + \mu (h(t, L) - h(t, 0))
\] (34)

then it follows from (8), (9), (10), (12) and definition (32) that the following differential equation holds for every classical solution of (7)–(10), (12) and all \( t > 0 \):
\[
\frac{d}{dt} \left( \tilde{v}(t) - \tilde{v}(t) \right) = -\beta (\tilde{z}(t) - \tilde{z}(t)) - \beta \int_0^t h(t, x)v(t, x)dx
\] (35)
Using (32), we can now estimate the liquid momentum
\[ p(t) := \int_0^t h(t, x) v(t, x) dx \]
\[ = Z(t) - \mu (h(t, L) - h(t, 0)) \]
by:

(1) either the quantity \( Z(t) - \mu (h(t, L) - h(t, 0)) \) which is going
to give an accurate estimate when \( |Z(t) - \hat{Z}(t)| \) is small,
(2) or by zero, which is going to give an accurate estimate
when \( \int_0^t h(t, x) v(t, x) dx \) is small.

Taking the convex combination of both estimates of the liquid
momentum we get the estimate
\[ \hat{p}(t) = \lambda Z(t) - \lambda \mu (h(L) - h(t, 0)) , \]
where \( \lambda \in [0, 1] \) is a constant. However, this estimate of the li-
quid momentum is not guaranteed to be a valid estimate. Eq. (35)
shows that the liquid momentum estimate \( \hat{p}(t) \) will converge to
\( p(t) \) only if \( \hat{p}(t) \to 0 \) as \( t \to +\infty \). Therefore, the liquid
momentum estimate cannot work in open-loop but can only work
in closed-loop. Only the combination of the applied feedback
and the provided estimates can guarantee that the momentum
estimate is successful (see Theorems 1 and 2 below).

The reader should notice that when \( \lambda = 0 \) then the filter (34)
is unutilized. Thus we may obtain four different dynamic output
feedback laws with:

(1) three ODEs when the full-order observer (30) is used and
\( \lambda \in [0, 1] \),
(2) two ODEs when the reduced-order observer (31) is used and
\( \lambda \in [0, 1] \),
(3) two ODEs when the full-order observer (30) is used and
\( \lambda = 0 \), and
(4) one ODE when the reduced-order observer (31) is used and
\( \lambda = 0 \).

4. Main results

Now we are in a position to state the main results of the present
work.

**Theorem 1 (Output Feedback with Full-Order Observer).** Let \( r \in [0, R] \) be an arbitrary constant, where \( R > 0 \) is the constant defined
in (21). Let \( \sigma, q, k, \gamma, \beta > 0 \), \( \lambda \in [0, 1] \) be arbitrary constants for
which
\[ \begin{align*}
2 \frac{\sigma}{\mu} L & > \sigma > 8 k + 2 \left( 2 \pi^2 \mu G^{-1} - (cr) \right) \rho \mu L^2 G^{-1} (cr) > \beta + 4 \sigma > 8 \sigma , \end{align*} \]
where \( \rho > 0 \) is defined by (19). Let \( S \) be the set defined by (25)
and definethefunctional \( V : S \to \mathbb{R} \) by (22), (23), (24). Let \( X \)
be the metric space defined by (27). Moreover, define the functional
\( \Phi : X \times \mathbb{R}^3 \to \mathbb{R} \) by the equation
\[ \Phi (\xi, w, h, v, \hat{\xi}, \hat{w}, \hat{v}) := \Psi (\xi, w, h, v) + \frac{\sigma q^2}{\gamma} \left( \hat{w} - w - \gamma (\hat{\xi} - \xi) \right)^2 \]
\[ + \frac{\lambda}{2} \left( \hat{z} - \int_0^t h(x) v(x) dx - \mu (h(L) - h(0)) \right)^2 . \]

Then there exist constants \( M \geq 1, \varphi > 0 \) with the following
property:

**Theorem 2 (Output Feedback with Reduced-Order Observer).** Let \( r \in [0, R] \) be an arbitrary constant, where \( R > 0 \) is the constant defined
in (21). Let \( \sigma, q, k, \gamma, \beta > 0 \), \( \lambda \in [0, 1] \) be arbitrary constants for
which
\[ \begin{align*}
2 \frac{\sigma}{\mu} L & > \sigma > 8 k + 2 \left( 2 \pi^2 \mu G^{-1} - (cr) \right) \rho \mu L^2 G^{-1} (cr) > \beta + 4 \sigma > 8 \sigma , \end{align*} \]
where \( \rho > 0 \) is defined by (19). Let \( S \) be the set defined by (25)
and definethefunctional \( V : S \to \mathbb{R} \) by (22), (23), (24). Let \( X \)
be the metric space defined by (27). Moreover, define the functional
\( \Psi : X \times \mathbb{R}^3 \to \mathbb{R} \) by the equation
\[ \Psi (\xi, w, h, v, \hat{\xi}, \hat{w}, \hat{v}) := \Phi (\xi, w, h, v) + \frac{\sigma q^2}{\gamma} \left( \hat{w} - w - \gamma (\hat{\xi} - \xi) \right)^2 \]
\[ + \frac{\lambda}{2} \left( \hat{z} - \int_0^t h(x) v(x) dx - \mu (h(L) - h(0)) \right)^2 . \]

Then there exist constants \( M \geq 1, \varphi > 0 \) with the following
property:

**Remark 1.** (i) Both Theorems 1 and 2 guarantee exponential
convergence to the desired equilibrium point. Moreover, both
estimates (39) and (42) are independent of \( z \) when \( \lambda = 0 \).
This is expected since the corresponding closed-loop systems are
independent of \( z \) when \( \lambda = 0 \).

(ii) It should be noticed that Theorem 1 in Karafyllis and Krstic
(2022b) had no condition whatsoever for the controller gain \( \sigma > 0 \)
in the feedback laws (38) and (41). However, Theorems 1 and 2
in the present work require the validity of inequality (36) which
shows that \( \sigma > 0 \) should be sufficiently large but cannot be
arbitrarily large. The reason for the existence of an upper bound
for \( \sigma > 0 \) is the existence of estimation errors in the present
work. Since we cannot implement the state feedback law (17)
but instead we try to approximate the feedback law (17) by
using appropriate estimates, the use of a large controller gain
\( \sigma > 0 \) will also magnify the effect of the estimation errors to
the closed-loop systems.

(iii) On the other hand, neither Theorem 1 nor Theorem 2 re-
quire any upper bound for the parameter \( \gamma > 0 \) that determines
the convergence rate to zero of the observer error for the full observer \((30)\) and the reduced observer \((31)\). However, a very large value for the observer gain \(\gamma > 0\) should not be used in practice due to the existence of possible measurement noise.

(iv) The gain \(\beta > 0\) of the filter \((34)\) has to be greater than \(4\sigma\) and less than \(\frac{2\sqrt{2}\mu G (1-\epsilon)}{m_H c^2 (1-\epsilon)} - 4\sigma\) (recall \((36)\)). Indeed, the use of a large filter gain \(\beta > 0\) will magnify the effect of the estimation errors to both closed-loop systems and this is the reason that the gain \(\beta\) has to be less than \(\frac{2\sqrt{2}\mu G (1-\epsilon)}{m_H c^2 (1-\epsilon)} - 4\sigma\).

(v) It should be noted that for every \(r \in [0, R]\) we can find constants \(\sigma, k, q, \beta, \gamma > 0\) and \(\lambda \in [0, 1]\) for which inequalities \((36)\) hold. First of all, notice that inequalities \((36)\) impose no condition on \(\gamma > 0\) and \(\lambda \in [0, 1]\). To show why we can always find constants \(\sigma, k, q, \beta > 0\) for which inequalities \((36)\) hold, notice that we can always pick \(\sigma > 0\) sufficiently small so that \(\sigma < \min\left(\frac{\lambda^2 \mu G (1-\epsilon)}{8 m_H c^2 (1-\epsilon)} , \frac{2\sigma^4}{\mu^2}\right)\). Then we can pick any \(\beta \in (4\sigma, \frac{3\pi^2 \mu G (1-\epsilon)}{2 m_H c^2 (1-\epsilon)}\) and any \(k, q > 0\) with \(\frac{k}{q} > \frac{a}{\gamma}\). Then all inequalities \((36)\) are valid.

