Stability Analysis for Time-Varying Systems With Asynchronous Sampling Using Contractivity Approach

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Abstract—We provide new sufficient conditions for a generalized exponential input-to-state stability condition to hold. Our proofs use innovative trajectory based contractivity and cooperative systems arguments. Another key ingredient is a new variant of Halanay’s inequality, which differs from prior Halanay inequalities because of its use of sampling. Our sufficient conditions allow arbitrarily long intervals between some consecutive sampling times, provided the intervals between consecutive sampling times are frequently small enough. This removes a significant limitation of prior results that relied on versions of Halanay’s inequality that did not contain sampling. Unlike standard stability conditions, the upper bounds for the norm of the current state in our less restrictive exponential input-to-state stability conclusions have suprema of disturbances over intervals that include earlier times than the initial times. We apply this letter to a broad class of time-varying continuous time systems with sampling.

Index Terms—Sampling, linear systems, stability.

I. INTRODUCTION

DYNAMICS having sampled data and discrete measurements play a fundamental role in current research in systems and controls. However, control and observer design and stability analysis for these systems can be difficult, especially under asynchronous (i.e., nonperiodic) sampling. This is because classical Lyapunov techniques provide conservative stability conditions in terms of upper bounds on the sizes of the sampling intervals.

To address this fundamental case, we propose a new stability analysis for systems with asynchronous sampling when a variant of the celebrated Halanay’s inequality [3] is satisfied. The usual Halanay’s inequality has the form 
\[ \dot{v}(t) \leq -cv(t) + d \sup_{t \in [t-T,t]} v(\ell) + \Delta(t) \] 
for nonnegative valued functions \( v \) and \( \Delta \) and positive constants \( c \) (called a decay rate), \( T \), and \( d \) (called a gain), where \( \Delta \) represents a disturbance. By contrast, our relaxed version has an integral instead of the sup, and it removes the usual Halanay’s inequality requirement that the decay rate is strictly larger than the gain.

Our results complement both Halanay’s and the trajectory based approaches, as well as our prior variants of Halanay’s inequality from [5] and linear Lyapunov functionals. The interval observers in our proof of our second theorem use tools that were developed, e.g., in the pioneering paper [1]. A key feature of our results is that they allow arbitrary large time intervals between consecutive sampling instants, provided the intervals between other consecutive sampling times are frequently small enough. Our results complement both Halanay’s and the trajectory based approaches, as well as our prior variants of Halanay’s inequality from [5] which did not allow the types of sampling that we allow here.

This letter is organized as follows. The main results are stated and proved in Sections II and III. An illustrative example is given in Section IV. We use standard notation, which is simplified when no confusion would arise. Set \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and \( \mathbb{Z}_0 = \mathbb{N} \setminus \{0\} \). The standard Euclidean 2-norm, and the corresponding matrix norm, are denoted by \( \| \cdot \| \) and \( \| \cdot \|_\infty \) is the usual sup norm. Let \( M \in \mathbb{R}^{n \times n} \) be a matrix with entry \( m_{ij} \) in its \( i \)th row and \( j \)th column. We use \( M^T \) to denote the transpose of \( M \), and \( D_M = M^T - M \).
\( V_1 \leq V_2 \) when \( v_{1,j} \leq v_{2,j} \) for all \( j \in \{1, \ldots, n\} \). A matrix is called nonnegative (resp., positive) provided all of its entries are nonnegative (resp., positive), and \( \mathrm{col}_j(M) \) is the \( j \)th column of any matrix \( M \). We let \( f(s^+) \) denote the left limit of a function \( f \) at a point \( s \), and we use the usual definition of nonnegative systems, e.g., from [4]. We consider sequences \( t_i \) such that \( t_0 = 0 \) that admit two constants \( \tau > 0 \) and \( \nu > 0 \) such that \( \nu \leq t_1 - t_i \leq \tau \) for all \( i \in \mathbb{Z}_0 \). We then also use the constants \( v_i = t_{i+1} - t_i \).

II. VARIANT OF HALANAY’S INEQUALITY

Let the sequences \( t_i \) and \( v_i \) satisfy the requirements from Section I. Consider a nonnegative valued function \( v : [0, +\infty) \rightarrow [0, +\infty) \) of class \( C^1 \) such that there are constants \( J \in \mathbb{N}, a \in \mathbb{R}, b \geq 0 \) and \( c > 0 \) and a piecewise continuous function \( \delta : [0, +\infty) \rightarrow [0, +\infty) \) such that

\[
\dot{v}(t) \leq -av(t) + (t - t_i)bv(t_i) + c \int_t^{t_i} v(\ell) d\ell
+ \delta(t) \quad \text{for all } t \in [t_i, t_{i+1}) \text{ and } i \geq J. \tag{1}
\]

We also use \( S = \sqrt{a^2 + 4c} \), and the function

\[
\mu(r) = \frac{1}{2} \left[ 1 + \frac{a}{S} + \frac{b(S-a)}{cS} \right] e^{-\frac{a-S}{2} r}
+ \frac{1}{2} \left[ 1 - \frac{a}{S} + \frac{b(S+a)}{cS} \right] e^{\frac{a+S}{2} r} - \frac{b}{c}. \tag{2}
\]

We assume the following, which can be interpreted to mean that \( \mu(v(t)) \) is frequently (but not necessarily always) in \((0, 1)\).

