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Transient-performance improvement with a new class of adaptive controllers*

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Abstract: Computable $\mathscr{L}_2$ and $\mathscr{L}_\infty$ performance bounds are derived for a recently proposed class of adaptive systems which show that, in addition to global stability and asymptotic tracking, a systematic improvement of transient performance can be achieved. The underlying linear nonadaptive controller is shown to possess a parametric robustness property, but for a large parameter uncertainty it requires high gain. A comparison between the adaptive and the nonadaptive performance bounds demonstrates that adaptation improves the overall performance without the undesirable effects of high gain.

Keywords: Adaptive control; transient performance; parametric robustness; performance improvement ratio; nonlinear control.

1. Introduction

In the absence of disturbances and unmodeled dynamics, the tracking error of most adaptive control schemes converges to zero, i.e., they satisfy the asymptotic performance requirement. In applications, however, the system's transient performance is often more important. Numerous simulations indicate that the transient response of adaptive systems may be unacceptable due to large initial swings. An example was presented in [12], where an extremely poor transient behavior occurs together with ideal asymptotic performance. It is therefore necessary that in the performance analysis of adaptive systems both transient and asymptotic behavior be addressed. Recently, such an analysis of transient performance has led to its improvement, suggested in [11] and this is further developed in [2]. The proposed modifications of adaptive controllers render the tracking error arbitrarily small in terms of both mean-square and $\mathscr{L}_\infty$ bounds. This important advance is conceptual, because the bounds derived in [2] depend on the normalizing signal and therefore are not a priori verifiable. Other efforts for estimating or improving transient performance are presented in [7, 9, 13].

In this paper we undertake a performance analysis for a new class of adaptive controllers [5, 6] which exhibited a transient behavior superior to other adaptive schemes in simulations. Here we demonstrate that the observed improvement of the transient performance is systematic. We prove this by deriving $\mathscr{L}_2$ and $\mathscr{L}_\infty$ bounds which show that all the error states of the adaptive system can be made arbitrarily small, except for

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the parameter error $\tilde{\theta}(t)$, which is bounded by a constant proportional to $\tilde{\theta}(0)$. The performance bounds are computable and informative: they are explicit functions of initial conditions and design parameters. Another remarkable property of the new class of adaptive controllers is that for a known bound on the uncertain parameters the stability can be guaranteed even without adaptation [6]. When the adaptation is switched off, the underlying linear nonadaptive controller satisfies $\mathscr{L}_\infty$ bounds which can be used for a performance comparison with the adaptive controller. The adaptation, although not necessary for stabilization when the bounds on the parametric uncertainties are known, results in a smaller $\mathscr{L}_\infty$ bound and achieves the asymptotic tracking that is not possible with nonadaptive controllers. It is also important that the new adaptive controllers avoid the use of high gain by which the nonadaptive controllers counteract large parameter uncertainties.

2. The new class of adaptive systems

The control objective is to track asymptotically a reference signal $y_r(t)$ with the output y of the plant

$$y(s)=\frac{B(s)}{A(s)}u(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}u(s). \tag{2.1}$$

We make the following assumptions about the plant and the reference signal.

Assumption 2.1. *The plant is minimum phase, i.e., the polynomial $B(s)=b_m s^m+\cdots+b_1 s+b_0$ is Hurwitz, and the plant order (n), relative degree $(\rho=n-m)$, and high-frequency gain b_m are known.*

Assumption 2.2. *The reference model is stable, has relative degree ρ, and its input $r(t)$ is piecewise continuous. Its initial conditions $y_r(0), \dot{y}_r(0), \ldots, y_r^{(\rho-1)}(0)$ are at the designer's disposal.*

To simplify our presentation we will consider the case where the high-frequency gain is known, $b_m=1$. The closed-loop adaptive system designed in [5, 6] consists of the plant (2.1), the input and output filters $\dot{\eta}=A_0\eta+e_n u$ and $\dot{\lambda}=A_0\lambda+e_n u$, and the parameter update law (2.6) to be defined below, where $e_n^{\mathrm{T}}=[0,\ldots,0,1]$ and $P_0A_0+A_0^{\mathrm{T}}P_0=-I$, $P_0=P_0^{\mathrm{T}}>0$, i.e., $K(s)=s^n+k_{n-1}s^{n-1}+\cdots+k_1 s+k_0 \triangleq \det(sI-A_0)$ is Hurwitz. In our analysis, the states of the closed-loop system are expressed in the error coordinates $\varepsilon\in\mathbb{R}^n$, $\tilde{\eta}\in\mathbb{R}^n$, $\tilde{\zeta}\in\mathbb{R}^m$, $z\in\mathbb{R}^\rho$, $\tilde{\theta}\in\mathbb{R}^{n+m}$. The filter states are represented by the state estimation error ε, and the output filter state error $\tilde{\eta}=\eta-\eta_r$:

$$\dot{\varepsilon}=A_0\varepsilon, \tag{2.2}$$

$$\dot{\tilde{\eta}}=A_0\tilde{\eta}+e_n z_1, \quad \tilde{\eta}(0)=0, \tag{2.3}$$

where $z_1=y-y_r$ is the tracking error. The zero dynamics ζ of the plant are represented by

$$\dot{\tilde{\zeta}}=A_b\tilde{\zeta}+b_b z_1, \quad \tilde{\zeta}(0)=0, \tag{2.4}$$

where $\tilde{\zeta}=\zeta-\zeta_r$, and $P_bA_b+A_b^{\mathrm{T}}P_b=-I$, $P_b=P_b^{\mathrm{T}}>0$. The ideal reference trajectories η_r and ζ_r satisfy $\dot{\eta}_r=A_0\eta_r+e_n y_r$ and $\dot{\zeta}_r=A_b\zeta_r+b_b y_r$, respectively, they are not implemented, and their initial values are chosen such that $\tilde{\eta}(0)=0$ and $\tilde{\zeta}(0)=0$. The properties of the closed-loop adaptive system are determined by the following ρth-order error system:

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\\ \vdots\\ \dot{z}_\rho\end{bmatrix}=\begin{bmatrix}-c_1-d_1 & 1 & 0 & 0 & \cdots & 0\\ -1 & -c_2-d_2(\partial\alpha_1/\partial y)^2 & 1+\sigma_{23}\Gamma\omega & \sigma_{24}\Gamma\omega & \cdots & \sigma_{2\rho}\Gamma\omega\\ 0 & -1-\sigma_{23}\Gamma\omega & -c_3-d_3(\partial\alpha_2/\partial y)^2 & 1+\sigma_{34}\Gamma\omega & \cdots & \sigma_{3\rho}\Gamma\omega\\ 0 & -\sigma_{24}\Gamma\omega & -1-\sigma_{34}\Gamma\omega & -c_4-d_4(\partial\alpha_3/\partial y)^2 & \cdots & \sigma_{4\rho}\Gamma\omega\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & -\sigma_{2\rho}\Gamma\omega & -\sigma_{3\rho}\Gamma\omega & -\sigma_{4\rho}\Gamma\omega & \cdots & -c_\rho-d_\rho(\partial\alpha_{\rho-1}/\partial y)^2\end{bmatrix}$$

$$\times \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \vdots \\ z_\rho \end{bmatrix} + \begin{bmatrix} 1 \\ -\partial\alpha_1/\partial y \\ \vdots \\ -\partial\alpha_{\rho-1}/\partial y \end{bmatrix} \omega^{\mathrm{T}}\tilde{\theta} + \begin{bmatrix} 1 \\ -\partial\alpha_1/\partial y \\ \vdots \\ -\partial\alpha_{\rho-1}/\partial y \end{bmatrix} \varepsilon_2, \tag{2.5}$$

where $c_i, d_i > 0$ are design parameters, ω is the 'regressor' to be defined later, σ_{ij} and $\partial\alpha_i/\partial y$ are nonlinear time varying functions which are smooth in all the state variables and piecewise continuous in time t. The parameter update law is

$$\dot{\tilde{\theta}} = -\Gamma\omega\left[1, -\frac{\partial\alpha_1}{\partial y}, \ldots, -\frac{\partial\alpha_{\rho-1}}{\partial y}\right]\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_\rho \end{bmatrix}, \tag{2.6}$$

where $\Gamma = \Gamma^{\mathrm{T}} > 0$.

Global uniform stability of the origin of the adaptive system (2.2)–(2.6), and the convergence of the states to the manifold $M = \{z = 0,\ \varepsilon = 0,\ \tilde{\eta} = 0,\ \tilde{\zeta} = 0\}$ were proved in [5] using the complete Lyapunov function $V = V_\rho + (1/k_\eta)|\tilde{\eta}|^2_{P_0} + (1/k_\zeta)|\tilde{\zeta}|^2_{P_b}$, $k_\eta, k_\zeta > 0$, where[1]

$$V_\rho = \frac{1}{2}|z|^2 + \frac{1}{d_0}|\varepsilon|^2_{P_0} + \frac{1}{2}|\tilde{\theta}|^2_{\Gamma^{-1}}, \tag{2.7}$$

and $(1/d_0) = \sum_{j=1}^{\rho}(1/d_j)$.

The closed-loop adaptive system is nonlinear, and is designed as such. However, with the adaptation turned off by setting $\Gamma = 0$ in the adaptive law (2.6), the uncertain parameter error $\tilde{\theta}$ is constant, the nonlinear terms $\sigma_{ij}\Gamma\omega$ vanish from (2.5) and, as explained in [6], the nonlinear terms $\partial\alpha_i/\partial y$ reduce to known constants, so that the whole closed-loop system becomes linear. In Section 3 we analyze the $\mathscr{L}_2$ and $\mathscr{L}_\infty$ performance of the adaptive system (2.2)–(2.6). We show that the design parameters c_i, d_i can be used to improve the transient performance. In Section 4 we analyze the $\mathscr{L}_\infty$ and mean-square performance of the underlying linear system without adaptation. Finally, in Section 5, we use the bounds derived to evaluate the advantages of adaptation qualitatively.

3. Performance of the adaptive system

In this section we prove that both the $\mathscr{L}_2$ and $\mathscr{L}_\infty$ norms of the states $z, \tilde{\eta}, \tilde{\zeta}$ of the adaptive system can be made arbitrarily small by a choice of the design parameters c_i, d_i, Γ.

From (2.3) and (2.4) we define transfer functions

$$W_{\tilde{\eta}}(s) \triangleq (sI - A_0)^{-1} e_n, \tag{3.1}$$

$$W_{\tilde{\zeta}}(s) \triangleq (sI - A_b)^{-1} b_b, \tag{3.2}$$

and denote the respective impulse responses by $w_{\tilde{\eta}}(t)$, $w_{\tilde{\zeta}}(t)$.

[1] Notation: The weighted Euclidean norms for vectors will be denoted by $|x|^2_P = x^{\mathrm{T}}Px$. The $\mathscr{L}_\infty$, $\mathscr{L}_2$ and truncated $\mathscr{L}_2$ norms for signals will be denoted by $\|\cdot\|_\infty$, $\|\cdot\|_2$ and $\|\cdot\|_{2,t}$, respectively. The $\mathscr{H}_\infty$ norms for transfer functions will be denoted by $\|\cdot\|_\infty$, and the $\mathscr{L}_1$ norms of their impulse responses by $\|\cdot\|_1$.

