Boundary Control of the Korteweg–de Vries–Burgers Equation: Further Results on Stabilization and Well–Posedness, with Numerical Demonstration

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***Abstract*—We consider the Korteweg–de Vries–Burgers (KdVB) equation on the interval [0,1]. Motivated by simulations resulting in modest decay rates with recently proposed control laws by Liu and Krstic which keeps some of the boundary conditions as homogeneous, we propose a strengthened set of feedback boundary conditions. We establish stability properties of the closed-loop system, prove well-posedness and illustrate the performance improvement by a simulation example.**

***Index Terms*—Global stabilization, Korteweg–de Vries–Burgers equation, nonlinear boundary feedback control.**

I. INTRODUCTION

The Korteweg–de Vries–Burgers (KdVB) equation is one of the simplest nonlinear mathematical models displaying the features of both dispersion and dissipation. It serves as a model of long waves in shallow water and some other physical phenomena. The usual and simplest setting in which the controlled and uncontrolled KdVB equation or the simpler KdV equation is considered is either the case of periodic boundary conditions (see, e.g. Bona *et al.* [3], Russel and Zhang [15]) or the case where the spatial domain is the whole real line (see, e.g. Biler [2], Bona and Smith [4]). As a next step in the analysis of a system it is natural to consider the controllability (see, e.g. Rosier [13]) and stabilization (see, e.g. Zhang [18]) on a bounded domain. In a recent work Liu and Krstic [11] consider a boundary feedback stabilization problem for a KdVB equation on a finite spatial interval. Our paper is motivated by simulations that show opportunity for considerable improvement of performance relative to the controllers in [11]. In this paper we propose a more aggressive control law that achieves better performance. Our control law can be implemented via any of the following three variables actuated at one boundary with w held at zero at the other boundary: $(w_x,\, w_{xx}), (w,\, w_x), (w,\, w_{xx})$. The uncontrolled versions of some of these problems are known not to be asymptotically stable. An example of a physical problem where our control law would be implementable is the water channel setup with boundary actuation discussed in Rosier [13]. In Section II we prove the existence and stability of solutions of the resulting boundary controlled KdVB equation. All the details of the calculations, including the ones omitted here due to space limitation, can be found at the authors' web pages. In Section III we provide a numerical example.

II. STABILIZATION

Consider the Korteweg–de Vries–Burgers equation

$$w_t - \epsilon w_{xx} + \delta w_{xxx} + w w_x = 0, \quad x \in [0,\, 1], \quad t > 0, \tag{1}$$

with $\epsilon,\, \delta > 0$ and with some initial data

$$w(x,\, 0) = w_0(x), \quad x \in [0,\, 1]. \tag{2}$$

Liu and Krstic [11] proposed the control law

$$w(0,\, t) = 0, \quad w_x(1,\, t) = 0, \quad w_{xx}(1,\, t) = -\frac{1}{3\delta} w^2(1,\, t), \tag{3}$$

and an improved version of it

$$w(0,\, t) = 0, \qquad w_x(1,\, t) = 0, \tag{4}$$

$$w_{xx}(1,\, t) = \frac{1}{\delta}\left(c + \frac{1}{9c} w^2(1,\, t)\right) w(1,\, t), \quad c > 0. \tag{5}$$

Unfortunately, as we shall see in Section III, the choice $w_x(1,t) = 0$ results in slow convergence to zero. For this reason, in this paper we seek and find a more aggressive boundary condition that also uses $w_x(1,t)$ for feedback:

$$w(0,\, t) = 0, \tag{6}$$

$$\begin{aligned} w_x(1,\, t) &= -g_1(w(1,\, t)) \\ &\triangleq -\frac{1}{\epsilon}\left(c + \frac{1}{9c} w^2(1,\, t)\right) w(1,\, t), \end{aligned} \tag{7}$$

$$\begin{aligned} w_{xx}(1,\, t) &= g_2(w(1,\, t)) \\ &\triangleq \frac{1}{\epsilon^2}\left(c + \frac{1}{9c} w^2(1,\, t)\right)^2 w(1,\, t). \end{aligned} \tag{8}$$

It is clear that, since (7) and (8) are invertible functions, this control law can be implemented via any of the following three variables at the 1-boundary: $(w_x,\, w_{xx}), (w,\, w_x), (w,\, w_{xx})$.

In order to formulate our problem as an abstract initial value problem we consider Hilbert spaces $X = L^2(0,\, 1)$, $H = H^1(0,\, 1)$, operator $\mathcal{A}$: $(\mathcal{D}(\mathcal{A}) \subset X) \to X^*$ given by

$$\mathcal{A}w = -\epsilon w_{xx} + \delta w_{xxx} + \tfrac{1}{2}(w^2)_x, \tag{9}$$

and domain

$$\mathcal{D}(\mathcal{A}) = \big\{ w \in H^3(0,\, 1) \,|\, w(0) = 0,\ w'(1) = -g_1(w(1)),\ w''(1) = g_2(w(1)) \big\}.$$

With the above notation our system (1), (2), (6)–(8) can be written in the form of

$$\frac{dw}{dt} + \mathcal{A}w = 0, \qquad w(0) = w_0. \tag{10}$$

Our main result is formulated in the following theorem.

Theorem 1: For any initial data $w_0 \in \mathcal{D}(\mathcal{A})$ *system (10) possesses a unique solution* $w(x,\, t) \in C(0,\, \infty;\, L^2(0,\, 1)) \cap C(0,\, \infty;\, H^1(0,\, 1))$ *with*

1) *Global exponential stability in the* L^2*-sense:*

$$\|w(t)\| \le \|w_0\| e^{-\epsilon t}, \qquad \forall\, t \ge 0, \tag{11}$$

Manuscript received October 15, 1998; revised February 8, 1999; September 20, 1999; and December 17, 1999. Recommended by Associate Editor, I. Lasiecka. This work was supported by grants from the Office of Naval Research, Air Force Office of Scientific Research, and from the National Science Foundation.

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Publisher Item Identifier S 0018-9286(00)06755-6.