Assumption 1: There are \( p \in \mathbb{N} \) and \( \mu \in (0, 1) \) such that

\[
\prod_{i=j}^{j+p} \mu(v_i) \leq \mu \quad \text{for all } j \in \mathbb{N}. \tag{3}
\]

See Remark 1 for the motivation for Assumption 1 and our example below, which explain how to check Assumption 1 (and why Assumption 1 is satisfied when \( a > 0 \)). We prove:

Theorem 1: Let \( v \) be a function defined above such that Assumption 1 is satisfied. Then there are positive constants \( a_i \) for \( i = 1 \) to \( 4 \) such that the inequality

\[
v(t) \leq a_1 e^{-\alpha s(t-s)} \sup_{t \in [t_i-t_j]} v(\ell) + a_4 \sup_{m \in [t_i-t_j]} \delta(m) \tag{4}
\]

holds for all \( s \geq a_3 \) and \( t \geq s \).

Proof: We can assume that (1) holds with equality for all \( t \in [t_i, t_{i+1}) \) and all \( i \geq J \). To understand why, notice that if (1) did not hold with equality, then it holds with \( \delta \) replaced by \( \delta - D \) for some nonnegative valued function \( D \) (where \( D \) can depend on \( v \) at each time). In that case, we would be able to replace \( \delta \) by \( \delta - D \) in the proof up through (21), and then we would be able to complete the proof as given below starting in (22) (with \( \delta \) instead of \( \delta - D \)) because of the nonnegative valuedness of the function \( \theta \) that we use in the proof. This would allow us to to obtain (4) with the original \( \delta \), again using the nonnegative valuedness of \( D \).

To adopt a representation which is inspired from [7, Sec. 1.5.3], we next define

\[
w_j(t) = v(t) + \int_{t_i}^{t} [v(\ell) + b(v(t_i))] d\ell \quad \text{for } j = 1 \text{ and } j = 2 \text{ for all } t \in [t_i, t_{i+1}) \text{ and } i \geq J, \tag{5}
\]

where

\[
\zeta_1 = \frac{a - \sqrt{a^2 + 4c}}{2c} \quad \text{and} \quad \zeta_2 = \frac{a + \sqrt{a^2 + 4c}}{2c}. \tag{6}
\]

Then

\[
\dot{w}_1(t) = -av(t) + (t - t_i)bv(t_i) + c \int_t^{t_i} v(\ell) d\ell
+ \zeta_1[c(v(t) + b(v(t_i))) + \delta(t)
= \left(\frac{a - S}{2} - a\right) v(t) + (t - t_i)b(v(t_i))
+ c \int_t^{t_i} v(\ell) d\ell + \zeta_1 b(v(t_i) + \delta(t)
= \left(\frac{a + S}{2} v(t) - \frac{2}{a + S} \int_t^{t_i} [c(v(t) + b(v(t_i))] d\ell \right)
+ \zeta_1 b(v(t_i) + \delta(t). \tag{7}
\]

Here and in the sequel, all equalities and inequalities should be understood to hold for all \( t \in [t_i, t_{i+1}) \) and all \( i \geq J \). Observe that

\[
-2 \frac{a}{a + S} = -2 \frac{a - \sqrt{a^2 + 4c}}{a^2 - (a^2 + 4c)} = -2 \frac{a - \sqrt{a^2 + 4c}}{2c} = \zeta_1. \tag{8}
\]

Thus,

\[
\dot{w}_1(t) = -\frac{a + S}{2} v(t) + \zeta_1 \int_t^{t_i} [c(v(t) + b(v(t_i))] d\ell
+ \zeta_1 b(v(t_i) + \delta(t)
= -\frac{a + S}{2} w_1(t) + \zeta_1 b(v(t_i) + \delta(t). \tag{9}
\]

Similarly, by replacing \( S \) by \(-S \) in the preceding argument, one can prove that

\[
\dot{w}_2(t) = -\frac{a - S}{2} w_2(t) + \zeta_2 b(v(t_i) + \delta(t). \tag{10}
\]

By integrating, we obtain

\[
w_1(t) = e^{-\frac{a-S}{2} (t-t_i)} w_1(t_i) + 1 - e^{-\frac{a+S}{2} (t-t_i)} \frac{b \zeta_1 b(v(t_i))}{a+S} \tag{11}
\]

By reorganizing terms, and noting that

\[
\frac{4b}{(a+S)^2} = \frac{4b(a-S)^2}{(a^2-S^2)^2} = \frac{4b(a-S)^2}{16c^2}, \tag{12}
\]

we obtain

\[
w_2(t) = \left[ e^{-\frac{a-S}{2} (t-t_i)} + \frac{1}{a+S} - e^{-\frac{a+S}{2} (t-t_i)} \right] w_1(t_i)
+ \int_t^{t_i} e^{\frac{a+S}{2} (t-\ell)} \delta(\ell) d\ell \tag{13}
\]

\[
= e^{-\frac{a-S}{2} (t-t_i)} + \frac{1}{a+S} - e^{-\frac{a+S}{2} (t-t_i)} \frac{b \zeta_1 b(v(t_i))}{a+S} \tag{14}
\]

