Extremum seeking for unknown scalar maps in cascade with a class of parabolic partial differential equations

Tiago Roux Oliveira¹ | Jan Feiling² | Shumon Koga³ | Miroslav Krstic³

¹Department of Electronics and Telecommunication Engineering, State University of Rio de Janeiro (UERJ), Rio de Janeiro, Brazil
²Institute for Systems Theory and Automatic Control (IST), University of Stuttgart, Stuttgart, Germany
³Department of Mechanical and Aerospace Engineering, University of California - San Diego (UCSD), San Diego, California, USA

Correspondence
Tiago Roux Oliveira, Department of Electronics and Telecommunication Engineering, State University of Rio de Janeiro (UERJ), Rua São Francisco Xavier, 524, office S018E, 20.550-900, Maracanã, Rio de Janeiro, RJ, Brazil.
Email: tiagoroux@uerj.br

Summary
We present a generalization of the scalar gradient extremum seeking (ES) algorithm, which maximizes static maps in the presence of infinite-dimensional dynamics described by parabolic partial differential equations (PDEs). The PDE dynamics contains reaction-advection-diffusion (RAD) like terms. Basically, the effects of the PDE dynamics in the additive dither signals are canceled out using the trajectory generation paradigm. Moreover, the inclusion of a boundary control for the PDE process stabilizes the closed-loop feedback system. By properly demodulating the map output corresponding to the manner in which it is perturbed, the ES algorithm maximizes the output of the unknown map. In particular, our parabolic PDE compensator employs the same perturbation-based (averaging-based) estimate for the Hessian of the function to be maximized applied in the previous publications free of PDEs. We prove local stability of the algorithm, real-time maximization of the map and convergence to a small neighborhood of the desired (unknown) extremum by means of backstepping transformation, Lyapunov functional and the theory of averaging in infinite dimensions. Finally, we present the generalization to the scalar Newton-based ES algorithm, which maximizes the map's higher derivatives in the presence of RAD-like dynamics. By modifying the demodulating signals, the ES algorithm maximizes the nth derivative only through measurements of the own map. The Newton-based ES approach removes the dependence of the convergence rate on the unknown Hessian of the higher derivative, an effort to improve performance and remove limitations of standard gradient-based ES. Numerical examples support the theoretical results.

Keywords
adaptive control, averaging in infinite dimensions, backstepping transformation, gradient extremum seeking, partial differential equations
1 | INTRODUCTION

Extremum seeking (ES) has received great attention in the control community, being recognized as one of the powerful methodologies in adaptive systems to face control problems where the plant is poorly modeled or its model is contaminated by severe uncertainties and unmodeled dynamics.

Despite of ES has been successfully employed to many engineering applications, the authors in Reference 11 pointed out the presence of delay as one limiting factor in the application of ES in practical situations. Although for ES with distinct input and/or output delays many predictor-based control designs have been developed since the paper, it was not until the results in Reference 11 that a rigorous ES solution has appeared to systems described by partial differential equations (PDEs). In particular, only first-order hyperbolic transport PDEs were originally assumed in Reference 11 to represent pure delays. This key idea has enabled the development of extensions to other classes of PDEs, such as those describing diffusion phenomenon studied in References 14 and 15. For instance, the former reference considered the Gradient version of the ES algorithm, while the Newton-based method was explored in the latter one.

However, in many chemical and biological processes, the actuation dynamics can be described not only by a diffusion process but possibly also by a reaction-advectio-diffusion (RAD) processes. For example, tubular, fluidized bed and fixed bed reactors, or the process of crystal growth from the melt, all exhibits this kind of behavior. This motivates us to extend the results in References 14 and 15 to ES with actuation dynamics governed by RAD-like parabolic PDEs, as illustrated in Figure 1.

In particular, we first focus on the gradient ES algorithm. The complete control design employing a compensator for the RAD-like actuation dynamics is developed via backstepping transformation by feeding back the estimates for the gradient and Hessian (first and second derivatives) of the unknown static map to be maximized. Our proofs for local stability of the closed-loop system and the convergence to a small neighborhood of the extremum are based on backstepping methodology for PDE control, the construction of a Lyapunov functional, and the use of averaging theorem for infinite-dimensional systems. Second, we consider the Newton-based ES algorithm. Some of the advantages of the Newton-based approach are its faster responses when compared to the gradient method as well as the possibility of optimizing in real-time not only the output of the convex map but also its arbitrarily higher derivatives, that is, maximizing map sensitivity. This is a generalization of the result by Reference 12 for pure advection PDEs to PDEs that also include diffusion and reaction, besides the advection.

To the best of our knowledge, this is the first paper that rigorously considers ES for maximizing unknown maps (or its higher derivatives) in the presence of actuation dynamics described by RAD-like parabolic PDEs.

1.1 | Notation and terminology

We denote the partial derivatives of a function $u(x, t)$ as $u_x(x, t) = \partial_x u(x, t)/\partial x, u_t(x, t) = \partial_t u(x, t) = \partial u(x, t)/\partial t$. The 2-norm of a finite-dimensional (ODE) state vector $\theta(t)$ is denoted by single bars, $|\theta(t)|$. In contrast, norms of functions of $x$ are denoted by double bars. We denote the spatial $L^2([0, D])$ norm of the PDE state $u(x, t)$ as $\|u(t)\|_{L^2([0, D])}^2 = \int_0^D u^2(x, t)dx$, where we drop the index $L^2([0, D])$ in the following, hence $\| \cdot \| = \| \cdot \|_{L^2([0, D])}$, if not otherwise specified. As defined in Reference 24, a vector function $f(t, \epsilon) \in \mathbb{R}^n$ is said to be of order $\Theta(\epsilon)$ over an interval $[t_1, t_2]$, if

![FIGURE 1](image-url) Gradient extremum seeking loop for maximizing unknown scalar maps $y = Q(\Theta)$ with infinite-dimensional input dynamics
∃ k, \bar{\epsilon} : |f(t, \epsilon)| \leq kc, \forall \epsilon \in [0, \bar{\epsilon}] and \forall t \in [t_1, t_2]. In most cases, we give no estimation of the constants k and \bar{\epsilon}, then \Theta(e) can be interpreted as an order of magnitude relation for sufficiently small \epsilon.

2 | PROBLEM STATEMENT

Standard scalar ES considers applications in which one want to maximize (or minimize) the output \( y \in \mathbb{R} \) of an unknown nonlinear static map \( Q(\Theta) \):

\[
y(t) = Q(\Theta(t)),
\]

by varying the input \( \Theta \in \mathbb{R} \) in real time.

In this sense and without loss of generality, let us consider the maximization of the output (1) using gradient ES, where the maximizing value of \( \Theta \) is denoted by \( \Theta^* \). We state our optimization problem as:

\[
\max_{\Theta \in \mathbb{R}} Q(\Theta).
\]

**Assumption 1.** Let \( Q^{(n)}(\cdot) \) be the \( n \)th derivative of a smooth function \( Q(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \). Now let us define

\[
\Theta_{\text{max}} = \{ \Theta \in \mathbb{R} \mid Q^{(1)}(\Theta) = 0, \ Q^{(2)}(\Theta) < 0 \},
\]

to be a collection of maxima where \( Q \) is locally concave. Now, assume there exists a unique \( \Theta^* \in \Theta_{\text{max}} \) such that \( \Theta_{\text{max}} \neq \emptyset \).

In the neighborhood of \( \Theta^* \), we can write \( Q(\Theta) \) in its quadratic form

\[
Q(\Theta) = y^* + \frac{H}{2}(\Theta - \Theta^*)^2,
\]

where \( y^* \in \mathbb{R} \) is the unknown extremum for \( Q(\Theta^*) \) and the scalar \( H < 0 \) is the unknown Hessian of the static map. We can switch from maximization to minimization problem by setting simply \( H > 0 \) in (4).

3 | GRADIENT ES EQUATIONS

In Figure 1, the actuation dynamics in the input of the map (1) is described by a RAD-like parabolic PDE according to:

\[
\Theta(t) = \alpha(0, t)
\]

\[
\alpha_t(x, t) = \epsilon \alpha_{xx}(x, t) + ba_\lambda(x, t) + \lambda a(x, t), \quad x \in [0, 1]
\]

\[
\alpha_x(0, t) = 0
\]

\[
\alpha(1, t) = \theta(t),
\]

with known coefficients of diffusion \( \epsilon > 0 \), advection \( b \geq 0 \), and reaction \( \lambda \geq 0 \). Here, we use the term RAD in a sense generalized relative to its common use in chemical engineering.\(^{18,25,26}\) Moreover, we state the following assumption, to assure stability of the actuator dynamics.

**Assumption 2.** We assume the following relationship of the coefficients in the RAD PDE (6):

\[
\frac{b^2}{4\epsilon} \geq \lambda.
\]

Furthermore, the propagated error dynamics through the RAD process are given by

\[
\dot{\theta}(t) = u(0, t)
\]
\( u_t(x, t) = \epsilon u_{xx}(x, t) + bu_x(x, t) + \lambda u(x, t), \quad x \in [0, 1] \) \hspace{1cm} (11)

\( u_x(0, t) = 0 \) \hspace{1cm} (12)

\( u(1, t) = U(t) \), \hspace{1cm} (13)

where \( \delta(t) \) is the propagated estimation error

\[ \tilde{\delta}(t) := \hat{\delta}(t) - \theta^* , \] \hspace{1cm} (14)

through the RAD domain, and \( \hat{\delta}(t) \) is the estimation of the optimal input \( \theta^* \). From the block diagram in Figure 1, the error dynamics can be written as

\[ \hat{\delta}(t) = \hat{\delta}(t) = U(t) . \] \hspace{1cm} (15)

In contrast to the actuation dynamics governed by diffusion PDEs,\(^{14,15} \) we have \( \theta^* \neq \Theta^* \) in general. The relation is given by

\[ \theta^* = \Theta^* e^{-\frac{b}{2\epsilon} \left[ \frac{b}{2\epsilon} \sinh(\tau) + \cosh(\tau) \right]}, \quad \tau = \sqrt{\frac{b^2}{4\epsilon^2} - \frac{\lambda}{\epsilon}}, \] \hspace{1cm} (16)

and shown in Figure 2, where we chose \( \Theta^* = 1 \).

