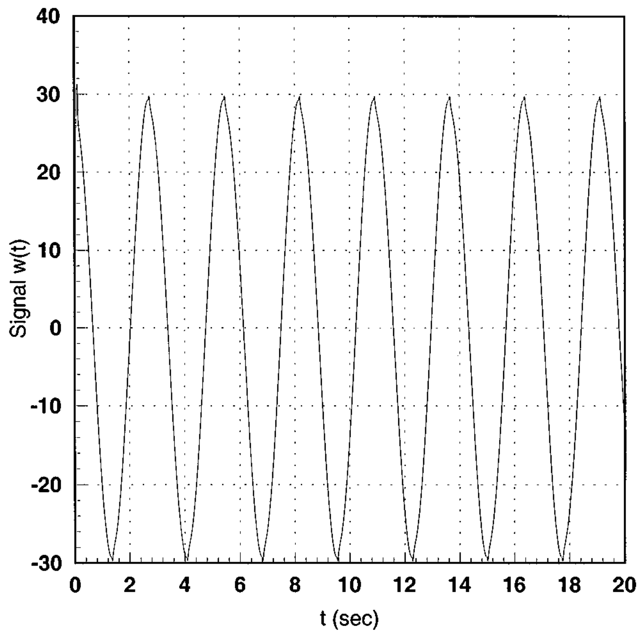

Fig. 4. Signal $w(t)$ acting as the output of backlash hysteresis.

We should mention that it is desirable to compare the control performance with and without considering the effects of hysteresis. Unfortunately, this comparison is not possible in this case as the control law (15)–(20) is designed for the entire cascade system.

VI. Conclusion

In this paper, a robust adaptive control architecture is proposed for a class of continuous-time nonlinear dynamic systems preceded by a backlash-like hysteresis, where the backlash-like hysteresis is modeled by a dynamic equation. By showing the properties of the hysteresis model, a robust adaptive control scheme is developed without constructing the hysteresis inverse. The new adaptive control law ensures global stability of the adaptive system and achieves both stabilization and tracking with excellent precision. Simulations performed on a simple nonlinear system illustrate and clarify the approach.

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Extremum Seeking for Limit Cycle Minimization

Hsin-Hsiung Wang and Miroslav Krstić

***Abstract*—In many physical problems, equilibrium stabilization is not possible and the controlled system is in a limit cycle. If the size of the limit cycle depends on some of the control parameters, then a reasonable objective would be to tune this parameter to minimize the size of the limit cycle. In this paper, we propose a method for achieving this. This method is an extension of our earlier result [13] on extremum seeking for equilibria. We illustrate the method with a Van der Pol oscillator example and present analysis for it using averaging and singular perturbations.**

***Index Terms*—Averaging, extremum seeking, limit cycles, singular perturbations.**

I. Introduction

Limit cycles occur in numerous areas of application. In particular, systems exist in which feedback control can only reduce the size of the limit cycle, but cannot completely eliminate it. The inability to remove the limit cycle and achieve equilibrium stabilization may be associated with actuator constraints, like magnitude and rate saturation. In this situation, the best control requirement is to enforce a stable, "smallest" limit cycle.

The method of "extremum seeking" has traditionally been used for searching for a minimum or a maximum of an *equilibrium map*. This method was an intensely studied topic between the 1940's and 1970's [2]–[5], [15], [19]–[21]. The most frequently cited references include the works by Kazakevich *et al.* [6]–[10], the survey by Sternby [24], and the book of Astrom and Wittenmark [1, Section 13.3]. Pioneering work on stability analysis based on averaging in an example of an extremum-seeking system dates back to Meerkov [16]–[18]. The first stability analysis for a problem with a *general nonlinear dynamical plant*

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H.-H. Wang is with the Department of Electrical Engineering, Oriental Institute of Technology, Panchiao, Taipei County, 220 Taiwan (e-mail: swang@www.ee.oit.edu.tw).

M. Krstić is with the Department of Applied Mechanics and Engineering Science, University of California, San Diego, La Jolla, CA 92093-0411 USA (e-mail: krstic@ucsd.edu).