2) *Global asymptotic and semi-global exponential stability in the H^1-sense: there exist $M > 0$ such that for any $0 \le \alpha < 1$*

$$\|w(t)\|_{H^1} \le \frac{M}{\sqrt{1-\alpha}} \|w_0\|_{H^1} e^{(M/\sqrt{1-\alpha})\|w_0\|_{H^1}^2} e^{-\alpha\epsilon t}, \quad \forall t \ge 0. \tag{12}$$

The same stability statements (reported in [11]) hold with control law (4) and (5). Since (11) and (12) are conservative energy estimates and M is a generic constant, they do not provide a good basis for comparison of the two controllers.

It is very important to understand the role of the parameter α in the H^1 estimate (12).[1] The larger α, the better the exponential decay rate is. At the same time the "overshoot," which is proportional to $1/\sqrt{1-\alpha}$, is a monotone increasing function of its argument α and it blows up at $\alpha = 1$. This decay rate dependent overshoot coefficient dominates the estimate on short time intervals, which shows again the need for numerical comparison.

Proof: In order to prove the stability results we use energy estimates.

First take the L^2-inner product of (1) with w to obtain

$$\int_0^1 (w_t - \epsilon w_{xx} + \delta w_{xxx} + w_x w) w \, dx = 0. \tag{13}$$

Using integration by parts, boundary conditions (6)–(8) and inequality

$$\int_0^1 w_x w^2 \, dx = \frac{1}{3} w^3(1, t) \le \frac{1}{18c} w^4(1, t) + \frac{c}{2} w^2(1, t) \tag{14}$$

we obtain from (13) estimate

$$\frac{d}{dt}\|w(t)\|^2 + 2\epsilon \|w_x\|^2 + \left(\frac{\delta c^2}{\epsilon^2} + c\right) w^2(1, t) + \left(\frac{1}{9c} + \frac{2\delta}{9\epsilon^2}\right) w^4(1, t) + \frac{\delta}{81c^2\epsilon^2} w^6(1, t) \le 0. \tag{15}$$

As a first consequence of (15) we obtain, using Poincaré's inequality, inequality

$$\frac{d}{dt}\|w(t)\|^2 \le -2\epsilon \|w_x(t)\|^2 \le -2\epsilon \|w(t)\|^2, \tag{16}$$

which implies (11). We now multiply (15) by $e^{2\alpha\epsilon t}$, where $0 \le \alpha < 1$ is arbitrary. Integration with respect to time and the use of (11) results in the inequality

$$\int_0^t e^{2\alpha\epsilon\tau} \left(\left(\|w_x(\tau)\|^2 + w^2(1, \tau) + w^4(1, \tau) \right) + w^6(1, \tau) \right) d\tau + e^{2\alpha\epsilon t} \|w(t)\|^2 \le \frac{1}{1-\alpha} \|w_0\|^2. \tag{17}$$

Next, we take the L^2-inner product of (1) with $-w_{xx}$ to obtain

$$-\int_0^1 (w_t - \epsilon w_{xx} + \delta w_{xxx} + w w_x) w_{xx} \, dx = 0. \tag{18}$$

[1]We thank a reviewer and associate editor for suggesting the use of this parameter. Our original analysis was for $\alpha = 1/2$.

The quadratic terms of (18) are integrated by parts and boundary conditions (6)–(8) are used again. For the last cubic term we obtain the estimate

$$\int_0^1 w_x w w_{xx} \, dx \le \frac{1}{2\epsilon} \|w_x(t)\|^4 + \frac{\epsilon}{2} \|w_{xx}(t)\|^2,$$

where we used the simple inequality $\|w\|_{L^\infty(0,1)} \le \|w_x\|$, which holds for $w \in H_0^1(0, 1)$. Introducing the notation

$$A(t) \equiv \frac{c}{\epsilon} w^2(1, t) + \frac{1}{18\epsilon c} w^4(1, t) + \|w_x(t)\|^2 \tag{19}$$

we obtain

$$\frac{d}{dt} A(t) + \epsilon \|w_{xx}(t)\|^2 \le M \left(w^2(1, t) + w^4(1, t) + w^6(1, t) + \|w_x(t)\|^2 \right) \times (1 + A(t)). \tag{20}$$

Omitting the nonnegative second term on the left, using definitions (19) and

$$b(t) \equiv e^{2\alpha\epsilon t} \left(w^2(1, t) + w^4(1, t) + w^6(1, t) + \|w_x(t)\|^2 \right) \tag{21}$$

furthermore multiplying (20) by $e^{2\alpha\epsilon t}$ we get

$$\frac{d}{dt}\left(e^{2\alpha\epsilon t} A(t) \right) \le M b(t) + M b(t) e^{2\alpha\epsilon t} A(t). \tag{22}$$

It follows now from Gronwall's inequality, estimate (17) and the definition of $b(t)$ that

$$\begin{aligned} e^{2\alpha\epsilon t} A(t) \le & \left(A(0) + \int_0^t M b(\tau) \, d\tau \right) \\ & \times \left(1 + \int_0^t M b(\tau) \exp\left(\int_\tau^t M b(s) \, ds \right) d\tau \right) \\ \le & \left(A(0) + \frac{M\|w_0\|^2}{1-\alpha} \right) \\ & + \left(A(0) + \frac{M\|w_0\|^2}{1-\alpha} \right) \frac{M\|w_0\|^2}{1-\alpha} \\ & \times e^{M\|w_0\|^2/(1-\alpha)}. \end{aligned} \tag{23}$$

Multiplying (23) by $e^{-2\alpha\epsilon t}$, taking the square root, and using the definition of $A(t)$ one more time we arrive at the inequality

$$\|w(t)\|_{H^1} \le k(\alpha) \|w_0\|_{H^1} e^{k(\alpha)\|w_0\|_{H^1}^2} e^{-\alpha\epsilon t} \tag{24}$$

where $k(\alpha) = M/\sqrt{1-\alpha}$. This proves (12), the semi-global exponential stability in the H^1-sense. Due to the general Sobolev embedding theorem $H^\ell(\Omega) \subset C^k(\overline{\Omega})$, which holds for $k \le \ell - n/2$, $\Omega \subset \mathbb{R}^n$, the solution $w(t, x)$ is continuous and bounded for all $t \ge 0$ and all $x \in [0, 1]$.