The perturbation signal \( S(t) \) is defined in order to solve the following \textit{trajectory generation problem} (chapter 12 of Reference 21) with Dirichlet boundary condition.

**Definition 1.** The perturbation signal \( S(t) \) is the boundary value of an inverse RAD process defined as follows:

\[ S(t) = \beta(1, t) \] \hspace{1cm} (17)

\[ \beta_t(x, t) = \epsilon \beta_{xx}(x, t) + b\beta_x(x, t) + \lambda \beta(x, t), \quad x \in [0, 1] \] \hspace{1cm} (18)

\[ \beta_x(0, t) = 0 \] \hspace{1cm} (19)

\[ \beta(0, t) = a \sin(\omega t), \] \hspace{1cm} (20)

with nonzero perturbation amplitude \( a \) and frequency \( \omega \).

**Figure 2** Relation between \( \theta^* \) and \( \Theta^* = 1 \) for different values of the diffusion coefficient \( \epsilon \). It can be observed that \( \theta^* \leq \Theta^* \) if \( \epsilon \leq 1 \). In the area where \( \theta^* = 0 \), Assumption 2 is not satisfied [Colour figure can be viewed at wileyonlinelibrary.com]
The explicit solution of (17) to (20) can be calculated as infinite sums:

\[ S(t) = e^{-b^2 \frac{t}{\epsilon}} \sum_{k=0}^{\infty} \frac{a_{2k}(t)}{(2k)!} + \frac{b}{2\epsilon} \frac{a_{2k}(t)}{(2k+1)!} \]

with \[ a_{2k} := \frac{a}{\epsilon^k} \sin(\omega t) \sum_{n=0}^{k} \left( \frac{k}{2n} \right) \xi^{k-2n-\frac{1}{2}} \cos(\omega t) \sum_{n=0}^{k} \left( \frac{k}{2n+1} \right) \xi^{k-2n-\frac{1}{2}} \]

\[ \xi := \frac{b^2}{4\epsilon} - \lambda \quad \text{and} \quad \left( \frac{y}{z} \right) := 0 \quad \text{for} \ y < z, \]

(21)

where the detailed derivation of (21) can be found in the Appendix.

Now, we define the multiplicative dithers \( M(t) \) and \( N(t) \) as

\[ M(t) = \frac{2}{a} \sin(\omega t), \quad N(t) = -\frac{8}{a^2} \cos(2\omega t), \]

(22)

and, consequently, the gradient and Hessian estimates for \( Q(\Theta) \) as:

\[ G(t) = M(t)y(t), \quad \hat{H}(t) = N(t)y(t). \]

(23)

### 4 | BOUNDARY CONTROL WITH BACKSTEPPING TRANSFORMATION FOR RAD COMPENSATION

In this section, we derive the RAD compensation controller. Therefore, we consider the PDE-ODE cascade (10)-(13) as illustrated in Figure 3.

First, we transform the system (10)-(13) with the transformation

\[ \Xi(x, t) = u(x, t)e^{\frac{b}{2\epsilon} x}, \]

(24)

into the reaction-diffusion system

\[ \delta(t) = \Xi(0, t) \]

(25)

\[ \Xi(x, t) = e^{\frac{b}{2\epsilon} x} - \xi \Xi(x, t), \quad x \in [0, 1] \]

(26)

\[ \Xi(0, t) = \frac{b}{2\epsilon} \Xi(0, t) \]

(27)

\[ \Xi(1, t) = U(t)e^{\frac{b}{2\epsilon}}, \]

(28)

1Another solution can be envisaged with an integral involving Bessel function. First convert RAD to RD with that \( \xi \) coefficient using the exponential scaling (28). Then, convert the RD equation into heat equation using backstepping. Then, solve trajectory generation explicitly for the heat equation, as in References 14 and 15. This will give us the state reference (12.18) and the input reference (12.19) in the book (chapter 12 of Reference 21), for the heat equation. Then you go back, using the inverse backstepping transform, to get the state and input references for the RD PDE. This is where the state reference (12.18) gets integrated against the Bessel kernel and you finally apply the inverse of the exponential scaling to get the state and input references for the RAD PDE.
with Robin and Dirichlet boundary conditions and \( \xi := b^2/(4e) - \lambda \geq 0 \). As with the usual procedure in backstepping, we propose the transformation of the system (25)-(28) into an exponentially stable target system.

**Proposition 1.** The backstepping transformation

\[
\begin{align*}
&w(x, t) = \bar{z}(x, t) - \int_0^x q(x, y)\bar{z}(y, t)dy - \gamma(x)\theta(t), \\
&\text{with the kernels} \\
&q(x, y) = K\frac{1}{e}\sqrt{\frac{\xi}{e}} \sinh\left(\sqrt{\frac{\xi}{e}}(x - y)\right) := \bar{K}m(x - y) \\
&\text{and} \\
&\gamma(x) = \bar{K} \cosh\left(\sqrt{\frac{\xi}{e}}x\right) + \frac{Kb}{2e}\sqrt{\frac{\xi}{e}} \sinh\left(\sqrt{\frac{\xi}{e}}x\right) := \bar{K}\gamma(x),
\end{align*}
\]

transforms the PDE-ODE cascade (10)-(13) into the exponentially stable target system

\[
\begin{align*}
&\dot{\theta}(t) = \bar{K}\theta(t) + w(0, t) \\
&w_t(x, t) = c\omega w_t(x, t) - \xi w(x, t), \quad x \in [0, 1] \\
&w_x(0, t) = \frac{b}{2e}w(0, t) \\
&w(1, t) = 0,
\end{align*}
\]

with \( \bar{K} < 0 \).

**Proof.** The proof is divided into two steps. First, the exponential stability of the target system (32)-(35) is proved. Second, the gain kernels \( q(x, y) \) and \( \gamma(x) \) are derived.

**Step 1:** Consider the Lyapunov-Krasovskii functional

\[
W(t) = \frac{1}{2} \theta^2(t) + \frac{1}{2} \int_0^1 w^2(x, t)dx.
\]

By taking the derivative w.r.t. time of (36) and integrating by parts, we get

\[
W(t) = \bar{K}\theta^2(t) + \theta(t)w(0, t) - \frac{b}{2}w^2(0, t) - c\|w_x(t)\|^2 - \xi e\|w(t)\|^2.
\]

Applying Young's inequality leads to

\[
W(t) \leq \left(\bar{K} + \frac{\gamma}{2}\right) \theta^2(t) + \left(\frac{1}{2\gamma} - \frac{b}{2}\right) w^2(0, t) - c\|w_x(t)\|^2 - \xi e\|w(t)\|^2.
\]

The parameters \( \gamma > 0 \) and \( \bar{K} < 0 \) can be chosen such that

\[
\exists \mu > 0 : W(t) \leq -\mu W(t),
\]

which shows the exponential stability of the target system (32)-(35).

**Step 2:** Differentiating the transformation (29) with respect to space \( x \) twice yields

\[
\begin{align*}
&w_x(x, t) = \bar{z}_x(x, t) - q(x, x)\bar{z}(x, t) - \int_0^x q_x(x, y)\bar{z}(y, t)dy - \gamma'(x)\theta(t),
\end{align*}
\]
\begin{align*}
    w_{xx}(x, t) &= \ddot{z}_{xx}(x, t) - \dddot{z}(x, t) \frac{d}{dx} q(x, x) - q(x, x) \dddot{z}(x, t) - q_{xx}(x, x) \dddot{z}(x, t) - \int_0^x q_{xx}(x, y) \ddot{z}(y, t) dy - \gamma''(x) \theta(t), \\
    \text{and with respect to time } t, \text{ gives} \\
    w_t(x, t) &= e \ddot{z}_{xx}(x, t) - eq(x, x) \dddot{z}(x, t) + \ddot{z}(x, t) \left[ eq_y(x, x) - \xi \right] + \dddot{z}(0, t) \left[ \frac{b}{2} q(x, 0) - eq_y(x, 0) - \gamma(x) \right] \\
    &\quad - \int_0^x \ddot{z}(y, t) \left[ eq_{yy}(x, y) - \xi q(x, y) \right] dy.
\end{align*}

By inserting (40)-(42) into the PDE (33), we obtain
\begin{align*}
    w_t(x, t) - cw_{xx}(x, t) - \xi w(x, t) &= 2 \ddot{z}(x, t) \frac{d}{dx} q(x, x) + \dddot{z}(0, t) \left[ \frac{b}{2} q(x, 0) - eq_y(x, 0) - \gamma(x) \right] + \\
    &\quad + \int_0^x \ddot{z}(y, t) \left[ eq_{xx}(x, y) - eq_{yy}(y, t) \right] dy + \theta(t) \left[ \gamma''(x) - \xi \gamma'(x) \right].
\end{align*}