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was presented in our recent work [12], [13], and it involves the use of both averaging and singular perturbations. The methods of deterministic extremum seeking have similarities with the stochastic approximation methods [14], [22], [23].

In this paper, we present the first extension of the extremum-seeking method to the case in which equilibrium operation is impossible (unstable) and the system is always in a *limit cycle*. The objective of the scheme is to reduce the size of the limit cycle to a minimum. Our algorithm is a slight variation on the standard extremum-seeking algorithm with an excitation signal, but the analysis is novel and incorporates a nontrivial sequence of steps involving averaging and singular perturbation methods.

We start in Section II with a scheme for general feedback systems in limit cycle. This scheme incorporates a block for detection of the "amplitude" of the limit cycle. In Section III, we apply the scheme to a Van der Pol oscillator example for which the simulations demonstrate the effectiveness of the scheme. Finally, in Section IV, we present stability/performance analysis (for the scheme with the Van der Pol example), which involves two steps of averaging with one step of singular perturbation analysis in between. The conclusions drawn are valid on $O(1)$ time intervals.

II. An Extremum-Seeking Scheme for Limit Cycle Minimization

We consider single-input–single-output systems of the form

$$\dot{x} = f(x, u)$$
$$y = h(x) \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, and f: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and h: $\mathbb{R}^n \to \mathbb{R}$ are smooth. Suppose we know a smooth control law

$$u = \alpha(x, \theta) \tag{2.2}$$

parameterized by a scalar parameter θ such that the closed-loop system

$$\dot{x} = f(x, \alpha(x, \theta)) \tag{2.3}$$

has a stable limit cycle corresponding to each θ. Our objective is to tune θ to minimize the "amplitude" of the limit cycle.

In order to employ the extremum-seeking scheme from [12], [13] to the problem of limit cycle minimization, only a small modification is needed. We add a detector block shown in Fig. 4 to the overall extremum seeking scheme in Fig. 1. The idea of the detector is simple, and we explain it first, before explaining the operation of the overall scheme in Fig. 1. We assume that the output of the system in a limit cycle is sinusoidal,[1] $y(t) = Y_0 + r\sin(\omega_0 t + \phi)$, where Y_0, ϕ are constants and r, ω_0 are positive constants. The high-pass filter is supposed to eliminate the DC component Y_0. The expected result, $r\sin(\omega_0 t+\phi)$, is squared to get $(r^2/2)(1+\cos(2\omega_0 t+\phi))$, and then passed through a low-pass filter to extract only $r^2/2$. The last block results in the amplitude of the limit cycle r. This idea is, of course, based on an assumption that $\omega_0 \gg \Omega_h, \Omega_l$.

The overall extremum-seeking scheme in Fig. 1 functions as follows. Suppose the limit cycle transients and the limit cycle oscillations are fast, so that the cascade of the plant and the limit cycle amplitude detector block can be regarded as a static nonlinear map $r(\theta)$ with a local minimum at θ^*. The excitation $a\sin\omega t$ will then create a periodic response in r. The high-pass filter $s/(s+\omega_h)$ would eliminate the "DC

[1]Assume that the limit cycle is almost purely harmonic; for a multiharmonic limit cycle, the method of this paper would result only in minimization of the first mode.

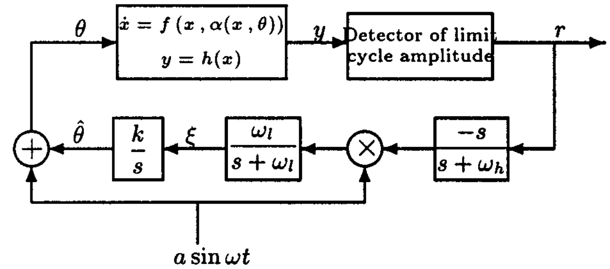

Fig. 1. Extremum-seeking scheme for limit cycle minimization. The x-system is assumed to be in a limit cycle for any constant θ [despite the use of feedback $\alpha(x, \theta)$].