Theorem 3.1 ($\mathscr{L}_2$ performance). *The $\mathscr{L}_2$ norms of the states $z, \tilde{\eta}, \tilde{\zeta}$ of the adaptive system (2.2)–(2.6) are bounded by*

$$\|z\|_2 \le \frac{1}{\sqrt{c_0}} \sqrt{V_\rho(0)}, \tag{3.3}$$

$$\|\tilde{\eta}\|_2 \le \frac{1}{\sqrt{c_0}} \sqrt{V_\rho(0)} \, \| W_{\tilde{\eta}} \|_\infty, \tag{3.4}$$

$$\|\tilde{\zeta}\|_2 \le \frac{1}{\sqrt{c_0}} \sqrt{V_\rho(0)} \, \| W_{\tilde{\zeta}} \|_\infty, \tag{3.5}$$

where $\| W_{\tilde{\eta}} \|_\infty$, $\| W_{\tilde{\zeta}} \|_\infty$ are independent of $c_0 = \min_{1 \le k \le \rho} c_k$.

Proof. As shown in [5], the derivative of V_ρ along the solutions of (2.2)–(2.6) is

$$\dot{V}_\rho \le -c_0|z|^2 - \frac{3}{4d_0}|\varepsilon|^2 \le -c_0|z|^2. \tag{3.6}$$

Since V_ρ is nonincreasing, we have

$$\|z\|_2^2 = \int_0^\infty |z(\tau)|^2 \, d\tau \le \frac{1}{c_0}[V_\rho(0) - V_\rho(\infty)] \le \frac{1}{c_0} V_\rho(0), \tag{3.7}$$

which implies (3.3). From (2.3) and (3.1) we get

$$\|\tilde{\eta}\|_2 \le \| W_{\tilde{\eta}} \|_\infty \|z_1\|_2 \le \frac{1}{\sqrt{c_0}} \sqrt{V_\rho(0)} \, \| W_{\tilde{\eta}} \|_\infty \tag{3.8}$$

and, from (2.4) and (3.2) we get

$$\|\tilde{\zeta}\|_2 \le \| W_{\tilde{\zeta}} \|_\infty \|z_1\|_2 \le \frac{1}{\sqrt{c_0}} \sqrt{V_\rho(0)} \, \| W_{\tilde{\zeta}} \|_\infty. \tag{3.9}$$

Remark 3.1. We point out that, although the initial states $z_2(0), \ldots, z_\rho(0)$ may depend on c_i, d_i, Γ (see [5]), they are at the designer's disposal. As explained in [5], by appropriately initializing the reference model or the filter states, $z(0)$ can be set to zero. The values used in such initialization and $\varepsilon(0)$, $\tilde{\theta}(0)$ are independent of c_i, d_i, Γ. Therefore, by setting $z(0) = 0$, we see that

$$V_\rho(0) = \frac{1}{d_0} |\varepsilon(0)|_{P_0}^2 + \frac{1}{2} |\tilde{\theta}(0)|_{\Gamma^{-1}}^2 \tag{3.10}$$

is a decreasing function of d_0 and Γ, independent of c_0.

The possibility to improve performance with the adaptation gain is particularly clear in the case $\varepsilon(0) = 0$ and $\Gamma = \gamma I$, when the $\mathscr{L}_2$ bounds of Theorem 3.1 become

$$\|z\|_2 \le \frac{1}{\sqrt{c_0 \gamma}} |\tilde{\theta}(0)|, \tag{3.11}$$

$$\|\tilde{\eta}\|_2 \le \frac{1}{\sqrt{c_0 \gamma}} |\tilde{\theta}(0)| \, \| W_{\tilde{\eta}} \|_\infty, \tag{3.12}$$

$$\|\tilde{\zeta}\|_2 \le \frac{1}{\sqrt{c_0 \gamma}} |\tilde{\theta}(0)| \, \| W_{\tilde{\zeta}} \|_\infty. \tag{3.13}$$

The error states $z, \tilde{\eta}, \tilde{\zeta}, \varepsilon$ are guaranteed to have good $\mathscr{L}_2$ performance. This is an improvement over [2], where only the *mean-square* tracking error $(1/t)\int_0^t [y(\tau) - y_r(\tau)]^2 \, d\tau$ of a modified MRAC scheme was proved to be arbitrarily small.

Another advantage of the bounds derived is that they are computable. The bound for $\|z\|_2$ is explicit, while the bound for $\|\tilde{\eta}\|_2$ involves $\|W_{\tilde{\eta}}\|_\infty$ which is known from (3.1). Only the factor $\|W_{\tilde{\zeta}}\|_\infty$ in the bound for the zero dynamics $\|\tilde{\zeta}\|_2$ depends on the unknown parameters $b_0, \ldots, b_{m-1}$. When these parameters belong to known intervals, $\|W_{\tilde{\zeta}}\|_\infty$ can be computed using [1].

For a more complete characterization of the performance achieved, we proceed to derive $\mathscr{L}_\infty$ norm bounds for the error states of the adaptive system (2.2)–(2.6). These bounds are also useful for a comparison with nonadaptive systems.