5. Can a robot move a glass of water without spilling out water and without measuring the water momentum?

We next consider the problem of the movement of a glass of water by means of a robotic arm, from near-rest to near-rest in finite (but not presupposed) time. The glass of water starting from an almost at rest state (both the glass and the water in the glass), is formulated as

\[
\left\| \left[0, w(0), h(0), -H^* \chi_{(0,1)}, v(0)\right] \right\|_{L} \leq \varepsilon \tag{43}
\]

where \(\varepsilon > 0\) is a small number (tolerance) and \(\| \cdot \|_{L}\) is the norm defined by \((28)\). The robotic arm should move the glass to a pre-specified position without spilling out water of the glass and having the water in the glass almost still at the final time, i.e., at the final time \(T > 0\) we must have

\[
\left\| [\xi(T), w(T), h(T)] - [-h^* \chi_{(0,1)}, v(T)] \right\|_{L} \leq \varepsilon \tag{44}
\]

In other words, we require the spill-free and slosh-free motion of the glass. The initial condition is \((\xi(0), w(0), h(0), v(0)) \in S\), where \(\xi(0) \neq 0\) (recall the definition \((25)\) of \(S\)). Our aim is to show that the glass of water can be transferred without spilling out of the glass, without residual end point sloshing, without measuring the glass velocity and without measuring the momentum of water in the glass — no matter how small the difference \(H_{max} - h^*\). The minimization of the time needed for the transition of the glass (a typical motion-planning, open-loop-control problem that cannot be solved by a smooth feedback law) is a completely different problem which is not studied in the present work.

The problem was solved in Karafyllis and Krstic (2022b) for a sufficiently small tolerance \(\varepsilon > 0\) by using the feedback law \((17)\). However, the feedback law \((17)\) requires the measurement both of the momentum of the water in the glass and the glass velocity. Both these two measurements are not easily obtained in practice and thus the problem of performing of spill-free and slosh-free transfer of the glass of water without measuring the momentum of the glass and without measuring the glass velocity is important.

Here, we solve the problem by applying Theorem 2 with \(\lambda = 0\) when the tolerance \(\varepsilon > 0\) is sufficiently small (but also Theorem 1 or different values of \(\lambda \in [0, 1]\) can be applied). More specifically, we require that \(\varepsilon > 0\) is small so that

\[
\varepsilon \leq \min\left(\frac{2\sigma^2}{\gamma}, \frac{H_{max} - h^*}{\sqrt{\lambda}}\right) \tag{45}
\]

Inequalities \((45)\) are exactly the same conditions for tolerance as in the case of the state feedback law \((17)\) that was studied in Karafyllis and Krstic (2022b). If inequalities \((45)\) hold then we can follow the following algorithm:

Step 1: Pick numbers \(r \in (0, R), q > 0\) with

\[
q \leq \max\left(\mu^2 \left(h^* - \varepsilon \sqrt{L}\right)^{-1}, \frac{3H_{max}}{2}\right) \quad \text{and} \quad \varepsilon^2 \max\left(\mu^2 \left(h^* - \varepsilon \sqrt{L}\right)^{-1}, \frac{3H_{max}}{2}\right) < r \text{ (due to (45)).}
\]

Step 2: Select \(\sigma, k, \gamma, \lambda > 0\) so that the inequalities \(\frac{2\sigma^4}{\gamma} > \sigma > \frac{8\kappa}{q}\) hold and so that

\[
\gamma \geq \frac{2\sigma q^2 \varepsilon^2}{r - \max\left(\mu^2 \left(h^* - \varepsilon \sqrt{L}\right)^{-1}, \frac{3H_{max}}{2}\right) \varepsilon^2}
\]

where \(R := \frac{2\sqrt{2}\pi}{\mu} \left(2h^* + \sqrt{H_{max}} \min\left(H_{max} - h^*, 0\right)\right)\).

Inequalities \((45)\) guarantee the inequality \(\xi(\varepsilon), w(0), h(0), v(0), \xi(0), z(0)) \leq r\) for \(\xi(0) = -\gamma \xi(0)\) and for arbitrary \(z(0) \in R\) (since \(\lambda = 0\) the filter \((34)\) is irrelevant). Indeed, this fact is a direct consequence of \((28), (40), (43), (45), (46)\) and Proposition 1 in Karafyllis and Krstic (2022b). More specifically, since \(\varepsilon < \min(\gamma \varepsilon \lambda, H_{max} - h^*)\) and \((\xi(0), w(0), h(0), v(0)) \in S\) satisfies \((43)\), Proposition 1 in Karafyllis and Krstic (2022b) implies the inequality

\[
V(\xi(0), w(0), h(0), v(0)) \leq \max\left(\mu^2 \left(h^* - \varepsilon \sqrt{L}\right)^{-1}, \frac{3H_{max}}{2}\right) q + 3q^2 k^2 \varepsilon^4(0)
\]

Since \(q \leq \max\left(\mu^2 \left(h^* - \varepsilon \sqrt{L}\right)^{-1}, \frac{3H_{max}}{2}\right) \varepsilon^2\), the above inequality, \((43)\) and definition \((40)\) gives the estimate (recall that \(\lambda = 0\)):

\[
\Psi(\xi(0), w(0), h(0), v(0), \xi(0), z(0)) = V(\xi(0), w(0), h(0), v(0)) + \frac{\sigma q^2}{\gamma} (\xi(0) - w(0) + \gamma \xi(0))^2
\]

\[
\leq \max\left(\mu^2 \left(h^* - \varepsilon \sqrt{L}\right)^{-1}, \frac{3H_{max}}{2}\right) \varepsilon^2 + \frac{3q^2 k^2 \varepsilon^2(0)}{2} + \frac{\sigma q^2}{\gamma} (\xi(0) - w(0) + \gamma \xi(0))^2
\]

The above inequality in conjunction with \((46)\) and the fact that \(|w(0)| \leq \left\| [0, w(0), h(0), -H^* \chi_{(0,1)}, v(0)] \right\|_{L} \leq \varepsilon\) (a consequence of \((28)\) and \((43)\)) guarantees that \(\Psi(\xi(0), w(0), h(0), v(0), \xi(0), z(0)) \leq r\) for \(\xi(0) = -\gamma \xi(0)\) and for arbitrary \(z(0) \in R\). Notice that since \(\frac{2\sigma^4}{\gamma} > \frac{8k}{q}\) \(\frac{2\sqrt{2}\pi \mu G (1-\epsilon)}{m_H c^2 (1-\epsilon)} - \sigma\) inequalities \((36)\) hold for appropriate \(\beta > 0\) but notice that \(\beta > 0\) in this case is irrelevant due to the fact that \(\lambda = 0\) and the filter \((34)\) is not used. Consequently, we are in a position to apply Theorem 2. Let \(M \geq 1, \psi > 0\) be the constants involved in \((42)\) and correspond to the selected parameters \(r \in (0, R), \lambda = 0\) and \(\sigma, k, \gamma, \beta > 0\).
Step 3: Set $T = \frac{1}{2} \ln \left( \frac{Mh(0)+2Mv}{2} \right)$ (notice that since $M \geq 1$ it follows that $T > \frac{1}{2} \ln 2 > 0$) and apply the feedback law (31), (41) with $\zeta(0) = -\gamma \xi(0)$, $\lambda = 0$ for $t \in [0, T]$. Inequalities (28), (42), (43) imply that estimate (44) holds. Indeed, we obtain from (42) for $\lambda = 0$:

$$
\| \xi(t) \|_X + \| w(t) \|_X + \| h(t) \|_X + \| h(x) \|_X + \| v(t) \|_X \leq M \exp (\gamma t) \left( \| \xi(0) \|_X + \| w(0) \|_X + \| h(0) \|_X + \| v(0) \|_X \right)
$$

