Mathematical Problems in Engineering
Volume 6, pp. 189–200
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Burgers' Equation with Nonlinear Boundary Feedback: H^1 Stability, Well-Posedness and Simulation

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(Received 17 June 1999)

We consider the viscous Burgers' equation under recently proposed nonlinear boundary conditions and show that it guarantees global asymptotic stabilization and semiglobal exponential stabilization in H^1 sense. Our result is global in time and allows arbitrary size of initial data. It strengthens recent results by Byrnes, Gilliam, and Shubov, Ly, Mease, and Titi, and Ito and Yan. The global existence and uniqueness of classical solutions follows from the general theory of quasi-linear parabolic equations. We include a numerical result which illustrates the performance of the boundary controller.

Keywords: Burgers' equation; Nonlinear boundary feedback; Global stability; Regularity

AMS Subject Classification: 93D20, 35B35, 35Q53

1 INTRODUCTION

Burgers' equation is a natural first step towards developing methods for control of flows. Recent references by Burns and Kang [1], Byrnes *et al.* [3,4], Ly *et al.* [12], and Ito and Yan [8] achieve progress in local stabilization and global analysis of attractors. The problem of global exponential stabilization in L^2 norm was first addressed by Krstić [9].

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This problem is non-trivial because for large initial conditions the quadratic (convective) term – which is negligible in a linear/local analysis – dominates the dynamics. Linear boundary conditions do not always ensure global exponential stability [4] or prevent finite blow-up [5] in the case of nonlinear reaction–diffusion equations. Nonlinear boundary conditions might cause finite blow-up [11], even for the simple heat equation [7].

With the introduction of cubic Neumann boundary feedback control we obtain a closed loop system which is globally asymptotically stable and semi-globally exponentially stable in H^1 norm and, hence in maximum norm whenever the initial data is compatible with the equation and the boundary conditions.

For clarity, our treatment does not include external forcing as in [3,4,8,12]. External forcing would preclude equilibrium stability but one could still establish appropriate forms of disturbance attenuation and regularity of solution.

2 PROBLEM STATEMENT AND MAIN RESULTS

Consider Burgers' equation

$$W_t - \epsilon W_{xx} + W W_x = 0, \tag{2.1}$$

where $\epsilon > 0$ is a constant, with some initial data

$$W(x,0) = W_0(x). \tag{2.2}$$

Our objective is to achieve set point regulation

$$\lim_{t\to\infty} W(x,t) = W_d, \quad \forall x \in [0,1], \tag{2.3}$$

where W_d is a constant, while keeping $W(x,t)$ bounded for all $(x,t) \in [0,1] \times [0,\infty)$. Without loss of generality we assume that $W_d \geq 0$. By defining the regulation error as $w(x,t) = W(x,t) - W_d$, we get the system

$$w_t - \epsilon w_{xx} + W_d w_x + w w_x = 0, \tag{2.4}$$

with initial data

$$w(x,0) = W_0(x) - W_d \equiv w_0(x). \tag{2.5}$$

We will approach the problem using nonlinear Neumann boundary control proposed in [9]

$$w_x(0,t) = \frac{1}{\epsilon}\left(c_0 + \frac{W_d}{2} + \frac{1}{9c_0}w^2(0,t)\right)w(0,t), \tag{2.6}$$

$$w_x(1,t) = -\frac{1}{\epsilon}\left(c_1 + \frac{1}{9c_1}w^2(1,t)\right)w(1,t), \tag{2.7}$$

where $c_0, c_1 > 0$.

The choice of w_x at the boundary as the control input is motivated by physical considerations. For example, in thermal problems one cannot actuate the temperature w, but only the heat flux w_x. This makes the stabilization problem non-trivial because, as Byrnes *et al.* [3] argue, homogeneous Neumann boundary conditions make any constant profile an equilibrium solution, thus preventing not only global but even local *asymptotic* stability. Even mixed linear boundary conditions can introduce multiple stationary solutions [2].

DEFINITION 1 *The zero solution of a dynamical system is said to be* globally asymptotically stable *in an $\mathcal{L}$ spatial norm if*

$$\|w(t)\|_{\mathcal{L}} \leq \beta(\|w_0\|_{\mathcal{L}}, t), \quad \forall t \geq 0, \tag{2.8}$$

where $\beta(\cdot,\cdot)$ is a class $\mathcal{KL}$ function, i.e., a function with the properties that

- *for fixed t, $\beta(r,t)$ is a monotonically increasing continuous function of r such that* $\beta(0,t) \equiv 0$;
- *for fixed r, $\beta(r,t)$ is a monotonically decreasing continuous function of t such that* $\lim_{t\to\infty}\beta(r,t) \equiv 0$.

