Output-Feedback Control of an Extended Class of Sandwiched Hyperbolic PDE-ODE Systems

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Abstract—Motivated by an engineering application in brake control of cable mining elevators, where the dynamics consist of a brake, a shock absorber, a time-varying-length cable, and a cage, we address a theoretical problem of control of a particular class of coupled hyperbolic PDEs sandwiched between a nonlinear ODE on the actuated side and a linear ODE on the opposite side, with a PDE domain that is time-varying. A state-feedback controller entering a single ODE state is designed to exponentially stabilize the overall system through several backstepping transformations. An observer which only uses the boundary values at the actuated side is constructed to recover all the states of the overall system, based on which a “collocated” type output-feedback control system is proposed. The global exponential stability of the closed-loop system, boundedness, and exponential convergence of the controller, are proved via Lyapunov analysis. The performance is investigated via numerical simulation.

Index Terms—Backstepping, boundary control, distributed parameter system, hyperbolic PDE.

I. INTRODUCTION

A. Motivation

BRAKE performance is one of important safety indexes of a mining cable elevator [29], [30], where the brake system consists of a drum brake, a shock absorber, a time-varying-length cable, and a cage. In the process of stoppages, especially in the emergence stop of a high-speed elevator, the acceleration of the cage changes rapidly, which would cause large vibrations and significant oscillations of stress in the cable. It would not only lead to passengers discomfort or injured by impact [27], but also produce premature fatigue problems [33] which require frequent inspections, costly repairs, and may result in cable fracture in a worst case. Some researchers installed additional brake devices, such as a magnetohydrological fluid damper [27] or a “safety gear” [19], between the cage and the rail guide to improve the brake performance of an elevator. It is a more economic and convenient way to design an appropriate brake control force applied at the drum brake to stop the cage smoothly through a shock absorber and a cable, without modifying the original structure of the mining cable elevator. This task can be mathematically abstracted as boundary control of coupled hyperbolic PDEs sandwiched between a cascade of a linear ODE and a nonlinear ODE on the actuated side and a linear ODE on the opposite side, with a PDE domain that is time-varying, for the dynamics of the aforementioned brake system.

B. Control of Coupled Hyperbolic PDEs

A time-varying-length cable connecting with a cage is a major element in the cable elevator, which can be modeled as a coupled hyperbolic PDE-ODE system on a time-varying domain, obtained from transforming a wave PDE-ODE system with in-domain damping through Riemann transformation [33]. Control of the coupled hyperbolic PDEs has received much attention in the recent years. Some successful methods to stabilize coupled transport PDEs in different directions can be found in [7], [9], [15], [21], and [28], on the basis of which some adaptive control designs for the hyperbolic PDEs with uncertain system parameters were also proposed in [4], [6], [37], and [38]. Moreover, some results about boundary control of coupled transport PDE-ODE systems were presented in [1], [3], [10], [12], [22], [26]. Boundary control designs of a class of infinite dimensional Port–Hamiltonian systems were proposed in [20] and [25]. The aforementioned papers focus on PDE systems on a fixed domain rather than a time-varying domain in accordance with the varying length of the cable. In a recent work [33], output-feedback control of a linear coupled hyperbolic PDE-ODE system on a time-varying domain was developed and applied into balancing control of a dual-cable mining cable elevator. However, all the above research considers control actuation directly flowing into the PDE boundary and ignoring the actuator dynamics. The dynamics of the drum brake and shock absorber have a significant influence on the brake performance of the mining cable elevator, so their dynamics, i.e., the actuator dynamics should be taken into consideration for the brake control design of the mining cable elevator, which produces a more challenging problem about control of a coupled hyperbolic PDE “sandwiched” system.

C. Control of “Sandwiched” PDE systems

Recently, some results about state-feedback and output-feedback control of coupled hyperbolic PDEs sandwiched between two ODEs were presented in [32] and [11], respectively, via the backstepping method. In addition to the coupled hyperbolic PDEs, control of transport PDE [5], [16], [17], viscous Burgers PDE [18] or heat PDE [35] sandwiched systems were also achieved successfully. However, these results only dealt with the problems where the PDE is on a fixed domain and sandwiched by linear ODEs. Control of a fixed-domain...
sandwiched wave PDE including a nonlinear ODE was presented in [34] by using the naive and straightforward backstepping method, which results in some high-order derivative terms in the resulting control law. Motivated by brake control of a mining cable elevator mentioned in Section I-A, a more challenging task dealt with in this article is output-feedback control design of a particular class of coupled hyperbolic PDEs sandwiched between a cascade of a linear ODE and a nonlinear ODE on the actuated side and a linear ODE on the opposite side, with a PDE domain that is time-varying, as displayed in Fig. 1. Moreover, the control input should be guaranteed as boundedness and exponential convergence to zero.

D. Main Contribution

1) Compared with recent results on boundary control of ODE-PDE-ODE systems [5], [11], [23], we not only consider a PDE on a time-varying domain, but also deal with a cascade of ODEs with nonlinearity in the actuation path of the PDE, and the global exponential stability of the closed-loop system, the exponential convergence of the designed control input are achieved.

2) As compared to our previous result about state-feedback control of ODE-coupled hyperbolic PDE-ODE sandwiched system on a fixed domain [32], this article solves a more challenging problem where a nonlinear ODE exists in the input channel of the PDE into which it enters and which is on a time-varying domain. Moreover, a “collocated” type observer-based output-feedback controller without the derivatives of states is proposed.

3) This is the first result of stabilizing such a particular class of coupled hyperbolic PDEs sandwiched between a nonlinear ODE on the actuated side and a linear ODE on the opposite side, where the PDE domain is time-varying and the control action enters a single ODE state. Even if the time-varying domain is reduced to a fixed domain, the theoretical result is new.

E. Organization

The rest of the article is organized as follows. The problem formulation is presented in Section II. State-feedback control design and stability analysis are proposed in Section III. Observer design of the overall system is present and the exponential stability of the observer error system is proved in Section IV. The exponential stability of the output-feedback closed-loop system is given in Section V. The simulation results are provided in Section VI. The conclusion and future work are presented in Section VII.

Notation: Throughout this article, the partial derivatives and total derivatives are denoted as: \( u_x(x, t) = \frac{\partial u}{\partial x}(x, t), u_t(x, t) = \frac{\partial u}{\partial t}(x, t), x \in \mathbb{R}^{n_1}, t \in \mathbb{R} \).

II. PROBLEM FORMULATION

The plant considered in this article is

\[
\begin{align*}
\dot{X}(t) &= AX(t) + Bv(0, t), \\
u_i(x, t) &= -p_1 u_x(x, t) + c_1 v(x, t), \\
v_i(x, t) &= p_2 v_x(x, t) + c_2 u(x, t), \\
\dot{u}(0, t) &= q\dot{v}(0, t) + CX(t), \\
\dot{v}(l(t), t) &= s_1(t) \\
\dot{s}_1(t) &= c_3 s_2(t) + f_1 \left( s_1(t), \int_0^{l(t)} u(x, t)dx \right) \\
\dot{s}_2(t) &= f_2(s_1(t), s_2(t), u(l(t), t)) + z(t) \\
\dot{z}(t) &= c_4 z(t) + ru(l(t), t) + U(t)
\end{align*}
\]

for \((x, t) \in [0, l(t)] \times [0, \infty], \) where \(X(t) \in \mathbb{R}^{n_1}, z(t) \in \mathbb{R} \) are ODE states, which describe the vibration dynamics of the cage and drum. The nonlinear ODE-S\((t) = [s_1(t), s_2(t)]^T \in \mathbb{R}^{2n_1} \) represents the shock absorber dynamics. \(u(x, t) \in \mathbb{R}, v(x, t) \in \mathbb{R} \) are states of the 2 \times 2 coupled hyperbolic PDEs, which model the vibration states of the cable. A \( \in \mathbb{R}^{n_n \times n}, B \in \mathbb{R}^{n_1 \times n_1}, C \in \mathbb{R}^{1 \times n_1} \) satisfy that the pair \([A, B] \) is controllable and \([A, C] \) is observable. \(c_1, c_2, c_3, c_4, r, q \in \mathbb{R} \) are arbitrary. \(p_1 \) and \(p_2 \) are arbitrary positive transport velocities. \(U(t) \) is the control input to be designed. The general nonlinear functions \(f_1 \) and \(f_2 \) can be unknown functions in the state-feedback control.

Functions \(f_1, f_2, \) and \(l(t) \) satisfy the following assumptions.

Assumption 1: \(f_1(0, 0) = 0 \) and \(f_2(0, 0, 0) = 0 \).

Assumption 2: \(f_1(x_1, x_2) \) and \(f_2(x_1, x_2, x_3) \) are continuously differentiable and globally Lipschitz in \((x_1, x_2) \) and \((x_1, x_2, x_3), \) respectively.

Assumption 3: \(l(t) \in C^2(0, \infty), l(t) \) is bounded: \(0 < l(t) \leq L, \forall t \geq 0 \).

Assumption 4: Velocity \(\dot{l}(t) \) of the moving boundary is bounded by

\[
|\dot{l}(t)| < \min\{p_1, p_2\}.
\]
system (without the right boundary condition):

\[
\begin{align*}
\dot{X}(t) &= (A + B\kappa)X(t) + B\beta(0, t) \\
\alpha(t, x) &= -p_1\alpha_x(x, t) + c_1\beta(x, t) \\
&- c_1 \int_0^x D(x, y)\alpha(y, t)dy \\
&- c_1 \int_0^x M(x, y)\beta(y, t)dy - c_1\mathcal{J}(x)X(t)
\end{align*}
\]  

(20)

where the row vector \(C_0 = C + q\gamma(0)\). Let us now consider the right boundary condition. Inserting \(x = l(t)\) into (11) and taking the derivative with respect to \(t\), we have

\[
\begin{align*}
\dot{\beta}(l(t), t) &= \dot{\phi}(l(t), t) - \dot{\gamma}(l(t))\dot{X}(t) \\
&= \dot{\gamma}(l(t))\dot{X}(t) - \int_0^{l(t)} \phi_x(l(t), y)u(y, t)dy \\
&- \int_0^{l(t)} \phi(l(t), y)v(y, t)dy - \gamma(l(t))\dot{X}(t).
\end{align*}
\]  

(24)

Using (5) and (6) to replace \(\phi(l(t), t)\) in (24), and then plugging the inverse transformations (18), (19) into (24) to replace \(u, v\) with \(\alpha, \beta\), through a change of the order of integration in a double integral, we get \(\dot{\beta}(l(t), t)\) as

\[
\begin{align*}
\dot{\beta}(l(t), t) &= c_3s_2(t) + f_2\left(\beta(l(t), t) - \int_0^{l(t)} D(l(t), y)\alpha(y, t)dy\right) \\
&- \int_0^{l(t)} M(l(t), y)\beta(y, t)dy - \mathcal{J}(l(t))X(t) \\
&+ \mathcal{F}(\beta(l(t), t), \beta(0, t), \alpha(l(t), t), \alpha(0, t), \beta(x, t), \alpha(x, t), X(t))
\end{align*}
\]  

(25)

where \(\mathcal{F}\) is a perturbation including \(\beta(l(t), t), \beta(0, t), \alpha(l(t), t), \alpha(0, t), \beta(x, t), \alpha(x, t), X(t)\). The complete expression of \(\mathcal{F}\) is shown in Appendix-A. Recalling (5), (7), (8) and (18), (19), yields

\[
\begin{align*}
\dot{s}_2(t) &= f_2\left(\beta(l(t), t) - \int_0^{l(t)} D(l(t), y)\alpha(y, t)dy\right) \\
&- \int_0^{l(t)} M(l(t), y)\beta(y, t)dy - \mathcal{J}(l(t))X(t) \\
&+ \mathcal{F}(\beta(l(t), t), \beta(0, t), \alpha(l(t), t), \alpha(0, t), \beta(x, t), \alpha(x, t), X(t))
\end{align*}
\]  

(26)

\[
\dot{z}(t) = c_4z(t) + r\alpha(l(t), t) + U(t).
\]  

(27)

Note that (25)–(27) are the right boundary condition of the intermediate system in the form of several ODEs regulated by the control input \(U(t)\).
Remark 1: (25)–(27) \((β(l(l,t), t), s₂(t), z(t))\) is a cascade of ODEs converted from (5)–(8) \((s₁(t), s₂(t), z(t))\) via transformation (10), (11). Equations (25) and (26) are second-order nonlinear ODE \((β(l(l,t), t), s₂(t))\) with perturbations \(F\). Equation (27) is a first-order linear ODE \(z(t)\) with a perturbation \(a(l(l,t), t)\).

