1. Introduction

Motivation: The Arctic sea ice has been studied intensively in the field of climate and geoscience. One of the main reasons is due to ice-albedo feedback which influences climate dynamics through the high reflectivity of sea ice. The other reason is the rapid decline of the Arctic sea ice extent in the recent decade shown in several observations. These observations motivate the investigation of future sea ice amount. Several studies have developed a computational model of the Arctic sea ice and performed numerical simulations of the model with initial sea ice temperature profile. However, the spatially distributed temperature in sea ice is difficult to recover in real-time using a limited number of thermal sensors. Hence, the online estimation of the sea ice temperature profile based on some available measurements is crucial for the prediction of the sea ice thickness.

Literature: A thermodynamic model for the Arctic sea ice was firstly developed in Maykut and Untersteiner (1971) (hereafter MU71), in which the authors investigated the correspondence between the annual cycle pattern acquired from the simulation and empirical data of Untersteiner (1961). The model involves a temperature diffusion equation evolving on a spatial domain defined as the sea ice thickness. Due to melting or freezing phenomena, the aforementioned spatial domain is time-varying. Such a model is called “Stefan problem” (Gupta, 2003) which is described by a parabolic partial differential equation (PDE) with a state-dependent moving boundary driven by a Neumann boundary value.

Refined models of MU71 have been suggested in the literature. For instance, Semtner (1976) proposed a numerical model to achieve faster and accurate computation of MU71 by discretizing the temperature profile into some layers and neglecting the salinity effect. An energy-conserving model of MU71 was introduced in Bitz and Lipscomb (1999) by taking into account an internal brine pocket melting on surface ablation and the vertically varying salinity profile. Their thermodynamic model was demonstrated by Bitz, Holland, Weaver, and Eby (2001) using a global climate model with a Lagrangian ice thickness distribution. Combining these two models, Winton (2000) developed an energy-conserving three-layer model of sea ice by treating the upper half of the ice as a variable heat capacity layer.

Remote sensing techniques have been employed to obtain the Arctic sea ice data in several studies. In Hall, Key, Casey, Riggs, and Cavalieri (2004), the authors suggested an algorithm to calculate sea ice surface temperature using the satellite measured brightness temperatures, which provided an excellent measurement of the actual surface temperature of the sea ice during the Arctic cold period. The Arctic sea ice thickness data were acquired in Kwok and Rothrock (2009) through a satellite called “ICESat” during 2003–2008 and compared with the data in Rothrock,
S.Koga and M.Krstic/Automatica112 (2020)108713

The satellite called “Cryosat” and “icethickness have been collected between 2010 and 2014 from 1958–2000. More recent data describing the evolution of the sea ice thickness were collected between 2010 and 2014 from the satellite called “Cryosat −2” (Kwok & Cunningham, 2015).

On the other hand, state estimation has been studied as a specific type of data assimilation which utilizes the numerical model along with the measured value. For finite dimensional systems associated with noisy measurements, a well-known approach is Kalman Filter. Another well-known method is the Luenberger-type state observer, which reconstructs the state variable from partially measured variables. For the application to sea ice, Fenty and Heimbach (2013) developed an adjoint-based method as an iterative state and parameter estimation for the coupled sea ice–ocean in the Labrador Sea and Baffin Bay to minimize an uncertainty-weighted model-data misfit in a least-square sense as suggested in Wunsch and Heimbach (2007), using Massachusetts Institute of Technology general circulation model (MITgcm) developed in Marshall, Adcroft, Hill, Perelman, and Heisey (1997).

In Fenty, Menemenlis, and Zhang (2015), the same methodology was applied to reconstruct the global ocean and ice concentration. Their sea ice model is based on the zero-layer approximation of Koga and Krstic (2017) by removing the availability of the temperature gradient at the ice–ocean interface. This paper improves the design of Koga and Krstic (2017) by removing the availability of the temperature gradient at the ice–ocean interface, which is nearly impossible to measure. The designed algorithm provides the temperature profile estimation in Arctic sea ice via the backstepping method. The observer utilizes the available measurements of the sea ice thickness and the ice surface temperature. A simplified model for the Arctic sea ice is formulated by neglecting the salinity effect, and the exponential convergence of its temperature estimation is ensured by using the designed observer. The simulation study for the proposed observer along with the original model is performed in order to investigate the convergence performance numerically. The simulation results illustrate that the estimated temperature profile converges uniformly to the actual sea ice temperature in three days, which is ten times faster than the straightforward open-loop estimator.