The trivial solution is said to be globally exponentially stable *when*

$$\beta(r,t) = kr\,\mathrm{e}^{-\delta t} \tag{2.9}$$

for some $k, \delta > 0$ *independent of* r *and* t, *and it is said to be* semi-globally exponentially stable *when*

$$\beta(r,t) = K(r)\,\mathrm{e}^{-\delta t}, \tag{2.10}$$

where $K(r)$ *is a continuous nondecreasing function with* $K(0) = 0$.

We use the following H^1-like norm in our stability analysis

$$\|w(t)\|_B = \sqrt{w(0,t)^2 + w(1,t)^2 + \|w_x(t)\|^2}. \tag{2.11}$$

We refer to [10] for the definition of Hölder type function spaces $\mathcal{H}^l([0,1])$ and $\mathcal{H}^{l,l/2}([0,1]\times[0,T])$, where $l > 0$ is typically noninteger. Smooth solutions of system (2.4), (2.6), (2.7) should clearly be compatible with the boundary conditions at $t = 0$ in some sense. For the definition of compatibility conditions of different order we refer to [10] again.

Our main result is the following theorem.

THEOREM 1 *Consider the system* (2.4), (2.6), (2.7). *For any* $T > 0$, $l > 0$, *and for any* $w_0 \in \mathcal{H}^{2+l}([0,1])$ *satisfying the compatibility condition of order* $[(l+1)/2]$ *there exists a unique classical solution* $w(x,t) \in \mathcal{H}^{2+l,1+l/2}([0,1]\times[0,T]) \subset C^{2,1}([0,1]\times[0,T])$ *with the following stability properties.*

(1) *Global exponential stability in the* L^q *sense: for any* $q \in [2,\infty)$ *there exists* $\delta(q) > 0$ *such that*

$$\|w(t)\|_{L^q} \le \|w_0\|_{L^q}\,\mathrm{e}^{-\delta t}, \quad \forall t \ge 0. \tag{2.12}$$

(2) *Global asymptotic and semi-global exponential stability in the* H^1 *sense: there exist* $k, \delta \in (0,\infty)$ *such that*

$$\|w(t)\|_B \le k\|w_0\|_B\,\mathrm{e}^{k\|w_0\|_B^2}\,\mathrm{e}^{-\delta t}, \quad \forall t \ge 0. \tag{2.13}$$

Since $\mathcal{H}^n = C^n$ for n integer, the theorem assumes initial data smoother than C^2 (but not necessarily as smooth as C^3). Specifically, the initial data need to satisfy

$$\sup_{x,y\in[0,1]} \frac{|w_0''(x) - w_0''(y)|}{|x-y|^l} < \infty \tag{2.14}$$

for some $l > 0$.

For solutions to be classical, besides C^{2+} smoothness of initial data, it is required that they satisfy the compatibility condition of order zero, i.e.,

$$w_0'(0) = \frac{1}{\epsilon}\left(c_0 + \frac{W_d}{2} + \frac{1}{9c_0} w_0^2(0)\right) w_0(0), \tag{2.15}$$

$$w_0'(1) = -\frac{1}{\epsilon}\left(c_1 + \frac{1}{9c_1} w_0^2(1)\right) w_0(1). \tag{2.16}$$

3 GLOBAL ASYMPTOTIC STABILITY

While irrelevant for finite-dimensional systems where all vector norms are equivalent, for PDEs, the question of the type of norm $\mathcal{L}$ with respect to which one wants to establish stability is a delicate one. Any meaningful stability claim should imply boundedness of solutions. We first establish global exponential stability in L^q for any $q \in [2, \infty)$, which does not guarantee boundedness. Then we show global asymptotic (plus local exponential) stability in an H^1-like sense which, by combining Agmon's and Poincaré's inequalities, guarantees boundedness.

Consider the Lyapunov function

$$V(w(t)) = \int_0^1 w^{2p}\,\mathrm{d}x = \|w^p(t)\|^2 = \|w(t)\|_{L^{2p}}^{2p}, \quad p \geq 1. \tag{3.1}$$