Through the backstepping transformations (10), (11), the original system-\((u(x,t), v(x,t), X(t), s₁(t), s₂(t), z(t))\) (1)–(8) is converted to the intermediate system-\((α(x,t), β(x,t), X(t), β(l(l,t), t), s₂(t), z(t))\) (20)–(23), (25)–(27). Next, we propose backstepping design for the ODEs \((β(l(l,t), t), s₂(t), z(t))\) (25)–(27) at the right boundary of the intermediate system.

B. Backstepping Transformation for ODEs (25)–(27)

The following backstepping transformation for the \((β(l(l,t), t), s₂(t))\) system (25), (26) is made:

\[
y₁(t) = β(l(l,t), t) \tag{28}
y₂(t) = s₂(t) + τ₁(t) \tag{29}
\]

where \(τ₁(t)\) to be defined in the following steps is the virtual control in the ODE backstepping method.

Step 1: We consider a Lyapunov function candidate as \(V₁ = \frac{1}{2}y₁(t)^2\). Taking the derivative of \(V₁\), recalling (22), (25), and (29), we obtain

\[
V₁'(t) = y₁(t) y₁(t) + y₁(t) β₁(l(l,t), t) = y₁(t) (c₁ y₂(t) - c₃ τ₁(t) + f₁ + F). \tag{30}
\]

The arguments of \(f₁\) and \(F\) are omitted in (30), which are the same as those in (25).

Define

\[
τ₁(t) = \frac{c₁}{c₃} y₁(t) \tag{31}
\]

where \(c₁\) is a positive constant to be determined later.

Substituting (31) into (29) yields

\[
V₁ = \frac{1}{2} y₁(t)^2 + c₃ y₁(t) y₂(t) + y₁(t) f₁ + y₁(t) F. \tag{32}
\]

Step 2: A Lyapunov function candidate for \(y₁(t), y₂(t)\) is considered as

\[
V₂ = V₁ + \frac{1}{2} y₂(t)^2 = \frac{1}{2} y₁(t)^2 + \frac{1}{2} y₂(t)^2. \tag{33}
\]

Taking the derivative of (33), we have

\[
V₂' = -c₁ y₁(t)^2 + c₃ y₁(t) y₂(t) + y₁(t) f₁ + y₁(t) F + y₂(t) F₂ + z(t) + τ₁ \tag{34}
\]

where (26) and (29) are used and the argument which is omitted of \(F₂\) is same as that in (26).

Step 3: Define a new variable \(E(t)\) as

\[
E(t) = z(t) + c₂ y₂(t) + c₃ y₁(t) \tag{35}
\]

where the positive constant \(c₂\) is to be determined later.

Inserting (35) into (34) to replace \(z(t)\), we have

\[
V₂ = -c₁ y₁(t)^2 - c₂ y₂(t)^2 + y₁(t) f₁ + y₁(t) F + y₂(t) E(t) + y₂(t) F₂ + \frac{c₁}{c₃} y₂(t) y₁(t). \tag{36}
\]

Using (35), then (27) can be written as

\[
\dot{E} = c₄ E(t) + rα(l(l,t), t) + c₂ y₂(t) + c₃ y₁(t) - c₄ c₂ y₂(t) - c₄ c₃ y₁(t) + U(t). \tag{37}
\]

Choosing \(U(t)\) in (37) as

\[
U(t) = -\bar{a}_0 E(t) - \rho α(l(l,t), t) + c₄ c₂ y₂(t) + c₄ c₃ y₁(t) \tag{38}
\]

we then have

\[
\dot{E} = -kₐ E(t) + c₂ y₂(t) + c₃ y₁(t) \tag{39}
\]

where \(kₐ = \bar{a}_0 - c₄ > 0\) by choosing the control gain \(\bar{a}_0\).

Through the transformations (10), (11), (28), (29), and (35), the original system-\((u(x,t), v(x,t), X(t), s₁(t), s₂(t), z(t))\) is converted to the target system-\((α(x,t), β(x,t), X(t), y₁(t), y₂(t), E(t))\) where the ODE states and PDE states are coupled. The exponential stability of the target system will be clear in the following Lyapunov analysis via choosing control parameters \(c₁, c₂, \bar{a}_0\).

C. Stability Analysis of State-Feedback Closed-Loop System

1) Controller: Substituting (10), (11), (28), (29), (31), (35) into (38), we get the controller expressed by the original states

\[
U(t) = -\bar{a}_0 z(t) + (c₄ - \bar{a}_0) c₂ s₂(t) - r u(t(l,t), t) + (c₄ - \bar{a}_0) \left(\frac{c₁ c₂}{\bar{c}} + c₃\right) (s₁(t) - \int_{t₀}^{t(l)} ψ(l(l,t), y) u(y,t) dy - \int_{t₀}^{t(l)} φ(l(l,t), y) v(y,t) dy - γ(l(l,t)) X(t) \right). \tag{40}
\]

The pending control parameters \(c₁, c₂, \bar{a}_0\) will be determined in the following stability analysis. Note that the control law (40) uses the signal \(u(l(l,t), t)\). In order to ensure the controller law is sufficiently regular, we will require the initial value \(u(x,0)\) to be in \(H^1(0,L)\) which is defined as \(H^1(0,L) = \{u|u ∈ L^2(0,L), u_x ∈ L^2(0,L)\}\), where \(L^2(0,L)\) is the usual Hilbert space and the positive constant \(L\) given in Assumption 3 is the maximum value of the time-varying PDE domain.

2) Stability of Closed-Loop System: Theorem 1: If initial values \((u(x,0), v(x,0)) ∈ H^1(0,L)\), for some \(c₁, c₂, \bar{a}_0\), the closed-loop system consisting of the plant (1)–(8) and the control law (40) is exponentially stable in the sense of that there exist constant positive \(γ₁, γ₂\) such that

\[
\Omega_1(t) ≤ τ₁ \Omega_0(t) e^{-γ₁ t} \tag{41}
\]

where

\[
\Omega_0(t) = ||u(., t)||^2 + ||v(., t)||^2 + |X(., t)|^2 + s₁(., t)^2 \tag{42}
\]

\(||u(., t)||^2\) is a compact notation for \(\int_{t₀}^{t(l)} u^2(x,t) dx\).

Proof: We start from studying the stability of the target system. The equivalent stability property between the target system and the original system is ensured due to the invertibility of the transformations (10), (11), (28), (29), and (35).

First, we study the stability proof of the target system via Lyapunov analysis of the PDE-ODE subsystem. Second, combining the Lyapunov analysis of ODEs in the input channel in Section III-B, Lyapunov analysis of the overall system is provided, where the control parameters \(c₁, c₂, \bar{a}_0\) in the control law (40) are determined.

a) Lyapunov analysis for the PDE-ODE subsystem-\((α(x,t), β(x,t), X(t))\): Consider now a Lyapunov function

\[
V₁(t) = X^T(t) P₁ X(t) + \frac{α₁}{2} \int_{t₀}^{t(l)} e^{λ₁ x} β(x,t)^2 dx + \frac{β₁}{2} \int_{t₀}^{t(l)} e^{-β₁ x} α(x,t)^2 dx \tag{43}
\]
where $P_i = P_i^T > 0$ is the solution to the Lyapunov equation 

$$P_1 (A + B_\kappa) + (A + B_\kappa)^T P_1 = -Q_1,$$ 

for some $Q_1 = Q_1^T > 0$.

The positive parameters $\alpha_1, b_1, \delta_1$ are chosen later.

Taking the derivative of $V_1(t)$, we arrive at

$$\dot{V}_1(t) \leq -\eta_1 |X(t)|^2 - \eta_2 \beta(0, t)^2 - \eta_3 \int_0^{t(t)} \beta(x, t)^2 \, dx$$

$$- \eta_4 \int_0^{t(t)} \alpha(x, t)^2 \, dx - \eta_5 \alpha(l(t), t)^2 + \eta_6 \beta(l(t), t)^2$$

(44)

where the detailed process of calculating $\dot{V}_1(t)$ is shown in Appendix-B, where the choices of $\alpha_1, b_1, \delta_1$ and the expressions of positive constants $\eta_1, \eta_2, \eta_3, \eta_4$ are also given. Defining $v_{\text{max}} = \max_{t \in [0, \infty)} \{|l(t)|\}$, we know $\eta_5 = (p_1 - v_{\text{max}}) \frac{\kappa}{a} e^{-\delta_1 L} > 0$ by recalling Assumption 4, and $\eta_6 = (p_2 + v_{\text{max}}) \frac{\kappa}{a} e^{-\delta_1 L} > 0$.

b) Lyapunov analysis for the overall system: Consider a Lyapunov function as

$$V(t) = V_1(t) + V_\theta(t) + \frac{1}{2} \mathcal{E}(t)^2.$$ 

(45)

Defining

$$\Omega_1(t) = \|\beta_1(t)\|^2 + \|\alpha(t)\|^2 + |X(t)|^2$$

$$+ y_1(t)^2 + y_2(t)^2 + \mathcal{E}(t)^2$$

(46)

we have

$$\theta_{1a} \Omega_1(t) \leq V(t) \leq \theta_{1b} \Omega_1(t)$$

(47)

for some positive constants $\theta_{1a}$ and $\theta_{1b}$.

Taking the derivative of (45), using (36), (39), and (44) with (A.1)-(A.8), recalling Assumptions 1, 2, 4, we have

$$\dot{V}(t) \leq -\lambda V(t) - \tilde{\eta}_0 \beta(0, t)^2 - \tilde{\eta}_1 \alpha(l(t), t)^2$$

(48)

for some positive $\lambda$, and $\tilde{\eta}_0, \tilde{\eta}_1$ are positive constants given as (C.12)-(C.13). The detailed process of calculating $\dot{V}(t)$ is shown in Appendix-C, where the choices of the control parameters $c_1, c_2, a_0$ in the ODE backstepping to tolerate the PDE perturbations are presented.