Organization: This paper is organized as follows: Section 2 describes the thermodynamic model of the Arctic sea ice in MU71 and introduces its simplification. Section 3 develops the state estimation design for the simplified model via a backstepping PDE observer and shows the exponential stability of the estimation error system. Section 4 illustrates the simulation of the designed observer applied to the original model of MU71. The paper ends with the concluding remarks and future direction in Section 5.

2. Thermodynamic model of sea ice

The thermodynamic model of MU71 describes the time evolution of the sea ice temperature profile in the vertical axis along with its thickness, which also evolves in time due to accumulation or ablation caused by energy balance.

2.1. Snow covered season model

Fig. 1 provides a schematic of the Arctic sea ice model. During the seasons other than summer (July and August), the sea ice is covered by snow, and the surface position of the snow also evolves in time. Let \( T_i(x, t) \) and \( T_s(x, t) \) denote the temperature profile of snow and sea ice, and \( h(t) \) and \( H(t) \) denote the thickness of snow and sea ice. The total incoming heat flux from the atmosphere is denoted by \( F_a \), and the heat flux from the ocean is denoted by \( F_w \). The Arctic sea ice model suggested by MU71 gives governing equations of a Stefan-type free boundary problem formulated as

\[
F_a - l_0 - \sigma (T_i(h(t), t) + 273)^4 + k_i \frac{\partial T_i}{\partial x} (h(t), t) = \left\{ \begin{array}{ll}
0, & \text{if } T_i(h(t), t) < T_{m1}, \\
-q h(t), & \text{if } T_i(h(t), t) = T_{m1},
\end{array} \right.
\]

for \( x \in (-h(t), 0) \), where

\[
\rho c, \frac{\partial T_i}{\partial t} = k_s \frac{\partial^2 T_i}{\partial x^2} (x, t), \forall x \in (-h(t), 0),
\]

for \( x \in (0, 0) \), where

\[
k_0 \frac{\partial T_i}{\partial x} (0, 0) = k_0 \frac{\partial T_s}{\partial x} (0, t),
\]

for \( x \in (0, H(t)) \), where

\[
F_i(H(t), t) = T_{m2},
\]

for \( x \in (H(t), t) - F_w \), where

\[
l_0, \sigma, \rho_s, \rho_i, c_0, k_0, \rho, T_{m1}, T_{m2}, \text{ and } q \text{ are solar radiation penetrating the ice, Stefan–Boltzmann constant, thermal conductivity of snow, density of snow, heat capacity of pure ice, thermal conductivity of pure ice, density of pure ice, melting point of surface snow, melting point of bottom sea ice, and latent heat of fusion, respectively. The total heat flux from the air } F_a \text{ includes the following terms } F_a = (1 - \alpha) F_e + F_s + F_i + F_t, \text{ where } F_e, F_s, F_i, \text{ and } F_t \text{ denote the incoming solar short-wave radiation, the long-wave radiation from the atmosphere and clouds, the flux of sensible heat, the latent heat in the adjacent air, and the surface albedo, respectively. The heat capacity and thermal conductivity of the sea ice are affected by the salinity as }
\]

\[
c_i(T_i, S(x)) = c_0 + \frac{\gamma S(x)}{T_i(x, t)^2}, \quad k_i(T_i, S(x)) = k_0 + \frac{\gamma S(x)}{T_i(x, t)}.
\]
where \( S(x) \) denotes the salinity in the sea ice. \( \gamma_1 \) and \( \gamma_2 \) represent the weight parameters. The thermodynamic model (1)–(7) allows us to predict the future thickness \( h(t), H(t) \) and the temperature profile \( (T_s, T_i) \) given the accurate initial data. However, from a practical point of view, it is not feasible to obtain a complete temperature profile due to a limited number of thermal sensors. To deal with the problem, the estimation algorithm is designed so that the state estimation converges to the actual state starting from an initial estimate.