For completeness we include here an existence proof based on the theory of monotone operators with locally Lipschitz perturbations [1], [6], [16] following the arguments in [10].[2]

We consider two operators, $\mathcal{A}_1$: $(\mathcal{D}(\mathcal{A}_1) \subset X) \to X$ given by $\mathcal{A}_1 w = -\epsilon_1 w_{xx} + \delta w_{xxx}$ with domain

$$\mathcal{D}(\mathcal{A}_1) = \{w \in H^3(0, 1) \mid w(0) = 0 \\ w'(1) = -g_1(w(1)), \quad w''(1) = g_2(w(1))\}$$

and $\mathcal{A}_2$: $(\mathcal{D}(\mathcal{A}_2) \subset X) \to X$ given by $\mathcal{A}_2 w = \lambda w - \epsilon_2 w_{xx} + d/dx\, f(w)$ with domain

$$\mathcal{D}(\mathcal{A}_2) = \left\{w \in H^2(0, 1) \mid w(0) = 0\ w'(1) = -g_1(w(1))\right\}.$$

Here $\epsilon_1, \epsilon_2 > 0$ with $\epsilon_1 + \epsilon_2 = \epsilon$ and $f(w) = w^2/2$.

Next we introduce a cut-off function of $f(y) = y^2/2$ and obtain the globally Lipschitz continuous function

$$f_K(y) = \begin{cases} y^2/2 & \text{if } |y| \le K, \\ K^2/2 & \text{if } |y| > K \end{cases} \tag{25}$$

with Lipschitz constant $L_K = K$. We define the nonlinear operator $\mathcal{A}_{2,K}$ corresponding to the cut-off version of $\mathcal{A}_2$ as

$$\mathcal{A}_{2,K} w = \lambda_K I - \epsilon_2 w_{xx} + \frac{d}{dx} f_K(w) \tag{26}$$

for some $\lambda_K \in \mathbb{R}$ with domain $\mathcal{D}(\mathcal{A}_{2,K}) = \mathcal{D}(\mathcal{A}_2)$.

First we consider the abstract, truncated Cauchy problem

$$\frac{dw}{dt} + (\mathcal{A}_K - \lambda_K I)w = 0, \qquad t > 0, \quad w(0) = 0 \tag{27}$$

where $\mathcal{A}_K w = \mathcal{A}_1 w + \mathcal{A}_{2,K}\ w = \lambda_K w - \epsilon w_{xx} + \delta w_{xxx} + d/dx f_K(w)$. We will show that problem (27) has a strong solution w_K for all $K > 0$, then we obtain a variational solution of the original problem as the limit of w_K, in an appropriate sense, as $K \to \infty$.

We have to show that $\mathcal{A}_K$ is m-accretive (maximal monotone or, with other words, $-\mathcal{A}_K$ is maximal dissipative) on X in order to use the Crandall–Liggett Theorem. First we show the monotonicity of $\mathcal{A}_K$ for some $\lambda_K \in \mathbb{R}$ by showing the monotonicity of $\mathcal{A}_1$ and $\mathcal{A}_{2,K}$ separately.

Using the explicit form of g_1 and g_2, and the fact that they are monotone functions we obtain

$$\langle \mathcal{A}_1 w - \mathcal{A}_1 v, w - v\rangle \ge \epsilon_1 \|w_x - v_x\|^2 \ge \epsilon_1 \|w - v\|^2. \tag{28}$$

As a result we obtain that the operator $\mathcal{A}_1$ is monotone on X and on $H \times H^*$. It is also maximal monotone, since its restriction to homogeneous boundary conditions is a linear, maximal monotone operator. Similarly

$$\langle \mathcal{A}_{2,K} w - \mathcal{A}_{2,K} v, w - v\rangle \\ \ge (\epsilon_2 - \gamma)\|w_x - v_x\|^2 + \left(\lambda_K - \frac{K}{2\gamma} - \frac{2K^2}{\gamma}\right)\|w - v\|^2. \tag{29}$$

[2]See Acknowledgment section at the end of the paper.

In deriving (29) we used integration by parts, the boundary conditions and the Cauchy–Schwarz inequality. We exploited the Lipschitz continuity of f_K, along with Young's inequality and the simple inequality $|\varphi^2(1)| \le 2\|\varphi_x\|\,\|\varphi\|$. We were able to drop a term containing g_1 from the estimates due to the monotonicity of g_1. Choosing $\gamma = \epsilon_2/2$ in (29) we obtain that $\mathcal{A}_{2,K}$ is monotone both on X and on $H \times H^*$ for λ_K large enough [for example for $\lambda_K \ge 2K(4K+1)/\epsilon_2$]. It also follows that $\mathcal{A}_{2,K}$ is hemicontinuous. Putting together $\mathcal{A}_1$ and $\mathcal{A}_{2,K}$ we obtain that $\mathcal{A}_K = \mathcal{A}_1 + \mathcal{A}_{2,K}$ is monotone on X and $H \times H^*$ and hemicontinuous on $H \times H^*$ for large ω_K. The operator $\mathcal{A}_K$ is also coercive on $H \times H^*$ since (28) and (29) imply (with $v = 0$) that

$$\lim_{\|w\|\to\infty} \frac{\langle \mathcal{A}_K w, w\rangle}{\|w\|_{H^1}} \ge \lim_{\|v\|\to\infty} \frac{\langle \mathcal{A}_{2,K} w, w\rangle}{\|w\|_{H^1}} \\ \ge \lim_{\|v\|\to\infty} \frac{\frac{\epsilon_2}{2}\|w\|^2_{H^1}}{\|w\|_{H^1}} = \infty.$$