With (43) and evaluating (40) and (41) at \( x = 0 \), we get the following conditions on the gain kernels \( q(x, y) \) and \( \gamma(x) \):
\begin{align*}
    \gamma''(x) &= \frac{\xi}{e} \gamma(x), \\
    \gamma(0) &= \bar{K}, \\
    \gamma'(0) &= \frac{\bar{K} b}{2e}, \\
    q_{xx}(x, y) - q_{yy}(x, y) &= 0, \\
    q(x, x) &= 0, \\
    q(x, 0) &= \frac{2e}{b} q_y(x, 0) + \frac{2}{b} \gamma(x).
\end{align*}

Solving the second-order ODE system (45)-(46) results in
\begin{equation}
    \gamma(x) = \bar{K} \gamma(x) = \bar{K} \cosh \left( \sqrt{\frac{\xi}{e}} x \right) + \frac{\bar{K} b}{2e} \sqrt{\frac{e}{\xi}} \sinh \left( \sqrt{\frac{\xi}{e}} x \right).
\end{equation}

Conditions (47)-(49) comprise a hyperbolic PDE of the Goursat type. Equations (47) and (48) lead to the convolutional-kernel ansatz
\begin{equation}
    q(x, y) = m(x - y),
\end{equation}
where \( m(\cdot) \) is a scalar function. With (49), we get the ODE
\begin{equation}
    m'(x) = -\frac{b}{2e} m(x) + \frac{1}{e} \gamma(x),
\end{equation}
to derive the kernel function \( q(x, y) \). Hence, using the ansatz \( m(x) = \exp(ax) \) and the variation of constant method, the gain kernel can be calculated to be
\begin{equation}
    q(x, y) = \bar{K} \tilde{m}(x - y) = \bar{K} \frac{1}{e} \sqrt{\frac{e}{\xi}} \sinh \left( \sqrt{\frac{\xi}{e}} (x - y) \right).
\end{equation}

This completes the proof.
Hence, the control law can be stated as

\[
U(t) = ke^{-\frac{b}{\varepsilon}} \left[ \bar{\gamma}(1)\theta(t) + \int_0^1 m(1-y)\bar{z}(y,t)dy \right],
\]

\[
\bar{\gamma}(1) = \cosh \left( \sqrt{\frac{\varepsilon}{\xi}} \right) + \frac{b^2}{2\varepsilon} \sinh \left( \sqrt{\frac{\varepsilon}{\xi}} \right),
\]

\[
\xi = \frac{b^2}{4\varepsilon} - \lambda \geq 0,
\]

\[
\bar{m}(1-y) = \frac{1}{\varepsilon} \sqrt{\frac{\varepsilon}{\xi}} \sinh \left( \sqrt{\frac{\varepsilon}{\xi}} (1-y) \right).
\]  \hspace{1cm} (54)

However, the control law in (54) is not applicable directly because we have no measurement on \( \theta(t) \). Thus, we introduce an important result of Reference 16: the averaged versions of the gradient and Hessian estimates in (23) are

\[
G_{av}(t) = H\theta_{av}(t), \quad \hat{H}_{av}(t) = H,
\]  \hspace{1cm} (55)

if at least a locally quadratic map as in (4) is considered. Regarding (55), we average (54) and choose \( \bar{K} = KH \) with \( K > 0 \), such that

\[
U_{av}(t) = ke^{-\frac{b}{\varepsilon}} \left[ \bar{\gamma}(1)G_{av}(t) + H \int_0^1 \bar{m}(1-y)\bar{z}_{av}(y,t)dy \right].
\]  \hspace{1cm} (56)

Due to technical reasons in the application of the averaging theorem for infinite-dimensional systems\(^{22}\) in the following stability proof, we introduce a low-pass filter to the controller. Finally, recalling (24), we get the average-based infinite-dimensional control law to compensate the RAD process:

\[
U(t) = \frac{c}{s+c} \left\{ Ke^{-\frac{b}{\varepsilon}} \left[ \bar{\gamma}(1)G(t) + \bar{H}(t) \int_0^1 e^{\frac{b}{\varepsilon}y} \bar{m}(1-y)u(y,t)dy \right] \right\}, \quad K > 0,
\]  \hspace{1cm} (57)

where constant \( c > 0 \) is chosen later. For notational convenience, we mix the time and frequency domain in (57), where the low-pass filter acts as an operator on the term between braces. Note that the RAD process of the actuation dynamics is known, that is, the coefficients \( c, b, \) and \( \lambda \).

**Remark 1.** Note that the feedback law (57) is consistent with the basic delay\(^{11}\) and diffusion\(^{14}\) cases. In fact, the delay and diffusion PDEs are special cases of this result, with \( c = \lambda = 0, \) and \( b > 0 \) for the former as well as \( \lambda = b = 0, \) and \( c > 0 \) for the latter.

## 5 | STABILITY AND CONVERGENCE

The proof of stability and convergence of the closed-loop system (10)-(13) with controller (57) is presented now.

### 5.1 | Average closed-loop system and its stability

Substituting (57) into (13) and reminding that \( G_{av}(t) = H\theta_{av}(t) \) from (55), we can write, for \( \omega \) sufficiently large, the average closed-loop system of (10)-(13) as:

\[
\dot{\theta}_{av}(t) = u_{av}(0, t)
\]  \hspace{1cm} (58)

\[
(u_{av})_t(x, t) = c(u_{av})_{xx}(x, t) + b(u_{av})_x(x, t) + \lambda u_{av}(x, t), \quad x \in [0, 1]
\]  \hspace{1cm} (59)

\[
(u_{av})_x(0, t) = 0
\]  \hspace{1cm} (60)
(u_{av})(1, t) = -cu_{av}(1, t) + cKHe^{-\frac{c}{\xi}} \left[ \tilde{\varphi}(1)\theta_{av}(t) + \int_{0}^{1} e^{-\frac{c}{\xi}y}M(1 - y)u_{av}(y, t)dy \right]. \tag{61}

In the next proposition, we show the exponential stability of the closed-loop average system.

**Proposition 2.** The average closed-loop system (58)-(61) is exponentially stable in the sense of the $H_1$-norm

$$\left( |\theta_{av}(t)|^2 + \int_{0}^{1} u_{av}^2(x, t)dx + \int_{0}^{1} (u_{av})^2_{x}(x, t)dx + u_{av}^2(1, t) + u_{av}^2(0, t) \right)^{1/2}. \tag{62}$$

**Proof.** With some abuse of notation, we also use $w(x, t)$ to denote the average transformed state, and from the backstepping transformation (29) in the variables $(u_{av}, w)$:

$$w(x, t) = e^{-\frac{c}{\xi}x}u_{av}(x, t) - \int_{0}^{x} \tilde{\varphi}(x, y)e^{-\frac{c}{\xi}y}u_{av}(y, t)dy - \tilde{\varphi}(x)\theta_{av}(t), \tag{63}$$

we can state its inverse (the complete derivation can be found in the Appendix) in the variables $(u_{av}, w)$ as

$$u_{av}(x, t) = e^{-\frac{c}{\xi}x}w(x, t) - e^{-\frac{c}{\xi}x} \int_{0}^{x} \tilde{p}(x, y)w(y, t)dy - e^{-\frac{c}{\xi}x}\tilde{\eta}(x)\theta_{av}(t), \tag{64}$$

with

$$\tilde{\varphi}(x) = KH\tilde{\varphi}(x) = KH \cosh \left( \sqrt{\frac{2}{e}}x \right), \tag{65}$$

$$\tilde{\varphi}(x) = KH\tilde{\varphi}(x) = KH \cosh \left( \sqrt{\frac{2}{e}}x \right), \tag{66}$$

$$\tilde{\varphi}(x) = KH\tilde{\varphi}(x) = KH \cosh \left( \sqrt{\frac{2}{e}}x \right), \tag{67}$$

which transforms the average closed-loop system (58)-(61) into the target system

$$\dot{\theta}_{av}(t) = KH\theta(t) + w(0, t) \tag{69}$$

$$w_{x}(x, t) = cw_{x}(x, t) - \xi w(x, t), \quad x \in [0, 1] \tag{70}$$

$$w_{x}(0, t) = \frac{b}{2e}w(0, t) \tag{71}$$

$$w_{1}(1, t) = -cw(1, t) - KHw(1, t) - (KH)^2 \left[ \tilde{\eta}(1)\theta_{av}(t) + \int_{0}^{1} \tilde{\eta}(1 - y)w(y, t) \right]. \tag{72}$$