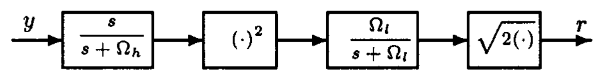

Fig. 2. Detector of limit cycle amplitude.

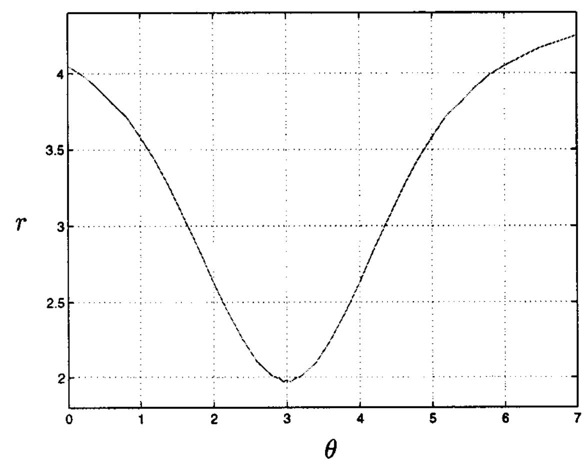

Fig. 3. Characteristic of the limit cycle "amplitude" r with respect to θ.

component" of r. Then, the multiplication by $a\sin\omega t$ would result in a signal that has a slow component proportional to $(a^2/2)(\partial y/\partial\hat{\theta})$, whereas the fast component would be eliminated by the low-pass filter $\omega_l/(s+\omega_l)$. Then, the integrator k/s would be acting approximately as a gradient update law driven by the sensitivity function, which tunes $\hat{\theta}$ to θ^*.

The design parameters of the entire scheme are selected as $\omega_0 \gg \Omega_h, \Omega_l \gg \omega \gg \omega_h, \omega_l, k$.

III. A Van der Pol Example

Consider a Van der Pol equation parameterized by θ as follows:

$$\ddot{x} + \epsilon[(x-x_0)^2 - 1 - (\theta-\theta^*)^2]\dot{x} + \mu^2(x-x_0) = 0 \tag{3.1}$$

where $\theta - \theta^*$ is a parameter that controls the amplitude of oscillation and x_0 is a parameter for the offset of x. The parameter μ controls the frequency of limit cycle oscillations, and ϵ controls the speed of the limit cycle transients (the attractivity of the limit cycle). We assume that θ^* is constant and θ is available as the input to the system. The system (3.1) will be in a limit cycle for any θ and θ^* [11]. This example is contrived to emulate problems in which feedback control can only reduce the size of a limit cycle, but cannot completely eliminate it.

We first study the relationship between the limit cycle amplitude and the parameter θ for the system (3.1). The relationship is shown in Fig. 3.

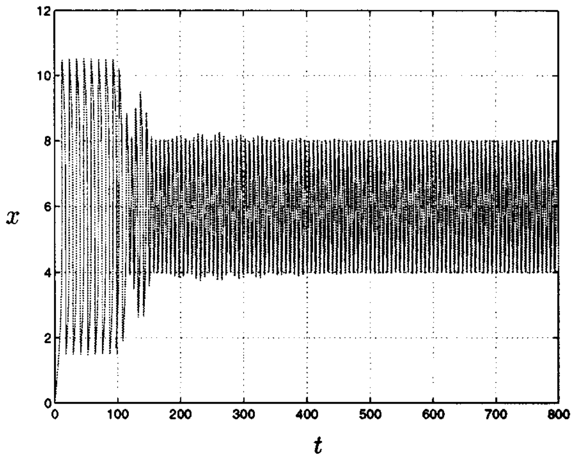

Fig. 4. Time response of state x of the Van der Pol system with large $\hat{\theta}(0)$.