We first give simple bounds on $\|z\|_\infty$ and $\|\tilde{\theta}\|_\infty$:

$$\|z\|_\infty \le \sqrt{2V_\rho(0)}, \tag{3.14}$$

$$\|\tilde{\theta}\|_\infty \le \sqrt{\bar{\lambda}(\Gamma)}\sqrt{2V_\rho(0)}. \tag{3.15}$$

Since $\dot{V}_\rho \le 0$, the bound (3.14) follows immediately from

$$2V_\rho(t) = |z(t)|^2 + \frac{2}{d_0}|\varepsilon(t)|^2_{P_0} + |\tilde{\theta}(t)|^2_{\Gamma^{-1}} \le 2V_\rho(0), \tag{3.16}$$

and the bound (3.15) is obtained by noting that

$$\frac{1}{\bar{\lambda}(\Gamma)}|\tilde{\theta}|^2 \le |\tilde{\theta}|^2_{\Gamma^{-1}} \le 2V_\rho(0). \tag{3.17}$$

For $\Gamma = \gamma I$, it further follows from (3.15) and (3.16) that

$$\|\tilde{\theta}\|_\infty \le \sqrt{\gamma}|z(0)| + \sqrt{\frac{2\gamma}{d_0}}|\varepsilon(0)|_{P_0} + |\tilde{\theta}(0)|. \tag{3.18}$$

In this way, $\|\tilde{\theta}\|_\infty$ is explicitly related to initial conditions and design parameters.

Theorem 3.2 ($\mathscr{L}_\infty$ performance). *The $\mathscr{L}_\infty$ norms of the states $z, \tilde{\eta}, \tilde{\zeta}$ of the adaptive system* (2.2)–(2.6) *are bounded by*

$$|z(t)| \le \frac{1}{\sqrt{c_0 d_0}} M + |z(0)|e^{-c_0 t}, \tag{3.19}$$

$$|\tilde{\eta}(t)| \le \left(\frac{1}{\sqrt{c_0 d_0}} M + |z(0)|e^{-c_0 t}\right)\|w_{\tilde{\eta}}\|_1, \tag{3.20}$$

$$|\tilde{\zeta}(t)| \le \left(\frac{1}{\sqrt{c_0 d_0}} M + |z(0)|e^{-c_0 t}\right)\|w_{\tilde{\zeta}}\|_1, \tag{3.21}$$

where

$$M \triangleq \frac{1}{2}\left\{\sqrt{\bar{\lambda}(\Gamma)}\sqrt{2V_\rho(0)}\,[\|h_\omega\|_1(\sqrt{2V_\rho(0)} + \|y_r\|_\infty) + \kappa_\omega] + \frac{1}{\sqrt{\underline{\lambda}(P_0)}}|\varepsilon(0)|_{P_0}\right\}, \tag{3.22}$$

and $\|w_{\tilde{\eta}}\|_1, \|w_{\tilde{\zeta}}\|_1, \|h_\omega\|_1, \kappa_\omega$ are independent of c_0, d_0.

Proof. Differentiating $\frac{1}{2}|z|^2 = \frac{1}{2}z^{\mathrm{T}}z$ along the solutions of (2.5) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|z|^2\right) = -\sum_{k=1}^{\rho} c_k z_k^2 - \sum_{k=1}^{\rho} d_k z_k^2 \left(\frac{\partial \alpha_{k-1}}{\partial y}\right)^2 - \sum_{k=1}^{\rho} z_k \frac{\partial \alpha_{k-1}}{\partial y}(\tilde{\theta}^{\mathrm{T}}\omega + \varepsilon_2)$$

$$= -\sum_{k=1}^{\rho} c_k z_k^2 - \sum_{k=1}^{\rho} d_k \left[\frac{\partial \alpha_{k-1}}{\partial y} z_k + \frac{1}{2d_k}(\tilde{\theta}^{\mathrm{T}}\omega + \varepsilon_2)\right]^2 + \left(\sum_{k=1}^{\rho}\frac{1}{4d_k}\right)(\tilde{\theta}^{\mathrm{T}}\omega + \varepsilon_2)^2$$

$$\leq -c_0|z|^2 + \frac{1}{4d_0}(\tilde{\theta}^{\mathrm{T}}\omega + \varepsilon_2)^2. \tag{3.23}$$

Upon multiplication by e^{2c_0t}, the last inequality becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(|z|^2\mathrm{e}^{2c_0t}) \leq \frac{1}{2d_0}(\tilde{\theta}^{\mathrm{T}}\omega + \varepsilon_2)^2\mathrm{e}^{2c_0t}. \tag{3.24}$$

Integrating (3.24) over $[0, t]$, we arrive at

$$|z(t)|^2 \leq |z(0)|^2\mathrm{e}^{-2c_0t} + \frac{1}{2d_0}\int_0^t \mathrm{e}^{-2c_0(t-\tau)}[\tilde{\theta}^{\mathrm{T}}\omega(\tau) + \varepsilon_2(\tau)]^2\,\mathrm{d}\tau, \tag{3.25}$$

and from this we obtain

$$|z(t)|^2 \leq |z(0)|^2\mathrm{e}^{-2c_0t} + \frac{1}{2d_0}\int_0^t \mathrm{e}^{-2c_0(t-\tau)}\mathrm{d}\tau \sup_{t\in[0,\infty)} |\tilde{\theta}^{\mathrm{T}}\omega(t) + \varepsilon_2(t)|^2$$

$$\leq |z(0)|^2\mathrm{e}^{-2c_0t} + \frac{1}{4c_0d_0}\|\tilde{\theta}^{\mathrm{T}}\omega + \varepsilon_2\|_\infty^2, \tag{3.26}$$

which in view of $|\varepsilon_2(t)| \leq [1/\sqrt{\underline{\lambda}(P_0)}]|\varepsilon(0)|_{P_0}$, gives

$$|z(t)| \leq \frac{1}{2\sqrt{c_0d_0}}\left(\|\tilde{\theta}\|_\infty\|\omega\|_\infty + \frac{1}{\sqrt{\underline{\lambda}(P_0)}}|\varepsilon(0)|_{P_0}\right) + |z(0)|\mathrm{e}^{-c_0t}. \tag{3.27}$$