Consider the Lyapunov-Krasovskii functional

$$V(t) = \frac{\theta_{av}^2(t)}{2} + \frac{a}{2} \int_{0}^{1} w^{2}(x, t)dx + \frac{d}{2} \int_{0}^{1} w_{x}^2(x, t)dx + \frac{e}{2} w^2(1, t) + \int_{0}^{1} w^2(0, t), \tag{73}$$

and its time derivative along with (58)-(61) and (69)-(72)
\[ \dot{V}(t) = KH\dot{\theta}_{av}(t) + \theta_{av}(t)w(0, t) - \frac{d}{2}w^2(0, t) - ae \int_0^1 w_x^2(x, t)dx - a\xi \int_0^1 w^2(x, t)dx + a\epsilon w(1, t)w_x(1, t) - d\epsilon \int_0^1 w_x^2(x, t)dx - \frac{db}{2c}w^2(0, t) + dw_x(1, t)w(1, t) + d\xi w_x(1, t)w(1, t) + f\theta(0, t)w_t(0, t), \]

where we choose \( e = db/(2c) \) and \( a, d, \epsilon, f > 0 \). Since the nonnegative terms of \( \dot{V} \) are the same as of the time derivative of the Lyapunov-Krasovkii functionals in References 14 and 15, we conclude following similar calculations that there exist \( c^* > 0 \) and \( \mu^* > 0 \) such that \( \dot{V}(t) \leq -\mu^*V(t) \) and the target system (69)-(72) is exponentially stable in \( H_1 \)-norm

\[ \left( |\dot{\theta}_{av}(t)|^2 + \int_0^1 \left[ w^2(x, t)dx + \int_0^1 w_x^2(x, t)dx + w^2(1, t) + w^2(0, t) \right] \right)^{1/2} \leq \Theta(1/\omega). \]  

for \( c > c^* \) sufficiently large in (57). Moreover, as in the proof of Theorem 1, step 4, of Reference 14, the exponential stability property in the variables \( (\theta_{av}, w) \) can be transferred into the original variables \( (\theta_{av}, u_{av}) \) by using the inverse transformation (64). Hence, the average closed-loop system (58)-(61) is also exponentially stable in the sense of the \( H_1 \)-norm (62). \( \Box \)

### 5.2 Invoking averaging theorem

The following proposition shows we can apply the averaging theorem for infinite-dimensional systems of section 2 of Reference 22 to the closed-loop system so that there exists an unique exponentially stable periodic solution in \( t \) of period \( \Pi := 2\pi/\omega \), denoted by \( \theta^{\Pi}(t), u^{\Pi}(t) \), satisfying \( \forall t \geq 0 \):

\[ \left( |\dot{\theta}(t)|^2 + \int_0^1 \left[ |u(t)|^2dx + \int_0^1 [u_x(t)]^2dx + [u(1, t)]^2 + [u(0, t)]^2 \right] \right)^{1/2} \leq \Theta(1/\omega). \]  

**Proposition 3.** The original closed-loop system (10)-(13) with the controller (57) can be written in the form

\[ \dot{z}(t) = FZ(t) + J(\omega t, Z(t)), \]  

where \( F \) generates an analytic semigroup, \( J(\omega t, Z(t)) \) satisfies the smoothness conditions of the averaging theorem (section 2 of Reference 22) and \( Z(t) \) is an infinite-dimensional state vector.

**Proof.** The state-transformation \( \nu(x, t) = u(x, t) - U(t) \) transforms the original closed-loop system (10)-(13) with the controller (57) into

\[ \dot{\theta}(t) = \nu(0, t) + U(t) \]  

\[ \nu_x(x, t) = \epsilon \nu_{xx}(x, t) + b

\nu(x, t) + \lambda \nu(x, t) - \phi(\theta, \nu, U, t) \]  

\[ \nu_x(0, t) = 0 \]  

\[ \nu(1, t) = 0 \]  

\[ \dot{U}(t) = \phi(\theta, \nu, U, t), \]  

with

\[ \phi(\theta, \nu, U, t) = -cU(t) + cK \left[ \overline{\theta}(1)G(t) + \dot{\theta}(t) \int_0^1 e^{\frac{\gamma}{2}y}M(1 - y)(\nu(y, t) + U(t))dy \right]. \]  

We write the PDE (78) as an evolutionary equation\(^7\) in the Banach space \( X := H_1([0, \Pi]) \)

\[ \dot{V}(t) = AV(t) - \phi(\theta, V, U, t), \quad t > 0 \]  

\( \Box \)
with
\[ A \varphi := \epsilon \frac{\partial^2 \varphi}{\partial x^2} + b \frac{\partial \varphi}{\partial x} + \lambda \varphi, \] (84)
\[ D(A) = \left\{ \varphi \in \mathcal{X} : \varphi, \frac{d \varphi}{dx} \in \mathcal{X}, \frac{d^2 \varphi}{dz^2} \varphi \in \mathcal{X}, \frac{d}{dz} \varphi(0) = 0, \varphi(1) = 0 \right\}. \] (85)

By introducing the linear operators
\[ \beta : \mathbb{R} \to \mathcal{X} \text{ s.t. } \beta \zeta := [\beta_1, \beta_2, \ldots]^T \zeta \quad \text{with } \beta_k = \int_0^1 \psi_k(x)dx, \quad \zeta \in \mathbb{R}, \] (86)
\[ B : \mathcal{X} \to \mathbb{R} \text{ s.t. } BV(t) := v(0, t), \] (87)
\[ \alpha^T : \mathcal{X} \to \mathbb{R} \text{ s.t. } \alpha^T V(t) := \sum_{k=1}^{\infty} \alpha_k v_k^*(t) \text{ with } \alpha_k = \int_0^1 e^{b \sqrt{m} (1 - y)} \tilde{\psi}_k(y)dy, \] (88)

where \( \psi_k \) are the eigenfunctions and \( \tilde{\psi}_k(x) \) are the adjoint eigenfunctions of \( A \), we arrive at the infinite-dimensional system of the form (76) with state vector \( Z(t) = [\theta(t)V(t)U(t)]^T \),
\[ F = \begin{bmatrix} 0 & B & 1 \\ 0 & A & c \beta \\ 0 & 0 & -c \end{bmatrix}, \] (89)
\[ J(\omega t, Z) = \begin{bmatrix} 0 \\ 0 \\ c \beta K \left[ G(t) + \dot{H}(t)g(Z(t)) \right] \\ c \left[ G(t) + \dot{H}(t)g(Z(t)) \right] \end{bmatrix}, \] (90)

where \( g(Z(t)) = \mu U(t) + \alpha^T V(t) \) and
\[ \mu = \int_0^1 e^{b \sqrt{m} (1 - y)}dy. \] (91)

Since \( A \) is a strongly elliptic operator,\(^{28}\) it is also a generator of an analytic semigroup (see Theorem 2.7 of Reference 29). The \( A \)-boundedness of the boundary operator \( B \) in (86) can be easily proven and is satisfied with the parameters \( \epsilon_1 = \max\{2, \epsilon, b, \lambda\} \), \( \epsilon_2 = 0 \) in the operator matrix theorem.\(^{30}\) Hence, the operator matrix \( F \) generates an analytic semigroup. Furthermore, \( J(\omega t, Z) \) in (89) is Fréchet differentiable in \( Z \), strongly continuous and periodic in \( t \) uniformly with respect to \( Z \). \hfill \Box

Hence, along with the exponential stability of the average closed-loop system (58)-(61), all assumptions to apply the averaging theorem for infinite-dimensional systems in section 2 of Reference 22 are satisfied. Thus, the closed-loop system (10)-(13) with controller (57) has an exponentially stable periodic solution \( Z^*(t) \) with \( \|Z^*(t)\| \leq O(1/\omega) \) that satisfies (75).

### 5.3 Convergence to a neighborhood of the extremum

Employing analogous of the procedure carried out in the proof of Theorem 1, step 6, of Reference 14, we can state the convergence of \( (\theta(t), \Theta(t), y(t)) \) to the neighborhood of the optimizers \( (\theta^*, \Theta^*, y^*) \) without further calculations:
\[ \limsup_{t \to \infty} |\theta(t) - \theta^*| = O \left( |a| \exp \left( \sqrt{\frac{\xi + \omega}{\epsilon}} + 1/\omega \right) \right), \quad \xi = \frac{b^2}{4\epsilon} - \lambda, \] (92)
\[
\limsup_{t \to \infty} |\Theta(t) - \Theta^*| = \mathcal{O}(|a| + 1/\omega), \tag{93}
\]
\[
\limsup_{t \to \infty} |y(t) - y^*| = \mathcal{O} \left( |a|^2 + 1/\omega^2 \right). \tag{94}
\]

The exponential term in (92) results from the order of \( S(t) \) in (21) by expanding the sums. Finally, (92) shows the ultimate convergence of the proposed ES approach with actuation dynamics governed with RAD PDEs of Figure 1 to a neighborhood of \((\theta^*, \Theta^*, y^*)\), by choosing \( \omega \) sufficiently large, and \( a \) sufficiently small.

### 6 | Extension to Newton-Based ES for Maximizing Higher Derivatives of Unknown Maps with Parabolic PDEs

Standard scalar gradient ES considers applications in which one wants to maximize (or minimize) the output \( y \in \mathbb{R} \) of an unknown nonlinear static map \( Q(\Theta) \), as described in Section 2.

In Reference 23, a new problem formulation was introduced for Newton-based ES: instead of optimizing the output of the map itself, the objective was to optimize any higher derivative of its output (assuming only the measurement of \( y \)). In this sense and without generality, let us consider the maximization of the \( n \)th derivative of the output (1), where the maximizing value of \( \Theta \) is denoted by \( \Theta^* \). We state our optimization problem as:

\[
\max_{\Theta \in \mathbb{R}} Q^{(n)}(\Theta). \tag{95}
\]

**Assumption 3.** Let \( Q^{(n)}(\cdot) \) be the \( n \)th derivative of a smooth function \( Q(\cdot): \mathbb{R} \to \mathbb{R} \). Now let us define

\[
\Theta_{\text{max}} = \{ \Theta | Q^{(n+1)}(\Theta) = 0, \quad Q^{(n+2)}(\Theta) < 0 \} \tag{96}
\]

to be a collection of maxima where \( Q^{(n)} \) is locally concave. Now assume that \( \Theta^* \in \Theta_{\text{max}} \) and \( \Theta_{\text{max}} \neq \emptyset \).