### 2.2. Simplification of the model

Before considering the state estimator design, we impose a simplification on the system (1)–(7). The effect of the salinity profile on the physical parameters is assumed to be sufficiently small so that it can be negligible, i.e. \( S(x) = 0 \). Therefore, the heat equation of the sea ice temperature (5) is rewritten as

\[
\frac{\partial T_i}{\partial t}(x, t) = D_i \frac{\partial^2 T_i}{\partial x^2}(x, t) + I_0 k_0 e^{-k x}, \quad \forall x \in (0, H(t)),
\]

where the diffusion coefficient is defined as \( D_i = k_0/\rho c_0 \). Next, we impose the following assumptions.

**Assumption 1.** The thickness \( H(t) \) is positive and upper bounded, i.e. there exists \( \hat{H} > 0 \) such that \( 0 < H(t) < \hat{H} \), for all \( t \geq 0 \).

**Assumption 2.** \( \dot{H}(t) \) is bounded, i.e., there exists \( M > 0 \) such that \( |\dot{H}(t)| < M \), for all \( t \geq 0 \).

According to Kwok and Rothrock (2009), the observation data of the sea ice’s thickness from the 1950s to 2008 show that the maximum value including the uncertainty is less than 5 [m]. Moreover, the largest variation of the thickness in a snow-covered season of a year essentially happens from December to March as an accumulation, and most of the literature shows at most 20 [cm] accumulation per month. Hence, conservatively it is plausible to set \( \hat{H} = 10 \) [m], and \( M = 50 \) [cm/month] = 1.9 \times 10^{-7} [m/s].

Mathematically, the existence of the classical solution of the simple Stefan problem given by (9) and (6)–(7) has been established in the literature. We refer the readers to follow Gupta (2003) and Koga, Diagne, and Krstic (2019) for the detailed explanation. The solution of the original sea ice model (1)–(7) has not been studied due to its high complexity.

### 3. State estimation design

In this section, we derive the estimation algorithm utilizing some available measurements and show the exponential convergence of the designed estimation to the simplified sea ice temperature. The ice thickness and surface temperature are measured in several studies (Hall et al., 2004; Kwok & Rothrock, 2009; Rothrock et al., 2008). It is indeed typical to check observability before observer design, at least for systems on a constant domain (see Moura, Chaturvedi, & Krstic, 2014 for instance). Here, we start with the observer design that is accompanied by a proof of exponential stability, which ensures the states’ detectability.

#### 3.1. Observer structure

Suppose that the sea ice thickness and the ice surface temperature are obtained as measurements \( \hat{Y}_i(t) \) and \( \hat{Y}_s(t) \), i.e.

\[
\hat{Y}_i(t) = H(t), \quad \hat{Y}_s(t) = T_i(0, t).
\]

The state estimate \( \hat{T}_i \) of the sea ice temperature is governed by a copy of the plant (9) and (6)–(7) plus the error injection of \( H(t) \), namely, as follows:

\[
\frac{\partial \hat{T}_i}{\partial t}(x, t) = D_i \frac{\partial^2 \hat{T}_i}{\partial x^2}(x, t) + I_0 k_0 e^{-k x} - p_1(x, t) \left( \hat{Y}_i(t) - \hat{T}_i(t) \right), \forall x \in (0, H(t))
\]

\[
\hat{T}_i(0, t) = \phi_2(0) - p_2(t) \left( \hat{Y}_i(t) - \hat{T}_i(t) \right),
\]

\[
\hat{T}_i(H(t), t) = T_{m1} - p_3(t) \left( \hat{Y}_i(t) - \hat{T}_i(t) \right),
\]