Its time derivative is

$$\begin{aligned}
\dot{V} &= 2p \int_0^1 w^{2p-1}(\epsilon w_{xx} - W_d w_x - w w_x)\,\mathrm{d}x \\
&= 2p\left[-\epsilon(2p-1)\int_0^1 w^{2p-2} w_x^2\,\mathrm{d}x \right. \\
&\quad \left. + \epsilon w^{2p-1} w_x\Big|_0^1 - \frac{W_d}{2p} w^{2p}\Big|_0^1 - \frac{1}{2p+1} w^{2p+1}\Big|_0^1\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\epsilon 2p(2p-1)\|w^{p-1}(t)w_x(t)\|^2 \\
&\quad - 2pw^{2p}(0,t)\left[c_0 + \left(1-\frac{1}{p}\right)\frac{W_d}{2} - \frac{1}{2p+1}w(0,t) + \frac{1}{9c_0}w^2(0,t)\right] \\
&\quad - 2pw^{2p}(1,t)\left[c_1 + \frac{W_d}{2p} + \frac{1}{2p+1}w(1,t) + \frac{1}{9c_1}w^2(1,t)\right] \\
&\le -\epsilon 2p(2p-1)\|w^{p-1}(t)w_x(t)\|^2 - 2p\left[\frac{w^{2p}(0,t)}{c_0}\left(\frac{c_0^2}{2} + \frac{w^2(0,t)}{18}\right)\right. \\
&\quad \left. + \frac{w^{2p}(1,t)}{c_1}\left(\frac{c_1^2}{2} + \frac{w^2(1,t)}{18}\right)\right]. \qquad (3.2)
\end{aligned}
$$

From Poincaré's inequality it follows that

$$\|w^p(t)\|^2 \le 2(w^{2p}(0,t) + w^{2p}(1,t)) + (2p)^2\|w^{p-1}(t)w_x(t)\|^2. \qquad (3.3)$$

Thus we get

$$\dot{V} \le -\frac{2p-1}{2p}\epsilon' V, \qquad (3.4)$$

where $\epsilon' = \min\{\epsilon, c_0, c_1\}$. It then follows that

$$\|w(t)\|_{L^{2p}} \le \mathrm{e}^{-((2p-1)/(2p)^2)\epsilon' t}\|w_0\|_{L^{2p}}. \qquad (3.5)$$

Thus the solution $w(x,t) \equiv 0$ is globally exponentially stable in an L^q sense for any $q \in [2,\infty)$. Letting $p \to \infty$ in (3.5), we get

$$\operatorname*{ess\,sup}_{x\in[0,1]} |w(x,t)| \le \operatorname*{ess\,sup}_{x\in[0,1]} |w(x,0)|, \quad \forall t \ge 0. \qquad (3.6)$$

This result is not particularly useful for two reasons:

(1) The above estimate does not guarantee convergence to zero (it guarantees stability but not *asymptotic* stability).
(2) Without additional effort to establish continuity, with ess sup we cannot guarantee boundedness for all (but only for *almost* all) $x \in [0, 1]$.

For this reason, we turn our attention to the norm defined in (2.11). By combining Agmon's and Poincaré's inequalities, it is easy to see that

$$\max_{x\in[0,1]} |w(x,t)| \leq \sqrt{2}\|w(t)\|_B. \tag{3.7}$$

We will now prove global asymptotic stability in the sense of the B-norm. Let us start by rewriting (3.2) for $p=1$ as

$$k\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2 + \|w(t)\|_B^2 \leq 0, \tag{3.8}$$

where k is a generic positive constant independent of initial data and time, and by writing (3.5) as

$$\|w(t)\|^2 \leq \mathrm{e}^{-t/k}\|w_0\|^2. \tag{3.9}$$

Multiplying (3.8) by $\mathrm{e}^{t/(2k)}$, we get

$$\begin{aligned} k\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{t/(2k)}\|w(t)\|^2) + \mathrm{e}^{t/(2k)}\|w(t)\|_B^2 &\leq \tfrac{1}{2}\mathrm{e}^{t/(2k)}\|w(t)\|^2 \\ &\leq \tfrac{1}{2}\mathrm{e}^{-t/(2k)}\|w_0\|^2. \end{aligned} \tag{3.10}$$

Integrating from 0 to t yields

$$\int_0^t \mathrm{e}^{\delta t}\|w(\tau)\|_B^2\,\mathrm{d}\tau \leq k\|w_0\|^2, \tag{3.11}$$

where $\delta = 1/(2k) > 0$.