In the neighborhood of \( \Theta^* \), we can write \( Q^{(n)}(\Theta) \) in its quadratic form

\[
Q^{(n)}(\Theta) = y^* + \frac{H}{2}(\Theta - \Theta^*)^2, \tag{97}
\]

where \( y^* \in \mathbb{R} \) is the unknown extremum for \( Q^{(n)}(\Theta) \) and the scalar \( H < 0 \) is the unknown Hessian of the \( n \)th derivative of the static map. According to Reference 23, we switch from maximization to minimization by setting \( \text{sgn}(\Gamma_0) = \text{sgn}(Q^{(n+2)}(0)) \) with \( \Gamma_0 \) being the initial value of the Riccati filter to be discussed later on.

#### 6.1 | Newton-Based ES Equations

Basically, all problem formulations (5)-(21) constructed before repeat itself for the purpose of obtaining a Newton version of the proposed algorithm. The principal change is the redefinition of the signals \( M(t), N(t), \tilde{H}(t), G(t), \) and \( z(t) \). In this sense, we first recall the following auxiliary signal from Reference 23:

\[
Y_j(t) = C_j \sin \left( j\omega t + \frac{\pi}{4} (-1)^{n+1} (1 + (-1)^j) \right), \tag{98a}
\]

\[
C_j = \frac{2^{j-1}}{a^j} (-1)^F, \quad F = \frac{j - \left| \sin \left( \frac{j\pi}{2} \right) \right|}{2}. \tag{98b}
\]

In References 12 and 23, it was defined the signal \( \tilde{h}^{(j)}(t) := y(t)Y_j(t) \) by means of it was possible to obtain an estimate for the gradient of \( Q^{(n)}(\Theta) \) if the index \( j \) was chosen equal to \( j = n + 1 \), and for its Hessian if \( j = n + 2 \), respectively. Therefore, we define the multiplicative dithers \( M(t) \) and \( N(t) \) as

\[
M(t) = Y_{n+1}(t), \quad N(t) = Y_{n+2}(t), \tag{99}
\]
and, consequently, the gradient and Hessian estimates for $Q^{(n)}(\Theta)$ as:

$$G(t) = M(t)y(t), \quad \dot{H}(t) = N(t)y(t) .$$

(100)

Finally, let us define the signal

$$z(t) = \Gamma(t)G(t) ,$$

(101)

where $\Gamma(t)$ is updated according to the following Riccati differential equation:16

$$\dot{\Gamma} = \omega_r \Gamma - \omega_r \dot{H}^2 ,$$

(102)

with $\omega_r > 0$ being a design constant. Equation (102) generates an estimate of Hessian’s inverse ($H^{-1}$), avoiding inversions that may cross zero during the transient phase. The estimation error of Hessian’s inverse is defined as

$$\tilde{\Gamma}(t) = \Gamma(t) - H^{-1} ,$$

(103)

and its dynamic equation is written from (102) and (103) by

$$\dot{\tilde{\Gamma}} = \omega_r (\tilde{\Gamma} + H^{-1})(1 - \dot{H}(\tilde{\Gamma} + H^{-1})) .$$

(104)

According to Proposition 1, we are able to propose a Newton-based ES controller following the same backstepping transformation in (29)-(31), with $K = -K$ and $K > 0$.

Remind that the averaged versions of the gradient and Hessian estimates in (100) are (55) if at least a locally quadratic map as in (97) is considered.12,23 Hence, from (55) and $z(t)$ in (101), we can verify that

$$z_{av}(t) = \frac{1}{\Pi} \int_{0}^{\Pi} \Gamma M(\lambda)y d\lambda = \Gamma_{av}(t)H\theta_{av}(t) .$$

(105)

where $\Pi := 2\pi/\omega$, $\Gamma_{av}(t)$ and $\theta_{av}(t)$ denote the average versions of $\Gamma(t)$ and $\theta(t)$, respectively. Then, (105) can be written in terms of $\Gamma_{av}(t) = \Gamma_{av}(t) - H^{-1}$ as

$$z_{av}(t) = \theta_{av}(t) + \Gamma_{av}(t)H\theta_{av}(t) .$$

(106)

The second term in the right side of (106) is quadratic in $(\Gamma_{av}, \theta_{av})$, thus, the linearization of $\Gamma_{av}(t)$ at $H^{-1}$ results in the linearized version of (105) given by

$$z_{av}(t) = \theta_{av}(t) .$$

(107)

Regarding (107), we average (54) and choose $K = -K$ with $K > 0$, such that

$$U_{av}(t) = -Ke^{-\frac{\theta}{\Pi}} \left[ \tilde{\Gamma}(1)z_{av}(t) + \int_{0}^{1} e^{\frac{y}{\Pi}} \bar{m}(1 - y)z_{av}(y, t) dy \right] .$$

(108)

Analogously to (57) and recalling (24), we get the filtered average-based infinite-dimensional control law of (108) to compensate the RAD process:

$$U(t) = \frac{c}{s + c} \left\{ -Ke^{-\frac{\theta}{\Pi}} \left[ \tilde{\Gamma}(1)z(t) + \int_{0}^{1} e^{\frac{y}{\Pi}} \bar{m}(1 - y)u(y, t) dy \right] \right\} , \quad K > 0 ,$$

(109)

where $c > 0$ is an appropriate constant. Figure 4 shows an illustrative block diagram of the closed-loop system for the proposed Newton-based ES controller with PDEs in the input channel.
6.2 Stability analysis

The closed-loop system consists of the propagated error dynamics (10)-(13) with the controller (109) plus Hessian’s inverse error estimation dynamics (104). The following theorem summarizes the stability results of such a feedback system.

**Theorem 1.** Consider the closed-loop system (10)-(13) with controller (109) and suppose that Assumption 3 is satisfied for the map (1). For a sufficiently large \( c > 0 \), there exists some \( \omega > 0 \), such that \( \forall \omega > \omega \), the closed-loop system with states \( \Gamma(t), \vartheta(t), u(x,t) \), has a unique exponentially stable periodic solution in \( t \) of period \( \Pi := \frac{2\pi}{\omega} \), denoted by \( \Gamma_{\Pi}(t), \vartheta_{\Pi}(t), u_{\Pi}(x,t) \), satisfying \( \forall t \geq 0 \):

\[
\left( |\Gamma^{\Pi}(t)|^2 + |\vartheta^{\Pi}(t)|^2 + \int_0^1 [u^{\Pi}(x,t)]^2 dx + \int_0^1 [u_x^{\Pi}(x,t)]^2 dx + [u^{\Pi}(1,t)]^2 + [u^{\Pi}(0,t)]^2 \right)^{1/2} \leq \Theta(1/\omega). \tag{110}
\]

Furthermore,

\[
\limsup_{t \to \infty} |\theta(t) - \vartheta| = \Theta (|a| \exp \left( \sqrt{\frac{\xi + \omega}{\epsilon}} + 1/\omega \right)), \tag{111}
\]

\[
\limsup_{t \to \infty} |\Theta(t) - \vartheta| = \Theta (|a| + 1/\omega). \tag{112}
\]

**Proof.** The proof of Theorem 1 follows the same ideas carried out to prove Propositions 2 and 3 in Section 5 for the Gradient ES. For the sake of clarity on how we prove the stability and convergence properties, we present the structure of the proof in Figure 5, divided into six main steps.
The first step is to write the equations of the closed-loop system in the original variables \( \Gamma(t), \theta(t), u(x, t) \). Therefore, we expand all the equations to apply the averaging theory in order to obtain the closed-loop average system in the next step. Then, we use a backstepping transformation to first show the stability of the resulting target system with a suitable Lyapunov-Krasovskii functional and further the exponential stability of the average closed-loop system, due to the invertibility of the transformation. Invoking the averaging theorem for infinite-dimensional systems (see section 2 of Reference 22), we prove the exponential stability of the original closed-loop system. Finally, we can show the convergence of \((\theta(t), \Theta(t))\) to a neighborhood of the extremum \((\theta^*, \Theta^*)\) for \( t \to \infty \).

**Step 1: Original Closed-loop System**

Substituting (109) into (13), we can write the equations of the closed-loop system (10)-(13) and (104) as

\[
\dot{\theta}(t) = u(0, t) \tag{113}
\]

\[
\partial_t u(x, t) = cu_{av}(x, t) + bu(x, t) + \lambda u(x, t), \quad x \in [0, 1] \tag{114}
\]

\[
u(a, 0, t) = 0 \tag{115}
\]

\[
u(1, t) = U(t) \tag{116}
\]

\[
\dot{U}(t) = -cU(t) + c \left\{ -Ke^{-\frac{b}{2}} \left[ \tilde{\gamma}(1)z(t) + \int_0^1 e^{\frac{b}{2}y}m(1-y)u(y, t)dy \right] \right\} \tag{117}
\]

\[
\hat{\Gamma} = \omega_\gamma(\hat{\Gamma} + H^{-1})[1 - \hat{H}(\hat{\Gamma} + H^{-1})] \tag{118}
\]

**Step 2: Average Closed-loop System**

Reminding that \( z_{av}(t) = \theta_{av}(t) \) from (107), we can obtain, for \( \omega \) sufficiently large, the average closed-loop system of (113)-(117):

\[
\dot{\theta}_{av}(t) = u_{av}(0, t) \tag{119}
\]

\[
(u_{av})_\gamma(x, t) = \epsilon(u_{av})_\gamma(x, t) + b(u_{av})_\gamma(x, t) + \lambda u_{av}(x, t), \quad x \in [0, 1] \tag{120}
\]

\[
(u_{av})_\gamma(0, t) = 0 \tag{121}
\]

\[
(u_{av})_\gamma(1, t) = -cu_{av}(1, t) - cKe^{-\frac{b}{2}} \left[ \tilde{\gamma}(1)\theta_{av}(t) + \int_0^1 e^{\frac{b}{2}y}m(1-y)u_{av}(y, t)dy \right] \tag{122}
\]

On the other hand, the average model for Hessian’s inverse estimation error in (104), or equivalently (118), is

\[
\frac{d\bar{r}}{\Gamma_{av}(t)} = -\omega_\gamma \Gamma_{av}(t) - \omega_\gamma H_{av}(t) \tag{123}
\]

since \( H_{av} = \frac{1}{\Omega} \int_0^\Omega N(\lambda)y d\lambda = H \).