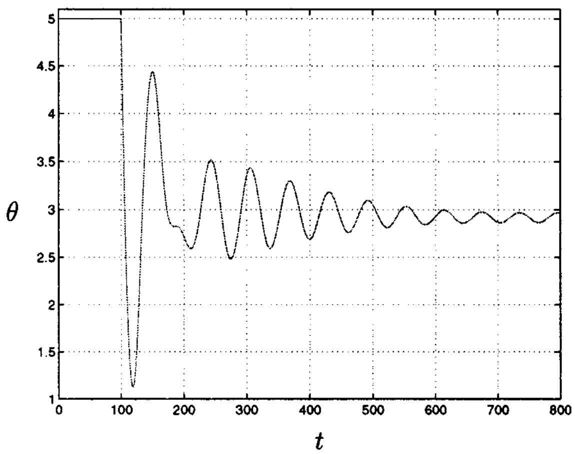

Fig. 5. Time response of parameter θ of the Van der Pol system with large $\hat{\theta}(0)$.

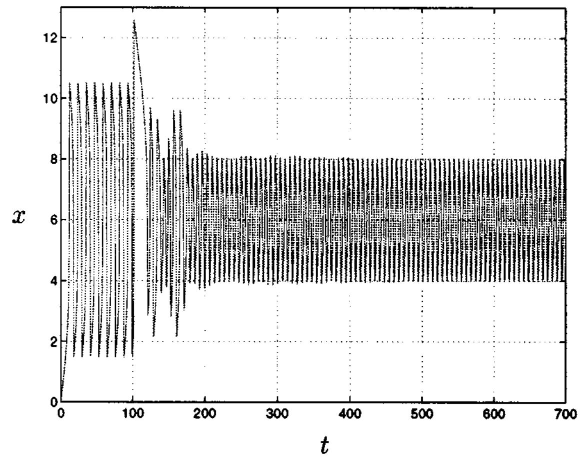

Fig. 6. Time response of state x of the Van der Pol system with small $\hat{\theta}(0)$.

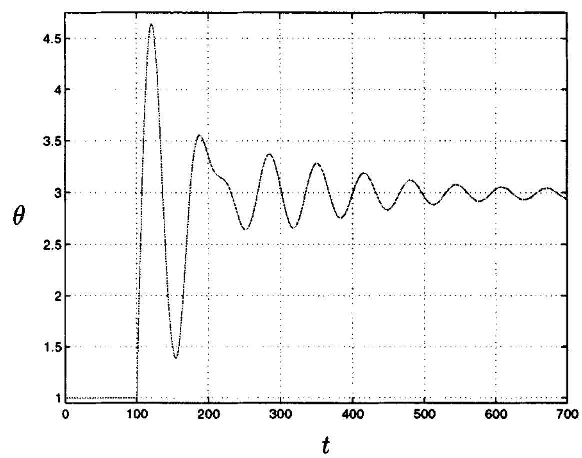

Fig. 7. Time response of parameter θ of the Van der Pol system with small $\hat{\theta}(0)$.

Because the characteristic has a minimum, we feed $-r$ to the input of the extremum-seeking block (see Fig. 1).

We perform simulations from both sides of the extremum. In both cases, we set $a = 0.1$, $\Omega_h = 0.75$, $\Omega_l = 0.02$, $\omega = 0.1$, $\omega_H = 0.02$, $k = 4$, $x_0 = 6$, $\theta^* = 3$, and $\epsilon = \mu = 1$. In the first case, we set the initial value of the integrator $\hat{\theta}(0) = 5$. We run the simulation without extremum seeking for 100 s and then start the extremum-seeking controller. The oscillation of x is shown in Fig. 4, and the process of convergence of the parameter θ to $\theta^* = 3$ is shown in Fig. 5. In the second case, we consider the initial value $\hat{\theta}(0) = 1$. The oscillation of x is shown in Fig. 6, and the process of convergence of θ to $\theta^* = 3$ is shown in Fig. 7. In both cases, the limit cycle is reduced to its minimal possible size.