The above inequality and the fact that $\zeta(0) = -\gamma \xi(0)$ in conjunction with definition (28) (which directly implies that $\| \xi(t) \|_X + \| w(t) \|_X + \| h(t) \|_X + \| v(t) \|_X \leq \frac{1}{\gamma} \| \xi(0) \|_X + \| w(0) \|_X + \| h(0) \|_X + \| v(0) \|_X$) gives the estimate

$$
\| \xi(T) \|_X + \| w(T) \|_X + \| h(T) \|_X + \| v(T) \|_X \leq M \exp (\gamma T) \left( \| \xi(0) \|_X + \| w(0) \|_X + \| h(0) \|_X + \| v(0) \|_X \right)
$$

Finally, inequality (43) and the fact that $|w(0)| \leq \| w(0) \|_X + \| h(0) \|_X$ imply the estimate

$$
\| \xi(t) \|_X + \| w(t) \|_X + \| h(t) \|_X + \| v(t) \|_X \leq M \exp (\gamma t) \left( \| \xi(0) \|_X + \| w(0) \|_X + \| h(0) \|_X + \| v(0) \|_X \right)
$$

Estimate (44) is a direct consequence of the above estimate and the fact that $T = \frac{1}{2} \ln \left( \frac{Mh(0)+2Mv}{2} \right)$.

The application of the above algorithm can guarantee that the robotic arm will move the glass to the pre-specified position without spilling out water of the glass, without residual end point sloshing, without measuring the glass level and without measuring the momentum of water in the glass - no matter how small the difference $H_{max} - h^* > 0$ is.

6. A novel numerical scheme and examples

Applying appropriate scaling and using dimensionless variables in the viscous Saint-Venant model (7)-(10), (12), we may assume (without any loss of generality) next that $L = g = h^* = 1$.

Using finite differences for (7), (8), (9) and (10) with time step $\Delta t > 0$, spatial discretization step $\Delta x = 1/n$, where $n \geq 2$ is an integer, and using the notation $h_i, v_i$ for the values of the numerical approximations of the liquid level and velocity, respectively, at time $t \geq 0$ and at position $x = i \Delta x$ with $i = 0, \ldots , n$, and the notation $h_i^+, v_i^+$ at time $t + \Delta t$, we obtain the following discrete-time system (numerical scheme):

$$
h_i^+ = h_i + \Delta t \frac{\Delta x}{2} \left( v_{i+1} - 4v_i + v_{i-1} \right),
$$

$$
h_i^+ = h_i + \Delta t \frac{\Delta x}{2} \left( v_{i+1} - v_{i+1} - v_{i-1} + \Delta t \ln \left( \frac{h_{i-1}}{h_{i+1}} \right) \right),
$$

for $i = 1, \ldots , n - 1$,

$$
h_i^+ = h_i + \Delta t \frac{\Delta x}{2} \left( 4v_{i+1} - v_{i+1} - v_{i-2} \right),
$$

and

$$
v_i^+ = \frac{1}{2} \left( \frac{2\Delta t \mu}{(\Delta x)^2} \right) v_i + \frac{\Delta t \mu}{(\Delta x)^2} \left( v_{i+1} + v_{i-1} \right) \left( v_{i+1} - v_{i+1} + \frac{\Delta t \mu}{(\Delta x)^2} + \frac{\Delta t \mu}{4(\Delta x)^2} \right) \left( v_{i+1} - v_{i+1} \right) \left( v_{i+1} - v_{i+1} - v_{i+1} - v_{i+1} + \frac{\Delta t}{\Delta x} \right) \left( v_{i+1} - v_{i+1} - v_{i+1} - v_{i+1} + \frac{\Delta t}{\Delta x} \right)
$$

for $i = 1, \ldots , n - 1$ with

$$
v_0^+ = v_0^+ = 0
$$

In the case of the output feedback with full-order observer the above model is combined with the control law

$$
f = -\sigma \left( 2\lambda z + (1 - 2\lambda) \mu (h_n - h_0) - q (\dot{w} + k \dot{x}) + q (\dot{\xi} - \dot{\xi}) \right) \right)
$$

where the velocity of the tank $w$, the position of the tank $\xi$, the velocity estimate $\hat{w}$ and the position estimate $\hat{\xi}$ at the next time instant are given by the following equations

$$
\xi^+ = \xi + \Delta t w - \frac{(\Delta t)^2}{2} f,
$$

$$
\hat{w}^+ = w - \Delta t f,
$$

$$
\xi^+ = \xi^+ + \exp(-\gamma \Delta t) \left( \cos \left( \frac{\Delta t}{\sqrt{2}} \right) \left( \hat{w} - w \right) + \gamma \left( \hat{\xi} - \xi \right) \right),
$$

$$
\dot{w}^+ = w^+ + \exp(-\gamma \Delta t) \left( \cos \left( \frac{\Delta t}{\sqrt{2}} \right) \left( \hat{w} - w \right) + \gamma \left( \hat{\xi} - \xi \right) \right),
$$

$$
\xi^+ = \xi^+ + \exp(-\gamma \Delta t) \left( \xi - w + \gamma \xi \right)
$$

Finally, inequality (43) and the fact that $|w(0)| \leq \| w(0) \|_X + \| h(0) \|_X$ imply the estimate

$$
\| \xi(t) \|_X + \| w(t) \|_X + \| h(t) \|_X + \| v(t) \|_X \leq M \exp (\gamma t) \left( \| \xi(0) \|_X + \| w(0) \|_X + \| h(0) \|_X + \| v(0) \|_X \right)
$$

Estimate (44) is a direct consequence of the above estimate and the fact that $T = \frac{1}{2} \ln \left( \frac{Mh(0)+2Mv}{2} \right)$.

The discretization schemes (47)-(51), (54) for the viscous Saint-Venant system under the output feedback controller with full order observer and (47)-(49), (52)-(54) for the viscous Saint-Venant system under the output feedback controller with reduced-order observer are non-standard, explicit, finite-diffERENCE schemes and they guarantee positivity of the liquid level (i.e., if $h_i > 0$ for $i = 0, \ldots , n$ then $h_i^+ > 0$ for $i = 0, \ldots , n$). Moreover, the discretization schemes (47)-(51), (54) and (47)-(49), (52)-(54) are accurate of order (2,1) (see Definition 2.3.3 on page 57 in the book Thomas, 1995). We have not been able to find similar discretization schemes in the literature and therefore, we have included the following results for completeness (their proofs are given in the following section). By the term “smooth solution” we mean a solution with derivatives of sufficiently large order (the existence of continuous derivatives of 4th order is sufficient).

**Theorem 3** (Numerical Discretization of the Viscous Saint-Venant System Under the Output Feedback with Full-Order Observer). Let constants $\sigma, q, k, \gamma, \beta > 0, \lambda \in [0, 1]$ be given. Let $T > 0$ be a given constant and let $\xi(t), w(t), h(t), v(t), x(t), \dot{x}(t), \xi(t), z(t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ with $t \in [0, T]$ be a classical smooth solution of the PDE–ODE system (7)-(10), (12), (30), (34), (38) with $L = g = h^* = 1$. Then there exists a constant $M > 0$ such that the following estimate holds for
all $\delta t \in (0, T)$, $\delta x = 1/n$

$$
\begin{align*}
&|\xi^+ - \xi(\delta t)| + |z^+ - z(\delta t)| + |\xi^+ - \xi(\delta t)| + |w^+ - w(\delta t)| \\
+ &\max_{i=0,\ldots,n} \left( |h_i^+ - h(\delta t, i\delta x)| + |v_i^+ - v(\delta t, i\delta x)| \right) \\
&\leq M \delta t (\delta t + (i\delta x)^2)
\end{align*}
$$

(55)

where $n \geq 2$ is an integer and $h_i^+, v_i^+$ for $i = 0, \ldots, n$, $w^+, z^+, \xi^+, \hat{w}^+$ are given by the difference scheme (47)-(51), (54) with $h_i = h(0, i\delta x), v_i = v(0, i\delta x)$, for $i = 0, \ldots, n$, $w = w(0)$, $v = \xi(0)$, $\xi = \hat{w}(0)$, $\hat{w} = \hat{w}(0)$ and $z = z(0)$.