**Step 3: Target System**

With some abuse of notation, we also use \( w(x, t) \) to denote the average transformed state, and from the backstepping transformation (29) in the variables \((u_{av}, w)\):

\[
w(x, t) = e^{\frac{b}{2}x}u_{av}(x, t) - \int_0^x \tilde{q}(x, y)e^{\frac{b}{2}y}u_{av}(y, t)dy - \tilde{\gamma}(x)\theta_{av}(t), \tag{124}
\]

we can state its inverse in the variables \((u_{av}, w)\) as

\[
u_{av}(x, t) = e^{-\frac{b}{2}x}w(x, t) - e^{-\frac{b}{2}x} \int_0^x \tilde{p}(x, y)w(y, t)dy - e^{-\frac{b}{2}x}\tilde{q}(x)\theta_{av}(t), \tag{125}
\]

with

\[
\tilde{q}(x, y) = -Km(x - y) = -K \frac{1}{\epsilon} \sqrt{\frac{\epsilon}{\xi}} \sinh \left( \sqrt{\frac{\xi}{\epsilon}}(x - y) \right), \tag{126}
\]
\[ \ddot{\tilde{x}}(x) = -K \tilde{y}(x) = -K \cosh \left( \sqrt{\frac{\xi}{\varepsilon}} x \right) - \frac{Kb}{2e} \sqrt{\frac{\varepsilon}{\xi}} \sinh \left( \sqrt{\frac{\xi}{\varepsilon}} x \right), \] (127)

\[ \ddot{\tilde{y}}(x, y) = -K \tilde{n}(x - y) = \frac{K}{\sqrt{\xi} - K} \sinh \left( \sqrt{\frac{\xi}{\varepsilon}} (x - y) \right), \] (128)

\[ \ddot{\eta}(x) = -K \eta(x) = K \cosh \left( \sqrt{\frac{\xi}{K}} x \right) + \frac{K}{2e\sqrt{\xi} - K} \sinh \left( \sqrt{\frac{\xi}{\varepsilon}} x \right), \] (129)

which transforms the average closed-loop system (119)-(122) into the target system

\[ \dot{\theta}_{av}(t) = -K \dot{\theta}_{av}(t) + w(0, t) \] (130)

\[ w_{av}(x, t) = cw_{av}(x, t) - \xi w_{av}(x, t), \quad x \in [0, 1] \] (131)

\[ w_{av}(0, t) = \frac{b}{2e} w(0, t) \] (132)

\[ w_{av}(1, t) = -cw(1, t) + Kw(1, t) - K^2 \left[ \eta(1) \dot{\theta}_{av}(t) + \int_0^1 \tilde{y}(1 - y)w(y, t) \right]. \] (133)

**Step 4: Exponential Stability of the Target System in \( H_1 \)-norm**

Consider the Lyapunov-Krasovskii functional

\[ V(t) = \frac{\dot{\theta}_{av}^2(t)}{2} + \frac{a}{2} \int_0^1 w^2(x, t)dx + \frac{d}{2} \int_0^1 w_{av}^2(x, t)dx + \frac{e}{2} w^2(1, t) + \frac{f}{2} w^2(0, t), \] (134)

and its time derivative along with (119)-(122) and (130)-(133)

\[ \dot{V}(t) = -K \dot{\theta}_{av}(t) + \dot{\theta}_{av}(t)w(0, t) - \frac{a}{2} w^2(0, t) - ac \int_0^1 w_{av}^2(x, t)dx - a \xi \int_0^1 w^2(x, t)dx + acw(1, t)w_{av}(1, t) + \\
- de \int_0^1 w_{av}^2(x, t)dx - \frac{db}{2e} w^2(0, t) + dw_{av}(1, t)w(1, t) + dw_{av}(0, t)w(0, t) + fw(0, t)w(1, t), \] (135)

where we choose \( e = db/(2e) \) and \( a, d, e, f > 0 \). Since the nonnegative terms of \( V \) are the same as of the time derivative of the Lyapunov-Krasovskii functionals in References 14 and 15, we conclude the following similar calculations that there exist \( c^* > 0 \) and \( \mu^* > 0 \) such that \( \dot{V}(t) \leq -\mu^* V(t) \) and the target system (130)-(133) is exponentially stable in \( H_1 \)-norm

\[ \left( |\dot{\theta}_{av}(t)|^2 + \int_0^1 w^2(x, t)dx + \int_0^1 w_{av}^2(x, t)dx + w^2(1, t) + w^2(0, t) \right)^{1/2}, \] (136)

for \( c > c^* \) sufficiently large in (109).

**Step 5: Local Exponential Stability of the Average Closed-Loop System**

As in the proof of Theorem 1, step 4, of Reference 14, the exponential stability property in the variables \( (\theta_{av}, w) \) can be transferred to the original variables \( (\tilde{\theta}_{av}, \tilde{u}_{av}) \) by using the inverse transformation (125). Hence, the average closed-loop system (119)-(122) is also exponentially stable in the sense of the \( H_1 \)-norm

\[ \left( |\tilde{\theta}_{av}(t)|^2 + \int_0^1 \tilde{u}_{av}^2(x, t)dx + \int_0^1 (\tilde{u}_{av})_{av}^2(x, t)dx + \tilde{u}_{av}^2(1, t) + \tilde{u}_{av}^2(0, t) \right)^{1/2}. \] (137)

In addition, the linearized version for the average model of Hessian’s inverse estimation error in (123) around \( \tilde{f}_{av}(t) = 0 \) is given by
\[
\frac{d\hat{\Gamma}_{av}(t)}{dt} = -\omega_r \hat{\Gamma}_{av}(t),
\]

which is also exponentially stable for \(\omega_r > 0\).

Hence, the average of the full closed-loop system (119)-(122) and (138) is exponentially stable in the sense of the \(H_1\)-norm

\[
\left( |\Gamma_{av}(t)|^2 + |\theta_{av}(t)|^2 + \int_0^1 u^2_{av}(x, t) dx + \int_0^1 (u_{av})_x^2(x, t) dx + u^2_{av}(1, t) + u^2_{av}(0, t) \right)^{1/2}.
\]

**Step 6: Local Exponential Stability of the Original Closed-Loop System via Averaging Theorem**

The state-transformation \(v(x, t) = u(x, t) - U(t)\) transforms the original closed-loop system (113)-(117) with the controller (109) into

\[
\begin{align*}
\dot{\theta}(t) &= v(0, t) + U(t) \\
v_1(x, t) &= ev_{ax}(x, t) + bv_x(x, t) + \lambda v(x, t) - \phi(\theta, v, U, t) \\
v_2(0, t) &= 0 \\
v(1, t) &= 0 \\
\dot{U}(t) &= \phi(\theta, v, U, t),
\end{align*}
\]

with

\[
\phi(\theta, v, U, t) = -cU(t) - cK \left[ \overline{f}(1)z(t) + \int_0^1 e^{z+1} m(1 - y) (v(y, t) + U(t)) dy \right].
\]

We write the PDE (141) as an evolutionary equation in the Banach space \(\mathcal{X} := H_3([0, 1])\)

\[
V(t) = AV(t) - \hat{\phi}(\theta, V, U) + H(t), \quad \forall t > 0,
\]

with an appropriate operator \(A\) and function \(\hat{\phi}\). Thus, we arrive at the infinite-dimensional system of the form

\[
\dot{Z}(t) = FZ(t) + J(\omega t, Z(t)),
\]

with infinite-dimensional state vector \(Z(t) = [\theta(t) V(t) U(t) \Gamma(t)]^T\),

\[
F = \begin{bmatrix} 0 & B & 0 \\ 0 & A & c\beta \\ 0 & 0 & -c & 0 \end{bmatrix}, \quad J(\omega t, Z) = \begin{bmatrix} 0 \\ -c\beta K \left[ z(t) + g(Z) \right] \\ -cK \left[ z(t) + g(Z) \right] \\ \omega_r H^{-1} - \omega_r \hat{H}(t) (\Gamma + H^{-1}) \end{bmatrix},
\]

where \(g(Z) = \mu U(t) + a^TV(t)\) and \(\mu = \int_0^1 e^{z+1} m(1 - y) dy\), for adequate linear operators \(B, a,\) and \(\beta\). Since \(A\) is a strongly elliptic operator, it is also a generator of an analytic semigroup (see Theorem 2.7 of Reference 29). The \(A\)-boundedness of the boundary operator \(B\) can be easily proven and is satisfied with the parameters \(\epsilon_1 = \max \{ 2, c, b, \lambda \}, \epsilon_2 = 0\) in the operator matrix theorem. Hence, the operator matrix \(F\) generates an analytic semigroup and \(J(\omega t, Z(t))\) satisfies the smoothness conditions of the averaging theorem (section 2 of Reference 22) and along with the exponential stability of the average closed-loop system (119)-(122), all assumptions to apply the averaging theorem for infinite-dimensional systems in section 2 of Reference 22 are satisfied. Thus, the closed-loop system (10)-(13) with controller (109) has an exponentially stable periodic solution \(Z^H(t)\) with \(\|Z^H(t)\| \leq \mathcal{O}(1/\omega)\) that satisfies (110).