It was shown in [6] that

$$\omega = \left[\frac{s+k_1}{K(s)}[s^{n-1},\ldots,s,1], \frac{(s+k_1)A(s)}{K(s)B(s)}[s^{m-1},\ldots,s,1]\right]^{\mathrm{T}} y + \omega_0(t) \triangleq H_\omega(s)y + \omega_0(t), \tag{3.28}$$

where $|\omega_0(t)| \leq \kappa_\omega\mathrm{e}^{-\sigma t}$ is the response due to the initial conditions of the filters $\eta(0)$, $\lambda(0)$. Observe that κ_ω, σ depend only on the plant and filter parameters and not on c_0, d_0, Γ. Now, using $y = z_1 + y_r$ and (3.14), we get

$$\|\omega\|_\infty \leq \|h_\omega\|_1(\|z_1\|_\infty + \|y_r\|_\infty) + \kappa_\omega\mathrm{e}^{-\sigma t} \leq \|h_\omega\|_1(\sqrt{2V_\rho(0)} + \|y_r\|_\infty) + \kappa_\omega. \tag{3.29}$$

Substituting (3.29) into (3.27), and using (3.15) we obtain

$$|z(t)| \leq \frac{1}{\sqrt{c_0d_0}}\frac{1}{2}\left\{\sqrt{\bar{\lambda}(\Gamma)}\sqrt{2V_\rho(0)}\,[\|h_\omega\|_1(\sqrt{2V_\rho(0)} + \|y_r\|_\infty) + \kappa_\omega] + \frac{1}{\sqrt{\underline{\lambda}(P_0)}}|\varepsilon(0)|_{P_0}\right\} + |z(0)|\mathrm{e}^{-c_0t}$$

$$= \frac{1}{\sqrt{c_0d_0}}M + |z(0)|\mathrm{e}^{-c_0t}. \tag{3.30}$$

From (2.3) we have

$$\|\tilde{\eta}\|_\infty \leq \|w_{\tilde{\eta}}\|_1\|z_1\|_\infty \leq \left(\frac{1}{\sqrt{c_0d_0}}M + |z(0)|\mathrm{e}^{-c_0t}\right)\|w_{\tilde{\eta}}\|_1, \tag{3.31}$$

and from (2.4) we have

$$\|\tilde{\zeta}\|_\infty \le \|w_{\tilde{\zeta}}\|_1 \|z_1\|_\infty \le \left(\frac{1}{\sqrt{c_0 d_0}} M + |z(0)| e^{-c_0 t}\right) \|w_{\tilde{\zeta}}\|_1 . \tag{3.32}$$

With the initialization $z(0) = 0$, the expression (3.10) for $V_\rho(0)$ and (3.22) show that M is a decreasing function of d_0 independent of c_0.

Thus, the entire state $z, \tilde{\theta}, \varepsilon, \tilde{\eta}, \tilde{\zeta}$ is guaranteed to have good $\mathscr{L}_\infty$ performance. This is an improvement over [2], where only the tracking error of a modified MRAC scheme was proved to be arbitrarily small.

Since M in (3.30) depends on $\|h_\omega\|_1$, the bounds (3.19)–(3.21) require computation of $\|h_\omega\|_1$, $\|w_{\tilde{\eta}}\|_1$ and $\|w_{\tilde{\zeta}}\|_1$. Although $\|h_\omega\|_1$ and $\|w_{\tilde{\zeta}}\|_1$ depend on uncertain parameters, we can employ the procedure of [1] to compute their $\mathscr{H}_\infty$ norms and then apply the well-known inequality $\|g\|_1 \le (2n+1)\|G\|_\infty$, where $G(s)$ is a stable transfer function, n its McMillan degree, and $g(t)$ its impulse response.

Let us now give a special but more revealing form of the above $\mathscr{L}_\infty$ bounds.

Corollary 3.1. *In the case $z(0) = 0$, $\varepsilon(0) = 0$, $\kappa_\omega = 0$ and $\Gamma = \gamma I$, the $\mathscr{L}_\infty$ bounds of Theorem 3.2 become*

$$\|z\|_\infty \le \frac{|\tilde{\theta}(0)| \, \|h_\omega\|_1}{2\sqrt{c_0 d_0}} \left(\|y_r\|_\infty + \frac{1}{\sqrt{\gamma}} |\tilde{\theta}(0)|\right), \tag{3.33}$$

$$\|\tilde{\eta}\|_\infty \le \frac{|\tilde{\theta}(0)| \, \|h_\omega\|_1}{2\sqrt{c_0 d_0}} \left(\|y_r\|_\infty + \frac{1}{\sqrt{\gamma}} |\tilde{\theta}(0)|\right) \|w_{\tilde{\eta}}\|_1, \tag{3.34}$$

$$\|\tilde{\zeta}\|_\infty \le \frac{|\tilde{\theta}(0)| \, \|h_\omega\|_1}{2\sqrt{c_0 d_0}} \left(\|y_r\|_\infty + \frac{1}{\sqrt{\gamma}} |\tilde{\theta}(0)|\right) \|w_{\tilde{\zeta}}\|_1 . \tag{3.35}$$

The assumption $z(0) = 0$, $\varepsilon(0) = 0$, $\kappa_\omega = 0$ is satisfied in the particular case where the initial conditions of the plant and the filter states are zero and the system is driven by $r(t)$.

The form of bounds in Corollary 3.1 clarifies the dependence of the $\mathscr{L}_\infty$ performance on the parameter uncertainty $|\tilde{\theta}(0)|$ and the design parameters c_0, d_0 and γ. Any increase in those parameters results in an improvement of the $\mathscr{L}_\infty$ performance. It is of interest to observe that d_0, present in the $\mathscr{L}_\infty$ bounds (3.33)–(3.35), is absent from the $\mathscr{L}_2$ bounds (3.11)–(3.13). This is consistent with the 'peak-shaving' ability of the nonlinear damping terms observed in [5].