\[
+ \int_t^{t_i} e^{\frac{a+S}{2} (t-\ell)} \delta(\ell) d\ell \tag{15}
\]

\[
= \left( 1 + \frac{4b}{(a+S)^2} \right) e^{-\frac{a-S}{2} (t-t_i)} - \frac{4b}{(a+S)^2} w_1(t_i). \tag{16}
\]
\[ + \int_{t_i}^1 e^{\frac{S}{2}}(t-\ell) \delta(\ell) d\ell \]

\[ = G_1(t-t_i) w_1(t_i) + \int_{t_i}^1 e^{\frac{S}{2}}(t-\ell) \delta(\ell) d\ell, \quad (13) \]

where

\[ G_1(r) = \left( 1 + \frac{b(a-S^2)}{4\ell^2} \right) e^{-\frac{S}{2r}} - \frac{b(a-S^2)}{4\ell^2}. \quad (14) \]

Similarly, by replacing \( S \) by \( -S \) in the preceding argument,

\[ w_2(t) = G_2(t-t_i) w_2(t_i) + \int_{t_i}^1 e^{\frac{S}{2}}(t-\ell) \delta(\ell) d\ell, \quad (15) \]

where

\[ G_2(r) = \left( 1 + \frac{b(a+S)^2}{4\ell^2} \right) e^{-\frac{S}{2r}} - \frac{b(a+S)^2}{4\ell^2}. \quad (16) \]

As an immediate consequence, we have

\[ v(t) + \zeta_1 \int_{t_i}^t [ce(\ell) + bv(\ell)] d\ell \]

\[ = G_1(t-t_i) v(t_i) + \int_{t_i}^t e^{\frac{S}{2}}(t-\ell) \delta(\ell) d\ell \]

\[ = G_2(t-t_i) v(t_i) + \int_{t_i}^t e^{\frac{S}{2}}(t-\ell) \delta(\ell) d\ell. \quad (17) \]

and

\[ v(t) + \zeta_2 \int_{t_i}^t [ce(\ell) + bv(\ell)] d\ell \]

\[ = G_1(t-t_i) v(t_i) + \int_{t_i}^t e^{\frac{S}{2}}(t-\ell) \delta(\ell) d\ell. \quad (18) \]

From (17)-(18), we deduce that

\[ (\zeta_2 - \zeta_1) v(t) = \zeta_2 G_1(t-t_i) v(t_i) - \zeta_1 G_2(t-t_i) v(t_i) \]

\[ + \int_{t_i}^t \zeta_1 e^{\frac{S}{2}}(t-\ell) \delta(\ell) d\ell - \zeta_1 \int_{t_i}^t e^{\frac{S}{2}}(t-\ell) \delta(\ell) d\ell. \quad (19) \]

Since \( \zeta_2 - \zeta_1 = S/c \), we have

\[ \frac{S}{c} v(t) = [\zeta_2 G_1(t-t_i) - \zeta_1 G_2(t-t_i)] v(t_i) \]

\[ + \int_{t_i}^t \left[ \zeta_2 e^{\frac{S}{2}}(t-\ell) - \zeta_1 e^{\frac{S}{2}}(t-\ell) \right] \delta(\ell) d\ell. \quad (20) \]

Consequently,

\[ v(t) = \mu(t-t_i) v(t_i) + \int_{t_i}^t \theta(\ell-t) \delta(\ell) d\ell, \]

where

\[ \mu(r) = \frac{c}{S} \left[ \zeta_2 G_1(t-t_i) - \zeta_1 G_2(t-t_i) \right] \]

and

\[ \theta(r) = \frac{c}{S} \zeta_2 e^{\frac{S}{2}} - \zeta_1 e^{\frac{S}{2}}. \quad (21) \]

where \( \mu \) agrees with the function we defined in (2), and \( \theta \) is positive valued because \( c \) is positive valued, and because our formulas \( \zeta_2 = (a+S)/2c \) and \( \zeta_1 = (a-S)/2c \) from (6) imply that \( \mu(r) \) is bounded below by \( \frac{1}{4}(S - |a|) e^{a/2} - \frac{1}{4}(S + |a|) e^{-a/2} \), whereas \( \frac{1}{4}(S + |a|) e^{a/2} \) is bounded by \( \frac{1}{4}(S - |a|) e^{-a/2} \).

Setting \( \delta = c/S \left[ \zeta_2 e^{\frac{S}{2}} + \zeta_1 e^{\frac{S}{2}} \right] \), it follows that \( 0 \leq \delta(\ell) \leq \delta \) for all \( r \in [0, v] \) and that

\[ v(t_{i+1}) \leq \mu(v(t_i)) v(t_i) + \bar{\delta} \int_{t_i}^{t_{i+1}} \delta(\ell) d\ell. \quad (22) \]