We thus have

$$V(t) \leq V(0) e^{-\lambda t}.$$ 

(49)

It then follows that $\Omega_1(t) \leq \frac{\theta_{1b}}{\theta_{1a}} \Omega_1(0) e^{-\lambda t}$ by recalling (47).

Defining

$$\Xi(t) = \|u(\cdot, t)\|^2 + \|v(\cdot, t)\|^2 + |X(t)|^2$$

$$+ s_1(t)^2 + s_2(t)^2 + z(t)^2$$

(50)

applying Cauchy–Schwarz inequality and transformations (10), (11), (18), (19), (28), (29), and (35), it is straightforward to obtain

$$\theta_{1a} \Xi(t) \leq \Omega_1(t) \leq \theta_{1b} \Xi(t)$$

(51)

for some positive $\theta_{1a}$ and $\theta_{1b}$. Therefore, we have

$$\Xi(t) \leq \frac{\theta_{1b}}{\theta_{1a}} \Xi(0) e^{-\lambda t}.$$ 

(52)

Thus, (41) is achieved with

$$\Gamma_1 = \frac{\theta_{1b}}{\theta_{1a}} \theta_{1a}, \quad \lambda_1 = \lambda.$$ 

(53)

Then the proof of Theorem 1 is completed.

3) Exponential Convergence of Control Input: In Theorem 1, we have proved that all PDEs and ODEs are exponentially stable in the closed-loop system including the plant (1)-(8) and the controller (40). Moreover, next we would prove the controller $U(t)$ (40) in the closed-loop system is also bounded and exponentially convergent to zero.

Considering (40) and the exponential stability result proved in Theorem 1, the exponential convergence of the control input requires the exponential convergence of the signal $u(l(t), t)$ additionally, which can be obtained by proving the exponential stability estimate of $\|u_x(\cdot, t)\|^2 + \|v_x(\cdot, t)\|^2$.

Before proving the exponential convergence of the control input, we propose a lemma first.

Lemma 1: For any initial data $(u(x, 0), v(x, 0)) \in H^1(0, L)$, the exponential stability estimate of the closed-loop system $(u(x, t), v(x, t))$ is obtained in the sense of that there exist positive constants $\gamma_{1a}$ and $\lambda_{1a}$ such that

$$\|u_x(\cdot, t)\|^2 + \|v_x(\cdot, t)\|^2$$

$$\leq \gamma_{1a} (\Xi(t) + \|u_x(\cdot, 0)\|^2 + \|v_x(\cdot, 0)\|^2) e^{-\lambda_{1a} t} \quad (54)$$

where $\Xi(t)$ is given in (50).

The proof of Lemma 1 is shown in Appendix-D. Lemma 1 will be used in proving the exponential convergence and boundedness of the controller (40) in the following theorem.

Theorem 2: In the closed-loop system including the plant (1)-(8) and the controller $U(t)$ (40), there exist positive constants $\gamma_2$ and $\gamma_2$ making that $U(t)$ is bounded and exponentially convergent to zero in the sense of

$$|U(t)| \leq \gamma_2 (\|u_x(\cdot, 0)\|^2 + \|v_x(\cdot, 0)\|^2 + |X(0)|^2) + s_2(0)^2$$

$$+ s_2(0)^2 + z(0)^2 + \|u_x(\cdot, 0)\|^2 + \|v_x(\cdot, 0)\|^2) e^{-\lambda_{2a} t}.$$ \quad (55)

Proof: The proof is shown in Appendix-E.

IV. OBSERVER DESIGN AND STABILITY ANALYSIS

In Section III, a state-feedback controller which requires distributed states is designed to stabilize the original system exponentially. However, it is always difficult to measure the distributed states in practice. We propose an output-feedback control law which only requires measurements $u(l(t), t), v(l(t), t), z(t)$ at the controlled boundary of the PDE, i.e., a "collocated" type, based on a state observer designed in this section. In Section IV-A, the observer design is presented, where the observer gains are determined in two transformation processes from the observer error system to an intermediate observer error system, and then to a target observer error system. The exponential stability of the observer error system is proved in Section IV-B.

A. Observer Design

1) Structure of Observer and Error Dynamics: Using the measurements $u(l(t), t), v(l(t), t), z(t)$, the observer is designed as

$$\dot{\hat{X}}(t) = A \hat{X}(t) + B \hat{u}(0, t)$$

$$+ \Gamma_0(t) (u(l(t), t) - \hat{u}(l(t), t))$$

(56)

$$\hat{u}_t(x, t) = - p_1 \hat{u}_x(x, t) + c_1 \hat{v}(x, t)$$

$$+ \Gamma_1(t) (u(l(t), t) - \hat{u}(l(t), t))$$

(57)

$$\hat{v}_t(x, t) = \hat{p}_2 \hat{u}_x(x, t) + c_2 \hat{u}(x, t)$$

(58)
where $\Gamma_0(t), \Gamma_1(x,t), \Gamma_2(x,t), \mu_1, \mu_2, \mu_3$ are observer gains to be determined later. Note that the initial values $\bar{u}(x,0), \tilde{v}(x,0)$ are required to be in $H_1(0,L)$ to be consistent with Section III. Define observer errors as

$$
\begin{align*}
[\begin{bmatrix} \hat{X}(t), \bar{u}(x,t), \bar{v}(x,t) \end{bmatrix}], \bar{s}_1(t), \tilde{s}_2(t), \tilde{z}(t) &= [X(t), u(x,t), v(x,t), s_1(t), s_2(t), z(t)] \\
- [\hat{X}(t), \bar{u}(x,t), \bar{v}(x,t), \bar{s}_1(t), \tilde{s}_2(t), \tilde{z}(t)].
\end{align*}
$$

According to (56)--(63) and (1)--(8), the error dynamics can be obtained as

$$
\begin{align*}
\dot{\hat{X}}(t) &= A\hat{X}(t) + B\bar{v}(0,t) - \Gamma_0(t)\bar{u}(l(t),t) \\
\dot{\bar{u}}(x,t) &= -p_1\bar{u}(x,t) + c_1\bar{v}(x,t) - \Gamma_1(x,t)\bar{u}(l(t),t) \\
\dot{\bar{v}}(x,t) &= -p_2\bar{v}(x,t) + c_2\bar{u}(x,t) - \Gamma_2(x,t)\bar{u}(l(t),t) \\
\dot{\bar{v}}(0,t) &= q\bar{v}(0,t) + C\hat{X}(t) \\
\dot{\tilde{z}}(t) &= -k_z\tilde{z}(t).
\end{align*}
$$

where $k_z = \mu_3 - c_4 > 0$ by choosing the control parameter $\mu_3$, and

$$
\begin{align*}
\tilde{f}_1 &= f_1 \left( s_1(t), \int_0^{l(t)} u(y,t)dy \right) \\
&\quad - f_1 \left( \tilde{s}_1(t), \int_0^{l(t)} \bar{u}(y,t)dy \right) \\
\tilde{f}_2 &= f_2 \left( s_1(t), s_2(t), u(l(t),t) \right) \\
&\quad - f_2 \left( \tilde{s}_1(t), \tilde{s}_2(t), \bar{u}(l(t),t) \right).
\end{align*}
$$

Defining

$$
\tilde{S}(t) = [\tilde{s}_1(t), \tilde{s}_2(t)]^T
$$

(70), (71) can be rewritten as

$$
\dot{\tilde{S}}(t) = (A_s - BC_2)\tilde{S}(t) + \left[ \tilde{f}_1, \tilde{f}_2 \right]^T
$$

where

$$
A_s = \begin{bmatrix} 0 & c_3 \\ 0 & 0 \end{bmatrix}, \quad C_2 = [1, 0], \quad B = [\mu_1, \mu_2]^T.
$$

Note that $A_s - BC_2$ can be a Hurwitz matrix by choosing $B = [\mu_1, \mu_2]^T$, because $(A_s, C_2)$ is observable.

2) Transformation to Intermediate Observer Error System: In order to remove domain couplings in $\bar{v}$ (67) which affect the system stability [33], we apply the invertible backstepping transformation [33] for the PDE states $(\bar{u}, \tilde{v})$

$$
\begin{align*}
\hat{u}(x,t) &= \bar{u}(x,t) - \int_x^{l(t)} \tilde{\phi}(x,y)\tilde{\alpha}(y,t)dy \quad (78) \\
\tilde{v}(x,t) &= \bar{v}(x,t) - \int_x^{l(t)} \tilde{\psi}(x,y)\tilde{\alpha}(y,t)dy \quad (79)
\end{align*}
$$

to convert the error dynamics (65)--(72) to the intermediate observer error system as

$$
\begin{align*}
\dot{\tilde{X}}(t) &= A_s\tilde{X}(t) + B\tilde{\beta}(0,t) - B \int_0^{l(t)} \tilde{\psi}(0,y)\tilde{\alpha}(y,t)dy \\
&\quad - \Gamma_0(t)\tilde{\alpha}(l(t),t) \quad (80) \\
\tilde{\alpha}_t(x,t) &= -p_1\tilde{\alpha}_x(x,t) + \int_x^{l(t)} \tilde{M}(x,y)\tilde{\beta}(y,t)dy \\
&\quad + c_1\tilde{\beta}(x,t) \quad (81) \\
\tilde{\beta}_t(x,t) &= p_2\tilde{\beta}_x(x,t) + \int_x^{l(t)} \tilde{N}(x,y)\tilde{\beta}(y,t)dy \\
&\quad + \int_0^{l(t)} (\tilde{\phi}(0,y) - q\tilde{v}(0,y))\tilde{\alpha}(y,t)dy \quad (82) \\
\tilde{\tilde{\alpha}}(0,t) &= q\tilde{\beta}(0,t) + C\tilde{X}(t) \\
\tilde{\tilde{\beta}}(l(t),t) &= 0. \quad (83)
\end{align*}
$$

By matching (65)--(69) and (80)--(84), the kernel functions $\tilde{\phi}, \tilde{\psi}$ on $D_1 = \{0 \leq x \leq y \leq l(t)\}$ should satisfy

$$
\begin{align*}
-p_1\tilde{\phi}_x(x,y) - p_1\tilde{\phi}_y(x,y) - c_1\tilde{\psi}(x,y) &= 0 \quad (87) \\
\tilde{\psi}(x,x) &= \frac{c_2}{p_1 + p_2} \quad (88) \\
-p_1\tilde{\psi}_y(x,y) + p_2\tilde{\psi}_x(x,y) - c_2\tilde{\phi}(x,y) &= 0. \quad (89)
\end{align*}
$$

The boundary condition of $\tilde{\phi}$ is set as

$$
\tilde{\phi}(0,y) = q\tilde{v}(0,y) - CK_0(y)
$$

where $K_0(x)$ is shown later. The choice of (90) would be clear later.

