In this paper, we develop boundary output feedback control laws for traffic flow on two cascaded freeway segments connected by a junction. The macroscopic traffic dynamics are governed by the Aw–Rascle–Zhang (ARZ) model in which two subsystems of second-order nonlinear partial differential equations (PDEs) describe the evolution of traffic density and velocity on each segment. Due to the change of road access at the junction, different equilibria are considered for the two connected segments. To suppress stop-and-go traffic oscillations on the cascaded roads, we consider a ramp metering that regulates the traffic flow rate entering from the on-ramp to the mainline freeway. Different control designs are proposed such that the output feedback stabilization is realized with either the ramp metering located at the middle junction or the outlet with only boundary measurements of flow rate and velocity. The control objective is to simultaneously stabilize the upstream and downstream traffic to a given spatially-uniform constant steady-state. The distinct actuation locations motivate our design of two different delay-robust full state feedback control laws. The proposed designs are based on the PDE backstepping methodology and guarantee the exponential stability of the under-actuated network of two systems of two hyperbolic PDEs. Collocated boundary observers are proposed to construct output feedback controllers. Numerical simulations are performed to validate the control designs. Stabilization performance of the two output feedback is compared and evaluated with Proportional Integral (PI) boundary feedback controllers. Robustness to delays is also investigated.

1. Introduction

1.1. Macroscopic modeling of cascaded freeway traffic

Freeway traffic modeling and control have been intensively investigated over the past decades. Macroscopic models represent the traffic dynamics at an aggregate level and are widely used for freeway traffic control. Macroscopic modeling describes freeway traffic dynamics using aggregated state values which are easy to sense and actuate, leading to a particular interest in freeway traffic management. The macroscopic models predict the evolution of continuous traffic states in the temporal and spatial domain by employing hyperbolic PDEs to govern the dynamics of traffic density and velocity. The most widely-used macroscopic traffic PDE models include the classical first-order Lighthill–Whitham–Richards (LWR) model (Lighthill & Whitham, 1955; Richards, 1956) and the state-of-art second-order Aw–Rascle–Zhang (ARZ) model (Aw & Rascle, 2000; Zhang, 2002). The LWR model corresponds to the conservation law of the traffic density. It predicts the formation and propagation of traffic shockwaves on the freeway but fails to describe the stop-and-go oscillatory phenomenon (Flyn, Kasimov, Nave, Rosales, & Seibold, 2009), which causes unsafe driving conditions, increased fuel consumption, and delays in travel time. The second-order ARZ traffic model is then developed to describe this stop-and-go traffic by allowing a velocity PDE added to the LWR model. The ARZ traffic model presents as nonlinear second-order hyperbolic PDEs. More recently, the macroscopic road networks based on the ARZ family of models have been developed in Garavello and Piccoli (2006) and Herty and Rascle (2006). Considering the problem of suppressing the stop-and-go congested traffic on cascaded freeway segments, we adopt in this work the second-order macroscopic traffic network model (Herty & Rascle, 2006). The

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model conserves the mass and drivers' properties through the junction connecting two cascaded roads. As a result, a weak solution (in the sense of the conservative variables of the ARZ model) that guarantees the well-posedness of a closed-loop system for our control design. The model of two cascaded freeway segments can then be rewritten as a system of two interconnected hyperbolic PDE systems coupled by their boundaries.

1.2. Traffic boundary control strategies

Traffic control strategies have been developed and successfully implemented for the traffic management infrastructures, namely, ramp metering and varying speed limits (VSL). The flow rate from the on-ramp to mainline freeway is controlled by the ramp metering and velocity at a certain location is actuated by the VSL. Previous contributions of freeway traffic control strategies (Daganzo, 1995; Goatin, Göttlich, & Kolb, 2016; Gomes & Horowitz, 2006; Zhang & Ioannou, 2016) focus on controlling the spatially discretized approximation of LWR model, namely cell transmission model and its derivations, but the discretized systems sometimes exhibit discrepancies from the original continuous traffic PDE model. Li, Canepa, and Claudel (2016) develop an optimal control framework based on Hamilton–Jacobi formulation of the LWR model. Gugat, Herty, Klar, and Leugering (2005) consider adjoint-based optimization formulation for the control problem of a LWR-based traffic network by regulating nodes of the network. Boundary control designs have been developed for PDE modeling of freeway traffic in Bastin and Coron (2016), Karafyllis and Papageorgiou (2019), Zhang, Prieur, and Qiao (2019) and Yu and Krstic (2019). These control laws are restricted to control problem of traffic on one freeway segment which necessitates certain road homogeneity. Using the integration of ramp metering and VSLs, Zhang et al. (2019) first consider PI boundary control of a cascaded freeway traffic, which is modeled by the linearized homogeneous AR model. The static errors of boundary conditions are suppressed since the instabilities do not arise from the transport PDEs. The control design proposed in this paper differs in focusing on the oscillations generated by the in-domain traffic that can only be modeled by the inhomogeneous ARZ model. The PDE backstepping method is first used for this cascaded traffic freeway problem. More importantly, the control problem is underactuated in this work; feedback design is implemented for only one boundary among the four boundary conditions while all boundaries are actuated in Zhang et al. (2019). This boundary control problem of the cascaded freeway traffic based on the ARZ PDE model has not been studied to the authors' best knowledge.