As a result, by [1, Corollary 1.3, page 46], the operator $\mathcal{A}_K$: $H \to H^*$ (as well as $\mathcal{A}_K - \lambda I$) is surjective. Due to this result and the inclusion $H \subset X \subset H^*$, in order to show that the range of $\mathcal{A}_K - \lambda I$ is all of X, it suffices to show that if $f \in X$ and $w \in H$ satisfies

$$\mathcal{A}_K w - \lambda w = f \tag{30}$$

then $w \in \mathcal{D}(\mathcal{A}_K)$. Expanding and rearranging (30) we get

$$-\epsilon w_{xx} + \delta w_{xxx} = f + (\lambda - \lambda_K)w - \frac{d}{dx} f_K(w) \in X \tag{31}$$

and since we already know that $\mathcal{A}_1$ is maximal monotone for all $\epsilon_1, \delta > 0$, we obtain that $w \in \mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{A}_K)$. Hence, by Minty's Theorem [12] $\mathcal{A}_K$ is maximal monotone on X and by the Crandall–Liggett Theorem [6] problem (27) has a unique strong solution $w_K \in C(0, \infty;\ \mathcal{D}(\mathcal{A}_K)) \cap C^1(0, \infty;\ L^2(0, 1)) \subset C(0, \infty;\ H^3(0, 1)) \cap C^1(0, \infty;\ L^2(0, 1))$ for all $K > 0$.

Next, we establish the uniform boundedness of the sequence $\{w_K\}_{K>0}$ in the same way as the *a priori* estimates were obtained. Starting with the identity

$$\int_0^1 (w_{Kt} - \epsilon w_{Kxx} + \delta w_{Kxxx} + (f_K(w_K))_x) w_K\, dx = 0 \tag{32}$$

we estimate the last term as

$$\int_0^1 (f_K(w_K))_x w_K\, dx \\ = \int_{|w_K|<K} (f_K(w_K))_x w_K\, dx \\ = \tfrac{1}{3} \int_{|w_K|<K} (w_K^3)_x\, dx \le \tfrac{1}{3}\,|w_K(1, t)|^3$$

and with this we obtain the uniform in K *a priori* estimate (17) for w_K

$$\int_0^t e^{2\alpha\epsilon\tau}(\|w_{Kx}(\tau)\|^2 + w_K^2(1, \tau) + w_K^4(1, \tau) + w_K^6(1, \tau))\, d\tau \\ + e^{2\alpha\epsilon t}\|w_K(t)\|^2 \le \frac{1}{1-\alpha}\|w_0\|^2. \tag{33}$$

We obtain estimate (12) similarly.

Consider now two parameters K, L and the corresponding two solutions w_K, w_L of (27). For their difference $w = w_K - w_L$ we have

$$w_t - \epsilon w_{xx} + \delta w_{xxx} + (f_K(w_K))_x - (f_L(w_L))_x = 0 \tag{34}$$

$$w(x, 0) = 0, \qquad w(0, t) = 0, \tag{35}$$

$$\begin{aligned} w_x(1, t) &= g_1(w_L(1, t)) - g_1(w_K(1, t)) \\ &= w(1, t)\tilde{g}_1, \end{aligned} \tag{36}$$

$$\begin{aligned} w_{xx}(1, t) &= g_2(w_K(1, t)) - g_2(w_L(1, t)) \\ &= w(1, t)\tilde{g}_2 \end{aligned} \tag{37}$$

for all $x \in [0, 1]$, $t > 0$, where

$$\tilde{g}_1 = -\frac{c}{9\epsilon}\left(9 + w_K^2(1, t) + w_K(1, t)w_L(1, t) + w_L^2(1, t)\right) \tag{38}$$

and

$$\begin{aligned} \tilde{g}_2 = &\frac{c^2}{81\epsilon^2}\left(9 + w_L^2(1, t)\right)^2 + \frac{c^2}{81\epsilon^2} w_K(1, t) \\ &\times (w_K(1, t) + w_L(1, t))\left(w_K^2(1, t) + 18 + w_L^2\right). \end{aligned} \tag{39}$$

After taking the inner product of (34)–(37) with w we calculate

$$\epsilon \int_0^1 w_{xx} w\, dx \le C\tilde{g}_1^2 \|w\|^2 - \frac{3\epsilon}{4}\|w_x\|^2$$

and

$$\begin{aligned} \delta &\int_0^1 w_{xxx} w\, dx \\ &= \delta w_{xx} w|_0^1 - \delta \int_0^1 w_{xx} w_x\, dx \\ &= \frac{\delta c^2}{162\epsilon^2} w^2(1, t)\left(81 + 18\left(w_K^2(1, t) + w_L^2(1, t)\right)\right) \\ &\quad + \frac{\delta c^2}{162\epsilon^2} w^2(1, t)\left(w_K^2(1, t) - w_L^2(1, t)\right)^2 \\ &\quad + \frac{\delta c^2}{162\epsilon^2} w^2(1, t) w_K^2(1, t) w_L^2(1, t) + \frac{\delta}{2} w_x^2(0, t) \\ &\quad - \frac{\delta c^2}{9\epsilon^2} w^2(1, t) w_K(1, t) w_L(1, t). \end{aligned}$$

Here only the last term is not positive definite and it can be estimated as

$$\begin{aligned} &\frac{\delta c^2}{9\epsilon^2} w^2(1, t) w_K(1, t) w_L(1, t) \\ &\quad \le C(\|w_K\| + \|w_L\|)^2(\|w_{Kx}\| + \|w_{Lx}\|)^2\|w\|^2 + \frac{\epsilon}{4}\|w_x\|^2. \end{aligned}$$

Using the notation

$$\begin{aligned} L_1(t) = &C\left(\tilde{g}_1^2 + \|w_{Kx}\|^2 + \|w_L\|\,\|w_{Lx}\|\right) \\ &+ C(\|w_K\| + \|w_L\|)^2(\|w_{Kx}\| + \|w_{Lx}\|)^2 \end{aligned} \tag{40}$$

we get

$$\begin{aligned} &\frac{d}{dt}\|w(t)\|^2 + \epsilon\|w_x(t)\|^2 \\ &\quad \le L_1(t)\|w(t)\|^2 + \int_0^1 ((f_K(w_K))_x - (f_L(w_L))_x) \\ &\qquad \times (w_K - w_L)\, dx. \end{aligned}$$