$$
\begin{align*}
\tilde{M}(x,y) &= \int_x^y \tilde{\phi}(x,z)\tilde{M}(z,y)dz - c_1\tilde{\phi}(x,y) \quad (91) \\
\tilde{N}(x,y) &= \int_x^y \tilde{\psi}(x,z)\tilde{M}(z,y)dz + c_1\tilde{\psi}(x,y). \quad (92)
\end{align*}
$$

Observer gains $\Gamma_1(x,t)$ and $\Gamma_2(x,t)$ are obtained as

$$
\begin{align*}
\Gamma_1(x,t) &= \hat{l}(t)\tilde{\phi}(x,l(t)) - p_1\tilde{\phi}(x,l(t)) \quad (93) \\
\Gamma_2(x,t) &= \hat{l}(t)\tilde{\psi}(x,l(t)) - p_1\tilde{\psi}(x,l(t)). \quad (94)
\end{align*}
$$
3) Transformation to Target Observer Error System: In order to decouple the ODE (80) with the PDE state $\hat{x}$ ($\hat{\beta}$ reaches to zero after a finite time because of (84), (82)) and make the state matrix in the ODE (80) Hurwitz, where the observer gain $\Gamma_0(t)$ would be defined, we apply a transformation as

$$
\dot{Y}(t) = \dot{X}(t) - \int_0^{l(t)} K_0(x) \dot{\alpha}(x,t) dx
- \int_0^{l(t)} K_1(x) \dot{\beta}(x,t) dx
$$

(95)
to convert (80) into

$$
\dot{Y}(t) = (A - L_0 C) \dot{Y}(t) - \int_0^{l(t)} \left[ \int_0^x K_0(y) \dot{M}(y,x) dy + \int_0^x K_1(y) \dot{N}(y,x) dy \right] \dot{\beta}(x,t) dx
$$

(96)
where $A - L_0 C$ is a Hurwitz matrix by choosing $L_0$ recalling that $(A, C)$ is observable, and $K_0(x), K_1(x)$ are determined following.

Substituting (95) into (96), considering (80)–(84), using integration by parts and a change of the order of integration in a double integral, we have

$$
\left[ K_0(l(t)) \right] p_1 - \dot{l}(t) K_0(l(t)) - \Gamma_0(t) \right] \dot{\alpha}(l(t), t)
- \int_0^{l(t)} \left[ K_0'(x) p_1 - A K_0(x) + B \varphi(0, x) \right] \dot{\alpha}(x,t) dx
+ \left( L_0 - K_0(0) p_1 \right) \dot{\alpha}(0, t) + \int_0^{l(t)} \left[ -A K_0(x) c_1 + K_1'(x) p_2 \right] dx
$$

(97)
For (97) to hold, $K_0(x), K_1(x)$ should satisfy

$$
K_0'(x) p_1 - A K_0(x) + B \varphi(0, x) = 0
$$

(98)

$$
K_0(0) = \frac{L_0}{p_1}
$$

(99)

$$
K_1'(x) p_2 + (A - L_0 C) K_1(x) - K_0(x) c_1 = 0
$$

(100)

$$
K_1(0) = \frac{L_0 q + B \varphi(0, x) c_1}{p_2}
$$

(101)

Lemma 2: Equations (87)–(90), (98)–(101) of conditions of kernels $\hat{\phi}(x,y), \hat{\psi}(x,y), K_0(x), K_1(x)$ are well-posed.

Proof: After swapping positions of arguments as B.9-B.10 in [3], i.e., changing the domain $D_1$ to $D_2$ in conditions (87)–(90), (98), (99), $\varphi, \psi, K_0$ have the same form of the conditions (12)–(17) on the kernels $\varphi, \psi, \gamma$ which have been proved as well-posed in [32] and [36]. The explicit solutions of $K_1(x)$ are then easy to obtain considering the initial value problem (100), (101).

The observer gain $\Gamma_0(t)$ is obtained as

$$
\Gamma_0(t) = -\dot{l}(t) K_0(l(t)) + K_0(l(t)) p_1
$$

(102)
The target observer error system thus can be written as

$$
\dot{Y}(t) = (A - L_0 C) \dot{Y}(t) - \int_0^{l(t)} \left[ \int_0^x K(y) \dot{M}(y,x) dy + \int_0^x K_1(y) \dot{N}(y,x) dy \right] \dot{\beta}(x,t) dx
$$

(103)

$$
\dot{\beta}(x,t) = -p_1 \dot{\alpha}(x,t) + \int_0^x M(x,y) \dot{\beta}(y,t) dy + c_1 \dot{\beta}(x,t)
$$

(104)

$$
\dot{\beta}(x,t) = p_2 \dot{\alpha}(x,t) + \int_0^x N(x,y) \dot{\beta}(y,t) dy
$$

(105)

$$
\dot{\alpha}(0,t) = q \dot{\beta}(0,t) + C \dot{Y}(t) + \int_0^{l(t)} C K_1(y) \dot{\beta}(y,t) dy
$$

(106)

$$
\ddot{\beta}(l(t), t) = 0.
$$

(107)

$$
\ddot{S}(t) = (A_s - B C_2) \ddot{S}(t) + \begin{bmatrix} \ddot{f}_1, \ddot{f}_2 \end{bmatrix}^T
$$

(108)

$$
\ddot{z}(t) = -k_z \ddot{z}(t).
$$

(109)

The following theorem shows the exponential stability of the observer error system (65)–(72), which is obtained through the stability analysis of the target observer error system (103)–(109) and applying the invertibility of the transformations. Note that the initial data $(\bar{u}(x,0), \bar{v}(x,0))$ of the observer error system belongs to $H^1(0, L)$, which is defined by the initial conditions of the plant and the observer via (64).

B. Stability Analysis of Observer Error System

Theorem 3: Considering the observer system (56)–(63) with observer gains $\Gamma_0(t)$ (102), $\Gamma_1(x, t)$ (93), $\Gamma_2(x, t)$ (94), the observer error system (65)–(72) is exponentially stable in the sense of that there exist positive constants $\Upsilon_c, \lambda_c$ such that

$$
\dot{\Omega}_c(t) \leq \Upsilon_c \Omega_c(0) e^{-\lambda_c t}
$$

(110)

where

$$
\dot{\Omega}_c(t) = \| \ddot{\bar{u}}(\cdot, t) \|^2 + \| \ddot{\bar{v}}(\cdot, t) \|^2 + \| \ddot{X}(t) \|^2 + \dot{s}_1(t)^2
$$

(111)

$$
+ \dot{s}_2(t)^2 + \ddot{z}(t)^2.
$$

(112)

Proof: a) Analysis for the observer error subsystems $(\bar{u}(x, t), \bar{v}(x, t), \bar{X}(t), \ddot{z}(t))$: (108) $(\ddot{z}(t)$ is an exponentially stable ODE because of $k_z > 0$. From (105), (107), the $\ddot{\beta}$-dynamics is independent of $\dot{\alpha}$ and $\dot{\beta}(z, t) \equiv 0$ after $t_{f0} = \frac{L_0}{p_2^2}$, i.e., when the boundary condition (107) has propagated through the whole domain. The subsystem (103)–(107) becomes

$$
\ddot{Y}(t) = (A - L_0 C) \ddot{Y}(t)
$$

(113)

$$
\dot{\alpha}(x, t) = -p_1 \dot{\alpha}(x, t)
$$

(114)

for $t \geq t_{f0}$. $\ddot{Y}(t)$ is exponentially convergent to zero because $A - L_0 C$ in the ODE (112) is Hurwitz. Define

$$
V_0(a) = \ddot{Y}(t)^T P_a \ddot{Y}(t) + \frac{b_2}{2} \int_0^{l(t)} e^{-\sigma} \tilde{\alpha}(x, t)^2 dx
$$

(115)

where $b_2$ is a positive constant, and $P_a = P_a^T > 0$ is the solution to the Lyapunov equation $P_a (A - L_0 C) + (A - L_0 C)^T P_a = -Q_a$ for some $Q_a = Q_a^T > 0$.

Taking the derivative of $V_0(a)$ along (112)–(114), we have

$$
\dot{V}_0(a) \leq -\lambda_{\min}(Q_a) \ddot{Y}(t)^2 - p_1 b_2 \int_0^{l(t)} e^{-\sigma} \tilde{\alpha}(x, t) \tilde{\alpha}(x, t) dx
$$
Choosing $b_a < \frac{2 \min(Q_a)}{P_1 C_r}$ and recalling Assumption 4, yields
\begin{equation}
\dot{V}_a(t) \leq -\lambda_1 V_a(t) - \lambda_m \alpha(l(t), t)^2
\end{equation}
(117)
for some positive $\lambda_1, \lambda_m$. The exponential stability result in the sense of $|\dot{Y}(t)|^2 + \|\alpha(\cdot, t)\|^2 + \|\dot{\beta}(\cdot, t)\|^2$ is obtained.

**Remark 2:** Even though $\beta_a(x, t) \equiv 0$ and (112)–(114) holds for $t \geq t_f \Omega$ (if $\beta(0, 0) = 0$, they hold at $t = 0$), and the obtained exponential stability straightforwardly begins from $t = 0$, the obtained exponential stability also holds at the beginning $t = 0$, because any transient in the finite time $[0, t_f \Omega)$ can be bounded by an exponentially decay signal with arbitrary decay rate and an appropriate overshoot coefficient.

According to the invertible transformation (78)–(79), (95), we obtain the exponential stability in the sense of $|\dot{X}(t)|^2 + \|\dot{u}(\cdot, t)\|^2 + \|\dot{v}(\cdot, t)\|^2$.

**b) Analysis for the observer subsystem $\bar{S}(t)$:** Next we conduct the stability analysis for the ODE-$\bar{S}(t)$ (108). Consider a Lyapunov function
\begin{equation}
V_s(t) = \bar{S}(t)^T P_0 \bar{S}(t)
\end{equation}
(118)
where $P_0$ is a positive definite and symmetric solution of
\begin{equation}
(A_s - BC_2)^T P_0 + P_0 (A_s - BC_2) + \gamma_2^2 P_0^T P_0 + IT < 0
\end{equation}
(119)
with $\gamma_1^2 = \gamma_1^2 + 2\lambda_2^2$ and $\gamma_1^2$ being positive Lipschitz constants shown in Appendix-F. The existence of the solution $P_0$ of (119) and the procedure to define the observer gain $B$ are shown in [24, Sec. 4].

Taking the derivative of $V_s(t)$ (118), through the calculation process presented in Appendix-F where Assumption 2 is recalled, we achieve
\begin{equation}
\dot{V}_s(t) \leq -\sigma_s V_s(t) + \frac{\gamma_2^2}{\gamma_1^2 + 2\lambda_2^2} \|\alpha(\cdot, t)\|^2
\end{equation}
for some positive $\sigma_s$.