Theorem 4 (Numerical Discretization of the Viscous Saint-Venant System Under the Output Feedback with Reduced-Order Observer). Let constants $\sigma, q, k, \gamma, \beta > 0, \lambda \in [0, 1]$ be given. Let $T > 0$ be a given constant and let $\xi(t), w(t), h[t], v[t], \zeta(t), z(t) \in X \times \mathbb{R}^2$ with $t \in [0, T]$ be a classical smooth solution of the PDE–ODE system (7)-(10), (12), (31), (34), (41) with $L = g = h^+ = 1$. Then there exists a constant $M > 0$ such that the following estimate holds for all $\delta t \in (0, T)$, $\delta x = 1/n$

$$
\begin{align*}
&|\xi^+ - \xi(\delta t)| + |z^+ - z(\delta t)| + |\xi^+ - \xi(\delta t)| + |w^+ - w(\delta t)| \\
+ &\max_{i=0,\ldots,n} \left( |h_i^+ - h(\delta t, i\delta x)| + |v_i^+ - v(\delta t, i\delta x)| \right) \\
&\leq M \delta t (\delta t + (i\delta x)^2)
\end{align*}
$$

(56)

where $n \geq 2$ is an integer and $h_i^+, v_i^+$ for $i = 0, \ldots, n$, $w^+, z^+, \xi^+, \hat{w}^+$ are given by the difference scheme (47)-(49), (52)-(54) with $h_i = h(0, i\delta x), v_i = v(0, i\delta x)$, for $i = 0, \ldots, n$, $w = w(0)$, $\xi = \xi(0), \zeta = \zeta(0)$ and $z = z(0)$.

We have performed many simulations with $n = 100$ for various values of the parameters $\sigma, q, k, \gamma, \beta > 0, \lambda \in [0, 1]$. For each simulation we tested whether the inequalities

$$
\hat{\Phi}^+ \leq \Phi, \text{ for the output feedback with full observer}
$$

$$
\Psi^+ \leq \Psi, \text{ for the output feedback with reduced observer}
$$

were valid for all times, where

$$
\Phi = \frac{\sigma^2}{2} \xi^2 + \frac{q}{2} (w + k\xi)^2 + \frac{1}{2n} \sum_{i=1}^{n-1} h_i v_i^2 + \frac{1}{n} \sum_{i=1}^{n-1} (h_i - 1)^2
$$

$$
+ \frac{1}{2n} (h_0 - 1)^2 + \frac{1}{2n} (h_n - 1)^2
$$

$$
+ \frac{\sigma^2}{2} \left( \hat{\xi}^2 - \xi^2 \right) + \frac{q}{2} \gamma \left( \hat{w} - w - \gamma (\hat{\xi} - \xi) \right)^2
$$

$$
+ \frac{1}{2} \left( z - \frac{1}{n} \sum_{i=1}^{n-1} h_i v_i - \mu (h_n - h_0) \right)^2
$$

$$
+ \frac{1}{16} \sum_{i=1}^{n-1} h_i^{-1} \left( h_i v_i + \frac{\mu}{2} (h_{i+1} - h_{i-1}) \right)^2
$$

$$
+ \frac{\mu^2}{16} h_0^{-1} \left( 4h_1 - h_2 - 3h_0 \right)^2 + \frac{\mu^2}{16} h_n^{-1} \left( 3h_n + h_{n-2} - 4h_{n-1} \right)^2
$$

(57)

and $\hat{\Phi}^+, \hat{\Psi}^+$ are the values of the above functions with $h_i, v_i$ for $i = 0, 1, \ldots, n$ replaced by $h_i^+, v_i^+$, respectively, for $i = 0, 1, \ldots, n$, $\xi, w, z$ replaced by $\xi^+, w^+, z^+, \hat{\xi}, \hat{w}$ replaced by $\hat{\xi}^+, \hat{\psi}$, respectively, $\hat{\Phi}^+$ for the output feedback with full observer and $\xi$ replaced by $\xi^+$ for the output feedback with reduced observer. Notice that $\Phi, \Psi$ are numerical approximations of the values of the functionals $\Phi, \Psi$ defined by (37), (40), respectively (with all integrals evaluated by means of the trapezoidal rule and derivatives approximated by central differences). The proofs of Theorems 1 and 2 show that the functionals $\Phi, \Psi$ are strict Lyapunov functionals for the corresponding closed-loop systems and thus the inequality $\Phi^+ \leq \Phi$ or $\Psi^+ \leq \Psi$ must necessarily hold for each time, provided that the time step $\delta t > 0$ is appropriately selected. If there were a time for which the inequality $\Phi^+ \leq \Phi$ or $\Psi^+ \leq \Psi$ was not valid then the simulation was not accepted and we reduced the time step $\delta t > 0$.

In order to study the convergence of the state we used the following non-negative function

$$
\Omega := \left( \xi^2 + w^2 + \frac{1}{n} \sum_{i=1}^{n-1} (h_i - 1)^2 + \frac{1}{2n} (h_0 - 1)^2 \right.
$$

$$
+ \left. \frac{1}{2n} (h_n - 1)^2 \right)^{1/2}
$$

(59)

which is a numerical approximation (by means of the trapezoidal rule) of the square root of the functional

$$
\xi^2 + w^2 + \|h - \chi_{[0,1]}\|_2^2 + \|v\|_2^2
$$

We found that by increasing the controller gain $\sigma$ and the observer gain $\gamma$ we were able to guarantee faster convergence. The controller parameters $q, k, \beta > 0$ play a less significant role. The parameter $\lambda \in [0, 1]$ (the weight of the filter estimation) played a minor role and in many cases the time evolution of the state under the proposed output feedback laws with $\lambda = 0$ and $\lambda = 1$ was almost identical. Fig. 1 shows the time evolution of the function $\Omega$ defined by (59) with $\delta t = 10^{-4}$, $\mu = 0.2$, $\gamma = 1$, $\sigma = 9.5, q = 1.2, k = 1.5, \lambda = 0$ and

$$
\xi(0) = 2, w(0) = -2.2, h(0, x) = x + 0.5, v(0, x) = x^2 - x, \quad z(0) = 0.2
$$

and $\zeta(0) = -5$ for the output feedback with the reduced-order observer

$$
\xi(0) = 2, w(0) = -2.2, \hat{\xi}(0) = 2.3, \hat{u}(0) = -2.5, h(0, x) = x + 0.5, v(0, x) = x^2 - x, z(0) = 0.2
$$

for the output feedback with the full-order observer.

Fig. 2 shows the time evolution of the logarithms of the functionals $\Phi$ (for the case of output feedback with the full-order observer), $\Psi$ (for the case of output feedback with the reduced-order observer) defined by (57), (58), respectively. The plots clearly indicate exponential convergence. This is expected since the proofs of Theorems 1 and 2 show exponential convergence of the functionals $\Phi, \Psi$ defined by (37), (40), respectively (recall that $\Phi, \Psi$ are numerical approximations of the values of the functionals $\Phi, \Psi$). Fig. 3 shows the evolution of the level set profile and of the liquid velocity profile for the case of output feedback with the full-order observer. It is shown that the liquid level and the liquid velocity approach quickly the equilibrium although sloshing during an initial transient is apparent.