From here, using Gronwall's inequality we have

$$\begin{aligned} &\|w(t)\|^2 + \int_0^t \|w_x(\tau)\|^2\, d\tau \\ &\quad \le \int_0^t \int_0^1 ((f_K(w_K))_x - (f_L(w_L))_x)(w_K - w_L)\, dx\, d\tau \\ &\qquad \times \left(1 + \int_0^t L_1(\tau)\exp\left[\int_0^\tau L_1(s)\, ds\right] d\tau\right). \end{aligned}$$

Since the first factor above converges to zero according to Lemma 1 and the second factor is bounded according to (33), we obtain

$$\begin{aligned} &\|(w_K - w_L)(t)\|^2 \\ &\quad + \int_0^t \|w_{Kx}(\tau) - w_{Lx}(\tau)\|^2\, d\tau \xrightarrow{K, L \to \infty} 0. \end{aligned} \tag{41}$$

With (41) we obtain that

$$w_K \xrightarrow{K \to \infty} w \text{ in } C\left(0, T;\ L^2(0, 1)\right) \cap L^2\left(0, T;\ H^1(0, 1)\right). \tag{42}$$

Taking the inner product of (34)–(37) with $-w_{xx}$ we obtain

$$-\int_0^1 w_t w_{xx}\, dx = G(w_K, w_L)\frac{d}{dt}\left(\|w_x\|^2 + w^2(1, t)\right), \tag{43}$$

where $0 < G(w_K, w_L)$ depends also on the sign of $(d/dt)\|w_x\|^2$ and $(d/dt)w^2(1, t)$. We also have $\delta \int_0^1 w_{xxx} w_{xx}\, dx = (\delta/2)w^2(1, t)\tilde{g}_2 - (\delta/2)w_{xx}^2(0, t)$. We obtain

$$\begin{aligned} &G(w_K, w_L)\frac{d}{dt}\left(\|w_x(t)\|^2 + w^2(1, t)\right) \\ &\quad \le \frac{\delta}{2} w^2(1, t)\tilde{g}_2 + \int_0^1 ((f_K(w_K))_x - (f_L(w_L))_x) w_{xx}\, dx \end{aligned}$$

which can, in turn, be written as

$$\begin{aligned} &\frac{d}{dt}\left(\|w_x(t)\|^2 + w^2(1, t)\right) \\ &\quad \le L_2(t)\left(\|w_x(t)\|^2 + w^2(1, t)\right) \\ &\qquad + \int_0^1 ((f_K(w_K))_x - (f_L(w_L))_x) w_{xx}\, dx. \end{aligned}$$

From here, using Gronwall's inequality we obtain

$$\begin{aligned} &\|w_x(t)\|^2 + w^2(1, t) \\ &\quad \le \int_0^t \int_0^1 ((f_K(w_K))_x - (f_L(w_L))_x) w_{xx}\, dx\, d\tau \\ &\qquad \times \left(1 + \int_0^t L_2(\tau)\exp\left[\int_0^\tau L_2(s)\, ds\right] d\tau\right). \end{aligned}$$

Since the first factor converges to zero according to Lemma 2 and the second factor is bounded, we obtain

$$\|(w_{Kx} - w_{Lx})(t)\|^2 \to 0, \qquad \text{as } K, L \to \infty. \tag{44}$$

With this we obtain that

$$w_K \xrightarrow{K \to \infty} w \text{ in } C\left(0, T;\ L^2(0, 1)\right) \cap C\left(0, T;\ H^1(0, 1)\right), \tag{45}$$

where $w(x, t) \in C(0, T;\ L^2(0, 1)) \cap C(0, T;\ H^1(0, 1))$ is a variational solution of problem (10) satisfying stability estimates (11) and (12).

The uniqueness is obtained taking two assumed solutions w_1 and w_2 and subtracting the corresponding two systems from each other. Then, using the notation $w = w_1 - w_2$ we obtain

$$w_t - \epsilon w_{xx} + \delta w_{xxx} + w w_{1x} + w_2 w_x = 0, \quad x \in [0, 1], \quad t > 0,$$

$$\begin{aligned} &w(0, t = 0, \\ &w_x(1, t) = g_1(w_2(1, t)) - g_1(w_1(1, t)) = w(1, t)\tilde{g}_1, \\ &w_{xx}(1, t) = g_2(w_1(1, t)) - g_2(w_2(1, t)) = w(1, t)\tilde{g}_2, \end{aligned}$$

where $\tilde{g}_1$ and $\tilde{g}_2$ have the same form as in (38) and (39) except w_K and w_L are now replaced by w_1 and w_2 respectively. The calculations are also very similar to that of (38) and (39) except the cubic terms that can be estimated as $\int_0^1 w^2 w_{1x}\, dx \le \|w^2\|_{L^\infty}\|w_{1x}\| \le C\|w_{1x}\|^2\|w\|^2 + (\epsilon/8)\|w_x\|^2$ and $\int_0^1 w_2 w_x w\, dx \le C\|w_2\|\,\|w_{2x}\|\ \|w\|^2 + (\epsilon/8)\|w_x\|^2$. Then, using notation (40) with w_K and w_L are replaced by w_1 and w_2 we obtain

$$\frac{d}{dt}\|w(t)\|^2 \le L_1(t)\|w(t)\|^2. \tag{46}$$

From here Gronwall's inequality implies

$$\|w(t)\|^2 \le \|w_0\|^2 \exp\left(\int_0^t L_1(\tau)\, d\tau\right). \tag{47}$$

Inspecting $L_1(t)$ we observe that it is integrable and since $\|w_0\| = 0$ we obtain that $\|w(t)\| = 0$ for all $t \ge 0$, i.e. the solution of (10) is unique. □