4. Performance of the nonadaptive system

It is of interest to evaluate the performance achievable with the underlying linear controller resulting from setting $\Gamma = 0$ in the adaptive system (2.2)–(2.6). Here we investigate the $\mathscr{L}_\infty$ performance and the mean-square performance of this nonadaptive system.

Since the nonadaptive system is linear, without loss of generality we assume that all the initial conditions are zero. It is clear that this will not affect the conclusions about stability, and in the performance bounds this only amounts to neglecting the exponentially decaying terms.

Using Theorem 6.1 of [6], the following result is immediate.

Theorem 4.1 (stability and $\mathscr{L}_\infty$ performance). *The nonadaptive (NA) system (2.2)–(2.5) is asymptotically stable for*

$$2\sqrt{c_0 d_0} > |\tilde{\theta}| \, \|h_\omega\|_1 . \tag{4.1}$$

The $\mathcal{L}_\infty$ norms of the states of this system are bounded by

$$\|z^{\mathrm{NA}}\|_\infty \le \frac{|\tilde{\theta}|\,\|h_\omega\|_1}{2\sqrt{c_0 d_0} - |\tilde{\theta}|\,\|h_\omega\|_1}\,\|y_r\|_\infty, \tag{4.2}$$

$$\|\tilde{\eta}^{\mathrm{NA}}\|_\infty \le \frac{|\tilde{\theta}|\,\|h_\omega\|_1}{2\sqrt{c_0 d_0} - |\tilde{\theta}|\,\|h_\omega\|_1}\,\|y_r\|_\infty \|w_{\tilde{\eta}}\|_1, \tag{4.3}$$

$$\|\tilde{\zeta}^{\mathrm{NA}}\|_\infty \le \frac{|\tilde{\theta}|\,\|h_\omega\|_1}{2\sqrt{c_0 d_0} - |\tilde{\theta}|\,\|h_\omega\|_1}\,\|y_r\|_\infty \|w_{\tilde{\zeta}}\|_1. \tag{4.4}$$

The nonadaptive controller does not, in general, achieve asymptotic tracking, so we cannot talk about $\mathcal{L}_2$ performance in this case. However, it is possible to prove that the mean-square performance can be made accurate to the desired extent.

Theorem 4.2 (stability and mean-square performance). *The nonadaptive* (NA) *system* (2.2)–(2.5) *is asymptotically stable for*

$$2\sqrt{c_0 d_0} > |\tilde{\theta}|\,\|H_\omega\|_\infty. \tag{4.5}$$

The mean-square values of $z, \tilde{\eta}, \tilde{\zeta}$ are bounded by

$$\left(\frac{1}{t}\int_0^t |z^{\mathrm{NA}}(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} \le \frac{|\tilde{\theta}|\,\|H_\omega\|_\infty}{2\sqrt{c_0 d_0} - |\tilde{\theta}|\,\|H_\omega\|_\infty}\,\|y_r\|_\infty, \tag{4.6}$$

$$\left(\frac{1}{t}\int_0^t |\tilde{\eta}^{\mathrm{NA}}(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} \le \frac{\|w_{\tilde{\eta}}\|_1 |\tilde{\theta}|\,\|H_\omega\|_\infty}{2\sqrt{c_0 d_0} - |\tilde{\theta}|\,\|H_\omega\|_\infty}\,\|y_r\|_\infty, \tag{4.7}$$

$$\left(\frac{1}{t}\int_0^t |\tilde{\zeta}^{\mathrm{NA}}(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} \le \frac{\|w_{\tilde{\zeta}}\|_1 |\tilde{\theta}|\,\|H_\omega\|_\infty}{2\sqrt{c_0 d_0} - |\tilde{\theta}|\,\|H_\omega\|_\infty}\,\|y_r\|_\infty. \tag{4.8}$$

Proof. For $\Gamma = 0$, with initial conditions $z(0) = 0$, $\varepsilon(0) = 0$, we proceed from (3.25):

$$|z(t)|^2 \le \frac{1}{2d_0}\int_0^t \mathrm{e}^{-2c_0(t-\tau)}[\tilde{\theta}^{\mathrm{T}}\omega(\tau)]^2\,\mathrm{d}\tau. \tag{4.9}$$

Now, integrating (4.9) over $[0, t]$, we get

$$\int_0^t |z(\tau)|^2\,\mathrm{d}\tau \le \frac{1}{2d_0}\int_0^t\left[\int_0^\tau \mathrm{e}^{-2c_0(\tau-s)}[\tilde{\theta}^{\mathrm{T}}\omega(s)]^2\,\mathrm{d}s\right]\mathrm{d}\tau. \tag{4.10}$$

Changing the sequence of integration, (4.10) becomes

$$\begin{aligned}\int_0^t |z(\tau)|^2\,\mathrm{d}\tau &\le \frac{1}{2d_0}\int_0^t \mathrm{e}^{2c_0 s}[\tilde{\theta}^{\mathrm{T}}\omega(s)]^2\left(\int_s^t \mathrm{e}^{-2c_0\tau}\,\mathrm{d}\tau\right)\mathrm{d}s\\ &\le \frac{1}{2d_0}\int_0^t \mathrm{e}^{2c_0 s}[\tilde{\theta}^{\mathrm{T}}\omega(s)]^2\,\frac{1}{2c_0}\mathrm{e}^{-2c_0 s}\,\mathrm{d}s,\end{aligned} \tag{4.11}$$

because

$$\int_s^t \mathrm{e}^{-2c_0\tau}\,\mathrm{d}\tau = \frac{1}{2c_0}(\mathrm{e}^{-2c_0 s} - \mathrm{e}^{-2c_0 t}) \le \frac{1}{2c_0}\mathrm{e}^{-2c_0 s}.$$