Consider a Lyapunov function
\begin{equation}
V_c(t) = V_s(t) + R_{\alpha} V_a(t).
\end{equation}
(120)

V. STABILITY OF OUTPUT-FEEDBACK CLOSED-LOOP SYSTEM

Replacing all the original states in the state-feedback controller (40) by the observer states, the output-feedback controller can be written as
\begin{equation}
\begin{aligned}
U_{of}(t) &= -\bar{u}_o \ddot{z}(t) + (c_4 - \bar{u}_o) \ddot{\xi}_2 \ddot{s}_2(t) - r \ddot{u}(l(t), t) + (c_4 - \bar{u}_o) \left( \ddot{\xi}_1 \ddot{\xi}_2 \frac{c_3}{c_3} (s_1(t) - \int_0^{l(t)} \psi(l(t), y) \ddot{u}(y, t) dy + \int_0^{l(t)} \phi(l(t), y) \ddot{v}(y, t) dy - \gamma(l(t)) \ddot{X}(t) \right).
\end{aligned}
\end{equation}
(123)

The output-feedback closed-loop system consists of the plant (1)–(8), the observer (56)–(63), and the controller (123).
1) There exist positive constants $\Upsilon_4$ and $\lambda_4$ such that

$$\Omega(t) \leq \Upsilon_4 \Omega(0) e^{-\lambda_4 t}$$

where

$$\Omega(t) = \|\dot{\theta}(\cdot, t)\|^2 + \|\ddot{\theta}(\cdot, t)\|^2 + \|\dddot{X}(\cdot, t)\|^2 + \dot{\delta}_1(t)^2 + \dot{\delta}_2(t)^2$$

$$+ \dot{z}(t)^2 + \|\dot{u}(\cdot, t)\|^2 + \|u(\cdot, t)\|^2 + |X(t)|^2$$

$$+ \dot{s}_1(t)^2 + \dot{s}_2(t)^2 + z(t)^2.$$  \hfill (125)

2) The output-feedback controller (123) is bounded and exponentially convergent to zero.

Proof: 1) Rewrite (56)–(63) as

$$\dot{X}(t) = A\dot{X}(t) + B\dot{v}(0, t) + \Gamma_0(t)\dot{u}(l(t), t)$$

$$u_t(x, t) = -p_1u_x(x, t) + \zeta(x, t)$$

$$v_t(x, t) = p_2v_x(x, t) + e_2(x, t)$$

$$\dot{u}(0, t) = q\dot{v}(0, t) + CX(t)$$

$$\dot{l}(t) = \dot{s}_1(t) + s_1(t)$$

$$\dot{\delta}_1(t) = c_3\dot{\delta}_2(t) + f_1\left(\dot{s}_1(t), \int_0^{l(t)} \dot{u}(y, t) dy\right)$$

$$+ \mu_1\dot{s}_1(t)$$

$$\dot{\delta}_2(t) = f_2(\dot{s}_1(t), \dot{s}_2(t), \ddot{u}(l(t), t)) + \dot{z}(t) + \ddot{z}(t)$$

$$+ \mu_2\dot{s}_1(t)$$

$$\dot{z}(t) = c_4\dot{z}(t) + r\dot{u}(l(t), t) + r\ddot{u}(l(t), t)$$

$$+ \mu_3\dot{z}(t) + U_{of}(t)$$

which has the same structure with the original system (1)–(8) plus the injections $\dot{u}(l(t), t), \dot{s}_1(t), \dot{z}(t)$. Applying transformations (10), (11), (18), (19), (28), (29), and (35) (note that all the state variables should be able to be handled “as” $\dot{u}$), through same steps in Section III, we can arrive at the target system $-\dot{\alpha}, \dot{\beta}, \dot{X}, \ddot{y}_1, \dot{y}_2, \dot{E}, \ddot{u}(l(t), t), \ddot{z}(t), \dot{s}_1(t), \dot{s}_1(t)$, the main body of the exponential stability target system in state-feedback design, plus several observer error injections $\dot{u}(l(t), t), \ddot{z}(t)$, $\dot{s}_1(t), \dot{s}_1(t)$. Recalling Theorem 3 and Lemma 3, we have $\dot{u}(l(t), t), \dot{s}_1(t), \dot{s}_1(t)$ are exponentially convergent to zero. According to (70), (F.2), we have

$$\dot{s}_1(t)^2 \leq 3c_3^2\dot{s}_2(t)^2 + 3f_1^2 + 3\mu_1^2\dot{s}_1(t)^2$$

$$\leq 3c_3^2\dot{s}_2(t)^2 + 3\gamma_1^2\|\dot{\alpha}(\cdot, t)\|^2 + 3(\mu_1^2 + \gamma_1^2)\dot{s}_1(t)^2.$$  \hfill (134)

Thus, $\dot{s}_1(t)$ is also exponentially convergent to zero recalling Theorem 3.

Define a Lyapunov function as

$$V_{of} = \dot{X}(t)^T P_\delta \dot{X}(t) + \frac{\alpha_1}{2} \int_0^{l(t)} e^{\delta_1 x} \dot{\beta}(x, t)^2 dx + \frac{1}{2} \dot{E}(t)^2$$

$$+ \frac{\beta_1}{2} \int_0^{l(t)} e^{-\delta_1 x} \dot{\alpha}(x, t)^2 dx + \frac{1}{2} \dot{y}_1(t)^2 + \frac{1}{2} \dot{y}_2(t)^2$$

$$+ R_{Vc} V_{ec}(t) + R_z\dot{z}(t)^2$$

where $\alpha_1, \beta_1, \delta_1, R_{Vc}, R_z$ are positive constants, and $P_2 = P_2^T > 0$ being the solution to the Lyapunov equation $P_2(A + B\lambda) + (A + B\lambda)^T P_2 = -Q_2$ for some $Q_2 = Q_2^T > 0$.

Through the same steps in Theorem 1, using (134), (122), (109), we obtain $V_{of} \leq -\lambda_{of} V_{of}(t)$ for some positive $\lambda_{of}$. We then obtain $\dot{\Omega}_4(t) \leq \dot{\Upsilon}_4 \dot{\Omega}_4(0) e^{-\lambda_{of} t}$, where $\dot{\Omega}_4(t) = \|\ddot{\alpha}(\cdot, t)\|^2 + \|\ddot{\beta}(\cdot, t)\|^2 + \|\ddot{X}(\cdot, t)\|^2 + y_1(t)^2 + y_2(t)^2 + 2\|\dot{E}(t)\|^2 + 2\|\dot{E}(t)\|^2 + 2\|\ddot{E}(t)\|^2$, for some positive $\dot{\Upsilon}_4$. Applying all transformations and their inverses, through same steps with (50)–(52), we have

$$\dot{\Omega}(t) \leq \dot{\Upsilon}_{4b} \dot{\Omega}(0) e^{-\lambda_{of} t}$$

where $\dot{\Upsilon}_{4b}$ is a positive constant, and

$$\dot{\Omega}(t) = \|\ddot{\alpha}(\cdot, t)\|^2 + \|\ddot{\beta}(\cdot, t)\|^2 + \|\ddot{X}(\cdot, t)\|^2$$

$$+ \dot{s}_1(t)^2 + \dot{s}_2(t)^2 + z(t)^2 + \|\dot{u}(\cdot, t)\|^2 + \|\ddot{u}(\cdot, t)\|^2$$

$$+ \|\dot{X}(\cdot, t)\|^2 + \ddot{s}_1(t)^2 + \ddot{s}_2(t)^2 + \ddot{z}(t)^2.$$

Then recalling (64) and applying Cauchy–Schwarz inequality, we thus obtain (124).

2) In order to prove the boundedness and exponential convergence of the output-feedback controller (123), considering the above exponential stability results in 1), the exponential convergence to zero of $\dot{u}(l(t), t)$ is required additionally. It can be obtained by the exponential stability estimate in the sense of $\|\dddot{u}_2(\cdot, t)\| + \|\dddot{v}_2(\cdot, t)\|$ which can be proved through same steps as Lemma 1 with recalling Lemma 3 and Theorem 3. Then through the same steps in Theorem 2, we have the output-feedback controller (123) is bounded and exponentially convergent to zero as well.

The proof of Theorem 4 is completed.

VI. SIMULATION

Consider the system given by

$$\dot{X}(t) = 0.4X(t) + v(0, t)$$

$$u_t(x, t) = -u_x(x, t) + 0.5v(x, t)$$

$$v_t(x, t) = v_t(x, t) + 0.5u(x, t)$$

$$u(0, t) = v(0, t) + X(t), \quad v(l(t), t) = s_1(t)$$

$$\dot{s}_1(t) = s_2(t) + s_1(t)^2 + \int_0^{l(t)} u(x, t) dx$$

$$\dot{s}_2(t) = s_1(t) s_2(t) + u(l(t), t) + z(t)$$

$$\dot{z}(t) = 0.5z(t) + u(l(t), t) + U(t)$$

$x \in [0, l(t)], l(t)$ is a preknown function decreasing from $l(0) = 1$ to 0.2 during 10 s, as shown in Fig. 3. The initial values are given as $u(x, 0) = 3 \sin(4\pi x), v(x, 0) = 3 \sin(4\pi x), X(0) =$
Open-loop responses of $\hat{u}(\cdot, t)$ and $\hat{v}(\cdot, t)$ at the opposite boundary converge to zero under the proposed output-feedback controller. The simulation results are shown following.

The proposed design in this article can be applied into brake systems, to verify the effectiveness of the proposed controller and observer. A simulation example is conducted to recover the overall system only using available boundary values at the actuated side, to build a “collocated” type observer-based output-feedback controller. The exponential stability results of the closed-loop system and the observer error system, and the boundedness and exponential convergence of the control input are proved in this article. A simulation example is conducted to verify the effectiveness of the proposed controller and observer. The proposed design in this article can be applied into brake control of a mining cable elevator.

\[ u(0, 0) - \delta(0, 0), \quad s_1(0) = \delta(l(0), 0), \quad s_2(0) = z(0) = 0. \]

The initial values of the observer are given as $\hat{u}(x, 0) = u(x, 0) + 0.2 \sin(2\pi l(0) - x)$, $\hat{v}(x, 0) = v(x, 0) + 0.2 \sin(2\pi l(0) - x)$, $\hat{X}(0) = \hat{u}(0, 0) - \hat{v}(0, 0)$, $\hat{s}_1(0) = \hat{v}(l(0), 0)$, $\hat{s}_2(0) = s_2(0) + 0.5$, $\hat{Z}(0) = Z(0) + 0.5$, where the additional terms are initial observer errors.

The simulation is performed by the finite-difference method for the discretization in time and space after converting the time-varying domain PDE to a fixed domain PDE via introducing $\xi = \frac{x}{L}$, and then the time step and space step are chosen as 0.001 and 0.02 respectively. Kernels (12)–(17), (87)–(90), (98), (99) used in the control input are also solved by the finite difference method. The control parameters are chosen as $c_1 = 80$, $c_2 = 150$, $\alpha = 350$, $\kappa = -10$, $L_0 = 10$, $\mu_1 = \mu_2 = \mu_3 = 5$. The simulation results are shown following.