\[
\dot{\hat{Y}}_i(t) = p_4(t) \left( \hat{Y}_i(t) - \hat{H}(t) \right) + \beta \frac{\partial \hat{T}_i}{\partial x}(\hat{Y}_i(t), t) - \frac{F_w}{q},
\]

where \( \beta := \frac{\lambda}{\bar{q}} \). Next, we define the estimation error states as

\[
\hat{T}(x, t) := -(\hat{T}_i(x, t) - \hat{T}_i(t)), \quad \hat{H}(t) := H(t) - \hat{H}(t),
\]

where the negative sign is added to be consistent with the model developed in Koga, Diagne, and Krstic (2019) for the liquid phase. Subtraction of the observer system (11)–(14) from the system (9) and (6)–(7) yields the estimation error system as

\[
\frac{\partial \hat{T}}{\partial t}(x, t) = D_i \frac{\partial^2 \hat{T}}{\partial x^2}(x, t) - p_1(x, t) \hat{H}(t), \quad \forall x \in (0, H(t))
\]

\[
\hat{T}(0, t) = -p_2(t) \hat{H}(t),
\]

\[
\hat{T}(H(t), t) = -p_3(t) \hat{H}(t),
\]

\[
\dot{\hat{H}}(t) = -p_4(t) \hat{H}(t) - \beta \frac{\partial \hat{T}}{\partial x}(\hat{H}(t), t).
\]

Our goal is to design the observer gains \( p_1(x, t), p_2(t), p_3(t), p_4(t) \) so that the temperature error \( \hat{T} \) converges to zero. The main theorem of this paper is stated as follows.

**Theorem 3.** Let Assumptions 1 and 2 hold. Consider the estimation error system (16)–(19) with the design of the observer gains

\[
p_1(x, t) = \frac{c \lambda x I_1(z)}{z} + \left( \frac{\epsilon H(t)}{D_i} - \frac{3}{\beta} \right) \lambda^2 \chi^2 I_2(z) z^2 + \frac{\bar{\lambda} \chi^2 I_2(z)}{D_i \beta z^2},
\]

\[
p_2(t) = 0,
\]

\[
p_3(t) = -\frac{\lambda}{2\beta} H(t) - \epsilon,
\]

\[
p_4(t) = \epsilon - \frac{1}{2} \left( \lambda H(t)^2 - \frac{8D_i}{\beta} \right) + \frac{\epsilon \lambda}{2D_i} H(t),
\]

where \( \lambda > 0, c > 0, \) and \( \epsilon > 0 \) are positive free parameters, \( z \) is defined by

\[
z := \sqrt{\lambda H(t)^2 - x^2},
\]

where \( \lambda := \frac{1}{\bar{q}} \), and \( I_j(\cdot) \) denotes the modified Bessel function of the \( j \)th kind. Then, there exist positive constants \( c^* > 0 \) and \( \bar{M} > 0 \) such that, for all \( c > c^* \), the norm

\[
\Phi(t) := \int_0^{H(t)} \hat{T}(x, t)^2 \, dx + \frac{H(t)^2}{\bar{M}}
\]

satisfies the following exponential decay:

\[
\Phi(t) \leq \bar{M} \Phi(0) e^{-\min(\lambda, c^*) t},
\]

namely, the origin of the estimation error system is exponentially stable in the spatial \( L_2 \) norm.
The technical novelty of the proposed observer is that the design only requires the measured surface temperature and the thickness, which is practically implementable. On the other hand, the estimation design developed in Koga, Diagne, and Krstic (2019) and Koga and Krstic (2017) additionally requires the temperature gradient at the liquid–solid (i.e., ice–ocean) interface, which is nearly impossible to measure.

Remark 1. The observer gains (20)–(23) include the thickness $H(t)$, so the gains are not precomputed offline, but are easily calculated online, along with the state estimation. Owing to the slow dynamics of the sea ice model, the computation time is much less than the time step size as we see in Section 4, which enables the real-time computation of the proposed observer.

Remark 2. The measurements (10) are assumed to be noiseless; however, in practice, the measured data accompany with some noise. Preferably the observer needs pre-filtering to deal with the noisy measurements.