Boundary control of the network of hyperbolic PDEs has been intensively studied over the past years. Despite many theoretical results in the literature, boundary control of PDE networks remains a challenging research topic. This is because these systems are underactuated. For practical consideration, only the PDEs located at some nodes of the network can be actuated. To tackle this problem, multiple approaches have been proposed: PI boundary controllers for fully actuated networks (Bastin & Coron, 2013), flatness based design of feedback control laws for tree-like transmission networks (Schmuck, Woittennek, Gensor, & Rudolph, 2014). While the backstepping approach has been successfully applied to design boundary controls for a large class of hyperbolic PDE system (Anfinsen & Aamo, 2019; Coron, R.Vazquez, Krstic, & Bastin, 2013; Deutscher, 2017a; DI Meglio, Vazquez, & Krstic, 2013), the considered system always have (at least) one boundary which is fully actuated. Recently Auriol, DI Meglio, and Bribiesca-Argomedo (2019) develop backstepping-based state feedback control laws for the underactuated network of hyperbolic PDEs. This class of system is used to model the dynamics of many industrial applications, including water networks in open-channels, communication networks, and gas networks in pipeline. In this paper, we will develop backstepping PDE output feedback controllers for the underactuated traffic network system which has never been studied before.

In the authors’ previous work (Yu & Krstic, 2019), backstepping boundary control laws for ramp metering are designed to suppress the stop-and-go traffic oscillations on the freeway either upstream or downstream of the ramp. As shown in Fig. 1, the traffic flow rate is actuated through on-ramp traffic lights so that either the upstream or the downstream traffic is stabilized. Such control design cannot stabilize the two segments simultaneously, and distinct traffic scenarios appearing on the cascaded segments are not addressed by the model. In the conference versions of our work, we propose a full-state feedback control from the middle boundary (Yu, Auriol, & Krstic, 2020a) and an anti-collocated output feedback controller (Yu, Auriol, & Krstic, 2020b).

This paper’s contributions are twofold: novel PDE backstepping output feedback controllers and observers are developed for under-actuated hyperbolic PDE system which has not been studied before in theory for such class of system; the advanced backstepping control design is first applied for cascaded freeway traffic. This paper introduces output-feedback control designs that simultaneously stabilize the traffic on two cascaded segments modeled as an underactuated fourth-order PDE system. The actuation and measurement are only taken from either the middle junction or the outlet. Four output-feedback controllers are proposed, as shown in Table 1: two collocated and two anti-collocated ones. They are robust to input delays. A conclusive comparison is also provided between the PI and backstepping controllers for the stabilization of the cascaded freeway traffic.

### Notation

For any function of two variables \( f(x, t) \) defined on \( x \in [0, L], t \in [0, \infty) \), the \( L^2 \)-norm is denoted as \( \|f\|^2 := \int_0^L f(x, t)^2 \, dx \).

For \( L^2 \) function, \( f(x, t) \in L^2([0, T]) \) denotes \( \int_0^T f(x, t)^2 \, dt < \infty \), and \( f(x, t) \in L^2([0, L]) \) denotes \( \int_0^L f(x, t)^2 \, dx < \infty \). For any bounded set \( \xi \) of \( \mathbb{R}^2 \), we denote \( \mathcal{F}(\xi) \) the set of bounded real functions on \( \xi \). This set is a Banach space for the sup-norm. We define the following sets: \( \mathcal{F}_1 = \{ (x, \xi) \in [0, L]^2, \xi \geq x \}, \mathcal{F}_2 = \{ (x, \xi) \in [0, L]^2, \xi \leq x \}, \mathcal{F}_3 = \{ (x, \xi) \in [L, \infty)^2, \xi \geq x \}, \mathcal{F}_4 = \{ (x, \xi) \in [L, \infty)^2, \xi \leq x \}, \mathcal{F}_5 = \{ (x, \xi) \in [-L, 0]^2, \xi \geq x \}, \mathcal{F}_6 = \{ (x, \xi) \in [-L, 0]^2, \xi \leq x \} \), and \( \mathcal{F} = \{ (x, \xi) \in [-L, 0] \times [0, L] \times [0, L) \times [0, L] \times [0, L] \} \).

### Table 1

<table>
<thead>
<tr>
<th>Actuator/sensor location</th>
<th>Sensor x = 0</th>
<th>Sensor x = L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actuator at x = 0</td>
<td>Collocated</td>
<td>Anti-collocated</td>
</tr>
<tr>
<td>Actuator at x = L</td>
<td>Anti-collocated</td>
<td>Collocated</td>
</tr>
</tbody>
</table>

### Fig. 1

Traffic flow on upstream and downstream roads of a junction with actuation either from the junction or the outlet.

### 2. Problem statement

We consider a road network that consists of two connected road segments with unidirectional traffic flow and different road...
conditions, as shown in Fig. 1. The two segments are assumed to be the same length for simplicity of notation. The spatial scaling can be easily made for different lengths.