Thus, for all \( j \geq 0 \), we have

\[ v(t_{j+p+1}) \leq \mu(v_{j+p}) v(t_{j+p}) + \bar{\delta} I(0) \]

\[ \mu(v_{j+p}) v(t_{j+p}) \leq \mu(v_{j+p}) \mu(v_{j+p-1}) + \mu(v_{j+p}) \bar{\delta} I(1) \]

\[ \vdots \]

\[ j+p \]

\[ \mu(v_{j+p}) v(t_{j+p}) \leq \mu(v_{j+p}) + \mu(v_{j+p}) \bar{\delta} I(p) \quad (23) \]

\[ \mu(v_{j+p}) v(t_{j+p}) \leq \mu(v_{j+p}) \bar{\delta} I(p) \]

\[ \int_{t_{j+p-1}}^{t_{j+p}} \delta(\ell) d\ell \]

and we omit the dependence of \( I(q) \) on \( j \) and \( p \) to simplify notation. By moving the left side terms to the right side in the last \( p \) of the inequalities in (23), then adding the results to the first inequality in (23) and cancelling terms, we get

\[ v(t_{j+p+1}) \leq \mu(v_{j+p}) v(t_{j+p}) + \bar{\delta} I(0) \]

\[ + \mu(v_{j+p}) \bar{\delta} I(1) + \cdots + \mu(v_{j+p}) \bar{\delta} I(p) \]

\[ \leq \mu(v_{j+p}) v(t_{j+p}) + \bar{\delta} \sigma(j, p) \int_{t_{j+p-1}}^{t_{j+p+1}} \delta(\ell) d\ell \]

\[ \sigma(j, p) = 1 + \mu(v_{j+p}) + \cdots + \mu(v_{j+p}) \quad (26) \]

admits a bound \( \bar{\delta} > 0 \) that is independent of \( j \) (because of our bound \( v \) on the \( v(t_j) \)’s), and where we use the nonnegative valuedness of \( \mu \) and the nonnegativity of \( \bar{\delta} \) to collect all of the integrals from the \( I(q) \) terms in the last integral in (25) (which justifies the last inequality in (25) and explains why there are no \( I(q) \’s in (26)’s). This yields a constant \( C \) such that

\[ v(t_{j+p+1}) \leq \mu(v_{j+p}) v(t_{j+p}) + C \int_{t_{j+p-1}}^{t_{j+p+1}} \delta(\ell) d\ell \quad (27) \]

for all \( j \). It follows from Assumption 1 that

\[ v(t_{j+p+1}) \leq \Pi(v(t_j)) + C \int_{t_{j+p-1}}^{t_{j+p+1}} \delta(s) ds \quad (28) \]

Also, we can apply an integrating factor to (1) to conclude that for all \( t \in [t_i, t_{i+1}] \) and \( i \geq J \), we have

\[ v(t) \leq e^{\frac{a}{2} \bar{v}} v(t) \left( bv(t) + \sup_{e(t_i,t)} \delta(\ell) \right) + e^{\frac{a}{2} \bar{v}} v(t_i) \]

\[ + e^{\frac{a}{2} \bar{v}} \int_{t_i}^t \delta(s) ds \]

\[ \sup_{e(t_i,t)} \delta(\ell) \]

so when \( a \neq 0 \), Gronwall’s inequality gives a constant \( \bar{v} > 0 \) such that

\[ v(t) \leq \bar{v} \left( v(t_i) + \sup_{e(t_i,t)} \delta(\ell) \right) \quad (30) \]

for all \( t \in [t_i, t_{i+1}] \) and \( i \geq J \).
Let $k \in \mathbb{N}$ be such that $k > -\ln(\bar{\nu})/\ln(\bar{\mu})$. Then, since $\bar{\mu} \in (0, 1)$, the constant $q = \bar{\nu}^k \bar{\nu}$ satisfies $q < 1$. From (28), we deduce that there is a constant $\bar{d} > 0$ such that

$$v(t_{j+k(p+1)}) \leq \bar{d} v(t_j) + \int_{t_j}^{t_{j+k(p+1)-1}} \delta(s) ds.$$  \hfill (31)

Let us take $t \in [t_{j+k(p+1)}, t_{j+k(p+1)-1}]$ with $j \geq J$ such that $t_{j+k(p+1)} \geq [k(p+1)+1]\bar{\nu}$. Then (30)-(31) give

$$v(t) \leq \bar{d} v(t_j) + \int_{t_j}^{t_{j+k(p+1)-1}} \delta(s) ds + \sup_{(32)} \delta(\ell).$$

Now, observe that our choice of $\bar{\nu}$ in Section I implies that $t \leq t_{j+k(p+1)} \leq t_j + \sup [k(p+1)+1] 2 \bar{\nu}$. It follows that

$$v(t) \leq q \sup_{t \in [t_j, t_j+1]} v(t) + \sup_{t \in [t_j, t_j+1]} \delta(\ell).$$ \hfill (33)

If we now apply [6, Lemma 1] to the functions $v(t) = v(s+\ell)$ and $\delta(\ell) = \sup_{t \in [s, s+1]} \delta(r)$, we deduce that

$$v(t) \leq e^{(\ln q)(t-s)} \sup_{t \in [s, s+1]} v(t) + \frac{\sup_{t \in [s, s+1]} \delta(r)}{1 - q}.$$ \hfill (34)

if $t \geq s$. Since $q < 1$, we can conclude. \hspace{1cm} \Box

Remark 1: Using Taylor’s polynomials, we can find a continuous function $\phi$ such that for all $r \in \mathbb{R}$, we have

$$\mu(r) = \mu(0) + \mu'(0)r + r^2 \phi(r).$$ \hfill (35)