Lemma 1: Under the hypotheses of Theorem 1

$$\int_0^t \int_0^1 ((f_K(w_K))_x - (f_L(w_L))_x)(w_K - w_L)\, dx\, d\tau \xrightarrow{K, L \to \infty} 0$$

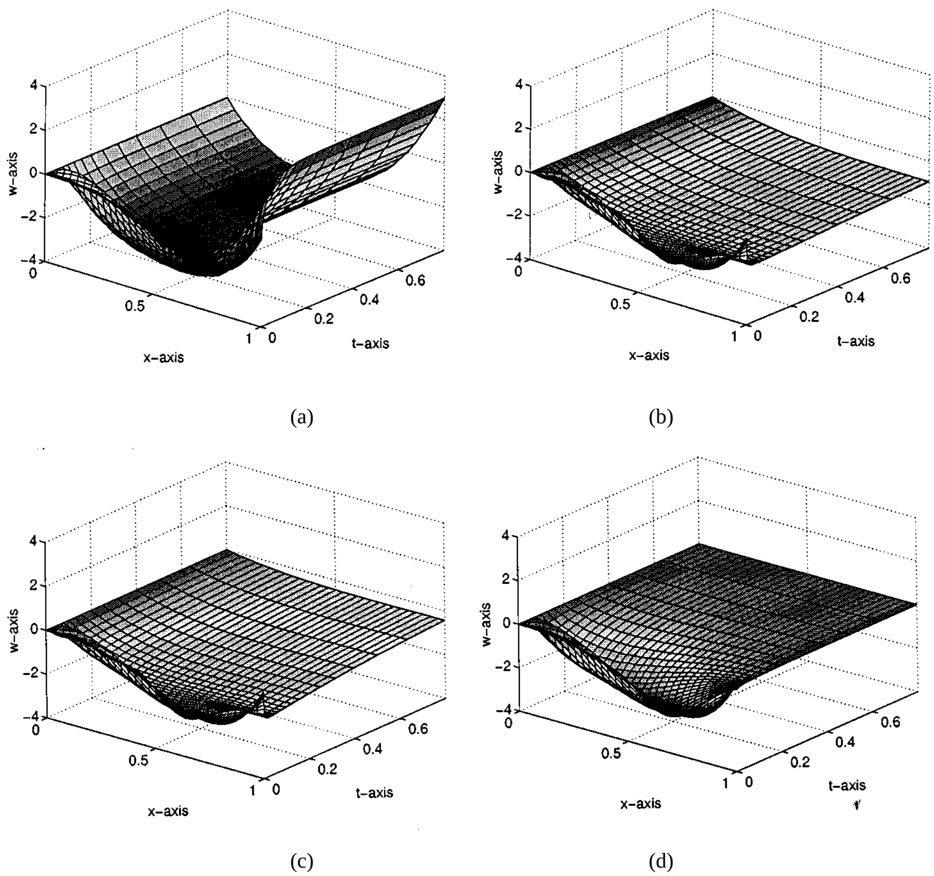

Fig.1. Comparison of norms. $\cdots$ uncontrolled, ooo controlled second derivative (quadratic), - - - controlled second derivative (cubic), — two derivatives controlled

for any $t > 0$.

Proof: Let us use the notation $w = w_K - w_L$ and $\Omega_K = \{x \in [0,\, 1]\colon |w_K(x)| > K\}$. The measure of Ω_K can be estimated as

$$\text{mes}\,\Omega_K \le K^{-6}\|w_K\|_{L^6}^6 \le 4K^{-6}\|w_{Kx}\|^2\|w_K\|^4, \tag{48}$$

where we used the classical multiplicative inequality (55) with $r = m = 2$ and $q = 6$. We have

$$\begin{aligned}&\int_0^t\int_0^1 ((f_K(w_K))_x - (f_L(w_L))_x)\,(w_K - w_L)\,dx\,d\tau\\ &\quad\le 2\left(\int_0^t \|w_{Lx}\|^2\,d\tau\right)^{1/2}\left(\int_0^t\int_{\Omega_K}|w_K|^4\,dx\,d\tau\right)^{1/2}\end{aligned} \tag{49}$$

where, without loss of generality, we assumed that $\int_0^t\int_{\Omega_K} w_K^4\,dx\,d\tau \ge \int_0^t\int_{\Omega_K} w_L^4\,dx\,d\tau$. In (49) the first factor on the right-hand side is bounded according to estimate (33). For the second factor we have

$$\begin{aligned}&\int_0^t\int_{\Omega_K}|w_K|^4\,dx\,d\tau\\ &\quad= \int_0^t \|w_K\|_{L^6}^4(\text{mes}\,\Omega_K)^{1/3}\,d\tau\\ &\quad\le K^{-2}\int_0^t \|w_{Kx}\|^2\|w_K\|^4\,d\tau \xrightarrow{K\to\infty} 0.\end{aligned}$$ □

Lemma 2: Under the hypotheses of Theorem 1

$$\int_0^t\int_0^1 ((f_K(w_K))_x - (f_L(w_L))_x)w_{xx}\,dx\,d\tau \xrightarrow{K,\,L\to\infty} 0 \tag{50}$$

for any $t > 0$.