Now we take the L^2-inner product of (2.4) with $-w_{xx}$,

$$-\int_0^1 w_t w_{xx}\,\mathrm{d}x + \epsilon\int_0^1 w_{xx}^2\,\mathrm{d}x - W_d\int_0^1 w_x w_{xx}\,\mathrm{d}x - \int_0^1 w w_x w_{xx}\,\mathrm{d}x = 0. \tag{3.12}$$

The estimation of the various terms follows:

$$
\begin{aligned}
-\int_0^1 & w_t w_{xx}\,\mathrm{d}x \\
&= -w_t w_x\big|_0^1 + \int_0^1 w_{xt} w_x\,\mathrm{d}x \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_x(t)\|^2 + \frac{w_t(1,t)}{\epsilon}\left(c_1 w(1,t) + \frac{1}{9c_1} w^3(1,t)\right) \\
&\quad + \frac{w_t(0,t)}{\epsilon}\left(c_0 w(0,t) + \frac{W_d}{2} w(0,t) + \frac{1}{9c_0} w^3(0,t)\right) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c_1}{\epsilon} w^2(1,t) + \frac{1}{18c_1\epsilon} w^4(1,t) + \frac{(2c_0 + W_d)}{2\epsilon} w^2(0,t)\right. \\
&\qquad\qquad \left. + \frac{1}{18c_0\epsilon} w^4(0,t) + \|w_x(t)\|^2\right),
\end{aligned} \tag{3.13}
$$

$$
W_d \int_0^1 w_x w_{xx}\,\mathrm{d}x \le W_d\|w_x(t)\|\,\|w_{xx}(t)\| \le \frac{W_d^2}{\epsilon}\|w_x(t)\|^2 + \frac{\epsilon}{4}\|w_{xx}(t)\|^2, \tag{3.14}
$$

$$
\begin{aligned}
\int_0^1 w w_x w_{xx}\,\mathrm{d}x &\le \|w(t)\|_{L^\infty} \int_0^1 |w_x w_{xx}|\,\mathrm{d}x \\
&\le \|w(t)\|_{L^\infty}\|w_x(t)\|\|w_{xx}(t)\| \\
&\le \frac{1}{\epsilon}\|w(t)\|_{L^\infty}^2\|w_x(t)\|^2 + \frac{\epsilon}{4}\|w_{xx}(t)\|^2 \\
&\le \frac{2}{\epsilon}\|w_x(t)\|^2\|w(t)\|_B^2 + \frac{\epsilon}{4}\|w_{xx}(t)\|^2.
\end{aligned} \tag{3.15}
$$

Using the notation

$$
\begin{aligned}
A(t) &= \frac{c_1}{\epsilon} w^2(1,t) + \frac{1}{18c_1\epsilon} w^4(1,t) + \frac{(2c_0 + W_d)}{2\epsilon} w^2(0,t) \\
&\quad + \frac{1}{18c_0\epsilon} w^4(0,t) + \|w_x(t)\|^2,
\end{aligned} \tag{3.16}
$$

and substituting (3.13)–(3.15) into (3.12) we obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}A(t) + \frac{\epsilon}{2}\|w_{xx}(t)\|^2 \le \frac{W_d^2}{\epsilon}\|w(t)\|_B^2 + \frac{2}{\epsilon}\|w(t)\|_B^4, \tag{3.17}
$$

and hence

$$\dot{A}(t) \leq k\|w(t)\|_B^2 + k\|w(t)\|_B^2 A(t). \tag{3.18}$$

Multiplying by $\mathrm{e}^{\delta t}$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\delta t}A(t)) &\leq k\mathrm{e}^{\delta t}\|w(t)\|_B^2 + \delta\mathrm{e}^{\delta t}A + k\|w(t)\|_B^2\mathrm{e}^{\delta t}A(t)\\ &\leq k\mathrm{e}^{\delta t}\|w\|_B^2 + k\|w\|_B^2\mathrm{e}^{\delta t}A(t).\end{aligned} \tag{3.19}$$

By Gronwall's inequality, we get

$$\begin{aligned}\mathrm{e}^{\delta t}A(t) &\leq \left[A(0) + k\int_0^t \mathrm{e}^{\delta\tau}\|w(\tau)\|_B^2\,\mathrm{d}\tau\right]\mathrm{e}^{k\int_0^t \|w(\tau)\|_B^2\,\mathrm{d}\tau}\\ &\leq [A(0) + k\|w_0\|^2]\mathrm{e}^{k\|w_0\|^2}.\end{aligned} \tag{3.20}$$

Thus

$$\begin{aligned}A(t) &\leq (A(0) + k\|w_0\|^2)\mathrm{e}^{k\|w_0\|^2}\mathrm{e}^{-\delta t}\\ &\leq k(\|w_0\|_B^2 + \|w_0\|_B^4)\mathrm{e}^{k\|w_0\|_B^2}\mathrm{e}^{-\delta t}\\ &\leq k\|w_0\|_B^2\,\mathrm{e}^{k\|w_0\|_B^2}\mathrm{e}^{-\delta t},\end{aligned} \tag{3.21}$$

which implies

$$\|w(t)\|_B \leq k\|w_0\|_B\,\mathrm{e}^{k\|w_0\|_B^2}\mathrm{e}^{-\delta t/2}. \tag{3.22}$$