Then simple calculations give $\mu(0) = 1$ (by the relation $\frac{1}{\pi} (b(S-a)(cS)) + \frac{1}{\pi} (b(S+a)(cS)) - b/c = 0$ and cancelling to $a(S)$ terms) and $\mu'(0) = -\frac{\bar{d}^2}{\bar{d}^2} \left( \frac{1}{4} + a(S) + b(S-a)(cS) + b(S-a)(cS) \right) = \frac{1}{4} \left( -S + 2a \right) - a$. Thus, when $a > 0$, then there are constants $\ell > 0$, $\bar{r} > 0$, and $\mu_0 \in (0, 1)$ such that $\mu(r) \in (0, \mu_0)$ when $e \in [\ell, \bar{r}]$ (but Assumption 1 is not satisfied when $a \leq 0$). Given any constant $\bar{v} > 0$, consider a sequence $t_i$ such that there is a $k \in \mathbb{N}$ such that $k > 2$ and $v_{i+k} = \bar{v}$ for all $i \in [0, k-2]$ and $v_{k-1} = \bar{v}$. Then

$$\prod_{i=j}^{j+k-1} \mu(v_i) \leq \mu(\bar{v}) \mu_{0}^{k-1}.$$ \hfill (36)

Then Assumption 1 is satisfied provided there is a constant $\omega \in (0, 1)$ such that

$$\mu(\bar{v}) \mu_{0}^{k-1} \leq \omega,$$ \hfill (37)

which is equivalent to

$$k \geq 1 + \ln \left( \frac{\omega}{\mu(\bar{v})} \right).$$ \hfill (38)

For any $R > 0$, there are $k \in \mathbb{N}$ such that $R$ is satisfied, so no constraint on $\bar{v}$ is imposed, but larger sampling intervals $[t_i, t_{i+1})$ require more frequent small sampling intervals.

## III. APPLICATION

### A. Statement of Result

Consider the system

$$\dot{x}(t) = M(t)x(t) + N(t)x(t) + \gamma(t)$$ \hfill (39)

where $x$ is valued in $\mathbb{R}^n$, $M : [0, +\infty) \to \mathbb{R}^{n \times n}$ and $N : [0, +\infty) \to \mathbb{R}^{n \times n}$ are continuous functions that are bounded in norm by the constants $\bar{m} > 0$ and $\bar{p} > 0$ respectively, and $\gamma : [0, +\infty) \to \mathbb{R}^n$ is a piecewise continuous bounded function that can represent uncertainty.

To motivate (39), note that systems of the type (39) are encountered in many circumstances. For instance, a system $\dot{x}(t) = A(t)x(t) + B(t)u(t) + d(t)$ with the discrete output $y(t) = Cx(t)$ and the input $u(t) = F(t)y(t)$ gives

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + d(t)$$ \hfill (40)

Also, if one wants to use an observer of the form $\dot{x}(t) = (A(t)x(t) + B(t)u(t) - G(t)C\hat{x}(t) - \gamma(t))$, then one obtains the error equation

$$\dot{e}(t) = A(t)e(t) - G(t)C\hat{x}(t) - d(t)$$ \hfill (41)

for $e = \hat{x} - x$, which also is of the form (39).

Having motivated (39), we next provide conditions ensuring generalized input-to-state stability type properties of the system (39). We introduce the simplifying notation $H(t) = M(t) + N(t)$ and the following assumption, in terms of the matrices $D_M, R_M, M^+ \text{ and } M^t$ we defined in Section I.

**Assumption 2:** There exist a $C^1$ function $P = (P_1, \ldots, P_n) : [0, +\infty) \to \mathbb{R}^n$ and positive constants $p, \bar{p}$, and $p$ such that

$$P(t_i) + [D_M(t_i) + R_M(t_i) + R_M(t_i)]^T P(t_i) \leq -p P(t_i)$$ \hfill (42)

and $p \leq P_i(t) \leq \bar{p}$ for $k = 1, 2, \ldots, n$ \hfill (43)

hold for all $t \geq 0$.

We also choose the positive constants

$$d_1 = \frac{2|P_{\infty}| \bar{m}}{\bar{p}}$$ \hfill (44)

and $S_0 = \sqrt{\bar{d}^2 + 4d_1}$ and the function

$$\kappa(r) = \frac{1}{2} \left( 1 + \frac{p}{S_0} \right) e^{-\frac{p}{S_0} r} + \frac{1}{2} \left( 1 - \frac{p}{S_0} \right) e^{-\frac{p}{S_0} - \frac{d_2}{d_2}}$$ \hfill (45)

which agrees with the $\mu$ formula (2) when $a = p, b = d_2$, and $c = d_1$. The following is an analog of Assumption 1.

**Assumption 3:** There are two constants $q \in \mathbb{N}$ and $\bar{R} \in (0, 1)$ such that for all $j \in \mathbb{N}$, the inequality

$$\sum_{i=j}^{j+q} \mu_i(v_i) \leq \bar{R}$$ \hfill (46)

is satisfied.