Finally, it should be noticed that simulations showed that for both output feedback schemes and for fixed controller parameters $\sigma, q, k, \gamma, \beta > 0, \lambda \in [0, 1]$ there is an optimal value $\mu_{opt}$ of the viscosity coefficient $\mu$ for which the convergence of the function $\Omega$ defined by (59) becomes fastest. It is important to mention
that $\mu_{opt}$ is different for each output feedback scheme and for different combinations of the controller parameters $\sigma$,$q$,$k$,$\gamma$,$\beta$ and $\lambda$.

7. Proofs

We first give the proof of Theorem 1.

**Proof of Theorem 1.** Let $r \in [0, R]$ be given (arbitrary) and let constants $\sigma$, $q$, $k$, $\gamma$, $\lambda \in [0, 1]$ for which (36) holds (but otherwise arbitrary) be given. Consider a classical solution of the PDE–ODE system (7)–(10), (12) with (30), (34), (38) that satisfies

$$\Phi(\xi(0), w(0), h(0), v(0), \hat{\xi}(0), \hat{w}(0), z(0)) \leq r$$

Using Lemma 2 in Karafyllis and Krstic (2022b), (7), (30), (35) and definitions (32), (37), we obtain for all $t > 0$:

$$\dot{\Phi} = -\mu g \int_0^h h(x)v(x)dx - \mu \int_0^h \mu h(x)v(x)dx - qkx^2$$

$$+ qk(w + k\xi)^2 + f(2\tilde{z} - \mu (h(L) - h(0))) - q(w + k\xi))$$

$$- \sigma q^2 \left(\tilde{\xi} - \xi\right)^2 - 2\sigma q^2 \left(\tilde{w} - w - \gamma \left(\tilde{\xi} - \xi\right)\right)^2$$

$$- \beta \lambda (z - \bar{z})^2 - \beta \lambda (z - \bar{z}) \int_0^h h(x)v(x)dx$$

Eq. (38) and definition (32) implies that

$$f = -\sigma (2\tilde{z} - \mu (h(L) - h(0))) - q(w + k\xi)) - 2\sigma \lambda (z - \bar{z})$$

$$+ \sigma q \left(\tilde{w} - w - \gamma \left(\tilde{\xi} - \xi\right)\right) + 2\sigma (1 - \lambda) \int_0^h h(x)v(x)dx$$

for all $t > 0$. Combining Eqs. (60), (61), we obtain for all $t > 0$:

$$\Phi = -\mu g \int_0^h h(x)v(x)dx - \mu \int_0^h \mu h(x)v(x)dx - qkx^2$$

$$+ qk(w + k\xi)^2 - \sigma (2\tilde{z} - \mu (h(L) - h(0))) - q(w + k\xi))$$

$$- \sigma q^2 \left(\tilde{\xi} - \xi\right)^2 - 2\sigma q^2 \left(\tilde{w} - w - \gamma \left(\tilde{\xi} - \xi\right)\right)^2$$

$$- \beta \lambda (z - \bar{z})^2 - \beta \lambda (z - \bar{z}) \int_0^h h(x)v(x)dx$$

$$- 2\sigma (1 - \lambda) (2\tilde{z} - \mu (h(L) - h(0))) - q(w + k\xi))$$

$$+ \sigma q \left(\tilde{w} - w - \gamma \left(\tilde{\xi} - \xi\right)\right) + 2\sigma (1 - \lambda) \int_0^h h(x)v(x)dx$$

Using the Young inequalities

$$- (z - \bar{z}) \int_0^h h(x)v(x)dx \leq \frac{1}{2} (z - \bar{z})^2 + \frac{1}{2} \left(\int_0^h h(x)v(x)dx\right)^2$$

$$- (z - \bar{z}) (2\tilde{z} - \mu (h(L) - h(0))) - q(w + k\xi))$$

$$\leq (z - \bar{z})^2 + \frac{1}{2} (2\tilde{z} - \mu (h(L) - h(0))) - q(w + k\xi)^2$$
Define for each $t \geq 0$:

$$h_{\text{max}}(t) := \max_{0 \leq x \leq L} (h(t,x)) , \ h_{\text{min}}(t) := \min_{0 \leq x \leq L} (h(t,x)) > 0$$ (74)

The fact that $h_{\text{min}}(t) > 0$ follows from the fact that $h \in C^1([0, +\infty) \times [0, L]; (0, +\infty))$ (which implies that for every $t \geq 0$, $h(t)$ is a continuous positive function on $[0, L]$). Applying Wirtinger’s inequality and using definition (74), we obtain for each $t \geq 0$:

$$\int_0^l h(x)v^2(x)dx \leq h_{\text{max}}(t) \int_0^l v^2(x)dx \leq \frac{L^2h_{\text{max}}}{\pi^2} \int_0^l v^2(x)dx$$ (75)

Combining (73) with (75), we get for all $t > 0$:

$$\Phi \leq -\mu \left( g - \sigma \frac{\mu L}{2} \right) f_0^l h_2^2(x)dx + \frac{\sigma}{\pi^2} \left( \frac{L}{2} - 2\sigma \right) (z - \xi)^2 + \left( \frac{\bar{\beta}^2}{2} + 2\alpha(1 - \lambda) \right) \left( f_0^l h_2(x)v(x)dx \right)^2$$ (76)

Inequality (76) in conjunction with the facts that $\beta > 4\alpha$, $\sigma \frac{\alpha}{\beta} - k > 0$, $g - \frac{\mu L}{2} > 0$ (recall (36)) shows that the following implication holds:

"If $t > 0$ and $\frac{2\mu \pi^2}{(\beta + 4\sigma) L h_{\text{min}}^2} \geq \frac{h_{\text{max}}(t)}{h_{\text{min}}(t)}$, then $\Phi \leq 0$" (77)

The fact that $h \in C^1([0, +\infty) \times [0, L]; (0, +\infty))$ implies that both mappings $t \rightarrow h_{\text{max}}(t)$, $t \rightarrow h_{\text{min}}(t) > 0$ defined by (74) are continuous (see Proposition 2.9 on page 21 in Freeman & Kokotovic, 1996). Consequently, the mapping $t \rightarrow \frac{h_{\text{max}}(t)}{h_{\text{min}}(t)} \geq 1$ is continuous with $\frac{h_{\text{max}}(0)}{h_{\text{min}}(0)} \leq \frac{\bar{C}}{C}$. The latter inequality is a consequence of the fact that (recall definition (37))

$$V(\xi(0), w(0), h(0), v(0)) \leq \Phi(\xi(0), w(0), h(0), v(0), \dot{\xi}(0), \dot{w}(0), z(0)) \leq r$$

and (29). Since $\frac{2\mu \pi^2}{(\beta + 4\sigma) L^2 h_{\text{min}}^2} > \frac{\gamma}{C} > r$ (recall (36)) it follows that $\frac{h_{\text{max}}(t)}{h_{\text{min}}(t)} \geq \frac{2\mu \pi^2}{(\beta + 4\sigma) L^2 h_{\text{min}}^2}$. Therefore, by continuity there exists $T > 0$ such that $\frac{h_{\text{max}}(t)}{h_{\text{min}}(t)} \geq \frac{2\mu \pi^2}{(\beta + 4\sigma) L^2 h_{\text{min}}^2}$ for all $t \in [0, T]$.