IV. Analysis

To simplify the analysis, we replace the amplitude detector block with a quadratic function. We also drop the low-pass filter from the extremum-seeking scheme to make the proof as simple as possible.[2] The resulting extremum-seeking scheme with the Van der Pol system is shown in Fig. 8. Denoting $\tilde{\theta} = \hat{\theta} - \theta^*$ and $y = x - x_0$, the system can be written as

$$\ddot{y} - \epsilon\left(1 + (\tilde{\theta} + a\sin\omega t)^2 - y^2\right)\dot{y} + \mu^2 y = 0 \tag{4.1}$$

$$\dot{\eta} = (y^2 - \eta)\omega_h \tag{4.2}$$

$$= -ka(y^2 - \eta)\sin\omega t. \tag{4.3}$$

Before we start the analysis, we outline its intended main result. It is well known [11, Sect. 8.4] that, for $\tilde{\theta} = 0$ and $a = 0$, if μ/ϵ is sufficiently large, we have

$$\sqrt{y(t)^2 + \frac{\dot{y}(t)^2}{\mu^2}} = 2 + O\left(\frac{\epsilon}{\mu}\right) + \text{exp. decaying terms} \tag{4.4}$$

over and $O(1)$ time interval. In this section, we will show that the extremum-seeking scheme guarantees that (4.4) holds with an

$$O\left(a + \frac{ka}{\omega} + \frac{\omega_h}{\omega} + \frac{\omega}{\epsilon}\right) \text{ error} \tag{4.5}$$

provided $\sqrt{(x(0) - x_0)^2 + \dot{x}(0)^2/\mu^2} - 2$, $\hat{\theta}(0) - \theta^*$, $\eta(0) - 2 - a^2$ and the quantities

$$\frac{\epsilon}{\mu}, \frac{\omega}{\epsilon}, \frac{ka}{\omega}, \frac{\omega_h}{\omega} \tag{4.6}$$

[2]The conclusions of the analysis without these simplifications would be the same, but the analysis would be much more complicated.

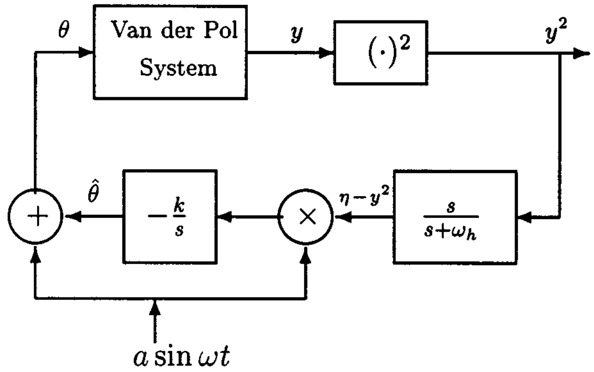

Fig. 8. Simplified extremum-seeking scheme for limit cycle minimization.

are all sufficiently small, which means that, over an $O(1)$ time interval, the solutions will be locally exponentially converging to a small neighborhood of the "smallest" limit cycle. The smallness condition (4.6) implies that the adaptation gain ka and the filter cutoff frequency ω_h should be an order of magnitude smaller than the excitation frequency ω, which should be an order of magnitude smaller than the limit cycle damping ϵ, which should be an order of magnitude smaller than the limit cycle frequency μ. This ordering of $\mu, \epsilon, \omega, k, a$, and ω_h ensures the following separation of time scales:

- limit cycle oscillations faster than;
- limit cycle transients faster than;
- excitation signal oscillations faster than;
- extremum-seeking filter transients.

For notational simplicity, in this section, we take $\epsilon = O(1)$.

We are now ready to commence our analysis. To represent the Van der Pol system (4.1) in polar coordinates, let

$$y = r\sin\phi, \qquad \dot{y} = \mu r\cos\phi. \tag{4.7}$$

Then, we have

$$\dot{r} = \epsilon r\cos^2\phi\left[1 + \left(\tilde{\theta} + a\sin\omega t\right)^2 - r^2\sin^2\phi\right] \tag{4.8}$$

$$= \mu - \epsilon\cos\phi\sin\phi\left[1 + \left(\tilde{\theta} + a\sin\omega t\right)^2 - r^2\sin^2\phi\right]. \tag{4.9}$$