To handle the discrete-time measurements in practice as in Petrus, Chen, Bentsman, and Thomas (2017), the designed observer should be discretized in time such as Euler or Runge–Kutta methods so that the estimation can be computed at every sampling of the discrete-time measurements. The free parameters $\lambda$, $c$, and $\varepsilon$ have their physical units [1/s], [1/s], and [1/C/m], respectively. Hence we can see the consistency of the physical units in the estimation error system (16)–(19) together with (20)–(23).

3.2. Gain derivation via state transformation

The backstepping is a well-known method to design the observer gains for PDEs (Krstic, 2009b; Krstic & Smyshlyaev, 2008). Hereafter, the partial derivatives of a variable in $t$ and $x$ are denoted as the variable with a subscript of $t$ and $x$, respectively. For the estimation error system (16)–(19), we apply the following invertible transformations:

$$
\tilde{T}(x, t) = w(x, t) - \int_x^{H(t)} q(x, y)w(y, t)dy
$$

(27)

$$
\tilde{w}(x, t) = \tilde{T}(x, t) - \int_x^{H(t)} r(x, y)\tilde{T}(y, t)dy
$$

(28)

which map the estimation error system (16)–(19) into the following target system:

$$
w_1(x, t) = D_1w_0(x, t) - \lambda w(x, t)
$$

(29)

$$\tilde{w}(0, t) = 0,
$$

(30)

$$w(H(t), t) \equiv \tilde{w}(H(t), t),
$$

(31)

$$\tilde{H}(t) = - cH(t) - \beta w_1(H(t), t),
$$

(32)

where $f(x, H(t))$ is to be determined. Taking the first and second spatial derivatives of the transformation (27), we get

$$\tilde{T}_t(x, t) = w_t(x, t) + q(x, x)w(x, t)
$$

(33)

$$\tilde{T}_{xx}(x, t) = w_{xx}(x, t) + q(x, x)w_0(x, t) + \left(q(x, x) + \frac{d}{dx}q(x, x)\right)w(x, t)
$$

$$\int_x^{H(t)} q_0(x, y)w(y, t)dy - \psi_0(x, H(t))\tilde{H}(t).$$

Next, taking the time derivative of (27) along the solution of the target system (29)–(32), using integration by parts, and substituting the boundary condition (31), we get

$$\tilde{T}_t(x, t) = D_1w_0(x, t) + D_1q(x, x)w_0(x, t)
$$

$$- (\lambda + D_1q(x, x))w(x, t)
$$

$$+ (\beta\psi(x, H(t)) - D_1q(x, H(t)))w_0(x, H(t), t)
$$

$$+ (\tilde{D}_1\psi_0(x, H(t)) + c\psi(x, H(t)))\tilde{H}(t)
$$

$$+ \int_x^{H(t)} (\lambda q(x, y) - D_1q_0(x, y))w(y, t)dy
$$

$$- \tilde{H}(t)\tilde{H}(t)\left(\varepsilon q(x, H(t)) + \psi_0(x, H(t))q(x, y)\right)f(y, H(t))dy
$$

$$+ f(x, H(t)) - \int_x^{H(t)} q(x, y)f(y, H(t))dy. \quad (35)
$$

Thus, by (34) and (35), we have

$$\tilde{T}(x, t) = D_1\tilde{T}_x(x, t) + p_1(x, t)\tilde{H}(t)
$$

$$= \left(\lambda + 2D_1\frac{d}{dx}q(x, x)\right)w(x, t)
$$

$$+ (\beta\psi(x, H(t)) - D_1q(x, H(t)))w_0(x, H(t), t)
$$

$$+ (\tilde{D}_1\psi_0(x, H(t)) + c\psi(x, H(t)))\tilde{H}(t)
$$

$$+ \int_x^{H(t)} (\lambda q(x, y) - D_1q_0(x, y))w(y, t)dy
$$

$$- \tilde{H}(t)\tilde{H}(t)\left(\varepsilon q(x, H(t)) + \psi_0(x, H(t))\right)f(y, H(t))dy
$$