2.1. ARZ PDE traffic network model

The evolution of traffic density and velocity on the downstream road segment and the upstream road segment are modeled by the following ARZ model.

\[
\rho_t + \rho v_x = 0, \\
\rho_i(v_i + p_i) + \rho(v_i(v_i + p_i)) = -\frac{\rho_i(v_i - V(\rho_i))}{\tau_i},
\]

where PDE states \( \rho_i, v_i : [0, L] \times [0, \infty) \to \mathbb{R}^+ \) and \( i \in \{1, 2\} \) represents downstream and upstream road respectively. The labeling of freeway segments is chosen as the reverse direction of traffic flow but same as the propagation direction of the control signal. The traffic pressure \( p_i(\rho_i) \) is defined as an increasing function of the density \( \rho_i \) to represent the overall drivers’ property, reflecting their change of driving behavior to the increase of density. The positive constant \( v_m \) represents the maximum velocity and the positive constant \( \rho_{m,i} \) is the maximum density defined as the number of vehicles per unit length. The equilibrium density–velocity relation \( V(\rho_i) \) is given by \( V(\rho_i) = v_m - p_i(\rho_i) \) for both segments, which assumes the same maximum velocity for the two segments when there are no vehicles on the road \( \rho_i = 0 \). We define the following variable

\[
w_i = v_i + p_i(\rho_i),
\]

which is interpreted as traffic “friction” or drivers’ property (Fan & Seibold, 2013). This property transports in the traffic flow with vehicle velocity, representing the heterogeneity of individual driver with respect to the equilibrium density–velocity relation \( V(\rho_i) \). The maximum velocity \( v_m \) is assumed to be the same for the two road segments while the maximum density \( \rho_{m,i} \) and coefficient \( \gamma_i \) are allowed to vary. The positive constant \( \tau_i \) is the relaxation time that represents the time scale for traffic velocity \( v_i \) adapting to the equilibrium density–velocity relation \( V(\rho_i) \). We denote the traffic flow rate on each road as \( q_i = \rho_i v_i \). The equilibrium flow and density relation, also known as the fundamental diagram, is then given by \( Q(\rho_i) = \rho_i V(\rho_i) = \rho_i v_m (1 - \rho_i/\rho_{m,i})^{\gamma_i/\gamma_i} \).

We consider the situation that the upstream road segment for \( x \in [-L, 0] \) has more lanes than the downstream for \( x \in [0, L], \) in which congested traffic is usually formed up from downstream to upstream road. The equilibrium density and velocity, flow relations are shown Fig. 2. The maximum density \( \rho_{m,2} > \rho_{m,1} \). The maximum driving speed \( v_m \) is assumed to be the same for the two segments. The critical density \( \rho_c \) segregates the free and congested regimes of traffic states. The critical density is given by \( \rho_{c,i} = \rho_{m,i}/(1 + \gamma_i)^{1/\gamma_i} \) such that \( Q(\rho)_{\rho=\rho_{c,i}} = 0 \). The traffic is defined as free when the density satisfies \( \rho < \rho_{c,i} \), and as congested when \( \rho > \rho_{c,i} \) is satisfied. For the free traffic, oscillations around the steady states will be damped out fast. For the congested traffic, there are two directional waves on road with one being the velocity oscillation propagating upstream and the other one being the density oscillation propagating downstream with the traffic.

Regarding the boundary conditions connecting the two PDE systems, the Rankine–Hugoniot condition is satisfied at the junction such that the weak solution exists for the cascaded system (1)–(2). This condition implies piecewise smooth solutions and corresponds to the conservation of the mass and of the drivers’ properties defined in (3) at the junction. Thus the flux and drivers’ property are assumed to be continuous across the boundary conditions at \( x = 0, \) that is

\[
\rho_1(0, t)v_1(0, t) = \rho_2(0, t)v_2(0, t),
\]

\[
w_2(0, t) = w_1(0, t).
\]

For open-loop system, we assume a constant inflow \( q^* \) entering the inlet boundary \( x = -L \) and a constant outflow \( q^* \) at the outlet boundary for \( x = L \):

\[
q_1(-L, t) = q^*,
\]

\[
q_2(L, t) = q^*.
\]

The control problem we solve consists on stabilizing the traffic flow in both the upstream and downstream road segments with a single actuator. We only present the control and estimation results for control input \( u(t) \) either from the middle junction \( x = 0 \) or from the outlet \( x = L \), as shown in Fig. 1. Actuation at the inlet \( x = -L \) is a less challenging control problem that can be solved following Yu and Krstic (2019) by reducing the traffic inflow. Other boundary conditions remain to be the same with the open-loop system in (4)–(7).

Ramp metering control \( U_b(0) \) from the junction \( x = 0 \): The traffic flow entering from the junction to the mainline road is controlled by \( u_b(0) \). Given the flux continuity condition, the boundary condition at the junction is

\[
q_1(0, t) = q_2(0, t) + u_b(0),
\]

where the downstream segment flow consists of the inflow from the mainline upstream segment and the actuated traffic flow from the on-ramp.

Ramp metering control \( U_t(t) \) from the outlet \( x = L \): The downstream outflow at \( x = L \) is actuated by \( u_t(t) \).

\[
q_1(L, t) = q^* + u_t(t),
\]

where the outflow rate is equal to the summation of the on-ramp metering flow and the constant mainline flow. In what follows, when we implement one choice of control input, the other control input is equal to zero. It should be noted that the designed controllers \( u_b \) in (35) and \( u_t \) in (49) are the flow rate perturbations around a nominal flow rate.