Comparing Fig. 4 which shows the open-loop responses of $\|u(\cdot, t)\|$, $\|v(\cdot, t)\|$ and Fig. 5 which gives the closed-loop responses of $\|u(\cdot, t)\|$, $\|v(\cdot, t)\|$, as one can observe, in the latter case convergence to zero is achieved, whereas the states grow unbounded in the former case. According to Fig. 6, we see that the responses of the ODE-$z(t)$, the nonlinear ODE-$s_1(t)$, $s_2(t)$ and the ODE-$X(t)$ at the opposite boundary converge to zero under the proposed output-feedback controller. Moreover, in Figs. 7 and 8, it can be observed that the proposed observer converge to the actual plant for both PDE and ODE states. Note that because $\delta(l(t), t)$ and $z(t)$ are measurable, $\hat{s}_1(t)$ and $\hat{z}(t)$ are at a small magnitude and fast convergent to zero, the curves of which are omitted here due to the space limit. In Fig. 9, it is shown that the observer-based output-feedback control input is bounded and convergent to zero.

### VII. Conclusion

In this article, we address the output-feedback control problem for a particular class of coupled hyperbolic PDEs sandwiched between a cascade of an ODE and a nonlinear ODE on the actuated side and a linear ODE on the opposite side, with a PDE domain that is time-varying. First, a state-feedback control design is proposed to exponentially stabilize the overall system via a series of transformations, and then a state-observer is constructed to recover the overall system only using available boundary values at the actuated side, to build a “collocated” type observer-based output-feedback controller. The exponential stability results of the closed-loop system and the observer error system, and the boundedness and exponential convergence of the control input are proved in this article. A simulation example is conducted to verify the effectiveness of the proposed controller and observer. The proposed design in this article can be applied into brake control of a mining cable elevator.
In the future work, it is of interest to extend the control design to a more complicated and practical case with some system parameters being unknown, where an adaptive design should be developed.

**APPENDIX**

**A. Expression of $\mathcal{F}$**

$$\mathcal{F}(\beta(l(t), \beta(0,t), \alpha(l(t), t), \alpha(0,t), \beta(x,t), \alpha(x,t), X(t)))$$

$$= h_1(l(t))\beta(l(t), t) + h_2(l(t))\beta(0,t)$$

$$+ h_3(l(t))\alpha(l(t), t) + h_4(l(t))\alpha(0,t)$$

$$+ \int_{0}^{l(t)} h_5(l(t), y)\beta(y,t)dy$$

$$+ \int_{0}^{l(t)} h_6(l(t), y)\alpha(y,t)dy + H_7(l(t))X(t) \quad \text{(A.1)}$$

where

$$h_1(l(t)) = -p_2\phi(l(t), l(t)) - \dot{i}(t)\phi(l(t), l(t)) \quad \text{(A.2)}$$

$$h_2(l(t)) = p_2\phi(l(t), 0) - \gamma(l(t))B \quad \text{(A.3)}$$

$$h_3(l(t)) = p_1\psi(l(t), l(t)) - \dot{i}(t)\psi(l(t), l(t)) \quad \text{(A.4)}$$

$$h_4(l(t)) = -p_1\psi(l(t), 0) \quad \text{(A.5)}$$

$$h_5(l(t), y) = \left( p_2\phi(l(t), l(t)) + \dot{i}(t)\phi(l(t), l(t)) \right) \mathcal{M}(l(t), y)$$

$$+ p_2\phi_\gamma(l(t), y) - c_1\psi(l(t), y) - \dot{i}(t)\phi_x(l(t), y)$$

$$- \int_{y}^{l(t)} \left( p_2\phi_\gamma(l(t), z) - c_1\psi(l(t), z) \right) \mathcal{M}(z,y)dz \quad \text{(A.6)}$$

$$h_6(l(t), y) = p_1\psi_y(l(t), y) + c_2\phi(l(t), y) + \dot{i}(t)\psi_x(l(t), y)$$

$$- \left( p_2\phi(l(t), l(t)) + \dot{i}(t)\phi(l(t), l(t)) \right) \mathcal{D}(l(t), y)\right)$$

$$- \int_{y}^{l(t)} \left( p_2\phi_\gamma(l(t), z) - c_1\psi(l(t), z) \right) \mathcal{D}(z,y)dz \quad \text{(A.7)}$$

$$H_7(l(t)) = \left( p_2\phi(l(t), l(t)) + \dot{i}(t)\phi(l(t), l(t)) \right) \mathcal{J}(l(t))$$

$$- \gamma(l(t))A - \dot{i}(t)\gamma'(l(t)) - (p_2\phi(l(t), 0) - \gamma(l(t))B) \mathcal{J}(0)$$

$$- \int_{0}^{l(t)} \left( p_2\phi_\gamma(l(t), y) - c_1\psi(l(t), y) \right)$$

$$- \dot{i}(t)\phi_x(l(t), y) \mathcal{J}(y)dy. \quad \text{(A.8)}$$

**B. Calculation of $\dot{V}_1(t)$**

Taking the time derivative of (43) along (20)–(23), we obtain

$$\dot{V}_1(t) \leq -\lambda_{\min}(Q_1)[X(t)]^2 + 2X^TP_1B\dot{\beta}(0,t)$$

$$+ \frac{p_2^2}{2}a_1\delta_1\beta(l(t), t)^2 - \frac{p_2^2}{2}a_1\beta(0,t)^2$$

$$+ \frac{a_1l(t)}{2}e^{\delta_1l(t)\beta}(l(t), t)^2 + \frac{b_1l(t)}{2}e^{-\delta_1l(t)\alpha}(l(t), t)^2$$

$$- \frac{p_2^2}{2}\delta_1a_1\int_{0}^{l(t)} e^{\delta_1x\beta}(x,t)^2dx$$

$$- \frac{p_2^2}{2}\delta_1b_1\int_{0}^{l(t)} e^{-\delta_1x\alpha}(x,t)^2dx$$

$$- \frac{p_2^2}{2}b_1e^{-\delta_1l(t)\alpha}(l(t), t)^2 + \frac{p_1}{2}b_1\alpha(0,t)^2$$

$$+ b_1c_1\int_{0}^{l(t)} e^{-\delta_1x\alpha}(x,t) \left( \beta(x,t) - \int_{0}^{x} \mathcal{D}(x,y)\alpha(y,t)dy \right)$$

$$- \int_{0}^{x} \mathcal{M}(x,y)\beta(y,t)dy - \mathcal{J}(x)X(t) \text{ dx.} \quad \text{(B.1)}$$

Recalling (23), using Young’s inequality and Cauchy–Schwarz inequality for the last part in (B.1) yields the existence of $\xi > 0$ such that

$$\dot{V}_1(t) \leq - \left( \frac{1}{2}\lambda_{\min}(Q_1) - p_1b_1 |C_0|^2 \right) |X(t)|^2$$

$$- \left( \frac{p_1}{2} + \frac{\xi}{p_2} \right) \beta(0,t)^2$$

$$- \left( \frac{p_1}{2} - \frac{b_1\beta}{\lambda_{\min}(Q_1)} - b_1\xi \right) e^{-\delta_1L\int_{0}^{l(t)} \alpha(x,t)^2\text{ dx}}$$

$$- (p_1 - \dot{i}(t))\frac{b_1}{2}e^{-\delta_1l(t)\alpha}(l(t), t)^2$$

$$+ (p_2 + \dot{i}(t))\frac{a_1}{2}e^{\delta_1l(t)\beta}(l(t), t)^2 \quad \text{(B.2)}$$

where

$$\xi = \max \left\{ \frac{1}{4} [D(1 + L) + \mathcal{M}], L\bar{\beta}^2c_1^2, \frac{1}{2} c_1 \bar{D}L \right\} \quad \text{(B.3)}$$

and

$$\mathcal{D} = \max_{0 \leq y \leq x \leq L} \{|\mathcal{D}(x,y)|\} \quad \text{(B.4)}$$

$$\mathcal{M} = \max_{0 \leq y \leq x \leq L} \{|\mathcal{M}(x,y)|\} \quad \text{(B.5)}$$

$$\mathcal{J} = \max_{0 \leq x \leq L} \{|\mathcal{J}(x)|\} \quad \text{(B.6)}$$

Choose parameters $b_1, \delta_1, a_1$ to satisfy

$$0 < b_1 < \frac{\lambda_{\min}(Q_1)}{2p_1|C_0|^2} \quad \text{(B.7)}$$

$$\delta_1 > \max \left\{ \frac{1}{2} p_2, \frac{2\xi + \frac{\xi b_1}{\lambda_{\min}(Q_1)}}{p_2} \right\} \quad \text{(B.8)}$$

$$a_1 > \max \left\{ \frac{8 |P_1B|}{p_2\lambda_{\min}(Q_1)} + 2q_2^2, \frac{p_1}{2p_2\delta_1} + \frac{2b_1\xi}{p_2\delta_1^2} \right\} \quad \text{(B.9)}$$

we arrive at (44), where

$$\eta_1 = \frac{1}{2} \lambda_{\min}(Q_1) - p_1b_1 |C_0|^2 > 0 \quad \text{(B.10)}$$

$$\eta_2 = \frac{p_2}{2} a_1 - p_1b_1q^2 - \frac{4 |P_1B|}{\lambda_{\min}(Q_1)} > 0 \quad \text{(B.11)}$$

$$\eta_3 = \frac{p_2}{2} \delta_1a_1 - b_1\xi - b_1\frac{\xi}{\delta_1} > 0 \quad \text{(B.12)}$$
\[
\eta_4 = \left( \frac{p_2 b_1}{2} - b_1 \xi - \frac{\xi b_1^2}{\lambda_{\min}(Q_1)} - b_1 \xi \right) e^{-\delta t - L} > 0.
\]  
(B.13)

C. Calculation of \( \dot{V}(t) \)

Taking the time derivative of (45) and recalling (39) (36), and (44) with (A.1)–(A.8), we have

\[
\dot{V} \leq -\eta_1 X(t)^2 - \eta_2 \beta(0,t)^2 - \eta_3 \int_0^{l(t)} \beta(x,t)^2 \, dx \\
- \eta_4 \int_0^{l(t)} \alpha(x,t)^2 \, dx - \eta_5 \alpha(l(t),t)^2 + \eta_6 \beta(l(t),t)^2 \\
- \bar{c}_1 y_1(t)^2 - \bar{c}_2 y_2(t)^2 + y_1(t) f_1 \\
y_3(l(t)) \alpha(l(t),t) + h_4(l(t)) \alpha(0,t) \\
+ \int_0^{l(t)} h_5(l(t),y) \beta(y,t) \, dy + \int_0^{l(t)} h_6(l(t),y) \alpha(y,t) \, dy \\
+ H_f(l(t)) X(t) + y_2(t) E(t) + y_2(t) f_2 + \frac{\bar{c}_1}{c} y_2(t) y_1(t) \\
- k_2 E(t)^2 + E(t) \bar{c}_2 y_2(t) + E(t) c_3 y_1(t).
\]

(C.1)

Recalling Assumptions 1 and 2 and (28), (29), (31), we have

\[
f_1^2 \leq \gamma_f f_1 (4y_1(t)^2 + 4||\beta(\cdot,t)||^2) \\
+ 5||\alpha(\cdot, t)||^2 + 4|X(t)|^2
\]

(C.2)

\[
f_2^2 \leq \gamma_f f_2 \left( \left( 4 + \frac{2\bar{c}_2^2}{c^4} \right) y_1(t)^2 + 4||\beta(\cdot,t)||^2 + 4||\alpha(\cdot,t)||^2 \right)
\]

+ 4|X(t)|^2 + 2y_2(t)^2 + \alpha(l(t),t)^2

(C.3)

where \( \gamma_f, \gamma_f f_2 \) are positive constants depending on kernels \( D, M, J \). The omitted arguments of \( f_1, f_2 \) are same as those in (25) and (26).