Proof: We have, similarly as in Lemma 1

$$\begin{aligned}&\int_0^t\int_0^1 ((f_K(w_K))_x - (f_L(w_L))_x)w_{xx}\,dx\,d\tau\\ &\quad\le \left(\int_0^t \|w_{xx}(\tau)\|^2\,d\tau\right)^{1/2}\\ &\qquad\times\left(\int_0^t\int_{\Omega_K\setminus\Omega_L}|w_L|^2|w_{Lx}|^2\,dx\,d\tau\right)^{1/2}.\end{aligned}$$

The boundedness of the first factor above can be obtained integrating (20) which holds for all w_K uniformly in K. For the second factor we have

$$\begin{aligned}&\int_0^t\int_{\Omega_K\setminus\Omega_L}|w_L|^2|w_{Lx}|^2\,dx\,d\tau\\ &\quad\le\left(\int_0^t\int_{\Omega_K\setminus\Omega_L}|w_L|^4\,dx\,d\tau\right)^{1/2}\left(\int_0^t \|w_{Lx}\|_{L^4}^4\,d\tau\right)^{1/2}.\end{aligned} \tag{51}$$

We already know from inequality (50) of Lemma 1 that the first factor on the right hand side of inequality (51) converges to zero as $K,\, L \to \infty$. The second factor is estimated with the help of inequality (54) as

$$\begin{aligned}\int_0^t \|w_{Lx}\|_{L^4}^4\,d\tau &\le Mt\max_{\tau\in[0,\,t]}\|w_x(\tau)\|^4\\ &\quad+ M\left(\int_0^t \|w_{xx}(\tau)\|^2\,d\tau\right)^{1/2}\sqrt{t}\max_{\tau\in[0,\,t]}\|w_x(\tau)\|^6\end{aligned} \tag{52}$$

where M is a generic constant. Since each expression is finite in (52) for any $t > 0$, the result of Lemma 2 follows. □

Lemma 3: For any $w \in H^1(0,\,1)$ *and* $2 \le q \le \infty$ *we have*

$$\|w\|_{L^q} \le \gamma_1\|w\| + \gamma_2\|w_x\|^\alpha\|w\|^{1-\alpha} \tag{53}$$

where $\alpha = 1/2 - 1/q$, $\gamma_1 = 2^{1+\alpha}$ *and* $\gamma_2 = 2^{(1/2)}6^{(\alpha/2)}$. *We also have*

$$\|w\|_{L^q}^2 \le 2^{1+2\alpha}\|w\|^2 + 2^{1+\alpha}3^\alpha\|w_x\|^{2\alpha}\|w\|^{2-2\alpha}. \tag{54}$$

Proof: This is a one-dimensional extension of a classical inequality (see, e.g., [9, Theorem 2.2, pp 62])

$$\|w\|_{L^q} \le \beta \|w_x\|_{L^m}^{\alpha} \|w\|_{L^r}^{1-\alpha}, \tag{55}$$

which holds for $w \in W_m^1[a, b]$, $m \ge 1$ with $w(a) = 0$, where $r \le q \le \infty$, $\alpha = ((1/r) - (1/q))\,(1 - (1/m) + (1/r))^{-1}$ and $\beta = (1 + ((m-1)/m)r)^{\alpha}$. The proof is very similar to that of [5, Lemma 2.2] and hence it is omitted. □

III. A Numerical Example

In this section we compare three controllers through a numerical example: controller (3), controller (4) and (5) and controller (6)–(8). A comparison is also made relative to the uncontrolled system consisting of the KdVB equation (1) and boundary conditions

$$w(0, t = 0, \quad w_x(1, t) = w_0'(1), \quad w_{xx}(1, t) = w_0''(1).$$

The local existence of a solution to the uncontrolled system is obvious and can be proven for example using Galerkin's method.

As a consequence of the third derivative in x and first derivative in t, it is necessary to use very small time steps ($\approx 10^{-9}$) in order to balance the very small number in the denominator resulting from the cube of the spatial step. We are able to compensate in a certain extent the very small time steps by rescaling the equation, i.e. compressing the time domain. We consider, from the above reason, the scaled equation

$$u_t - \epsilon' u_{xx} + \delta' u_{xxx} + puu_x = 0, \qquad x \in [0, 1], \quad t > 0 \tag{56}$$

with some initial data

$$u(x, 0) = u_0(x), \qquad u_0(0) = 0 \tag{57}$$

and in the controlled case with boundary condition

$$u(0, t) = 0 \tag{58}$$

$$u_x(1, t) = -\frac{p}{\epsilon'}\left(c + \frac{1}{9c}\,u^2(1, t)\right)u(1, t) \tag{59}$$

$$u_{xx}(1, t) = \frac{p^2}{\epsilon'^2}\left(c + \frac{1}{9c}\,u^2(1, t)\right)^2 u(1, t) \tag{60}$$

where ϵ', δ', c and p are positive constants. The transformation $u(x, t) \equiv w(x, pt)$ shows the equivalence of system (1), (2), (6)–(8) to (56)–(60) with $\epsilon \equiv \epsilon'/p$ and $\delta \equiv \delta'/p$.

Our numerical simulation is based on a fully discrete, implicit scheme of second order accuracy, using three time level quadratic approximation in time and central difference scheme in space, derived using the finite volume method (see, e.g. [7]).

As an example, we consider the (KdVB) equation (56) with parameters $\epsilon' = 1$, $\delta' = 10$, $p = 100$ and with initial function

$$u_0(x) = 20x^3(x - 1.001). \tag{61}$$

The time step we use is $k = 10^{-9}$ with final time $T = 10^{-2}$, and spatial step $h = 5 \times 10^{-3}$. The scaling $p = 100$ corresponds to an unscaled Korteweg–de Vries–Burgers system with parameters $\epsilon = 0.01$, $\delta = 0.1$ on a time interval [0, 1]. In the controlled case the control gain was $c = 0.1$. As we can see in Fig. 1, the uncontrolled solution seems to converge to a nontrivial stationary solution. All three controlled systems converge to zero [Fig. 1(b)–(d)], but when the first derivate is kept at zero at $x = 1$ and only the second derivative is controlled by feedback, cases (b) and (c), show poor convergence relative to our controller (59) and (60). In fact, Fig. 2 shows that the differences between the rates of convergence are significant both in the L^2 and in the H^1 sense.