2.2. Linearized model in the Riemann coordinates

We are concerned with the congested traffic and assume that steady states of the two segments \( (\rho^*_1, v^*_1), (\rho^*_2, v^*_2) \) are in the congested regime, which is the only one of theoretical control interest among all four traffic scenarios including free and free, free and congested, congested and free, congested and congested. The boundary conditions (4) and (5) are satisfied, i.e.,

\[
\rho^*_1 v^*_1 = \rho^*_2 v^*_2 = q^*.
\]
where the steady state velocities satisfy the equilibrium density–velocity relation \( v^*_i = V(\rho^*_i) \). According to (3), the constant driver’s property in (11) implies that we have the same maximum velocity \( v_p \) for the two segments (which corresponds to our initial assumption): \( v^*_1 + p^*_1 = v^*_2 + p^*_2 = v_p \), where \( p^*_1 = p_1(\rho^*_1) \).

We linearize the ARZ based traffic network model \((\rho_i, v_i)\) in (1), (2) with the boundary conditions (4), (5), (6), (7) around the steady states \((\rho^*_i, v^*_i)\). In order to obtain simplify the model for control design, the linearized model is then rewritten into the Riemann variables and then a invertible spatial transformation is applied

\[
\tilde{w}_1 = \exp \left( \frac{\chi}{\tau_1 \nu_1} \right) \left( \frac{\lambda \rho^*_1}{\nu_1} \left( \rho_1(\nu_1 - \rho^*_1 v^*_1) + \frac{1}{\tau_i} \left( v_i - v^*_i \right) \right) \right),
\]

\[
\tilde{v}_i = v_i - v^*_i,
\]

where the constant coefficients \( r_i \) are defined as \( r_i = -\frac{\nu_i}{\gamma p^*_i - v^*_i} \). The for congested regime we have \( \gamma p^*_i - v^*_i > 0 \). the velocity variations \( \tilde{v}_i(x, t) \) transport upstream which means the action of velocity acceleration or deceleration is repeated from the leading vehicle to the following vehicle.

With such a change of variable, the linearized system with the controlled boundary conditions (8) and (9) writes as

\[
\begin{align*}
\partial_t \tilde{w}_1 + v^*_i \partial_x \tilde{w}_1 &= 0, \\
\partial_t \tilde{v}_i - (\gamma \rho^*_i - v^*_i) \partial_x \tilde{v}_i &= c_i(x) \tilde{w}_1, \\
\tilde{w}_1(0, t) &= \tilde{w}_2(0, t), \\
\tilde{v}_1(L, t) &= \int \frac{L}{\tau_1 \nu_1} \tilde{w}_1(L, t) \\
&+ 1 - \frac{1}{\tau_i} U_1(t), \\
\tilde{w}_2(-L, t) &= \exp \left( \frac{-L}{\tau_2 \nu_2} \right) \tilde{v}_2(-L, t), \\
\tilde{v}_2(0, t) &= \frac{\rho_2}{\tau_2} \tilde{v}_1(0, t) + r_2 (1 - \delta) \tilde{w}_2(0, t) \\
&+ 1 - \frac{1}{\tau_2} U_0(t),
\end{align*}
\]

where the spatially varying coefficients \( c_i(x) \) are defined as \( c_i(x) = -\frac{1}{\tau_i} \exp \left( -\frac{x}{\nu_i \tau_i} \right) \). and the constant coefficient \( \delta = \gamma \rho^*_i - v^*_i > 0 \). The constant \( \delta \) represents the ratio related to the traffic pressure of the segments. Derivation of the linearization and the spatial transformation is straightforward to obtain by following Yu and Krstic (2019) and thus are omitted here. The control diagram is shown in Fig. 3 for the transformed system (14)-(19). The well-posedness (in the weak sense) of the linearized system (14)-(19) is stated in Bastin and Corson (2016, Theorem A.4). The control operator is admissible: in presence of \( L^2 \) control inputs and for any initial conditions \((\tilde{w}_0), (\tilde{v}_0) \in (L^2([0, L])^2 \) and \((\tilde{w}_0), (\tilde{w}_0) \in (L^2([0, L])^2 \), there is only one \( L^2 \)-solution. It is shown in Yu and Krstic (2019) that only marginal stability holds for the open-loop system of one segment.

Our objective is to design the control law \( U_0(t) \) or \( U_1(t) \) to stabilize the system (14)-(19) in the sense of the \( L^2 \)-norm. Note that we could obtain more regular solutions (strong solutions) by imposing some additional regularity conditions on the initial conditions or the coupling terms, and adding compatibility conditions (see Bastin & Corson, 2016 for instance). We make the following non-restrictive assumption so that the proposed feedback laws have some (delay)-robustness margins.

\[
U_0(t) = \frac{\rho^*_i}{1 - \rho_2} \left( \int_{-L}^L K^{ww}(0, \xi) \tilde{w}_1(\xi, t) + K^{vv}(0, \xi) \tilde{v}_1(\xi, t) d\xi \right) - \delta \frac{r_2}{\tau_1} \int_{-L}^L K^{ww}(0, \xi) \tilde{w}_1(\xi, t) + K^{vv}(0, \xi) \tilde{v}_1(\xi, t) d\xi.
\]

More details of the control design can be found in Yu et al. (2020a). This control input is a \( L^2 \) function. For any initial conditions \((\tilde{w}_0), (\tilde{w}_0) \in (L^2([0, L])^2 \) and \((\tilde{w}_0), (\tilde{w}_0) \in (L^2([0, L])^2 \), there is only one \( L^2 \)-solution to the closed-loop system for (14)-(19) with (21). Moreover, since the kernels are bounded functions, our control operator is a linear bounded operator. Consequently, it is a continuous operator. Thus the control law \( U_0 : [0, T] \to \mathbb{R} \) is continuous. The event-triggered control in Espita, Yu, and Krstic (2020) provides a way to implement the continuous-time controllers into digital forms of traffic lights on ramp by updating the input values only when needed. It is strictly proper as it is only composed of integral terms. Following the ideas of Auriol et al. (2019), we can prove that it is robust with respect to delays in the actuation and uncertainties on the parameters. We have the following theorem.
Theorem 1. Consider the PDE system (14–19) with the feedback law $U_0$ defined in (21). Then, for any $L^2$ initial condition $(\bar{u}_i(\cdot, 0), \bar{v}_i(\cdot, 0))$, the closed-loop system is exponentially stable at the origin.