The overall system is shown in (4.2) and (4.3) and (4.8) and (4.9). We treat t as a state and use ϕ as an independent variable. Then, the whole system can be represented as

$$\frac{dr}{d\phi} = \frac{\epsilon}{\mu}\,\frac{r\cos^2\phi\left[1 + \left(\tilde{\theta} + a\sin\omega t\right)^2 - r^2\sin^2\phi\right]}{1-\Delta} \tag{4.10}$$

$$\frac{d\tilde{\theta}}{d\phi} = -\frac{k}{\mu}\,\frac{a(r^2\sin^2\phi - \eta)\sin\omega t}{1-\Delta} \tag{4.11}$$

$$\frac{d\eta}{d\phi} = \frac{\omega_h}{\mu}\,\frac{r^2\sin^2\phi - \eta}{1-\Delta} \tag{4.12}$$

$$\frac{dt}{d\phi} = \frac{1}{\mu}\,\frac{1}{1-\Delta} \tag{4.13}$$

where

$$\Delta \triangleq \frac{\epsilon}{\mu}\cos\phi\sin\phi\left[1 + \left(\tilde{\theta} + a\sin\omega t\right)^2 - r^2\sin^2\phi\right]. \tag{4.14}$$

Now, averaging with respect to ϕ for $1/\mu$ small, we obtain

$$\frac{dr^a}{d\phi} = \frac{\epsilon}{\mu}\,r^a\left[\frac{1 + (\tilde{\theta}^a + a\sin\omega t^a)^2}{2} - \frac{(r^a)^2}{8}\right] \tag{4.15}$$

$$\frac{d\tilde{\theta}^a}{d\phi} = -\frac{k}{\mu}\,a\sin\omega t^a\left(\frac{(r^a)^2}{2} - \eta^a\right) \tag{4.16}$$

$$\frac{d\eta^a}{d\phi} = \frac{\omega_h}{\mu}\left(\frac{(r^a)^2}{2} - \eta^a\right) \tag{4.17}$$

$$\frac{dt^a}{d\phi} = \frac{1}{\mu}. \tag{4.18}$$

Note that $t^a = (\phi/\mu)$ in this average system. Denote $\phi_\tau = (\omega\phi/\mu)$. By using the relationship $\omega \gg \omega_h, k$, (4.15)–(4.17) can be expressed as

$$\omega\,\frac{dr^a}{d\phi_\tau} = \epsilon r^a\left[\frac{1 + (\tilde{\theta}^a + a\sin\phi_\tau)^2}{2} - \frac{(r^a)^2}{8}\right] \tag{4.19}$$

$$\frac{d\tilde{\theta}^a}{d\phi_\tau} = -\frac{k}{\omega}\,a\sin\phi_\tau\left(\frac{(r^a)^2}{2} - \eta^a\right) \tag{4.20}$$

$$\frac{d\eta^a}{d\phi_\tau} = \frac{\omega_h}{\omega}\left(\frac{(r^a)^2}{2} - \eta^a\right). \tag{4.21}$$

This system is in the standard singular perturbation form [recall the smallness condition (4.6)].

The next step in our analysis is to study the system (4.19)–(4.21). We freeze r^a in (4.19) at its "quasi-steady-state" value

$$(r^a)^2 = 4\left[1 + (\tilde{\theta}^a + a\sin\phi_\tau)^2\right]. \tag{4.22}$$

Substituting (4.22) into (4.20) and (4.21), we obtain the "reduced model"

$$\frac{d\tilde{\theta}_r^a}{d\phi_\tau} = -\frac{k}{\omega}\,a\sin\phi_\tau\left[2 + 2(\tilde{\theta}_r^a + a\sin\phi_\tau)^2 - \eta_r^a\right] \tag{4.23}$$

$$\frac{d\eta_r^a}{d\phi_r} = \frac{\omega_h}{\omega}\left[2 + 2(\tilde{\theta}_r^a + a\sin\phi_\tau)^2 - \eta_r^a\right]. \tag{4.24}$$