$$+ f(x, H(t)) - \int_x^{H(t)} q(x, y)f(y, H(t))dy. \quad (36)
$$

Substituting $x = 0$ and $x = H(t)$ into (27), we get

$$\tilde{T}(0, t) + p_2(t)\tilde{H}(t) = - \int_0^{H(t)} q(x, y)f(y, H(t))dy
$$

$$+ (p_1(t) - \psi(0, H(t)))\tilde{H}(t), \quad (37)
$$

$$\tilde{H}(H(t), t) + p_3(t)\tilde{H}(t) = \varepsilon - \psi(H, H) + p_3(t)\tilde{H}(t). \quad (38)
$$

Moreover, substituting $x = H(t)$ into (33) yields

$$\tilde{H}(t) + p_4(t)\tilde{H}(t) + \beta\tilde{T}_x(t, t) = (p_4(t) - c
$$

$$+ \beta(\varepsilon q(H(t), H(t)) - \psi_0(H(t), H(t))))\tilde{H}(t). \quad (39)
$$

Therefore, for Eqs. (16)–(19) to hold, the gain kernel functions must satisfy the following conditions:

$$q_0(x, y) - q_0(y, x) = - \tilde{\lambda}q(x, y),
$$

(40)

$$\frac{d}{dx}q(x, x) = - \frac{\lambda}{2}\varepsilon q(0, y) = 0,
$$

(41)

$$\beta\psi(x, H(t)) = D_1q(x, H(t)),
$$

(42)

and the observer gains must satisfy

$$p_1(t) = - D_1(\varepsilon q(x, H(t)) + \psi_0(x, H(t))) - c\psi(x, H(t)),
$$

(43)

$$p_2(t) = \psi(0, H(t)),
$$

(44)

$$p_3(t) = \psi(H(t), H(t)) - \varepsilon,
$$

(45)

$$p_4(t) = c - \beta(\varepsilon q(H(t), H(t)) - \psi_0(H(t), H(t))).
$$

(46)

and the function $f(x, H(t))$ must satisfy

$$f(x, H(t)) + \varepsilon q(x, H(t)) + \psi_0(x, H(t)) = - \int_x^H q(x, y)f(y, H(t))dy.
$$

(47)
The solutions to (40)–(42) are uniquely given by
\[ q(x, y) = -\frac{\lambda}{\beta} J_1 \left( \sqrt{\lambda(y^2 - x^2)} \right), \quad (48) \]
\[ \dot{\psi}(x, H(t)) = -\frac{\lambda}{\beta} J_1 \left( \sqrt{\lambda(y^2 - x^2)} \right), \quad (49) \]
where \( z \) is defined by (24). Then, using (48)–(49), the conditions (43)–(46) are led to the explicit formulations of the observer gains given as (20)–(23). In the similar manner, the conditions for the gain kernel functions of the inverse transformation (28) are given by
\[ r(x, y) = -r(y, x) = \tilde{\lambda} r(x, y), \quad (50) \]
\[ \frac{d}{dx} r(x, x) = 2 \tilde{\lambda}, \quad r(0, 0) = 0, \quad (51) \]
\[ \beta \phi(x, H(t)) = D r(x, H(t)), \quad (52) \]
and, the function \( f(x, H(t)) \) is obtained by
\[ f(x, H(t)) = r(x, H(t)) \phi(y, H(t)) + \phi(y, H(t)). \quad (53) \]
The solutions to (50)–(52) are given by
\[ r(x, y) = \tilde{\lambda} x J_1 \left( \sqrt{\lambda(y^2 - x^2)} \right), \quad \phi(x, H) = \frac{\lambda}{\beta} J_1 \left( \sqrt{\lambda(y^2 - x^2)} \right), \quad (54) \]
where \( J_1 \) is Bessel function of the first kind. Using the solutions (54), the function \( f(x, H(t)) \) is obtained explicitly by (53), which also satisfies the condition (47). Hence, the transformation from \((\tilde{T}, \tilde{H})\) to \((w, \tilde{H})\) is invertible.