The definition of exponential convergence can be found, e.g., in Bastin and Coron (2016, Definition 3.1).

3.2. Feedback law $U_i(t)$ with flow rate control from $x = l$

We now consider that the available actuation is located at the outlet $x = l$. Our approach is adjusted from Auriol et al. (2019) by improving two successive backstepping transformations in Auriol et al. (2019). We use here only one transformation, as it is known that such a transformation exists.

Proof. We start by assessing the existence of $\bar{K}^{uv}_i$ and $\bar{K}^{vw}_i$ using Coron et al. (2013). The rest of the proof is based on an induction argument and is adjusted from the one given in Auriol et al. (2019, Lemma 2). Let us define $\chi = \frac{\gamma_0 p_i^* - v_i^*}{\gamma_0}$ and let us define the sequence $x_k$ by $x_k = \min(\chi \times k, 1)$.

Let us now define the following triangular domains defined for $k \geq 1$.

$$\mathcal{A}_k = \{(x, \xi) \in [0, 1] \times [-1, 0], \xi \leq -\frac{1}{\chi} (x - x_{k-1})\}$$

$$\mathcal{B}_k = \{(x, \xi) \in [0, 1] \times [-1, 0], \xi \geq -\frac{1}{\chi} (x - x_{k-1})\}$$

We denote by $\bar{K}_i$ the corresponding kernels after this change of variables. Again, we can apply Di Meglio et al. (2018, Theorem 3.2) to prove the existence of the kernels $M^v_i$ and $M^w_i$ on $\mathcal{A}_i$. This implies the existence of $\bar{K}_i$ on $\mathcal{A}_i$. We then iterate the procedure on the intervals $[x_{k-1}, x_k]$ to conclude the proof. □

The kernels here are bounded functions (instead of continuous functions) since we decided to apply the results from Di Meglio et al. (2018, Theorem 3.2). This theorem has been stated in a more general framework where the kernels may present some discontinuities. However, these discontinuities occur along the characteristic lines and do not have any consequence on the backstepping transformation. Adjusting the proof given in Di Meglio et al. (2018), it is possible to show that the kernels are piecewise continuous functions whose discontinuities occur along the characteristic lines. More regularity can be obtained, if necessary, by increasing the regularity of the coupling coefficients $c_i$. The transformation (23)–(25) maps the original system (14)–(19) to the following decoupled target system,

$$\begin{align*}
\partial_t \alpha_i + v_i^* \partial_x \alpha_i &= 0, \\
\partial_t \beta_i - (\gamma_0 p_i^* - v_i^*) \partial_x \beta_i &= 0, \\
\alpha_i(0, t) &= \alpha_{2i}(0, t), \\
\beta_i(L, t) &= r_i \exp \left( -\frac{L}{r_i v_i^*} \right) \alpha_{2i}(L, t), \\
\alpha_{2i}(-L, t) &= \exp \left( -\frac{L}{r_i v_i^*} \right) \frac{1}{r_i} \beta_{2i}(-L, t), \\
\beta_{2i}(0, t) &= \delta_{r_i^2} r_i \beta_i(0, t) + r_i (1 - \delta) \alpha_{2i}(0, t). 
\end{align*}$$

The control input $U_i(t)$ is obtained as

$$U_i(t) = \frac{\rho_i^*}{1 - r_i} \left( \int_{0}^{t} \bar{K}^{uv}_i(L, \xi) w_1(\xi, t) d\xi + \bar{K}^{vw}_i(L, \xi) v_1(\xi, t) d\xi \right) + \int_{-1}^{0} M^v_i(L, \xi) w_2(\xi, t) d\xi + M^w_i(L, \xi) v_2(\xi, t) d\xi.$$  

We have the following theorem.

Theorem 3. Consider the PDE system (14–19) with the feedback law $U_i$ defined in (41). Then, for any $L^2$ initial condition $(\bar{u}_i(\cdot, 0), \bar{v}_i(\cdot, 0))$, the closed-loop system is exponentially stable at the origin.
4. Boundary observer designs

The control laws designed in the previous section require the value of the state all over the spatial domain. Therefore we design boundary observers which either rely on the measurement of traffic states from the junction or from the outlet.