We next prove by contradiction that $\frac{h_{\text{max}}(t)}{h_{\text{min}}(t)} \geq \frac{2\mu \pi^2}{(\beta + 4\sigma) L^2 h_{\text{min}}^2}$ for all $t \geq 0$. Assume the contrary, i.e. that there exists $t_0 \geq 0$ such that $\frac{h_{\text{max}}(t)}{h_{\text{min}}(t)} < \frac{2\mu \pi^2}{(\beta + 4\sigma) L^2 h_{\text{min}}^2}$. Therefore, the set $A := \{ t \geq t_0 : \frac{h_{\text{max}}(t)}{h_{\text{min}}(t)} \geq \frac{2\mu \pi^2}{(\beta + 4\sigma) L^2 h_{\text{min}}^2} \}$ is non-empty and bounded from below. Thus we can define $t^* = \inf(A)$. By definition, it holds that $t^* \geq T > 0$ and $\frac{h_{\text{max}}(t)}{h_{\text{min}}(t)} < \frac{2\mu \pi^2}{(\beta + 4\sigma) L^2 h_{\text{min}}^2}$ for all $t \in [0, t^*)$. By continuity of the mapping $t \rightarrow \frac{h_{\text{max}}(t)}{h_{\text{min}}(t)}$, we obtain that $\frac{h_{\text{max}}(t)}{h_{\text{min}}(t)} = \frac{2\mu \pi^2}{(\beta + 4\sigma) L^2 h_{\text{min}}^2}$. Moreover, since $\frac{h_{\text{max}}(t)}{h_{\text{min}}(t)} \geq \frac{2\mu \pi^2}{(\beta + 4\sigma) L^2 h_{\text{min}}^2}$ for all $t \in [0, t^*)$, it follows from implication (77) that $\Phi \leq 0$ for all $t \in [0, t^*)$. By continuity of the mapping $t \rightarrow \Phi(\xi(t), w(t), h(t), v(t), \dot{\xi}(t), \dot{w}(t), z(t))$, we obtain that

$$\Phi(\xi(t^*), w(t^*), h(t^*), v(t^*), \dot{\xi}(t^*), \dot{w}(t^*), z(t^*)) \leq \Phi(\xi(0), w(0), h(0), v(0), \dot{\xi}(0), \dot{w}(0), z(0)) \leq r$$.
On the other hand, the above inequality and (37), (29) imply that
\[ \frac{h_{\text{max}}(t^2)}{C_{\text{max}}(t^2)} \leq \frac{2\omega + \frac{2\mu \pi^2}{(\beta + 4\sigma)\sigma^2}}{\beta + 4\sigma} \]
which contradicts the equation
\[ \frac{h_{\text{max}}(t^2)}{C_{\text{max}}(t^2)} = \frac{2\omega + \frac{2\mu \pi^2}{(\beta + 4\sigma)\sigma^2}}{\beta + 4\sigma} \]
for all \( t \geq 0 \). We conclude from implication (77) that \( \Phi \neq 0 \) for all \( t \geq 0 \). By continuity of the mapping \( t \to \Phi \left( \xi(t), w(t), h(t), v(t), \dot{\xi}(t), \dot{w}(t), z(t) \right) \) and definition (37), we obtain that
\[ V \left( \xi(t), w(t), h(t), v(t) \right) \leq \Phi \left( \xi(t), w(t), h(t), v(t), \dot{\xi}(t), \dot{w}(t), z(t) \right) \]
(78)
\[ \leq \Phi \left( \xi(0), w(0), h(0), v(0), \dot{\xi}(0), \dot{w}(0), z(0) \right) \]
for all \( t \geq 0 \). Hence, \( \Phi(t, w(t), h(t), v(t)) \in X \) for all \( t \geq 0 \) (recall definitions (25), (27)). Moreover, (29) implies that \( \frac{h_{\text{max}}(t^2)}{C_{\text{max}}(t^2)} \leq \frac{C_{\text{max}}(t^2)}{C_{\text{max}}(t^2)} \) for all \( t \geq 0 \). The previous inequality in conjunction with
\[ \Phi \leq -\omega \left( h^{\text{max}}(t) \right) dx + \int_0^1 \frac{1}{\beta + 4\sigma} \sigma q^2 \left( \frac{\xi - \xi_0}{\gamma} \right)^2 + \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \dot{\xi}_0 \right)^2 + \left( w - w_0 + \frac{\lambda}{2} (z - z_0) \right)^2 ) \]
where
\[ \omega := \min \left( \mu - \frac{\sigma q L}{2}, \frac{\beta + 4\sigma}{2\pi^2}, \frac{2\mu \pi^2}{(\beta + 4\sigma)\sigma^2} \right) \]
(80)
It follows from Lemma 3 in Karafyllis and Krstic (2022b), definition (37) and (79) that the following estimate holds for all \( t > 0 \):
\[ \frac{d}{dt} \Phi \left( \xi(t), w(t), h(t), v(t), \dot{\xi}(t), \dot{w}(t), \dot{z}(t) \right) \]
\[ \leq -\omega \max \left( \frac{1}{\gamma}, \frac{1}{\gamma}, \frac{1}{\gamma} \right) \Phi \left( \xi(t), w(t), h(t), v(t), \dot{\xi}(t), \dot{w}(t), \dot{z}(t) \right) \]
(81)
where \( \Gamma : [0, R] \to (0, +\infty) \) is a non-decreasing function.
Since \( \Gamma : [0, R] \to (0, +\infty) \) is non-decreasing and since \( V \left( \xi(t), w(t), h(t), v(t) \right) \leq 0 \) for all \( t \geq 0 \) (recall (78)), we obtain from (81) the following estimate for all \( t > 0 \):
\[ \frac{d}{dt} \Phi \left( \xi(t), w(t), h(t), v(t), \dot{\xi}(t), \dot{w}(t), \dot{z}(t) \right) \]
\[ \leq -\omega \max \left( \frac{1}{\gamma}, \frac{1}{\gamma}, \frac{1}{\gamma} \right) \Phi \left( \xi(t), w(t), h(t), v(t), \dot{\xi}(t), \dot{w}(t), \dot{z}(t) \right) \]
(82)
By continuity of the mapping \( t \to \Phi \left( \xi(t), w(t), h(t), v(t), \dot{\xi}(t), \dot{w}(t), \dot{z}(t) \right) \), the differential inequality (82) implies the following estimate for all \( t \geq 0 \):
\[ \Phi \left( \xi(t), w(t), h(t), v(t), \dot{\xi}(t), \dot{w}(t), z(t) \right) \]
\[ \leq \exp \left( -\omega \max \left( \frac{1}{\gamma}, \frac{1}{\gamma}, \frac{1}{\gamma} \right) \right) \times \Phi \left( \xi(0), w(0), h(0), v(0), \dot{\xi}(0), \dot{w}(0), z(0) \right) \]
(83)
Estimate (83) in conjunction with Lemma 4 in Karafyllis and Krstic (2022b) and the inequalities
\[ \frac{\sigma q^2}{\gamma} \left( \frac{\xi - \xi_0}{\gamma} \right)^2 + \left( \dot{\xi} - \dot{\xi}_0 \right)^2 \]
\[ \leq \frac{\sigma q^2}{\gamma^2} \left( \frac{\xi - \xi_0}{\gamma} \right)^2 + \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \dot{\xi}_0 \right)^2 \]
\[ \leq \frac{\sigma q^2}{\gamma} \left( \frac{\xi - \xi_0}{\gamma} \right)^2 + \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \dot{\xi}_0 \right)^2 \]
\[ \leq \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \xi_0 \right)^2 + \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \dot{\xi}_0 \right)^2 \]
\[ \leq \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \xi_0 \right)^2 + \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \dot{\xi}_0 \right)^2 \]
\[ \leq \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \xi_0 \right)^2 + \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \dot{\xi}_0 \right)^2 \]
\[ \leq \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \xi_0 \right)^2 + \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \dot{\xi}_0 \right)^2 \]
\[ \leq \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \xi_0 \right)^2 + \frac{\sigma q^2}{\gamma} \left( \dot{\xi} - \dot{\xi}_0 \right)^2 \]
implies the following estimate for all \( t \geq 0 \):
\[ \Phi(t) \leq \Phi(0) \exp \left( -\frac{\omega t}{\gamma} \max \left( 1, \frac{1}{\gamma} \right) \right) \]
(84)
where \( G_i : [0, R] \to (0, +\infty), i = 1, 2 \), are non-decreasing functions and
\[ \Phi(t) := \left( \xi(t), w(t), h(t) \right)_X \left( \frac{1}{\gamma}, \frac{1}{\gamma}, \frac{1}{\gamma} \right) \]
(85)
Since \( G_i : [0, R] \to (0, +\infty), i = 1, 2 \), are non-decreasing functions and since \( V \left( \xi(t), w(t), h(t), v(t) \right) \leq 0 \) for all \( t \geq 0 \) (recall (78)), we obtain from (84) the following estimate for all \( t \geq 0 \):
\[ \Phi(t) \leq \Phi(0) \exp \left( -\frac{\omega t}{\gamma} \max \left( 1, \frac{1}{\gamma} \right) \right) \]
(86)
Estimate (39) is a direct consequence of estimate (86) and definition (85). The proof is complete. \( \rightleftharpoons \)