3.3. Stability analysis

In this section, we prove the exponential stability of the origin of the estimation error system (16)–(19) in the spatial \( L_2 \) norm. First, we show the exponential stability of the origin of the target system (29)–(32). We consider the following Lyapunov functional
\[ V = \frac{1}{2} \| w \|^2 + \frac{\epsilon}{2\beta} \tilde{H}(t)^2, \quad (55) \]
where \( \| w \| \) denotes the spatial \( L_2 \) norm of \( w \), defined by \( \| w \| := \sqrt{\int_0^{\tilde{H}(t)} w(x, t)^2 dx} \). Taking the time derivative of (55) together with the solution of (29)–(32) yields
\[ \dot{V} = -D \| w \|^2 - \lambda \| w \|^2 - \frac{\epsilon C}{\beta} \tilde{H}(t)^2 + \frac{\tilde{H}(t)}{2} \tilde{\lambda} \tilde{H}(t)^2 \]
\[ - \tilde{\lambda} \tilde{H}(t) \int_0^{\tilde{H}(t)} w(x, t) \phi(y, H(t)) dx. \quad (56) \]
Applying Young’s and Cauchy–Schwarz inequalities to the last term in (56) with the help of Assumption 2, and choosing the gain parameter \( c \) to satisfy
\[ c > \frac{\beta M e^{-\frac{\lambda}{\epsilon}}}{\epsilon \lambda} + \beta M e, \quad (57) \]
one can obtain the following inequality:
\[ \dot{V} \leq -min[\lambda, c] \| V \|. \quad (58) \]
Applying comparison principle to the differential inequality (58), we get
\[ V(t) \leq V(0) e^{-min[\lambda, c] t}. \quad (59) \]
Hence, the target system (29)–(32) is exponentially stable at the origin. Due to the invertibility of the transformations (27) and (28), there exist positive constants \( M > 0 \) and \( \tilde{M} > 0 \) such that for the norm \( \Phi(t) \) defined in (25) the inequalities hold \( M \Phi(t) \leq V(t) \leq \tilde{M} \Phi(t) \). Hence, we obtain (26) by defining \( M = M \tilde{M} / \tilde{M} \), which completes the proof of Theorem 3. Note that the designed backstepping observer achieves faster convergence with a possibility of causing overshoot since the overshoot coefficient \( M \tilde{M} / \tilde{M} \) is a monotonically increasing function in the observer gains’ parameters \( (\lambda, c) \).

4. Numerical simulation

While we have focused on the simplified PDE (9) to derive a rigorous proof of the proposed state estimation design (11)–(14) with observer gains given by (20)–(23), simulation studies are performed by applying the estimation design to the original thermodynamic model (1)–(7) including salinity. For the computation, we use boundary immobilization method and finite difference semi-discretization (Kutluay, Bahadir, & Özdes, 1997) with 100-point mesh in space, and the resulting approximated ODEs are calculated by using MATLAB ode15 solver.

Input Parameters: The input parameters are taken from Maykut and Untersteiner (1971) in SI units and Table 1 shows the monthly averaged values of heat fluxes coming from the atmosphere for each month. Table 2 shows the physical parameters of snow and sea ice. Following Bittz and Lipscomb (1999), the salinity profile is described by
\[ S(x) = A \left[ 1 - \cos \left( \pi \left( \frac{x}{H(t)} \right) \right) \right], \]
where \( A = 1.6, n = 0.407, \) and \( m = 0.573. \)