Because $\omega_h, k \ll \omega$, the system (4.23) and (4.24) is in the form to which the averaging method is applicable [11, Sect. 8.3]. The average model of (4.23) and (4.24) is

$$\frac{d\tilde{\theta}_r^{aa}}{d\phi_\tau} = -2\,\frac{k}{2\pi\omega}\,a\int_0^{2\pi}\sin\phi_\tau \cdot\left[2 + 2(\tilde{\theta}_r^{aa} + a\sin\phi_\tau)^2 - \eta_r^{aa}\right]d\phi_\tau \tag{4.25}$$

$$\frac{d\eta^{aa}}{d\phi_\tau} = \frac{\omega_h}{2\pi\omega}\int_0^{2\pi} \cdot\left[2 + 2(\tilde{\theta}_r^{aa} + a\sin\phi_\tau)^2 - \eta_r^{aa}\right]d\phi_\tau. \tag{4.26}$$

Performing the integrations, the average system becomes

$$\frac{d\tilde{\theta}_r^{aa}}{d\phi_\tau} = -2\,\frac{ka^2}{\omega}\,\tilde{\theta}_r^{aa} \tag{4.27}$$

$$\frac{d\eta_r^{aa}}{d\phi_\tau} = \frac{\omega_h}{\omega}\left[-\left(\eta_r^{aa} - 2 - a^2\right) + 2\left(\tilde{\theta}_r^{aa}\right)^2\right]. \tag{4.28}$$

Define

$$\tilde{\eta}_r^{aa} = \eta_r^{aa} - (2 + a^2). \tag{4.29}$$

Then, the average system is

$$\frac{d\tilde{\theta}_r^{aa}}{d\phi_\tau} = -2\,\frac{ka^2}{\omega}\,\tilde{\theta}_r^{aa} \tag{4.30}$$

$$\frac{d\tilde{\eta}_r^{aa}}{d\phi_\tau} = \frac{\omega_h}{\omega}\left[-\tilde{\eta}_r^{aa} + 2\left(\tilde{\theta}_r^{aa}\right)^2\right]. \tag{4.31}$$

The Jacobian at the average equilibrium $\tilde{\theta}_r^{aa} = \tilde{\eta}_r^{aa} = 0$ is

$$J_r^{aa} = \begin{bmatrix} -2\,\dfrac{k}{\omega}\,a^2 & 0 \\ 0 & -\dfrac{\omega_h}{\omega} \end{bmatrix}. \tag{4.32}$$

Obviously, J_r^{aa} is Hurwitz, which implies that the average equilibrium is exponentially stable. Then, according to the averaging theorem [11, Theorem 8.3], all solutions $(\tilde{\theta}_r^a(\phi_\tau), \tilde{\eta}_r^a(\phi_\tau))$ exponentially converge to an $O(\delta)$-neighborhood of the origin, where

$$\delta = \frac{\max\{ka, \omega_h\}}{\omega}. \tag{4.33}$$

Because (4.23)–(4.24) is the reduced model of the singularly perturbed system (4.19)–(4.21), by the Tikhonov-type theorem on the infinite interval [11, Theorem 9.4], we have that

$$\tilde{\theta}^a(\phi_\tau) - \tilde{\theta}_r^a(\phi_\tau) = O(\omega) \tag{4.34}$$

$$r^a(\phi_\tau) - 2\left[1 + \left(\tilde{\theta}_r^a(\phi_\tau) + a\sin\phi_\tau\right)^2\right]^{1/2} \xrightarrow{\exp} O(\omega) \tag{4.35}$$

because it is easy to verify that the boundary layer model

$$\frac{dr_b}{dt} = \epsilon\left(r_b + 2\sqrt{1+\theta^2}\right)\left(\frac{1+\theta^2}{2} - \frac{\left(r_b + 2\sqrt{1+\theta^2}\right)^2}{8}\right) \tag{4.36}$$

has an exponentially stable equilibrium at $r_b = 0$ for all θ. The above conclusions imply that

$$\tilde{\theta}^a(\phi_\tau) \xrightarrow{\exp.} O(\delta + \omega) \tag{4.37}$$

$$r^a(\phi_\tau) \xrightarrow{\exp.} 2 + O(a + \delta + \omega). \tag{4.38}$$

Because (4.19)–(4.21) is the average system of (4.10)–(4.13), from the averaging theorem, it follows that

$$\tilde{\theta}(\phi) \longrightarrow O\left(\delta + \omega + \frac{1}{\mu}\right) \tag{4.39}$$

$$r(\phi) \longrightarrow 2 + O\left(a + \delta + \omega + \frac{1}{\mu}\right) \tag{4.40}$$