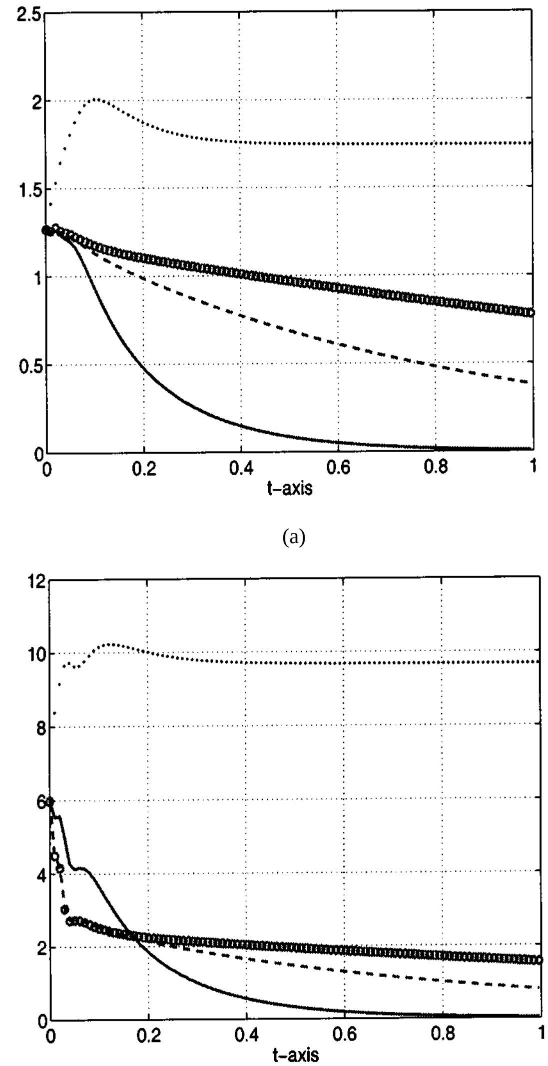

Fig. 2. Comparison of Solutions

Acknowledgment

The authors would like to thank Associate Editor I. Lasiecka for suggesting, and providing them with detailed technical insight on, the use of the method of monotone operators with locally Lipschitz perturbations.

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Constrained Approximate Controllability

Jerzy Klamka

***Abstract*— In the paper, constrained approximate controllability for linear dynamical systems described by abstract differential equations with unbounded control operator is considered. Using methods of spectral analysis for linear self-adjoint operators and general constrained controllability results given in the paper [20], necessary and sufficient conditions of the constrained approximate controllability for the piecewise polynomial controls with values in a given cone are formulated and proved. Moreover, as illustrative examples constrained approximate boundary controllability of one-dimensional distributed parameter dynamical systems described by partial differential equations of parabolic type with Dirichlet and Neumann boundary conditions are investigated. The constrained controllability conditions obtained in the paper represent an extension of the unconstrained controllability results given in [5] and [6].**

***Index Terms*—Controllability, discrete–time systems, distributed parameter systems, linear systems.**

I. Introduction

In recent years, controllability problems for different kinds of dynamical systems have been considered in many publications. The extensive list of publications containing more than 500 positions can be found in the monograph [6]. Most literature in this direction so far has been concerned, however, with the so-called unconstrained controllability problems. Only a few papers deal with the so-called constrained controllability problems, i.e., with the case when the control functions are restricted to take their values in a prescribed admissible set [1]–[4], [7]–[10], [12], [15]–[18], [20] . The papers [3], [4], [7], and [8] contain results concerning constrained controllability problems for linear retarded dynamical systems, and in the paper [10], constrained boundary approximate controllability of parabolic-type partial differential equations is discussed. Moreover, it should be also stressed that up to now constrained boundary controllability problems for distributed parameter dynamical systems with piecewise polynomial controls have not been considered in the literature. In order to fill this gap, the present paper studies in detail the constrained approximate controllability problems for linear systems described by abstract differential equations with unbounded control operator and piecewise polynomial controls. The main purpose of the paper is to formulate and prove necessary and sufficient conditions for the so-called constrained approximate controllability using some general results given in the paper [20]. It should be mentioned that the paper [20] does not contain any results concerning boundary controllability of distributed parameter systems. Moreover, it will be pointed out that in the special cases we can easily obtain from general results the computable constrained approximate controllability criteria. Finally, simple numerical examples that illustrate the general theory will be presented. In these examples, computable necessary and sufficient conditions for constrained approximate boundary controllability of linear distributed parameter dynamical system described by partial differential equation of parabolic type with Dirichlet and Neumann boundary conditions are presented.

Manuscript received June 10, 1998; revised December 2, 1998 and February 25, 1999. Recommended by Associate Editor, I. Lasiecka.

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Publisher Item Identifier S 0018-9286(00)06097-9.

II. Preliminaries

Let X be Hilbert space and $U = R^M$. We consider abstract linear continuous-time system defined on X and described by the following differential equation:

$$x'(t) = Ax(t) + Bu(t) \tag{2.1}$$

where A: $X \supset D(A) \to X$ is a generator of a strongly continuous semigroup of bounded linear operators $S(t)$: $X \to X$, for $t > 0$.

The linear control operator $B = [b_1, b_2, \cdots, b_m, \cdots, b_M]$ where $b_m \in X_{-1}$, for $m = 1, 2, \cdots, M$ is bounded from R^M into X_{-1}.

Here, X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} = \|(sI - A)^{-1}x\|x$ for some $s \in \rho(A)$, where $\rho(A)$ is a resolvent set for the operator A. The operator A extends to a generator of a strongly continuous semigroup $S(t)$ on X_{-1}. Furthermore, $X \subset X_{-1}$ with continuous and dense embedding, and $(sI - A)^{-1}$ is an isomorphism between the spaces X_{-1} and X. Notice that the above equality defines equivalent norms for different $s \in \rho(A)$ and so X_{-1} is independent of s. Also, we have the scalar product on X_{-1} given by [19]:

$$\langle f, g\rangle_{.X_{-1}} = \langle (sI - A)^{-1}f, (sI - A)^{-1}g\rangle_{D(A)}, \quad \text{for } f, g \in X_{-1}$$

and it satisfies the identities

$$\langle f, g\rangle_{X_{-1}} = \langle f, (sI - A)^{-1}g\rangle_X = \langle (sI - A)^{-1}f, g\rangle_X.$$

Observe that the operator B does not have its range in X, and, moreover, we do not even assume admissibility of the operator B [13], [14]. It should be pointed out that the majority of linear distributed parameter