Similar to the procedure carried out in Theorem 1, step 6, of Reference 14 and along with (110), it is not difficult to show

\[
\limsup_{t \to \infty} |\hat{\theta}(t)| = \mathcal{O}(1/\omega).
\]
From (14) and Figure 4, we can write \( \theta(t) - \theta^* = \tilde{\theta}(t) + S(t) \), and recalling \( S(t) \) in (20) is of order \( O \left( |a| \exp \left( \sqrt{\frac{\omega}{\epsilon}} \right) \right) \), we finally get with (111). The convergence of the propagated actuator \( \Theta(t) \) to the optimizer \( \Theta^* \) is easier to prove. The relation among the propagated estimation error \( \theta(t) \), the propagated input \( \Theta(t) \), and the optimizer of the static map \( \Theta^* \) is given by

\[
\theta(t) + a \sin(\omega t) = \Theta(t) - \Theta^*.
\]

(150)

Using (150) and taking the absolute value, one has

\[
|\Theta(t) - \Theta^*| = |\theta(t) + a \sin(\omega t)|.
\]

(151)

As in the convergence proof of the parameter \( \theta(t) \) to the optimal input \( \theta^* \) above, we write (151) in terms of the periodic solution \( \tilde{\theta}^P(t) \) and follow the same steps by applying Young’s inequality and \( \theta(t) - \tilde{\theta}^P(t) \to 0 \) exponentially by the averaging theorem (see section 2 of Reference 22). Hence, along with (110), we finally get (112).

7 | NUMERICAL SIMULATIONS

In the next, we present three numerical tests in order to illustrate the applicability of the proposed algorithms for real-time optimization of static maps (1) subject to infinite-dimensional input dynamics (5)-(8). The first one shows a maximization problem (2) using a quadratic map (4), whereas the second one brings comparison results between gradient and Newton-based ES for flatter maps, highlighting the convergence speed of each one. After that, the third example presents the maximization (95) of higher derivatives for an unknown map (96) satisfying the quadratic condition in (97).

7.1 | Gradient ES

The numerical simulation for the gradient ES with actuation dynamics governed by RAD PDEs is performed with the parameters listed in Table 1. The simulation results are shown in Figure 6 for the nonlinear map (4).

We observe the convergence of \( (y(t), \theta(t), \Theta(t)) \) to a neighborhood of \( (y^*, \theta^*, \Theta^*) \). The main difference to the pure diffusion case in References 14 and 15 is \( \theta^* \neq \Theta^* \), shown theoretically in (16). The estimated optimal input parameter \( \tilde{\theta}(t) \) converges to \( \theta^* = 1.885 \), which is the same calculated with (16) for the parameters used in the simulation.

<table>
<thead>
<tr>
<th><strong>TABLE 1</strong> Simulation parameters for ES with actuation dynamics governed by a reaction-advection-diffusion PDE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symbol</td>
</tr>
<tr>
<td>Controller parameters</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>System parameters</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
Consider a quadratic static map as in the previous section, but now we are going to change the Hessian value to $H = -0.2$ in order to obtain a flatter map (1)-(4). This change will allow us to check that faster results can be expected from Newton-based algorithm when compared with classical gradient ES design since the latter has convergence rates which are dependent of the Hessian. The remaining parameters are the same shown in Table 1. In particular, the estimate of $H^{-1} = -5$ in the proposed Newton-based ES is given by the solution of the Riccati equation (102), which was implemented with $\omega_r = 0.5$ and initial condition $\Gamma(0) = -2$.

The simulation results of the closed-loop with Newton-based ES are illustrated in Figures 7 to 9. The convergence of the gradient and Hessian's inverse estimates are given in Figures 7 and 8, respectively. In particular, the exact estimation of $\Gamma(t) \to H^{-1}$ allows us to cancel the Hessian $H$ and thus guarantee convergence rates that can be arbitrarily assigned by the user. In Figure 9, we observe $y$ converging to a neighborhood of the optimum $y^*$. The improved performance by the Newton-based controller can be seen by comparing with the simulation result of the proposed Gradient ES. As
expected, Newton algorithm converges to the extremum faster than the Gradient scheme, even in the presence of actuation dynamics governed by a RAD-like parabolic PDE.

**FIGURE 7** Newton-based ES with a RAD-like PDE in actuation dynamics: Gradient estimate $G(t)$ and control signal $U(t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 8** Newton-based ES with a RAD-like PDE in actuation dynamics: Hessian's inverse estimate $\Gamma(t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 9** Newton-based ES versus Gradient ES: time response of the output $y(t)$ subject to an actuator RAD-like PDE [Colour figure can be viewed at wileyonlinelibrary.com]
The static map \( y = Q(\Theta) = \Theta y^* + (H/6)(\Theta - \Theta^*)^3 \)
and its first derivative \( Q^{(1)}(\Theta) = y^* + (H/2)(\Theta - \Theta^*)^2 \), with \( H = -0.2 \)
and maximum at \( \Theta^* = 2 \) [Colour figure can be viewed at wileyonlinelibrary.com]

Newton-based ES maximizing the first derivative \( Q^{(1)}(\Theta) = y^* + (H/2)(\Theta - \Theta^*)^2 \) with a RAD-like PDE in the actuation dynamics of \( y = Q(\Theta) \). Parameter \( \Theta(t) \) converging to \( \Theta^* = 2 \) [Colour figure can be viewed at wileyonlinelibrary.com]

### 7.3 Maximization of higher derivatives of unknown maps with Newton-based ES

The numerical test for the Newton-based ES of higher derivative map (4) with actuation dynamics governed by RAD PDEs is performed with the same parameters listed in Table 1, except for the Hessian which was chosen \( H = -0.2 \). The original map is chosen as \( y = Q(\Theta) = \Theta y^* + (H/6)(\Theta - \Theta^*)^3 \), which is shown in Figure 10 with dashed line. The objective is to maximize the first derivative of the map \( Q^{(1)}(\Theta) \), which has an extremum at \( \Theta^* = 2 \)—see the solid line in Figure 10. The simulation result of the closed-loop with the control law (109) is shown in Figure 11. As expected, we observe that the variable \( \Theta \) converges to a neighborhood of the optimum \( \Theta^* \).

### 8 CONCLUSIONS

In this article, we presented and proved local stability of the proposed gradient ES algorithm based on boundary control for actuation dynamics governed by RAD PDEs. The proposed method maximizes in real-time the output of an unknown static nonlinear map employing perturbation-based estimates of its gradient and Hessian. The resulting approach guarantees exponential convergence of the system input to a small neighborhood around the maximizer point where extremum occurs, despite the presence of the PDE. A rigorous proof is given in terms of backstepping transformation and averaging analysis in infinite dimensions for locally quadratic objective functions. The proposed approach has also broad applicability in practice since the presence of PDE models is often listed as a major limiting factor in the application of ES.
controllers in some practical situations. As a further contribution, we considered the extension to scalar Newton-based ES designs in order to alleviate the scaling issues associated with gradient descent algorithms and yield significant improvements in transient performance and faster convergence rates. The proposed method maximizes arbitrary $n$th derivatives of an unknown static map. The only available measurement is from the map output itself and not of its derivatives.

As a future research, an interesting topic for investigation is the extension of our ES approach for systems with distributed effects (e.g., distributed input/sensor delays [31] or counter-convection dynamics [32]).

ACKNOWLEDGEMENTS
Tiago Roux Oliveira would like to thank the Brazilian Funding Agencies CNPq, CAPES, and FAPERJ.