**Proof of Theorem 2.** Let \( r \in [0, R) \) be given (arbitrary) and let constants \( \sigma, q, k, \gamma > 0, \lambda \in [0, 1] \) for which (36) holds (but otherwise arbitrary) be given. Consider a classical solution of the PDE–ODE system (7)–(10), (12) with (31), (34), (41) that satisfies \( \Psi(\xi(0), w(0), h(0), v(0), \xi(0), \dot{\xi}(0), \dot{w}(0), \dot{z}(0)) \leq r \). We define
\[ \ddot{\zeta} := w - \gamma \dot{\xi} \]
(87)
It follows from (7) and definition (87) that the following differential equation holds for all \( t > 0 \):
\[ \ddot{\zeta} \]
(88)
A direct consequence of (31) and (88) is the following differential equation for all \( t > 0 \):
\[ \frac{d}{dt} (\zeta - \ddot{\zeta}) = -\gamma (\zeta - \ddot{\zeta}) \]
(89)
Using Lemma 2 in Karafyllis and Krstic (2022b), (7), (35), (89) and definitions (32), (40), (87) we obtain for all \( t > 0 \):
\[ \ddot{\zeta} = -\mu g \int_0^t h^{\text{max}}(x) dx - \mu \int_0^t h(x) v^2(x) dx - q k^2 \xi^2 + q k \xi \]
(90)
Eq. (41) and definitions (32) and (87) imply that
\[ f = -\sigma \left( 2 \zeta - \mu (h(t) - h(0)) \right) \]
(91)
for all $t > 0$. Combining Eqs. (90), (91), we obtain for all $t > 0$:

$$\dot{\Psi} = -\mu \int_0^t h_2(x) dx - \mu \int_0^t h(x) v_2(x) dx - qk^2 \xi^2 + qk(\xi + (\gamma + k)\xi) < 0,$$

Using inequality (63), inequalities (64), (66) with $w = \tilde{\xi} + \gamma \xi$ (recall (87)), and the following inequality

$$\dot{\Psi} \leq -\mu \int_0^t h_2(x) dx - \mu \int_0^t h(x) v_2(x) dx - qk^2 \xi^2 + qk(\xi + (\gamma + k)\xi) < 0,$$

we obtain from (92) the following differential inequality that holds for all $t > 0$:

$$\dot{\Psi} \leq -\mu \int_0^t h_2(x) dx - \mu \int_0^t h(x) v_2(x) dx - qk^2 \xi^2 + qk(\xi + (\gamma + k)\xi) < 0,$$

Using inequality (68) with $w = \tilde{\xi} + \gamma \xi$ (recall (87)) along with definitions (87), (32), we obtain from (94) for all $t > 0$:

$$\dot{\Psi} \leq -\mu \int_0^t h_2(x) dx - \mu \int_0^t h(x) v_2(x) dx - qk^2 \xi^2 + qk(\xi + (\gamma + k)\xi) < 0,$$

Using definition (70) and the facts that $\beta > 4\sigma$, $\lambda \in [0, 1]$, $h(L-h(0)) = \int_0^t h(x) dx$, we obtain from (95) for all $t > 0$:

$$\dot{\Psi} \leq -\mu \int_0^t h_2(x) dx - \mu \int_0^t h(x) v_2(x) dx - qk^2 \xi^2 + qk(\xi + (\gamma + k)\xi) < 0,$$

Inequalities (72) combined with (96) give for all $t > 0$:

$$\dot{\Psi} \leq -\mu \int_0^t h_2(x) dx - \mu \int_0^t h(x) v_2(x) dx - qk^2 \xi^2 + qk(\xi + (\gamma + k)\xi) < 0,$$

Using definition (74) and the inequalities (75) and (97), we get for all $t > 0$:

$$\dot{\Psi} \leq -\mu \int_0^t h_2(x) dx - \mu \int_0^t h(x) v_2(x) dx - qk^2 \xi^2 + qk(\xi + (\gamma + k)\xi) < 0,$$

Exploiting Lemma 3 in Karafyllis and Krstic (2022b), (98) and definition (40) and using similar arguments as in the proof of Theorem 1, we are in a position to get the following estimates

$$\Psi(t) < \Psi(0) \exp \left( \frac{-\omega t}{\max(1, \Gamma(r))} \right) \max \left( \frac{G_2(V(\xi(0), w(0), h(0), v(0)), \sigma \gamma^2, 1)}{G_1(V(\xi(t), w(t), h(t), v(t))) \sigma \gamma^2, 1} \right) \min \left( \frac{1}{G_1(r)}, \frac{1}{G_2(r)} \right) \right)$$

where $\omega$ is given by (80) and $\Gamma^\ast : [0, R] \to (0, +\infty)$ is a non-decreasing function. Estimate (99) in conjunction with Lemma 4 in Karafyllis and Krstic (2022b) and definition (40) implies the following estimate for all $t \geq 0$:

$$\Psi(t) < \Psi(0) \exp \left( \frac{-\omega t}{\max(1, \Gamma(r))} \right) \max \left( \frac{G_2(V(\xi(0), w(0), h(0), v(0)), \sigma \gamma^2, 1)}{G_1(V(\xi(t), w(t), h(t), v(t))) \sigma \gamma^2, 1} \right) \min \left( \frac{1}{G_1(r)}, \frac{1}{G_2(r)} \right) \right)$$

Estimation (42) is a direct consequence of estimate (102) and definition (101). The proof is complete. $\Box$

**Proof of Theorem 3.** Let constants $\sigma, q, k, \gamma, \beta > 0, \lambda \in [0, 1]$ be given. Let $T > 0$ be a given constant and let $(\xi(t), w(t), h(t), v(t), \tilde{\xi}(t), \tilde{w}(t), z(t)) \in X \times \mathbb{R}^3$ with $t \in [0, T]$ be a classical smooth solution of the PDE–ODE system (7)–(10), (12), (30), (34), (38) with $L = g = h^* + L$. Let $n \geq 2$ be an integer and define $dx = 1/n$. Let $\delta t \in (0, T)$ be a given constant and consider $h^*, v^*$ for $i = 0, \ldots, n, w^*, \xi^*, \tilde{\xi}, \tilde{w}^*$ given by the difference scheme (47)–(52), (55) with $h_i = h(0, i \delta x)$, $v_i = v(0, i \delta x)$, for $i = 0, \ldots, n, w = w(0)$, $\xi = \xi(0)$, $\tilde{\xi} = \tilde{\xi}(0)$, $\tilde{w} = \tilde{w}(0)$ and $z = z(0)$. Define:

$$u(t, x) = \ln b(h(t, x)),$$

for $t \in [0, T], x \in [0, 1]$.