In the next subsection, we prove:

**Theorem 2:** Let the system (39) satisfy Assumptions 2-3. Then there are positive constants $\beta_i$ for $i = 1$ to $4$ such that all solutions of the system (39) satisfy

$$||x(t)|| \leq \beta_1 e^{-\beta_2(t-s)} \sup_{m \in [x_{-\beta_4}, x]} |x(m)| + \beta_3 \sup_{m \in [x_{-\beta_4}, t]} |\gamma(m)|$$ \hfill (47)

for all $s \geq \beta_4$ and $t \geq s$. 

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B. Proof of Theorem 2

For all \( t \in [t_i, t_{i+1}) \), we can rewrite the system (39) as
\[
\dot{x}(t) = H(t)x(t) + N(t) [x(t_i) - x(t)] + \gamma(t)
\]
\[
= H(t)x(t) - N(t) \int_{t_i}^{t} \dot{x}(\ell) d\ell + \gamma(t)
\]
\[
= H(t)x(t) - N(t) \int_{t_i}^{t} [M(\ell)x(\ell) + N(\ell)(x(t_i) - x(\ell)) + \gamma(\ell)] d\ell + \gamma(t).
\]
(48)

Here and in the sequel, statements involving \( i \) should be understood to hold for all \( i \in \mathbb{Z}_0 \). Thus
\[
\dot{x}(t) = H(t)x(t) + \int_{t_i}^{t} Z(t, \ell)x(\ell) d\ell + K(t)x(t_i) + \sigma(t)
\]
(49)
for all \( t \in [t_i, t_{i+1}) \), where
\[
K(t) = -N(t) \int_{t_i}^{t} \gamma(\ell) d\ell + \gamma(t),
\]
and \( \sigma(t) = -N(t) \int_{t_i}^{t} \gamma(\ell) d\ell + \gamma(t) \).
(50)

We study the system (49) through the following dynamics:
\[
\begin{cases}
\dot{x}(t) = \left( D_H(t) + R_H(t)^+ \gamma(t) - R_H(t)^- \gamma(t) \right) x(t)
+ \int_{t_i}^{t} \left[ Z(t, \ell)^+ \gamma(\ell) - Z(t, \ell)^- \gamma(\ell) \right] d\ell
+ K(t)^+ \gamma(t) - K(t)^- \gamma(t) + \sigma(t)^+ - \sigma(t)^-
\end{cases}
(51)
\]
for all \( t \in [t_i, t_{i+1}) \). Let \( s \geq \overline{\nu} \) and let us consider a solution of (51) with the initial conditions
\[
\bar{x}(t) = 0, \quad \bar{x}(t) = 0, \quad \bar{x}(t) = 0 \quad \bar{x}(t) = \bar{x}(t_i)
\]
(53)

Then we prove in the Appendix that
\[
\bar{x}(t) \geq 0, \quad \bar{x}(t) \leq 0, \quad \bar{x}(t) \leq x(t) \leq \bar{x}(t)
\]
(54)

for all \( t \geq s \) where \( \bar{x}(t) = \bar{x}(t_i) - \bar{x}(t) \).

Direct calculations imply that
\[
\dot{x}(t) = \hat{M}(t_i)x(t) + \int_{t_i}^{t} \hat{N}(t, \ell)\bar{x}(\ell) d\ell + K(t_i)\bar{x}(t_i) + \zeta(t)
\]
(55)

for all \( t \geq s \) with \( t \in [t_i, t_{i+1}) \), where \( M(t_i) = D_H(t) + R_H(t)^+ \gamma(t) - R_H(t)^- \gamma(t) \), \( \hat{N}(t, \ell) = Z(t, \ell)^+ \gamma(\ell) - Z(t, \ell)^- \gamma(\ell) \), \( K(t_i) = K(t_i)^+ - K(t_i)^- \), and \( \zeta(t) = \sigma(t)^+ - \sigma(t)^- \). By (53) and Assumption 2, it follows that along all solutions of (55), the time derivative of \( V(t, \bar{x}) = \bar{x}^TP(t) \) satisfies
\[
\dot{V}(t) \leq -pV(t, \bar{x}(t)) + P^T(t) \int_{t_i}^{t} \left[ Z(t, \ell)^+ \gamma(\ell) + \tilde{x}(\ell)^T K(t_i)\tilde{x}(t_i) + \zeta(t) \right] d\ell + \zeta(t)^TP(t)
\]
\[
\leq -pV(t, \bar{x}(t)) + d_1 \int_{t_i}^{t} V(t, \bar{x}(\ell)) d\ell
+ d_2(t - t_i)V(t_i, \bar{x}(t_i)) + \zeta(t)^TP(t)
\]
(56)

for all \( t \geq s \) with \( t \in [t_i, t_{i+1}) \), where the formulas for \( d_1 \) and \( d_2 \) from (44) followed because, for any \( n \times n \) matrix valued function \( \mathcal{M}(t, \ell) \) having nonnegative valued entries,
\[
P^T(t)\mathcal{M}(t, \ell)\bar{x}(\ell) = \sum_{j=1}^{n} p_j(t)\bar{x}(\ell) \leq \frac{|P^T(t)\mathcal{M}(t, \ell)|}{P} V(t, \bar{x}(\ell)) \leq \frac{P^T(t)\mathcal{M}(t, \ell)}{P} V(t, \bar{x}(\ell)).
\]
(57)