Now, the fact that $e^{2c_0 s}e^{-2c_0 s} = 1$ used in (4.11) yields

$$\int_0^t |z(\tau)|^2\,d\tau \le \frac{|\tilde\theta|^2}{4c_0 d_0}\int_0^t |\omega(\tau)|^2\,d\tau. \tag{4.12}$$

On the other hand, since $H_\omega(s)$ is stable and proper, the truncated $\mathscr{L}_2$ norms of ω and z are related as

$$\|\omega\|_{2,t} \le \|H_\omega\|_\infty \|y\|_{2,t} \le \|H_\omega\|_\infty(\|z\|_{2,t} + \|y_r\|_{2,t}). \tag{4.13}$$

From (4.12) and (4.13), by the small-gain theorem, $\mathscr{L}_2$ stability is guaranteed for $2\sqrt{c_0 d_0} > |\tilde\theta|\,\|H_\omega\|_\infty$, and the asymptotic stability is argued as in the proof of Theorem 6.1 of [6]. Substituting (4.13) into (4.12) and solving for $\|z\|_{2,t} = (\int_0^t |z(\tau)|^2\,d\tau)^{1/2}$, we get

$$\left(\int_0^t |z(\tau)|^2\,d\tau\right)^{1/2} \le \frac{|\tilde\theta|\,\|H_\omega\|_\infty}{2\sqrt{c_0 d_0} - |\tilde\theta|\,\|H_\omega\|_\infty}\|y_r\|_{2,t} \tag{4.14}$$

and (4.6) follows because $\|y_r\|_{2,t}^2 = \int_0^t |y_r(\tau)|^2\,d\tau \le \|y_r\|_\infty^2 t$. Inequalities (4.7) and (4.8) are immediate from (4.6) (recall that $\tilde\eta(0) = 0$, $\tilde\zeta(0) = 0$). □

Theorems 4.1 and 4.2 provide two different stability conditions (4.1) and (4.5), of which (4.5) is directly computable [1] and less conservative because $\|H_\omega\|_\infty \le \|h_\omega\|_1$.

Another usual way of expressing performance properties of a linear system is to examine the difference between the actual and the desired closed-loop transfer function. In the case of tracking, the desired transfer function is $y(s)/y_r(s) = 1$. The actual closed-loop transfer function of the nonadaptive system was derived in [6] as

$$\frac{y(s)}{y_r(s)} = G_c(s) = \frac{1}{1 + \dfrac{\beta_z(s)}{\alpha_z(s)}\tilde\theta^{\mathrm T} H_\omega(s)}. \tag{4.15}$$

In this expression, $\beta_z(s)/\alpha_z(s)$ is the transfer function from $\tilde\theta^{\mathrm T}\omega$ to z_1 in (2.5) when $\Gamma = 0$. This transfer function is stable, relative degree one, and with $\deg \alpha_z = \rho$. Its poles can be placed arbitrarily by using the design parameters c_i, d_i.

Theorem 4.3 (tracking performance). *In the nonadaptive system* (4.15), *the design parameters* c_i, d_i, $1 \le i \le \rho$ *can be chosen to satisfy, for any* $\delta_c > 0$, *the following tracking performance specification*:

$$|G_c(j\omega) - 1| < \delta_c \quad \forall \omega \in \mathbb{R}. \tag{4.16}$$

Proof. By setting $t = \infty$ in (4.12) we see that the induced $\mathscr{L}_2$ norm of $\beta_z(s)/\alpha_z(s)$ is not greater than $1/2\sqrt{c_0 d_0}$. This in turn means that $\|\beta_z/\alpha_z\|_\infty \le 1/2\sqrt{c_0 d_0}$, which implies $|\beta_z(j\omega)/\alpha_z(j\omega)| \le 1/2\sqrt{c_0 d_0}$, $\forall\omega \in \mathbb{R}$. From (4.15) we now have

$$|G_c(j\omega) - 1| = \left|\frac{\dfrac{\beta_z(j\omega)}{\alpha_z(j\omega)}\tilde\theta^{\mathrm T} H_\omega(j\omega)}{1 + \dfrac{\beta_z(j\omega)}{\alpha_z(j\omega)}\tilde\theta^{\mathrm T} H_\omega(j\omega)}\right| \le \frac{\dfrac{1}{2\sqrt{c_0 d_0}}|\tilde\theta|\,\|H_\omega\|_\infty}{1 - \dfrac{1}{2\sqrt{c_0 d_0}}|\tilde\theta|\,\|H_\omega\|_\infty}, \tag{4.17}$$

which is less than any δ_c provided that $c_0 d_0$ is sufficiently large:

$$2\sqrt{c_0 d_0} > \left(1 + \frac{1}{\delta_c}\right)|\tilde\theta|\,\|H_\omega\|_\infty. \tag{4.18}$$

As expected, the tracking condition (4.18) is more stringent than the corresponding stability condition (4.5). The required value of $2\sqrt{c_0 d_0}$ is increased by the factor $1 + (1/\delta_c)$ and tends to infinity as $\delta_c \to 0$. In this sense the underlying linear controller is a 'high-gain' controller which achieves a good tracking performance at the expense of an increase of the bandwidth of the closed-loop system.

5. Performance improvement due to adaptation

The global stability and tracking properties of the adaptive system established in [5, 6] and the bounds derived in the preceding sections, provide us with a data base for a semiquantitative performance comparison of the nonlinear adaptive system and its linear nonadaptive counterpart.