$$u_i = u(0, i \delta x), \text{ for } i = 0, \ldots, n$$

Using (47), (104), the following relations for $i = 1, \ldots, n - 1$

$$u_i(0, i \delta x) = \frac{1}{28x} \int_{i \delta x}^{(i+1) \delta x} \int_{i \delta x}^{i \delta x} u_{xx}(0, \eta) d\eta d\xi d\eta d\xi$$

$$- \frac{1}{28x} \int_{(i-1) \delta x}^{i \delta x} \int_{i \delta x}^{i \delta x} u_{xx}(0, \eta) d\eta d\xi d\eta d\xi$$

$$- \frac{1}{28x} \int_{(i-1) \delta x}^{i \delta x} \int_{i \delta x}^{i \delta x} u_{xx}(0, \eta) d\eta d\xi d\eta d\xi$$

(105)
Using the inequality \(|\exp(x) - \exp(y)| \leq \exp(x) \exp(|x - y|)\) for all \(x, y \in \mathbb{R}\) and the fact that \(|\ln(h(\delta t, i\delta x)) - \ln(h^\dagger(\delta t, i\delta x))| \leq ST (1 + T)\) (a consequence of (112) and the facts that \(\delta t \in (0, T)\) and \(\delta x = 1/n\)), we get from (112):

\[
\max_{\delta t \in [0, T]} \left( |h(i\delta x) - h(\delta t, i\delta x)| \right) \leq \max_{\delta t \in [0, T]} \left( |h[\delta t]| \right)_\infty
\]

\[
\times \exp \left( ST(1 + T) \right) S(\delta t) \left( (\delta x)^2 + (\delta t) \right)
\]

Using (30) and (7), we obtain the following formulas:

\[
\hat{\xi}(\delta t) - \xi(\delta t) = \exp(-\gamma \delta t) \left( \left( \hat{\xi}(0) - \xi(0) \right) \cos \left( \frac{\delta t}{\sqrt{2}} \right) + \sqrt{2} \left( \hat{\xi}(0) - \xi(0) \right) \sin \left( \frac{\delta t}{\sqrt{2}} \right) \right)
\]

\[
\hat{w}(\delta t) - w(\delta t) = \exp(-\gamma \delta t) \left( \left( \hat{w}(0) - w(0) \right) \cos \left( \frac{\delta t}{\sqrt{2}} \right) + \gamma \delta t \right)
\]

Consequently, we get from (51) and (114), (115):

\[
\left| \hat{\xi}^+ - \xi(\delta t) \right| + \left| \hat{w}^+ - \hat{w}(\delta t) \right| = \left| w^+ - w(\delta t) \right|
\]

Using (1) we obtain the following formulas:

\[
\hat{\xi}(\delta t) = \xi(0) + w(0) \delta t + \frac{(\delta t)^2}{2} f(0) + \int_0^{\delta t} \int_0^s f(\eta) d\eta d\eta d\delta t
\]

\[
w(\delta t) = w(0) - f(0) \delta t - \int_0^{\delta t} \int_0^s \left| \hat{\xi}(\delta t) \right| d\eta d\eta d\delta t
\]

Exploiting (117), (51) and the fact that \(\delta t \in (0, T)\), we get:

\[
\left| \hat{\xi}(\delta t) - \xi(\delta t) \right| \leq \frac{(\delta t)^2}{6} \max_{\delta t \in [0, T]} \left( |\hat{\xi}(s)| \right)
\]

\[
\left| w(\delta t) - w(\delta t) \right| \leq \frac{(\delta t)^2}{6} \max_{\delta t \in [0, T]} \left( |\hat{\xi}(s)| \right)
\]

Therefore, we get from (116), (118):

\[
\left| \hat{\xi}^+ - \xi(\delta t) \right| + \left| w^+ - w(\delta t) \right| + |\hat{\xi}^+ - \xi(\delta t)| + |\hat{w}^+ - \hat{w}(\delta t)|
\]

\[
= 2 \left| \hat{\xi}^+ - \xi(\delta t) \right| + 2 \left| w^+ - w(\delta t) \right|
\]

\[
\leq (\delta t)^2 \left( \frac{T}{3} + 1 \right) \max_{\delta t \in [0, T]} \left( |\hat{\xi}(s)| \right)
\]

Using (48), (103), (104), (105), (106), the fact that \(v_{i+1} - v_{i-1} = \int_{i\delta x}^{(i+1)\delta x} v_{0}(s) d\delta t\) for \(i = 1, \ldots, n - 1\), the following relations for \(i = 1, \ldots, n - 1\):

\[
h_i(0, i\delta x) = h_{i+1} - h_{i-1}
\]

\[
- \frac{1}{2\delta x} \int_{i\delta x}^{(i+1)\delta x} \int_{i\delta x}^{s} v_{0}(s) d\eta d\delta t d\delta t
\]

\[
\left| \hat{v}(\delta t, i\delta x) - \hat{v}(\delta t, i\delta x) \right| \leq S(\delta t) \left( (\delta x)^2 + (\delta t) \right)
\]

for \(i = 0, 1, \ldots, n\) and the fact that \(\delta x = 1/n\):

\[
\left| \ln \left( h(\delta t, i\delta x) \right) - \ln \left( h(\delta t, i\delta x) \right) \right| \leq S(\delta t) \left( (\delta x)^2 + (\delta t) \right)
\]

\[
\left| \ln \left( h(\delta t, i\delta x) \right) - \ln \left( h(\delta t, i\delta x) \right) \right| \leq S(\delta t) \left( (\delta x)^2 + (\delta t) \right)
\]

\[
\left| \hat{v}(\delta t, i\delta x) - \hat{v}(\delta t, i\delta x) \right| \leq S(\delta t) \left( (\delta x)^2 + (\delta t) \right)
\]
Consequently, we get from (53) and (129):
\[
\begin{align*}
|\mathbf{\zeta}(\delta t) - \mathbf{\zeta}^+| & = |\mathbf{w}(\delta t) - w^+ + \gamma (\mathbf{\xi}^+ - \mathbf{\xi}(\delta t))| \\
& \leq |\mathbf{w}(\delta t) - w^+| + \gamma |\mathbf{\xi}^+ - \mathbf{\xi}(\delta t)|
\end{align*}
\]
Using (1) we obtain the formulas (117), (118). Therefore, we get from (130), (118):
\[
\begin{align*}
|\mathbf{\xi}^+ - \mathbf{\xi}(\delta t)| & + |\mathbf{w}^+ - \mathbf{w}(\delta t)| + |\mathbf{\xi}^+ - \mathbf{\xi}(\delta t)| \\
& \leq (\gamma + 1) |\mathbf{\xi}^+ - \mathbf{\xi}(\delta t)| + 2 |\mathbf{w}^+ - \mathbf{w}(\delta t)| \\
& \leq (\delta t)^2 (\gamma + 1) \frac{3}{8} + 1 \max_{\mathbf{h} \in S} (|\mathbf{\xi}(\delta)|)
\end{align*}
\]
(123)

(8. Concluding remarks)

By applying the CLF methodology we have managed to achieve output feedback stabilization results for the viscous Saint-Venant liquid–tank system. As far as we know, this is the first paper in the literature that achieves output feedback stabilization of the nonlinear viscous Saint-Venant system.

The obtained results leave some open problems which will be the topic of future research:

1. The results were applied to classical solutions and it is of interest to relax this to weak solutions. It is also an open problem to show existence/uniqueness of (weak) solutions for the closed-loop system. To this purpose, ideas utilized in Sundbye (1996) can be employed.

2. The construction of CLFs which can allow the derivation of stability estimates in stronger spatial norms for the liquid level and velocity profiles.

Another more demanding problem that will be studied in the future is the spill-free, slosh-free and smash-free movement of a glass of water. This problem arises when we want to move the glass of water to a position which is close to a wall. In this case, we need to control the overshoot of the glass position error in order to avoid smashing the glass on the wall.

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