4.1. Observer with measurement $Y_0(t)$ at $x = 0$

Since it holds that $\dot{w}_2(0, t) = \frac{\gamma P^*_w}{\tau} \hat{w}_2(0, t) - \frac{1}{\rho^*_w} \hat{v}_2(0, t)$, we consider that the following measurement is available

$$Y_0(t) = \hat{w}_2(0, t).$$

The observer equation are proposed in Yu et al. (2020b). They are a copy of the original dynamics with output injection gains, which read as follows

$$\begin{align*}
\partial_t \hat{w}_1 + v_1^* \partial_x \hat{w}_1 &= -\phi_i(x)(\hat{w}_2(0, t) - \hat{w}_i(0, t)), \\
\partial_t \hat{v}_1 - (\gamma P^*_w - v_1^*) \partial_x \hat{v}_1 &= c_i(x)\hat{v}_i - v_i(x)(\hat{w}_2(0, t) - \hat{w}_i(0, t)),
\end{align*}$$

where $\hat{w}_i(0, t)$, $\hat{v}_i(0, t)$ are the estimates of the state variables $\tilde{w}_i(0, t)$, $\tilde{v}_i(0, t)$, respectively. The terms $\mu_i$ and $v_i$ are output injection gains which still have to be designed. They are piecewise continuous bounded functions, respectively defined on $[0, L]$ and $[-L, 0]$.

$$\begin{align*}
\dot{\hat{v}}_1(L, t) &= r_1 \exp \left(-\frac{L}{\tau_1 \rho^*_1} \right) \hat{v}_1(L, t) + \frac{1 - r_1}{\rho^*_1} U_1(t), \\
\dot{\hat{w}}_2(-L, t) &= \exp \left(-\frac{L}{\tau_2 \rho^*_2} \right) \hat{w}_2(-L, t) + \frac{1 - r_2}{\rho^*_2} U_0(t),
\end{align*}$$

where $\hat{w}_1(x, \cdot)$, $\hat{v}_1(x, \cdot)$, $\hat{w}_2(x, \cdot)$, $\hat{v}_2(x, \cdot)$ are the estimates of the state variables $\tilde{w}_1(x, \cdot)$, $\tilde{v}_1(x, \cdot)$, $\tilde{w}_2(x, \cdot)$, $\tilde{v}_2(x, \cdot)$, respectively. The terms $\mu_i$ and $v_i$ are output injection gains which still have to be designed. They are piecewise continuous bounded functions, respectively defined on $[0, L]$ and $[-L, 0]$.

Theorem 4. Consider the PDE system (43)–(48) with the output injection gains defined in (49)–(50). Then, for any $L^2$ initial condition $(\hat{w}_1(\cdot, 0), \hat{v}_1(\cdot, 0))$, the states $(\hat{w}_1, \hat{v}_1)$ exponentially converge to the states $(\tilde{w}_1, \tilde{v}_1)$.

Although the use of the trace operator in (43)–(50) imposes a loss of regularity, it is not a problem since the kernels that define the observer gains are regular enough ($H^1$ functions). Strong solutions remain possible by adding compatibility and regularity conditions.

4.2. Observer with measurement $Y_L(t)$ at $x = L$

Assume that the measurement available correspond to the values of $\tilde{w}_1$ and $\tilde{v}_1$ at the left side of the outlet $x = L$. Since we have $\tilde{w}_1(L, t) = \exp \left(-\frac{L}{\tau_1 \rho^*_1} \right) \hat{w}_1(L, t)$, we consider that the boundary measurement corresponds to

$$Y_L(t) = \hat{w}_1(L, t).$$

The observer system is given by

$$\begin{align*}
\partial_t \hat{u}_1 + v^*_1 \partial_x \hat{u}_1 &= -\mu(x)(\hat{w}_1(L, t) - \hat{w}_1(L, t)), \\
\partial_t \hat{v}_1 - (\gamma P^*_w - v^*_1) \partial_x \hat{v}_1 &= c_i(x)\hat{v}_i - v_i(x)(\hat{w}_1(L, t) - \hat{w}_1(L, t)),
\end{align*}$$

where $\hat{u}_1(0, t) = \hat{w}_2(0, t)$.
The well-posedness of this kernel PDE-system is guaranteed by the following lemma.

**Lemma 5.** Consider system (62)–(71). There exists a unique solution \( N_{1}^{kw}, N_{1}^{kw} \) in \( \mathcal{A}(\mathcal{F}_{1}) \), \( N_{2}^{kw}, N_{2}^{kw} \) in \( \mathcal{A}(\mathcal{F}_{2}) \) and \( F_{w}, F_{v} \) in \( \mathcal{A}(\mathcal{F}) \).

**Proof.** The well-posedness of the kernels \( \tilde{N}_{w}^{kw} \) and \( \tilde{N}_{v}^{kw} \) is proved following Vazquez, Krstic, and Coron (2011). Then we prove the well-posedness of the kernels \( F_{w}, F_{v} \) in \( \mathcal{A}(\mathcal{F}) \).

Let us now define the output injection gains \( \mu_{i} \) and \( v_{i} \) as

\[
\mu_{i}(x) = - v_{i} \tilde{N}_{1}^{kw}(x, L) + \int_{0}^{L} \mu_{1}(\xi) \tilde{N}_{1}^{kw}(x, \xi) d\xi, \tag{72}
\]

\[
v_{i}(x) = - v_{i} \tilde{N}_{1}^{kw}(x, L) + \int_{0}^{L} \mu_{1}(\xi) \tilde{N}_{1}^{kw}(x, \xi) d\xi, \tag{73}
\]

\[
\mu_{2}(x) = - v_{i} F_{w}(x, L) + \int_{0}^{L} \mu_{2}(\xi) \tilde{N}_{2}^{kw}(x, \xi) d\xi.
\]

\[
\mu_{2}(x) = - v_{i} F_{v}(x, L) + \int_{0}^{L} \mu_{2}(\xi) \tilde{N}_{2}^{kw}(x, \xi) d\xi.
\]

These output injection gains are perfectly defined: since (72) is a Volterra equation of second kind, it is invertible and we can obtain \( \mu_{1} \). Once \( \mu_{1} \) is obtained, then Eq. (74) becomes a Volterra equation and we can compute \( \mu_{2} \). Once \( \mu_{1} \) and \( \mu_{2} \) are obtained, the expressions of \( v_{1} \) and \( v_{2} \) are explicit. With this choice of injection gains, differentiating the transformations (58)–(59) and (60)–(61) with respect to time and space, it is straightforward to obtain that the convergence of the observer in the following theorem.