Given \( s \geq 0 \) and \( i \in \mathbb{Z}_0 \) such that \( t_i \geq s \), we apply Theorem 1 with the choices \( J = \overline{\nu}, a = p, b = d_2, \) and \( c = d_1 \) to
\[
v(t) = V(t, \bar{x}(\ell))
\]
(58)

which satisfies
\[
\dot{v}(t) \leq -pv(t) + d_1 \int_{t_i}^{t} v(\ell) d\ell + d_2(t - t_i)v(t_i) + \phi(t + t_2)^TP(t + t_2)
\]
(59)

for all \( t \in [t_i, t_{i+1}) \) and \( i \geq J \). Then Assumption 3 gives constants \( \theta_i > 0 \) such that
\[
V(t, \bar{x}(t)) \leq \theta_1 e^{-\theta_2(t-s)} \sup_{m \in [r-\theta_0, r]} V(m, \bar{x}(m)) + \theta_3 \sup_{m \in [r-\theta_0, r]} |\phi(m)|
\]
(60)

when \( t \geq r \geq \theta_0 \). Hence Assumption 2 provides constants \( \theta_i > 0 \) for \( i = 4 \) to 6 such that
\[
|\bar{x}(t)| \leq \theta_4 e^{-\theta_5(t-s)} \sup_{m \in [s, r]} |\bar{x}(m)| + \theta_6 \sup_{m \in [s, r]} |\phi(m)|
\]
(61)

when \( t \geq r \geq \theta_0 + s \).

Thus, combining (54) and (61) with \( r = \theta_0 + s \), it follows that when \( t \geq \theta_0 + s \), we have
\[
|x(t)| \leq \theta_7 e^{-\theta_8(t-s)} \sup_{m \in [s \theta_0 + s]} |\bar{x}(m)| + \theta_8 \sup_{m \in [s \theta_0 + s]} |\phi(m)|
\]
(62)

with \( \theta_7 = \theta_7 e^{\rho \theta_0} \). By applying variation of parameters to (55) on \( [t_i, t] \) for each \( t \in [t_i, t_{i+1}) \) and then applying Gronwall’s inequality repeatedly to the result on the intervals \( [t_i, t_{i+1}) \) for \( i = i - 1 \), one finds a constant \( C > 0 \) such that
\[
|\bar{x}(t)| \leq C \left( \sup_{m \in [s \theta_0 + s]} |\bar{x}(m)| + \sup_{m \in [s \theta_0 + s]} |\phi(m)| \right)
\]
(63)

for all \( \ell \in [s \theta_0 + s] \) and \( t \geq \theta_0 + s \) and \( s \geq \overline{\nu} \). Combining the last inequality in (63) with (62), we obtain
\[
|x(t)| \leq \theta_7 e^{-\theta_8(t-s)} C \sup_{m \in [s \theta_0 + s]} |\bar{x}(m)| + \theta_7 e^{-\theta_8(t-s)} C \sup_{m \in [s \theta_0 + s]} |\phi(m)| + \theta_8 \sup_{m \in [s \theta_0 + s]} |\phi(m)|
\]
(64)

when \( t \geq \theta_0 + s \). Thus
\[
|x(t)| \leq \theta_7 e^{-\theta_8(t-s)} \sup_{m \in [s \theta_0 + s]} |\bar{x}(m)| + \theta_8 \sup_{m \in [s \theta_0 + s]} |\phi(m)|
\]
(65)

when \( t \geq \theta_0 + s \) with \( \theta_8 = \theta_7 C + \theta_0 + \theta_8 \).
It now follows from (54) and (63) that there are constants \( \theta_{10} \) and \( \theta_{11} \) such that
\[
|x(t)| \leq \theta_{10} \sup_{m \in [s, -\tau, t]} |\tilde{x}(m)| + \theta_{11} \sup_{m \in [s, t]} |\varrho(m)|
\]
\[
\leq \theta_{10} e^{\theta_{10}(t-s)} \sup_{m \in [s, -\tau, t]} |\tilde{x}(m)| + \theta_{11} \sup_{m \in [s, t]} |\varrho(m)| \tag{66}
\]
for all \( t \in [s, \theta_0 + s] \). Setting \( \theta_13 = \max\{\theta_1, \theta_10 e^{\theta_{10} s}\} \) and \( \theta_{14} = \max\{\theta_1, \theta_{11}\} \), it follows that if \( t \geq s \geq 2 \tau \), then
\[
|x(t)| \leq \theta_{13} e^{-\theta_{13}(t-s)} \sup_{m \in [s, -\tau, t]} |\tilde{x}(m)| + \theta_{14} \sup_{m \in [s, s]} |\varrho(m)|. \tag{67}
\]
Since \( |\varrho(t)| \leq (2\overline{\nu} + 2) \sup_{\ell \in [s, -\tau, t]} |\gamma(\ell)| \) for all \( t \in [t_i, t_{i+1}) \) and \( i \in \mathbb{Z}_0 \), and since \( \tilde{x}(r) = \tilde{x}(r) - \varphi(r) = \varphi(r) + x(r) \) and so also \( |\tilde{x}(r)| \leq 2|x(r)| \) for all \( r \in [s, -\tau, s] \), the theorem now follows from (67).