The stability of the adaptive system is guaranteed to be global for any positive values of the design parameters c_0, d_0 and Γ. No a priori information is required about the parameter uncertainty. In contrast, the linear controller guarantees stability only if a bound on the parameter uncertainty is known and the value of $c_0 d_0$ is large enough to satisfy the stability condition (4.5).

Asymptotic tracking is achieved by the adaptive controller for any initial condition, any parameter uncertainty and any positive c_0, d_0 and Γ. The tracking error of the linear system can be reduced, but, in general, does not converge to zero. To make the tracking error small, the value of $c_0 d_0$ is required to be large. It can be shown that the increase of $c_0 d_0$ increases the bandwidth, which may be undesirable.

Transient performance of the adaptive system can be improved over that of the linear system without an increase of $c_0 d_0$. Performance improvement due to adaptation follows as a corollary from the bounds (3.33)–(3.35) and (4.2)–(4.4). We use the superscripts A and NA to denote the quantities in the adaptive and in the nonadaptive system, respectively, and assume the same parameter uncertainty, $\tilde{\theta}^{\mathrm{NA}} = \tilde{\theta}^{\mathrm{A}}(0) \triangleq \tilde{\theta}$. Then we measure the performance improvement due to adaptation using the performance ratio

$$R_{\mathscr{L}_\infty} \triangleq \frac{\dfrac{|\tilde{\theta}^{\mathrm{A}}(0)|\,\|h_\omega\|_1}{2\sqrt{c_0^{\mathrm{A}} d_0^{\mathrm{A}}}}\left(\|y_r\|_\infty + \dfrac{1}{\sqrt{\gamma}}|\tilde{\theta}^{\mathrm{A}}(0)|\right)}{\dfrac{|\tilde{\theta}^{\mathrm{NA}}|\,\|h_\omega\|_1}{2\sqrt{c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}} - |\tilde{\theta}^{\mathrm{NA}}|\,\|h_\omega\|_1}\|y_r\|_\infty} \tag{5.1}$$

between the $\mathscr{L}_\infty$ bounds (3.33)–(3.35) and (4.2)–(4.4). The improvement is achieved if the performance ratio is small: $R_{\mathscr{L}_\infty} \le \bar{R}_{\mathscr{L}_\infty} < 1$.

Corollary 5.1. *Let the initial conditions of $z, \varepsilon, \eta, \lambda$ be zero. Then with adaptation gain*

$$\gamma \ge \left[\frac{2\sqrt{c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}} - |\tilde{\theta}|\,\|h_\omega\|_1}{2\sqrt{c_0^{\mathrm{A}} d_0^{\mathrm{A}}}\,\bar{R}_{\mathscr{L}_\infty} - (2\sqrt{c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}} - |\tilde{\theta}|\,\|h_\omega\|_1)}\right]^2 \frac{|\tilde{\theta}|^2}{\|y_r\|_\infty^2}, \tag{5.2}$$

and $2\sqrt{c_0^{\mathrm{A}} d_0^{\mathrm{A}}}\,\bar{R}_{\mathscr{L}_\infty} > 2\sqrt{c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}} - |\tilde{\theta}|\,\|h_\omega\|_1$, the performance ratio $R_{\mathscr{L}_\infty}$ is no greater than $\bar{R}_{\mathscr{L}_\infty} < 1$.

From this corollary we can deduce two further advantages of the adaptive controller. First, the adaptation gain γ provides an additional degree of freedom with which the performance can be improved when $c_0^{\mathrm{A}} d_0^{\mathrm{A}}$ and $c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}$ are the same. Second, and more important, performance improvements can be achieved even with $c_0^{\mathrm{A}} d_0^{\mathrm{A}}$ smaller than $c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}$. In the presence of a large parameter uncertainty $\tilde{\theta}$, the nonadaptive controller must use $c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}$ sufficiently large to satisfy $2\sqrt{c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}} - |\tilde{\theta}|\,\|h_\omega\|_1 > 0$, thus increasing the bandwidth. From Corollary 5.1 it is clear that with the adaptive controller such an undesirable bandwidth increase can be avoided, because when both $\tilde{\theta}$ and $c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}$ are large, the condition $2\sqrt{c_0^{\mathrm{A}} d_0^{\mathrm{A}}}\,\bar{R}_{\mathscr{L}_\infty} > 2\sqrt{c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}} - |\tilde{\theta}|\,\|h_\omega\|_1$ can be satisfied with $c_0^{\mathrm{A}} d_0^{\mathrm{A}}$ much smaller than $c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}$. This confirms that adaptation is an efficient tool for reducing the effects of large parametric uncertainty without unacceptable widening of system bandwidth. For small parametric uncertainty, the linear controller is effective.

The improvement of performance due to adaptation was illustrated by simulations in [6] for $c_0^{\mathrm{A}} d_0^{\mathrm{A}} = c_0^{\mathrm{NA}} d_0^{\mathrm{NA}}$. While (5.2) shows the performance improvement only beyond a certain γ, the simulations indicate that the performance improvement is present for any $\gamma \geq 0$.

6. Conclusions

Our $\mathscr{L}_2$ and $\mathscr{L}_\infty$ bounds show that the performance of the new class of adaptive controllers proposed in [5, 6] can be made as good as desired. The use of design parameters for improvement of the transient performance is systematic. It is crucial that with adaptation this performance improvement, in the presence of large parameter uncertainty, can be achieved without large bandwidth required by the nonadaptive linear controller.

The performance bounds derived in this paper are for the ideal case, i.e., in the absence of disturbances and unmodeled dynamics. The problem of robustness with respect to such modeling errors is yet to be addressed, and robust update laws for the new class of adaptive systems will have to be developed. As in [2–4, 8], it is of interest to study the performance of the new adaptive controllers in the presence of unmodeled dynamics.

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