**Theorem 6.** Consider the PDE system (53)–(57) with the output injections gains defined in (72)–(73). Then, for any \( L^{2} \) initial condition, the closed-loop system with the controller (76) or (77) is exponentially stable at the origin. This implies the local convergence of the initial states of \( \rho_{i} \) and \( v_{i} \) to the steady states \( \rho_{i}^{*} \) and \( v_{i}^{*} \).

**6. Simulation results**

In this section, we first validate the control design with numerical simulations and compare the two colocated output feedback laws. Then we demonstrate the robustness of the proposed controllers to delays in the actuation path. In the end, our control design is compared with PI boundary controllers, which fully actuate the interconnected system. As stated in Table 1, there are four proposed output feedback controllers, but only the simulation results of the two colocated ones are conducted. The colocated controllers are the most relevant in practice since the anti-colocated sensor and actuator in the distance will have delays and errors caused by long-distance communication.

The length of each freeway segment is chosen to be \( L = 0.5 \) km so the total length of the two connected segments are 1 km. The simulation time is \( T = 12 \) min. The maximum speed limit is \( v_{m} = 40 \) m/s = 144 km/h. We consider 6 lanes for the downstream freeway segment 1. Assuming the average vehicle length is 5 m plus the minimum safety distance of 50% vehicle length, the maximum density of the road is obtained as \( \rho_{m,1} = 6/7.5 \) vehicles/m = 800 vehicles/km. The upstream segment has less functional lanes thus its maximum density is \( \rho_{m,2} = 700 \) vehicles/km. We take \( \gamma_{1} = 0.5 \). The steady states \( (\rho_{1}^{*}, v_{1}^{*}) \) and \( (\rho_{2}^{*}, v_{2}^{*}) \) are chosen respectively as \( 600 \) vehicles/km, \( 19 \) km/h and \( 488.6 \) vehicles/km, \( 23.8 \) km/h, both of which are in the congested regime and satisfy (10) and (11). The constant flow rate is \( q^{*} = \rho_{1}^{*} v_{1}^{*} = \rho_{2}^{*} v_{2}^{*} = 11640 \) vehicles/h, same for the two segments. If we consider the segment 1 with 6 lanes, then the averaged flow rate of each lane is 1940 vehicles/h/lane. The equilibrium steady state of the downstream road has higher density and lower velocity, thus is more congested than the upstream road. The relaxation time is \( t_{1} = 90 \) s and \( t_{2} = 60 \) s. We use sinusoidal initial conditions for flow rate and velocity field which represent the stop-and-go oscillations on the connected freeway and are highlighted in the figures with blue. The two-step Lax–Wendroff numerical scheme (LeVeque, 1992) is applied.

**6.1. Output feedback stabilization**

We consider in this traffic scenario that the downstream traffic in segment 1 is denser with slower velocity, compared with the upstream traffic in segment 2, as illustrated by the steady states. The closed-loop simulation with the colocated output feedback control input from the middle junction shows that the exponential convergence to the steady states is achieved simultaneously for the upstream and downstream segments in Fig. 4, where the actuated junction flow rate by the on-ramp metering is highlighted in red. The output feedback stabilization with the control input and measurement of velocity and flow rate from the outlet boundary is shown in Fig. 5. Comparing the two closed-loop simulations in Fig. 4 and Fig. 5, we find out that the outlet controller takes around the same convergence time but presents a larger transient before stabilizing the system. The controlled flow rate at the middle junction with ramp metering input \( U(t) \), highlighted in red in Fig. 4, first decreases such that less traffic is allowed into the downstream where traffic is denser. The controlled flow rate at the outlet with \( U(t) \), highlighted in red in Fig. 5, increases initially such that more traffic is discharged from the segment.
To further compare the two collocated output feedback stabilization results, the closed-loop performance is demonstrated with the temporal evolution of the state variables in the spatial averaged $L^2$-norm, defined as

$$S_q(t) = \left( \frac{1}{L} \int_X \left( \frac{q(x,t) - q^*}{q^*} \right)^2 \, dx \right)^{1/2},$$

and

$$S_v(t) = \left( \frac{1}{L} \int_X \left( \frac{v(x,t) - v^*}{v^*} \right)^2 \, dx \right)^{1/2},$$

where $X = [-L, 0] \cup [0, L]$ represents the spatial domain of the two segments. As shown in Fig. 6, the closed-loop convergence time of both output controllers are around the same at $t = 9$ min, whereas the output feedback controller at the outlet has a larger transient for all the state variables than the output feedback at the middle junction. At around $t = 2$ min, the blue highlighted line has a bigger overshoot than the red one. The ramp metering control input located at the downstream outlet is carried upstream by the propagation of velocity variations to mitigate traffic oscillations in both segments. In contrast, the ramp metering control input located at the middle junction works so that the actuated velocity variation at the junction travels upstream, and the actuated flow rate variations travel downstream with the traffic. Therefore, it takes a longer time for the control input to take effect on the upstream segment 2 when the output feedback is applied at the downstream outlet. The output feedback at the middle junction instantly starts stabilizing both the upstream segment 2 and downstream segment 1.