Applying Young’s inequality, Cauchy–Schwarz inequality into the ninth and tenth terms, and \( y_2(t) E(t) + y_2(t) f_2 + \frac{\bar{c}_1}{c} y_2(t) y_1(t) + E(t) \bar{c}_2 y_2(t) + E(t) c_3 y_1(t) \) in (C.1), where (26), (29), (31), (35) are used to rewrite \( y_2(t) \) as \( y_2(t) = f_2 + E(t) - \bar{c}_2 y_2(t) - c_3 y_1(t) + \frac{\bar{c}_1}{c} y_1(t) \), using (C.2)–(C.3) to replace the resulting \( f_1^2, f_2^2 \), recalling (23), (28) to rewrite the resulting \( \alpha(0,t)^2, \beta(l(t),t)^2 \), respectively, we have

\[
\dot{V} \leq -\left( \eta_1 - r_7 |H_f|^2 - 2r_8 h_2^2 |C_0|^2 - 4r_8 \gamma_1 f_1 - 4r_9 \gamma_f f_2 \\
- 4r_{11} \gamma_f^2 |X(t)|^2 - (\eta_2 - \bar{h}_2^2 r_2 - 2r_4 h_2^2 q^2) \beta(0,t)^2 \\
- \left( \eta_3 - r_5 h_2^2_{max} L - 4r_8 \gamma_f f_1 - 4r_9 \gamma_f f_2 \\
- 4r_{11} \gamma_f^2 \right) \int_0^{l(t)} \beta(x,t)^2 \, dx - \left( \eta_4 - r_6 h_2^2_{max} L \\
- 5r_8 \gamma_1 f_1 - 4r_9 \gamma_f f_2 - 4r_{11} \gamma_f^2 \right) \int_0^{l(t)} \alpha(x,t)^2 \, dx \\
- (\eta_5 - \bar{h}_2^2 r_3 - 4r_9 \gamma_f f_2 - r_{11} \gamma_f^2) \alpha(l(t),t)^2 \\
- \left( \bar{c}_1 - 1 - \frac{1}{4r_2} - \frac{1}{4r_3} - \frac{1}{4r_4} - \frac{1}{4r_5} - \frac{1}{4r_6} - \frac{1}{4r_7} - \eta_6 \\
- \frac{1}{4r_8} - 4r_8 \gamma_1 f_1 - \bar{h}_1 - \left( 4 + \frac{2\bar{c}_2^2}{c^4} \right) r_9 \gamma_f f_2 \\
- \left( 4 + \frac{2\bar{c}_2^2}{c^4} \right) r_{11} \gamma_f f_2 y_{11}(t)^2 \\
- \left( \frac{\bar{c}_1}{c_3} - \frac{c_1}{4} - \frac{c_2^2}{4r_11} - \frac{c_2^2 c_1^2}{4r_12 c_3^2} - \frac{c_3^2}{4r_13} \right) E(t)^2 \\
+ (r_{10} + r_{12} + r_{13}) \bar{y}_1(t)^2 \right)
\]

(C.4)

where \( r_1, \ldots, r_{13} \) are positive constants from using Young’s inequality,

\[
h_{5_{max}} = \max_{x \in [0, L], t \in [0, T]} \{|h_5(x,t,l(t))|\}
\]

(C.5)

\[
h_{6_{max}} = \max_{x \in [0, L], t \in [0, T]} \{|h_6(x,t,l(t))|\}
\]

(C.6)

and \( \bar{h}_1, \bar{h}_2, \bar{h}_4, \bar{H}_7 \) are maximum values of \( |h_1(l(t))|, |h_2(l(t))|, |h_3(l(t))|, |h_4(l(t))|, |H_f(l(t))| \) for \( l(t) \in [0, L] \) in (A.2)–(A.8).

According to (25), (28), (29), and (31) with (A.1)–(A.8), we have

\[
y_{11}(t)^2 \leq \xi_c \left( \bar{c}_1 y_1(t)^2 + y_1(t)^2 + y_2(t)^2 + f_1^2 + \beta(0,t)^2 \\
+ \alpha(l(t),t)^2 + \alpha(0,t)^2 + \beta(l(t),t)^2 \\
+ ||\alpha(\cdot, t)||^2 + X(t)^2 \right)
\]

(C.7)

for some positive constants \( \xi_c \) depending on kernels \( D, M, J \) and gains (A.2)–(A.8).

Inserting (C.7) into (C.4) to replace \( y_{11}(t)^2 \) with using (C.2), we arrive at

\[
\dot{V} \leq -\left( \eta_1 - r_7 |H_f|^2 - 2r_8 h_2^2 |C_0|^2 - 4r_8 \gamma_1 f_1 - 4r_9 \gamma_f f_2 \\
- 4r_{11} \gamma_f^2 - (r_{10} + r_{12} + r_{13}) \xi_c \right. \\
- 2(r_{10} + r_{12} + r_{13}) \xi_c |C_0|^2 \\
- 4r_{11} \gamma_f^2 - (r_{10} + r_{12} + r_{13}) \xi_c \\
- (1 + 2\bar{c}_2^2) (r_{10} + r_{12} + r_{13}) \xi_c \beta(0,t)^2 \left( \eta_3 - r_5 h_2^2_{max} L \\
- 4r_8 \gamma_f f_1 - 4r_9 \gamma_f f_2 - 4r_{11} \gamma_f^2 \right) \\
- 4r_{11} \gamma_f^2 \left( r_{10} + r_{12} + r_{13} \right) \xi_c \int_0^{l(t)} \beta(x,t)^2 \, dx - \left( \eta_4 - 5r_8 \gamma_1 f_1 \\
- r_6 h_2^2_{max} L - 4r_9 \gamma_f f_2 - 4r_{11} \gamma_f^2 \right) \left( r_{10} + r_{12} + r_{13} \right) \xi_c
\]
\[-5\gamma_1(x_0 + r_0 + r_1 + r_\xi) \int_0^{l(t)} \alpha(x, t)^2 \, dx - (\eta_5 - \bar{h}_3 r_3)\]

\[-r_9 \gamma_2 - r_{11} \gamma_2 - (r_0 + r_1 + r_\xi) \xi_\xi \alpha(l(t), t)^2\]

\[-\left(\bar{c}_1 - 1 - \frac{1}{4 r_2} - \frac{1}{4 r_3} - \frac{1}{4 r_4} - \frac{1}{4 r_5} - \frac{1}{4 r_6} - \frac{1}{4 r_7}\right)\]

\[-\eta_6 \frac{1}{4 r_6} - 4 r_8 \gamma_1 - \bar{h}_1 \left(4 + \frac{2c_1}{c_3}\right) (r_9 + r_{11} \gamma_2)\]

\[-\left(1 + 4 \gamma_1 \gamma_2 + \bar{c}_1 (r_0 + r_1 + r_\xi) \xi_\xi \right) y_1(t)^2\]

\[-\left(\bar{c}_2 - \frac{1}{4 r_9} - \frac{c_1^2}{4 c_3^2 r_10} \frac{3}{2} - 2 r_{11} \gamma_2 - 2 r_9 \gamma_2\right)\]

\[-\left(r_0 + r_1 + r_\xi \right) y_2(t)^2 - \left(\bar{a}_0 - c_4 - \bar{c}_2 - \frac{1}{2}\right)\]

\[-\frac{c_4^2}{4} - \frac{c_2^2}{4 r_11} - \frac{c_4^2 c_2^2}{4 r_11 c_3^2} - \frac{c_2^2}{4 r_11} - \frac{c_4 c_2}{4 r_11} \epsilon(t)^2\quad (C.8)\]

where \(k_7 = \bar{a}_0 - c_4 \) is recalled. Choosing small enough positive \(r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_\xi, r_0, r_2, r_1, r_\xi, r_\xi \) and making the control parameters \(c_1, c_2, a_0 \) to satisfy

\[c_1 > 1 + \frac{1}{4 r_2} + \frac{1}{4 r_3} + \frac{1}{4 r_4} + \frac{1}{4 r_5} + \frac{1}{4 r_7} + \eta_6\]

\[+ \frac{1}{4 r_8} + 4 r_8 \gamma_1 + \bar{h}_1\quad (C.9)\]

\[c_2 > \frac{1}{4 r_9} + \frac{c_1^2}{4 c_3^2 r_10} \frac{3}{2} + 2 r_{11} \gamma_2\]

\[+ 2 r_9 \gamma_2 + (r_0 + r_1 + r_\xi) \xi_\xi,\quad (C.10)\]

\[\bar{a}_0 > c_4 + \bar{c}_2 + \frac{1}{2} + \frac{c_4^2}{4} + \frac{c_2^2}{4 r_11} + \frac{c_4 c_2^2}{4 r_11 c_3^2}\]

\[+ \frac{c_2^2 c_4^2}{4} + \frac{c_3^2}{4} r_11\quad (C.11)\]

we obtain (48) with

\[\tilde{\eta}_0 = \eta_2 - \bar{h}_3 r_2 - 2 r_4 \bar{h}_4 q^2\]

\[+ (1 + 2 q^2) (r_0 + r_1 + r_\xi) \xi_\xi > 0\quad (C.12)\]

\[\tilde{\eta}_1 = \bar{h}_3 r_3 - r_9 \gamma_2 - r_{11} \gamma_2 > 0\]

\[+ (r_0 + r_1 + r_\xi) \xi_\xi > 0\quad (C.13)\]

\[D.\text{ Proof of Lemma 1}\]

Differentiating (21) and (22) with respect to \(x\), differentiating (23) with respect to \(t\), we have

\[\alpha_{xt}(x, t) = -p_1 \alpha_{xt}(x, t) + c_1 \beta_x(x, t) - c_1 J'(x) X(t)\]

\[-c_1 D(x, x, x) \alpha(x, t) - c_1 M(x, x) \beta(x, t)\]

\[c_1 \int_0^x D_x(x, y) \alpha(y, t) \, dy\]

\[-c_1 \int_0^x M_x(x, y) \beta(y, t) \, dy\quad (D.1)\]

\[\beta_{xt}(x, t) = p_2 \beta_{xt}(x, t)\]

\[\alpha_x(0, t) = -\frac{p_2}{p_1} \beta_x(0, t) - \frac{1}{p_1} (C_0 (A + B \kappa) - c_1 J(0)) X(t)\]

\[-\frac{1}{p_1} (C_0 B - c_1) \beta(0, t)\quad (D.3)\]

Defining

\[\tilde{\mathcal{A}} = \frac{b_2}{2} \int_0^{l(t)} e^{-\delta_2 x} \alpha_x(x, t)^2 \, dx + \frac{a_2}{2} \int_0^{l(t)} e^{\delta_2 x} \beta_x(x, t)^2 \, dx\quad (D.4)\]

where \(b_2\) is an arbitrary positive constant which can adjust the convergence rate and the positive constants \(\delta_2, a_2\) will be chosen later.