IV. ILLUSTRATION OF THEOREM 2

Let \( t_i \) be a sequence that satisfies the requirements from Section I. Consider the two-dimensional system
\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) + x_2(t_i) \\
\dot{x}_2(t) &= -2x_2(t) + x_1(t) - x_1(t_i).
\end{align*}
\tag{68}
\]
Then, with the notation of Section III, we choose
\[
M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\]
\[
H = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{69}
\]
Since \( R_{H}^- = 0 \), the equality \( H = D_H + R_H^+ + R_H^- \) holds. Therefore, \( (D_H + R_H^+)\gamma P = H' P = -P \). Thus, Assumption 2 is satisfied and \( \overline{p} = \underline{p} = \rho = 1, \overline{m} = 2 \), and \( \pi = 1 \). Moreover, \( d_1 = 4\sqrt{2} \) and \( d_2 = 2\sqrt{2} \) and
\[
\kappa(r) = \frac{1}{2} \left( 1 + \frac{d_2 (S_0 - 1)}{d_1 S_0} \right) e^{-\frac{\nu_2 S_0}{r}} + \frac{1}{2} \left( 1 + \frac{d_2 (S_0 + 1)}{d_1 S_0} \right) e^{-\frac{\nu_2 S_0}{r}} - \frac{d_2}{d_1}, \tag{70}
\]
where \( S_0 = 1 + 16\sqrt{2} \). Let us observe that \( \kappa(r) \) is approximated in any neighborhood of zero by \( 1 - \rho r \) (by using the approximation from Remark 1 with \( a = p \)). It follows that there are two constants \( \nu_2 > 0 \) and \( \kappa_2 \in (0, 1) \) such that \( \kappa(\nu_2) = \kappa_2 \).

We deduce that for any \( \tau > 0 \), one can satisfy Assumption 3. Indeed, let \( h \in \mathbb{N} \) be such that
\[
\ln(2k(\overline{\tau})) - \ln(2k_2(\overline{\tau})) + 1 \leq h \tag{71}
\]
and consider \( t_i \) such that \( \nu_l = \nu_2 \) for all \( l \in \{0, \ldots, h - 2\} \), \( \nu_{h-1} = \overline{\tau} \) and \( \nu_{h} \) is periodic of period \( h \). Then for all \( j \in \mathbb{Z}_0 \),
\[
\prod_{l=j}^{j+h-1} \kappa(\nu_l) = \kappa(\overline{\tau}) \kappa_{2}^{-\frac{1}{2}}. \tag{72}
\]
From (71), if follows that for all \( j \in \mathbb{Z}_0 \), we have
\[
\prod_{l=j}^{j+h-1} \kappa(\nu_l) \leq \frac{1}{2}, \tag{73}
\]
so (68) satisfies the conclusions of Theorem 2.

APPENDIX

TECHNICAL RESULT

We prove that our conditions (53) from our proof of Theorem 2 hold for all \( t \geq s \), in order to complete the proof of this theorem. Let \( z = -\overline{\tau} \). Then (51) gives
\[
\begin{align*}
\dot{x}(t) &= (D_H(t) + R_H(t)^+)\tilde{x}(t) + R_H(t)^-z(t) \\
&+ \int_{t_i}^{t} \left( \frac{\kappa(t)\gamma(\ell) + \kappa(t)^+\gamma(\ell) + \kappa(t)^-\gamma(\ell)}{P} + \frac{\kappa(t)\gamma(\ell) + \kappa(t)^+\gamma(\ell) + \kappa(t)^-\gamma(\ell)}{P} \right) d\ell, \tag{A.1}
\end{align*}
\]
for all \( t \in [t_i, t_{i+1}) \). Since \( D_H(t) + R_H(t)^+ \) is Metzler for all \( t \geq 0 \) and all of the other matrices in (A.1) are nonnegative valued, we can prove that (A.1) is nonnegative, by writing (A.1) in the form
\[
\dot{q}(t) = M_1(t)q(t) + \int_{t_i}^{t} M_2(\ell)q(\ell)d\ell + M_3(t)q(t_i) + \rho(t) \tag{A.2}
\]
for all \( t \in [t_i, t_{i+1}) \) and \( i \in \mathbb{Z}_0 \) where \( M_1 \) is Metzler for all \( t \geq 0 \), and where \( M_2, M_3, \) and \( \rho \) are nonnegative for all \( t \geq 0 \), then noting that each entry of the state transition matrix \( \Phi_{M_1} \) for \( M_1 \) is positive valued (e.g., by [4, Lemma 2]) to conclude that the positive orthant \((0, \infty)^{2d}\) is forward invariant for (A.2), and finally using a continuous dependence argument to conclude nonnegativity of (A.2). Since \( t - t_i \leq \overline{\tau} \) for all \( t \in [t_i, t_{i+1}) \), and since (52) gives
\[
\begin{align*}
\tilde{x}(m) &= x(m) \\
z(m) &= x(m) \tag{A.3}
\end{align*}
\]
we get \( \tilde{x}(t) \geq 0 \) and \( z(t) \geq 0 \) for all \( t \geq s \). By the same reasoning, we can use (49) and (51) to check that the dynamics for \( (\tilde{x}(t) - x(t), x(t) + z(t)) \) is nonnegative, so (A.3) gives \( \tilde{x}(t) - x(t) \geq 0 \) and \( x(t) + z(t) \geq 0 \) for \( t \geq s \).

REFERENCES