This proves global asymptotic stability in the sense of the B-norm with $\beta(r,t) = kr\mathrm{e}^{kr^2}\mathrm{e}^{-\delta t/2}$. It also shows semi-global exponential stability. The last estimate also guarantees that

$$\sup_{t\geq 0}\max_{x\in[0,1]}|w(x,t)| < \infty \tag{3.23}$$

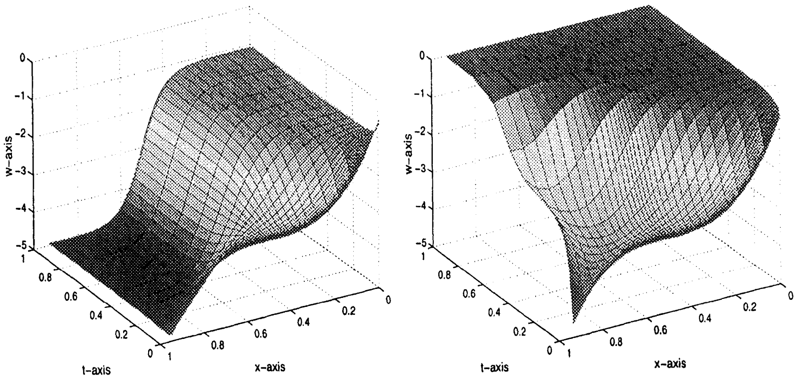
whenever $w(0,0)$, $w(1,0)$, and $\int_0^1 w_x(x,0)^2\,\mathrm{d}x$ are finite.

The existence of classical solutions follows from Theorem 7.4 in [10], Chapter V. This Theorem establishes, for a more general quasi-linear parabolic boundary value problem, the existence of a unique solution in the Hölder space of functions $\mathcal{H}^{2+l,1+l/2}([0,1]\times[0,T])$ for some $l>0$. Since $\mathcal{H}^{2+l,1+l/2}([0,1]\times[0,T])\subset C^{2,1}([0,1]\times[0,T])$, we obtain the existence of *classical* solutions for time intervals $[0,T]$, where $T>0$ is arbitrarily large. The proof in [10] is based on linearization of the system, and on application of the Leray–Schauder theorem on fixed points. It is important to note that a crucial step in the proof is establishing uniform *a priori* estimates for the system. These estimates are for the Hölder norms of solutions and hence are different from our Sobolev type energy estimates. The Hölder estimates establish boundedness of solutions, while our energy estimates establish stability. The existence of strong (but not necessarily classical) solutions was proved in [8] using a different method.

4 SIMULATION EXAMPLE

It is well known (see, e.g. [2,6]) that nonlinear problems, especially fluid dynamical problems, require extremely careful numerical analysis. Typically there is a trade-off between convergence, accuracy and numerical oscillation. This is the case in particular when the initial data is large relative to the viscosity coefficient ϵ in Burgers' equation. Higher order methods are preferred to lower order methods only when the time and/or spatial step sizes are sufficiently small, where the smallness is a delicate question. It is not the purpose of our paper to find the best approximation scheme for our problem, simply to demonstrate our theoretical results. Our numerical simulation is based on an unconditionally stable, fully implicit scheme of second order accuracy, using three time level quadratic approximation in time and central difference scheme in space. The simulations were carried out on various platforms using several different numerical packages (OCTAVE, SCILAB, MATLAB), and they show grid independence for sufficiently small time and spatial grid.

We consider first Burgers' equation (2.1) with zero Neumann boundary condition (uncontrolled system) and then the regulation error system (2.4)–(2.7) with $\epsilon=0.1$ and with initial data $w_0(x)=W_0(x)-W_d$, where $W_d=3$ and $W_0(x)=20(0.5-x)^3$. The uncontrolled system is shown in Fig. 1(a). The solution seems to converge to a nonzero "equilibrium" profile, although it eventually approaches zero, which



(a) Uncontrolled (b) Controlled, $c_0 = c_1 = 0.01$

FIGURE 1 Simulations of Burgers' equation with $\epsilon = 0.1$.

could be seen only for $t \gg 1$ (in fact, for some initial data, the numerical solution gets trapped into this profile and never converges to zero [2]). This unsatisfactory behavior is remedied by applying boundary feedback, as shown in Fig. 1(b).

Acknowledgement

This work was supported by grants from the Air Force Office of Scientific Research, the National Science Foundation and the Office of Naval Research.

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