Taking the derivative of (D.4) along (D.1), (D.2), we obtain

\[\dot{\mathcal{A}} = -\frac{p_1}{2} b_2 e^{-\delta_2 l(t)} \alpha_x(l(t), t)^2 + \frac{p_1}{2} b_2 \beta_x(0, t)^2\]

\[+ \frac{b_2 l(t)}{2} e^{-\delta_2 l(t)} \alpha_x(l(t), t)^2 - \frac{p_1}{2} b_2 \delta_2 \int_0^{l(t)} e^{-\delta_2 x} \alpha_x(x, t)^2 \, dx\]

\[+ (p_2 + l(t)) \frac{a_2}{2} e^{\delta_2 l(t)} \beta_x(l(t), t)^2 - \frac{p_2}{2} a_2 \beta_x(0, t)^2\]

\[-\frac{p_2}{2} a_2 \delta_2 \int_0^{l(t)} e^{\delta_2 x} \beta_x(x, t)^2 \, dx\]

\[-\int_0^{l(t)} b_2 e^{-\delta_2 x} \alpha_x(x, t) c_1 D(x, x, x) \alpha(x, t) \, dx\]

\[-\int_0^{l(t)} b_2 e^{-\delta_2 x} \alpha_x(x, t) c_1 M(x, x) \beta(x, t) \, dx\]

\[-\int_0^{l(t)} b_2 e^{-\delta_2 x} \alpha_x(x, t) c_1 \int_0^x D_x(x, y) \alpha(y, t) \, dy \, dx\]

\[-\int_0^{l(t)} b_2 e^{-\delta_2 x} \alpha_x(x, t) c_1 \int_0^x M_x(x, y) \beta(y, t) \, dy \, dx\]

\[-\int_0^{l(t)} b_2 e^{-\delta_2 x} \alpha_x(x, t) c_1 \beta(x, t) \, dx\]

\[-\int_0^{l(t)} b_2 e^{-\delta_2 x} \alpha_x(x, t) c_1 J(x) X(t) \, dx\quad (D.5)\]

Using Young’s inequality and Cauchy–Schwarz inequality into the last six terms in (D.5) yields the existence of \(\xi_2 > 0\) such that

\[\dot{\mathcal{A}}(t) \leq -\left(1 - \frac{p_1}{p_2} \right) \frac{b_2}{2} e^{-\delta_2 l(t)} \alpha_x(l(t), t)^2 + \frac{p_1}{2} b_2 \beta_x(0, t)^2\]

\[+ (p_2 + l(t)) \frac{a_2}{2} e^{\delta_2 l(t)} \beta_x(l(t), t)^2 - \frac{p_2}{2} a_2 \beta_x(0, t)^2\]

\[-\frac{p_1}{2} b_2 \delta_2 - \frac{2 \xi_2 b_2}{\delta_2} \int_0^{l(t)} e^{-\delta_2 x} \alpha_x(x, t)^2 \, dx\]

\[-\frac{p_2}{2} a_2 \delta_2 - \xi_2 b_2 \int_0^{l(t)} e^{\delta_2 x} \beta_x(x, t)^2 \, dx\]

\[+ \left(\xi_2 b_2 + \frac{\xi_2 b_2}{\delta_2}\right) \int_0^{l(t)} e^{-\delta_2 x} \alpha_x(x, t)^2 \, dx + \xi_2 b_2 |X(t)|^2\]

\[+ \left(\xi_2 b_2 + \frac{\xi_2 b_2}{\delta_2}\right) \int_0^{l(t)} e^{\delta_2 x} \beta_x(x, t)^2 \, dx\quad (D.6)\]
Note that $\alpha_2(0, t)^2$ in (D.6) can be replaced by

$$\alpha_2(0, t)^2 \leq \frac{1}{\beta_0} \left( C_0 B - c_1 \right)^2 \beta(t, 0)^2 + \frac{2}{P_1} \left( C_0 B - c_1 \right)^2 \beta(t, 0)^2$$

using Cauchy–Schwarz inequality into (D.3). Recalling (22), (25) with (A.1)–(A.8), (28), (29), (31), (C.2)

using Cauchy–Schwarz inequality, the positive term $(p_2 + \ell(t))^2$ can be replaced as

$$\xi_2 \eta_2(x, t)^2 + \xi_3 \delta(t) \xi_4 \beta(t, 0)^2 + \xi_5 \alpha(l(t), t)^2 + \xi_6 \alpha(0, t)^2$$

for some positive $\xi_i, i = 2, \ldots, 9$.

We propose a Lyapunov function

$$V_2(t) = \tilde A(t) + R_1 V(t).$$

Define the norm as

$$\Omega_2(t) = \| \beta_x(\cdot, t) \|^2 + \| \alpha_x(\cdot, t) \|^2 + \| \beta(\cdot, t) \|^2$$

for some positive $\theta_{2a}$ and $\theta_{2b}$.

Taking the derivative of (D.9) and recalling (D.6)–(D.8), (48), we then get

$$\dot{V}_2(t) = \tilde A(t) + R_1 V(t)$$

$$\leq - \left( p_1 - \ell(t) \right) \frac{b_2}{2} \mathrm{e}^{-\delta \ell(t)} \alpha(l(t), t)^2$$

$$- \left( p_2 - \frac{2}{2} \left( C_0 B - c_1 \right)^2 \beta(t, 0)^2$$

$$- \left( p_1 - \frac{2}{2} \left( C_0 B - c_1 \right)^2 \beta(t, 0)^2$$

$$- \left( R_1 \theta_{1a} - \frac{3b_2}{2b_2 p_1} \right) \beta(t, 0)^2$$

$$- \left( R_1 \theta_{1a} - \frac{3b_2}{2b_2 p_1} \right) \beta(t, 0)^2$$

$$- \left( R_1 \theta_{1a} - \frac{3b_2}{2b_2 p_1} \right) \beta(t, 0)^2$$

for some positive $\xi_\delta$.

Applying Cauchy–Schwarz inequality and recalling (4), (5), we have

$$|u(l(t), t)|^2 + |u(\cdot, t)|^2 + |u(\cdot, t)|^2$$

for some positive $\xi_\delta$.

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for some positive $\xi_\delta$. 

Similarily, differentiating (10), (11) with respect to $x$, together with (15), (16), using (10), (11), (18), (19), (28), (29), and (35), we have

$$\| \beta_x(\cdot, t) \|^2 + \| \beta(t, 0)^2 \|^2 + \| \beta(\cdot, t) \|^2$$

for some positive $\theta_{2a}, \theta_{2b},$ where $\Xi_1(t)$ is defined as $\Xi_1(t) = \Xi(t) + \| \beta_x(\cdot, t) \|^2 + \| \beta(t, 0)^2 \|^2 + \| \beta(\cdot, t) \|^2$. Therefore, one obtain

$$\| u_x(\cdot, t) \|^2 + \| u_x(\cdot, t) \|^2 \leq \Xi_1(t) \leq \Xi_1(0) e^{-\sigma \xi t}.$$ 

Thus (54) is obtained with

$$Y_{1a} = \frac{\theta_{2a} \theta_{2b}}{\theta_{2a} \theta_{2b}} \lambda_{1a} = \sigma_1.$$ 

The proof of Lemma 1 is completed.

**E. Proof of Theorem 2**

Applying Cauchy–Schwarz inequality into (40), we have

$$|U(t)|^2 \leq \xi_\delta \left( |s_1(t)|^2 + |s_2(t)|^2 + |z(t)|^2 + |X(t)|^2 \right)$$

$$+ |u(l(t), t)|^2 + |u(\cdot, t)|^2 + |u(\cdot, t)|^2$$

for some positive $\xi_\delta$.

Applying Cauchy–Schwarz inequality and recalling (4), (5), we have

$$|u(l(t), t)|^2 \leq |u(0, t)| + \sqrt{L} \| u_x(\cdot, t) \|^2$$

$$\leq |u(0, t)| + |C X(t)| + \sqrt{L} \| u_x(\cdot, t) \|^2$$

$$\leq |s_1(t)| + |q \sqrt{L} \| u_x(\cdot, t) \|^2$$

$$+ |C| |X(t)| + \sqrt{L} \| u_x(\cdot, t) \|^2.$$ 

(E.2)
Considering (E.1), (E.2), using Theorem 1 and Lemma 1, we have the control input (40) is bounded by
\[
|U(t)| \leq Y_2 \left( \Xi(0) + \|u_x(\cdot, 0)\|^2 + \|v_x(\cdot, 0)\|^2 \right)^{\frac{1}{2}} e^{-\gamma_2 t}. \tag{E.3}
\]
Then (55) is obtained by recalling (50). The proof of Theorem 2 is completed.

\[\frac{\dot{f}_1^2}{2} \leq \left( \gamma_1^2 + \gamma_2^2 \right) |s_1 - \tilde{s}_1|^2 + \gamma_2^2 \int_0^{\tilde{t}} (u(x, t) - \tilde{u}(x, t)) dx \right|^2 \leq \gamma_1^2 |s_1 - \tilde{s}_1|^2 + \gamma_2^2 \|\tilde{\alpha}(\cdot, t)\|^2 \tag{F.2}\]

\[\frac{\dot{f}_2^2}{2} \leq \left( \gamma_2^2 + \gamma_2^2 \right) |s_2 - \tilde{s}_2|^2 + \gamma_2^2 \|\tilde{\alpha}(\cdot, t)\|^2 \leq \left( \gamma_2^2 + \gamma_2^2 \right) \left| \tilde{S}(t) \right|^2 + \gamma_2^2 \|\tilde{\alpha}(\cdot, t)\|^2 \tag{F.3}\]

Thus, we have
\[
2\tilde{S}(t)^TP_0\tilde{f}(t) \leq \left( \gamma_1^2 + \gamma_2^2 \right) \left| P_0\tilde{S}(t) \right|^2 + \frac{1}{\gamma_1^2 + 2\gamma_2^2} \left| \dot{f}_1(t) \right|^2 \leq \left( \gamma_1^2 + \gamma_2^2 \right) \left| P_0\tilde{S}(t) \right|^2 + \frac{1}{\gamma_1^2 + 2\gamma_2^2} \left| \tilde{S}(t) \right|^2 \tag{F.4}\]

where Young’s inequality is used. Substituting (F.4) into (F.1), yields
\[
\dot{V}_x(t) \leq \tilde{S}(t)^T((\tilde{A} - BC_2)^TP_0 + P_0(\tilde{A} - BC_2))\tilde{S}(t) + \left( \gamma_1^2 + 2\gamma_2^2 \right) \tilde{S}(t)^TP_0\dot{S}(t) + \tilde{S}(t)^T\dot{S}(t) + \frac{\tilde{\gamma}_1^2}{\gamma_1^2 + 2\gamma_2^2} \left| \tilde{\alpha}(\cdot, t)\right|^2 + \frac{\tilde{\gamma}_2^2}{\gamma_1^2 + 2\gamma_2^2} \tilde{\alpha}(l(t), t)^2 \tag{F.5}\]

Recalling (119), we arrive (120).

\begin{thebibliography}{99}


\end{thebibliography}


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