The proposed output feedback controllers are robust to external boundary disturbances and delays in actuation path (due to Assumption 1, Auriol et al., 2019). Here we conduct a simulation for the closed-loop system with actuation constant delays $D_0$ and $D_L$ that are respectively 0 s, 30 s, 60 s, 120 s, where 0 s represents no delay and 120 s is the time length for the control input signal to traverse the two segments. Based on the definition in (78)–(79), we define an overall closed-loop performance index

$$S(t) = S_q(t) + S_v(t),$$

where $i = 1, 2$. Then the temporal evolution of $S(t)$ is plotted for the closed-loop system with the delayed collocated output feedback in Fig. 7.

6.2. Comparison with PI controllers

PI control has been applied for traffic control by ramp metering (Papageorgiou, Hadj-Salem, & Blosseville, 1991). For macroscopic second-order PDE model, Zhang et al. (2019) and Zhang and Wang (2019) developed PI boundary feedback controllers for the linearized ARZ model. For control of traffic on two cascaded freeway segments, boundary controllers are employed by Zhang and Wang (2019) including one ramp metering at inlet $x = -L$, one ramp metering and one VSL at middle junction $x = 0$, and one VSL at outlet $x = L$, as illustrated in Fig. 8. The controlled system is fully actuated since there are four boundary conditions and all of them are being actuated, whereas, in our design, only one boundary is actuated by ramp metering, either at the middle junction or at the outlet.
The four PI boundary controllers $R_{-L}, R_0, V_0, V_1$ are defined respectively for the controlled flow rate at inlet $x = -L$, the controlled flow rate at middle junction $x = 0$, the controlled velocity at middle junction $x = 0$ and the controlled velocity at outlet $x = L$. The fully actuated boundaries are defined as

$$q_2(-L, t) = R_{-L}(t), \quad v_2(0, t) = V_0(t),$$

$$q_1(0, t) = R_0(t), \quad v_1(L, t) = V_1(t),$$

where the boundary feedback controllers are given by

$$R_{-L}(t) = q^* + k_p^r \rho_2(0, t) + k_i^r \int_0^t \rho_2(0, \tau) d\tau,$$

$$V_0(t) = v_1^* + k_p^L \bar{v}_2(-L, t) + k_i^L \int_0^t \bar{v}_2(-L, \tau) d\tau,$$

$$R_0(t) = q^* + \tilde{L}_p^r \tilde{v}_1(L, t) + \tilde{L}_i^r \int_0^t \tilde{v}_1(L, \tau) d\tau,$$

$$V_1(t) = v_1^* + \tilde{L}_p^r \tilde{v}_1(0, t) + \tilde{L}_i^r \int_0^t \tilde{v}_1(0, \tau) d\tau,$$

where $k_p^r, k_i^r, k_p^L, k_i^L$ are tuning gains for the upstream segment 2, $L_p^r, L_i^r, \tilde{L}_p^r, \tilde{L}_i^r$ are tuning gains for the downstream segment 1 and $q^*, v_1^*$ are the steady states. We use the previous model parameters and conduct the simulation under the same initial conditions such that the PI controllers can be directly compared with the control design in this paper. The tuning gains are chosen to be $k_p^r = -55$, $k_i^r = -0.035$, $k_p^L = -0.6$, $k_i^L = -0.025$ and $L_p^r = -10$, $L_i^r = -0.035$, $\tilde{L}_p^r = -0.5$, $\tilde{L}_i^r = -0.005$.

The closed-loop system behavior is shown in Fig. 9 where the temporal evolution of the four PI control inputs are highlighted, including two ramp metering in red and two VSLs in green. We then compare the closed-loop performance of the PDE backstepping controller and the PI controllers with the evolution of state variables in the spatial averaged $L^2$-norm, defined with $S(t)$ in (80). In Fig. 10, the closed-loop performance with the ramp metering backstepping controller at middle junction $U_0(t)$ is plotted with the blue line, the one with the ramp metering backstepping controller at outlet $U_L(t)$ is plotted in red dotted line and the one with the four PI controllers is plotted with the yellow dashed line. We can see that the convergence time and the transient is about the same for $U_0(t)$ and four PI controllers. The outlet backstepping controller $U_L(t)$ takes a relatively larger time to stabilize the system.

7. Concluding remarks

We design stabilizing output feedback control laws that guarantee the simultaneous stabilization of the traffic flow on two cascaded roads around given steady states. The flow actuation is realized with the ramp metering at the junction or the downstream outlet. The observers are designed collocated by sensing traffic velocity and flow rate at the two locations. The proposed controllers are robust to actuation delays. A more comprehensive robust control design to model parameters, external boundary, and in-domain disturbances will be of future research interest. Comparing the two collocated output feedback controllers, the middle junction one presents faster convergence and a smaller transient than the outlet one. The trade-offs between the proposed PDE backstepping controller with the PI static output feedback controllers are also discussed.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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