ORCID
Tiago Roux Oliveira  https://orcid.org/0000-0002-2232-8715
Jan Feiling  https://orcid.org/0000-0003-1263-2403
Shumon Koga  https://orcid.org/0000-0002-5691-814X
Miroslav Krstic  https://orcid.org/0000-0002-5523-941X

REFERENCES


APPENDIX A. DERIVING THE PERTURBATION SIGNAL FOR THE ES CONTROL LOOP WITH ACTUATION DYNAMICS GOVERNED BY RAD PDES

Considering the system (17)-(20) and transform it, as in Section 4, with $\bar{p}(x, t) = \beta(x, t)e^{b\frac{x}{\varepsilon}}$ into an reaction-diffusion system, we have

\begin{align*}
S(t) &= \bar{p}(1, t)e^{-\frac{b}{\varepsilon}} \tag{A1} \\
\bar{p}(x, t) &= \epsilon \bar{p}_{\partial x}(x, t) - \xi \bar{p}(x, t), \quad x \in [0, 1] \tag{A2} \\
\bar{p}(0, t) &= \frac{b}{2\epsilon} a \sin(\omega t) \tag{A3} \\
\bar{p}(0, t) &= a \sin(\omega t), \tag{A4}
\end{align*}

with

$$\xi = \frac{b^2}{4\epsilon} - \lambda. \tag{A5}$$

As in example 12.2 in Reference 21, we postulate the full-state reference trajectory in the form

$$\bar{p}(x, t) = \sum_{k=0}^{\infty} a_k(t) \frac{x^k}{k!} \tag{A6}$$

From (A3) and (A4), we have

$$a_0(t) = a \sin(\omega t), \quad a_1(t) = \frac{ab}{2\epsilon} \sin(\omega t), \tag{A7}$$

and with the PDE (A2), we get

$$a_{k+2}(t) = \frac{1}{e} \dot{a}_k(t) + \xi a_k(t). \tag{A8}$$

Additionally, we observe the relation

$$a_{2k+1}(t) = \frac{b}{2\epsilon} a_{2k}. \tag{A9}$$

Calculating $a_{2k}, \ k = 1, \ldots, 4$, only depending on $a_0^{(k)}, \ k = 1, \ldots, 4$, we have
\[
\epsilon a_2 = a_0 + \xi a_0, \quad (A10)
\]
\[
\epsilon^2 a_4 = \ddot{a}_0 + 2\ddot{\xi} a_0 + \xi^2 a_0, \quad (A11)
\]
\[
\epsilon^3 a_6 = \dddot{a}_0 + 3\xi \dot{a}_0 + 3\xi^2 \dot{a}_0 + \xi^3 a_0, \quad (A12)
\]
\[
\epsilon^4 a_8 = \dddot{a}_0 + 4\xi \ddot{a}_0 + 6\xi^2 \dot{a}_0 + 4\xi^3 a_0 + \xi^4 a_0. \quad (A13)
\]

By the derivative-law of the sine, we can write
\[
a_0^{(2k)} = (-1)^k \omega^{2k} a_0, \quad a_0^{(2k+1)} = (-1)^k \omega^{2k} a_0
\]
and (A10)-(A13) can be rewritten as
\[
\frac{\epsilon}{a} a_2 = +\xi \sin(\omega t) + \omega \cos(\omega t), \quad (A15)
\]
\[
\frac{\epsilon^2}{a} a_4 = (\xi^2 - \omega^2) \sin(\omega t) + 2\omega \xi \cos(\omega t), \quad (A16)
\]
\[
\frac{\epsilon^3}{a} a_6 = (3\xi \omega^2 + \xi^3) \sin(\omega t) + (-\omega^3 + 3\xi^2 \omega) \cos(\omega t), \quad (A17)
\]
\[
\frac{\epsilon^4}{a} a_8 = (\omega^4 + 6\xi^2 \omega^2 + \xi^4) \sin(\omega t) + (4\xi \omega^3 + 4\xi^3 \omega). \quad (A18)
\]

Observations reveal that the coefficients of the sine and cosine terms of (A15)-(A18) are the same as of the modified Pascal triangle

| k = 0 |  | 1 |
| k = 1 | \omega | \xi |
| k = 2 | \omega^2 | 2\xi \omega | \xi^2 |
| k = 3 | \omega^3 | 3\xi \omega^2 | 3\xi^2 \omega | \xi^3 |
| k = 4 | \omega^4 | 4\xi \omega^3 | 6\xi^2 \omega^2 | 4\xi^3 \omega | \xi^4 |

where the bold coefficients belong to the sine and the other coefficients to the cosine terms. Since we have an iterative law to calculate \(a_{2k}\), the expansion for \(k > 4\) continues in the same way. Hence, with
\[
P_m(k) = \omega^m \prod_{j=m}^{k} \binom{j}{m} \xi^j, \quad (A19)
\]
we can determine the value in the m-th diagonal and k-th row, where the diagonal \(m = 0\) is \(\{1, \xi, \xi^2, \xi^3, \ldots\}\). Combining this result and defining \(\binom{y}{z} := 0\) for \(y < z\), we arrive at
\[
a_{2k} := \frac{a}{\epsilon^k} \sin(\omega t) \sum_{n=0}^{k} \binom{k}{2n} \xi^{k-2n} \omega^{2n} + \frac{a}{\epsilon^k} \cos(\omega t) \sum_{n=0}^{k} \binom{k}{2n+1} \xi^{k-2n-1} \omega^{2n+1}
\]
with \(\xi := \frac{b^2}{4\epsilon} - \lambda.\) \quad (A20)

Finally, with (A1) and (A6), we get
\[
S(t) = e^{-\frac{b}{2\epsilon} \sum_{k=0}^{\infty} \frac{a_{2k}(t)}{(2k)!} + \frac{b}{2\epsilon} \frac{a_{2k}(t)}{(2k+1)!}}, \quad (A21)
\]
APPENDIX B. INVERSE BACKSTEPPING TRANSFORMATION (RAD CASE)

In this part of the Appendix, we derive the inverse backstepping transformation of the backstepping transformation (29). Consider a candidate for the inverse backstepping transformation and its time and spatial derivatives of the backstepping transformation (29)

\[
\bar{z}(x, t) = w(x, t) - \int_0^x p(x, y)w(y, t)dy - \eta(x)\theta(t) \tag{A22}
\]

\[
\bar{z}_x(x, t) = \epsilon w_x(x, t) + w(x, t) [-\xi + \epsilon p_y(x, x)] - \epsilon p(x, x)w_x(x, t) + w(0, t) \left[ \frac{b}{2} p(x, 0) - \epsilon p_y(x, 0) - \eta(x) \right] - \overline{K} \eta(x)\theta(t)
\]

\[
- \int_0^x w(y, t) \left[ \epsilon p_{yy}(x, t) - \xi p(x, y) \right] dy \tag{A23}
\]

\[
\bar{z}_{xx}(x, t) = w_{xx}(x, t) - p(x, x)w_x(x, t) - \int_0^x p_x(x, y)w(y, t)dy - \eta'(x)\theta(t) \tag{A24}
\]

\[
\bar{z}_{xxx}(x, t) = w_{xxx}(x, t) - w(x, t) \frac{d}{dx}p(x, x)
\]

\[
+ w(0, t) \left[ \frac{b}{2} p(x, 0) - \epsilon p_y(x, 0) - \eta(x) \right]
\]

\[
+ \theta(t) \left[ -\eta(x)(\overline{K} + \xi) - \eta''(x) \right]
\]

\[
+ \int_0^x w(y, t) \left[ p_{xx}(x, y) - p_{yy}(x, y) \right] dy, \tag{A26}
\]

where \( p(x, y) \) is the gain kernel and \( \eta(x) \) is a function in \( x \). Note that the boundary value (27) and the integration by parts were used to derive (A23). Inserting (A22), (A23), and (A25) into the PDE (26)

\[
\bar{z}_t(x, t) - \epsilon \bar{z}_{xx}(x, t) + \xi \bar{z}(x, t) = 2\epsilon w(x, t) \frac{d}{dx}p(x, x)
\]

\[
+ w(0, t) \left[ \frac{b}{2} p(x, 0) - \epsilon p_y(x, 0) - \eta(x) \right]
\]

\[
+ \theta(t) \left[ -\eta(x)(\overline{K} + \xi) - \eta''(x) \right]
\]

\[
+ \int_0^x w(y, t) \left[ p_{xx}(x, y) - p_{yy}(x, y) \right] dy.
\]

and along with (A22), (A24), evaluated at \( x = 0 \) plus the boundary value (27), leads to the following conditions for the gain kernel \( p(x, y) \) and the function \( \eta(x) \):

\[
\eta(0) = -\overline{K}, \tag{A27}
\]

\[
\eta'(0) = -\overline{K} \frac{b}{2\epsilon}, \tag{A28}
\]

\[
\eta''(x) = \eta(x)(\overline{K} + \xi), \tag{A29}
\]

\[
p(x, 0) = \frac{2\epsilon}{b} p_y(x, 0) + \frac{2}{b} \eta(x), \tag{A30}
\]

\[
p(0, 0) = 0, \tag{A31}
\]

\[
p_{xx}(x, y) - p_{yy}(x, y) = 0. \tag{A32}
\]

The second-order ODE system (A27)-(A29) can be solved with the ansatz \( \eta(x) = A \cosh(\mu x) + B \sinh(\mu x) \) to

\[
\eta(x) = -\overline{K} \left( \cosh \left( \sqrt{\overline{K} + \xi} x \right) + \frac{b}{2\epsilon \sqrt{\overline{K} + \xi}} \sinh \left( \sqrt{\overline{K} + \xi} x \right) \right). \tag{A33}
\]
The hyperbolic PDE system of Goursat type (A30)-(A32) can be solved by the ansatz \( p(x, y) = m(x - y) \), where \( m(\cdot) \) is a scalar function. Additionally, by applying the variation of constants, we get

\[
p(x, y) = -\frac{\sqrt{K}}{\sqrt{K} + \xi} \sinh \left( \sqrt{K} + \xi(x - y) \right).
\]  

(A34)

Hence, we have the backstepping and its inverse transformation in the variables \((u, w)\)

\[
w(x, t) = e^{\frac{1}{2}x} u(x, t) - \int_0^x q(x, y) e^{\frac{1}{2}x} u(y, t) dy - \gamma(x) \theta(t),
\]  

(A35)

\[
u(x, t) = e^{-\frac{1}{2}x} w(x, t) - e^{-\frac{1}{2}x} \int_0^x p(x, y) w(y, t) dy - e^{-\frac{1}{2}x} \eta(x) \theta(t),
\]  

(A36)

with \( q(x, y) \) in (30) and \( \gamma(x) \) in (31).