[at least on an $O(\mu)$ interval for ϕ]. By an argument similar to that in [11, Theorem 8.4], we establish the same properties for $\tilde{\theta}$ and r as functions of time; i.e.,

$$\tilde{\theta}(t) \longrightarrow O\left(\delta + \omega + \frac{1}{\mu}\right) \tag{4.41}$$

$$r(t) \longrightarrow 2 + O\left(a + \delta + \omega + \frac{1}{\mu}\right) \tag{4.42}$$

[at least on an $O(1)$ interval for t]. This result, in turn, implies that

$$\sqrt{y(t)^2 + \frac{\dot{y}(t)^2}{\mu^2}} \longrightarrow 2 + O\left(a + \delta + \omega + \frac{1}{\mu}\right). \tag{4.43}$$

The last statement means that extremum seeking brings the limit cycle amplitude to within $O(a + \delta + \omega + (1/\mu))$ of its minimum.

A comment is in order about the statement that the results of this section are valid over an $O(1)$ time interval. Although this interval contains several periods of oscillation of the Van der Pol limit cycle, it contains only a fraction of a period of the excitation signal because $\omega \ll 1$, which would seem to weaken the results of the presented analysis. For this reason, it is important to understand where the $O(1)$ limitation on the time interval comes from. This limitation is typical for "weakly nonlinear oscillators" (see, e.g., [11, Sect. 8.4]). It is because the average model of the Van der Pol system is not exponentially stable, so that the approximation $\phi \approx \mu t$ is valid only on $O(1)$ time intervals. This time interval can be extended by taking into account higher order terms in $1/\mu$ (in our analysis, this would be done in the first averaging step). Because all of the rest of the analysis holds for infinite time (because the respective reduced models are all exponentially stable), the results can be made valid over time intervals of the order of the slowest time constants in the extremum-seeking loop. The simulation results presented in the previous section confirm that the conclusions of the analysis are valid for a time interval much longer than $O(1)$.

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Correction to "Asymptotic State Tracking in a Class of Nonlinear Systems Via Learning-Based Inversion"

Young-Hoon Kim and In-Joong Ha

In the above paper,[1] the biographies were printed without the authors' photographs. The revised biographies follow.

Young-Hoon Kim received the B.S., M.S., and Ph.D. degrees in control and instrumentation engineering from Seoul National University, Seoul, Korea, in 1992, 1994, and 1998, respectively.

He is presently a Member of Research Staff at the Nano Systems Laboratory, Samsung Advanced Institute of Technology, Suwon, Korea. His current research interests include nonlinear control theory and its application to high-precision servo systems for data storage devices.

In-Joong Ha received the Ph.D. degree in computer, information, and control engineering (CICE) from the University of Michigan, Ann Arbor, in 1985.

He is presently Professor in the School of Electrical Engineering, Seoul National University, Seoul, Korea. From 1985 to 1986, he was with General Research Laboratories, Troy, MI. From 1982 to 1985, he was a Research Assistant at the center for Research on Integrated Manufacturing, University of Michigan, Ann Arbor. From 1973 to 1981, he worked in the area of missile guidance and control at the Agency of Defense Development in Korea. His current research interest includes nonlinear control theory and its applications to missiles, electric machines, and high-precision servo systems for factory automation and multimedia.

Dr. Ha was the recipient of the 1985 Outstanding Achievement Award in the CICE program.

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Y.-H. Kim is with the Nano Systems Laboratory, Samsung Advanced Institute of Technology, Suwon 440-600 Korea.
I.-J. Ha is with the School of Electrical Engineering, Seoul National University, Seoul 151–742, Korea.
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