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Average monthly values for the energy fluxes.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symbol</td>
<td>( F_1 )</td>
</tr>
<tr>
<td>Unit</td>
<td>W/m²</td>
</tr>
<tr>
<td>Jan.</td>
<td>0</td>
</tr>
<tr>
<td>Feb.</td>
<td>0</td>
</tr>
<tr>
<td>Mar.</td>
<td>30.7</td>
</tr>
<tr>
<td>Apr.</td>
<td>160</td>
</tr>
<tr>
<td>May</td>
<td>286</td>
</tr>
<tr>
<td>Jun.</td>
<td>310</td>
</tr>
<tr>
<td>Jul.</td>
<td>220</td>
</tr>
<tr>
<td>Aug.</td>
<td>145</td>
</tr>
<tr>
<td>Sep.</td>
<td>59.7</td>
</tr>
<tr>
<td>Oct.</td>
<td>64.6</td>
</tr>
<tr>
<td>Nov.</td>
<td>0</td>
</tr>
<tr>
<td>Dec.</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Physical parameters of snow and sea ice.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symbol</td>
<td>Meaning</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Density (snow)</td>
</tr>
<tr>
<td>( k_s )</td>
<td>Conductivity (snow)</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Density (ice)</td>
</tr>
<tr>
<td>( c_p )</td>
<td>Heat capacity (ice)</td>
</tr>
<tr>
<td>( k_i )</td>
<td>Conductivity (ice)</td>
</tr>
<tr>
<td>( y_s )</td>
<td>Weight of heat cap.</td>
</tr>
<tr>
<td>( y_o )</td>
<td>Weight of conductivity</td>
</tr>
<tr>
<td>( I_o )</td>
<td>Solar radiation</td>
</tr>
<tr>
<td>( \kappa_i )</td>
<td>Penetration rate</td>
</tr>
<tr>
<td>( T_m1 )</td>
<td>Melting temp. (surface)</td>
</tr>
<tr>
<td>( T_m2 )</td>
<td>Melting temp. (bottom)</td>
</tr>
</tbody>
</table>
in Fig. 2(b). We can see that both of Fig. 2(a) and (b) have a good agreement with the simulation results shown in Maykut and Untersteiner (1971).

**Simulation Results of State Estimation:** The simulation results of temperature estimation $\hat{T}_i$ computed by (11)–(14) along with the available measurements obtained by the online calculation of (1)–(7) are shown in Fig. 3. As stated in Remark 2, we suppose that the noiseless measurements are available for the observer through pre-filtering in practice. Here the initial temperature profiles are formulated as $T_i(x, 0) = \frac{k_0}{k_s H_0} x + T_0$, $T_i(x, 0) = \frac{T_m - T_0}{H_0} \sin \left( \frac{4\pi}{H_0} x \right)$, where $T_0 = T_i(0, 0)$ which is obtained by solving fourth order algebraic equation from (1) and the input data, and $a$ is set as $a = 1 [C]$.

The estimated initial temperature is chosen as $\hat{T}_i(x, 0) = \frac{T_m - T_0}{H_0} \left( x^2 - 2dH_0x \right) + T_0$ with setting $d = 1/4$. Hence, the initial temperature estimate is lower than the actual temperature. This initial condition satisfies the boundary conditions (12) and (13). The initial state of the estimated ice thickness $\hat{H}(0)$ is set as that of the true thickness, i.e., $\hat{H}(0) = H(0)$, which is feasible because the thickness is actually measured. The design parameters ($\lambda$, $c$, $\varepsilon$) are selected as follows. First, we choose $\varepsilon \approx \beta$ for the norm (55) to be similarly weighted. Second, $\lambda$ is selected to be the inverse of a desired time constant (i.e., the time at 63% decay of the estimation error is achieved); here we set as one day, leading to $\lambda \approx \frac{1}{24} \times 3600 = 1.2 \cdot 10^{-5}$. Third, $c$ is selected sufficiently larger than $\lambda$ so that the decay rate $\min(\lambda, c)$ is not reduced and (57) is satisfied. Finally, these parameters are varied around these reference values until we observe a smooth and sufficiently fast convergence. Throughout the simulation, we see that the minimum value of the time step size in ode solver is more than 1 min, while the computation time of each time update is less than 0.1 s, which shows its real-time implementability as addressed in Remark 1.

Fig. 3 illustrates the contour plots of the simulation results of $T_i(x, t)$ and $\hat{T}_i(x, t)$ for open-loop estimation in (a) by setting all the observer gain to be zero, and the proposed estimation in (b)–(d) with observer gains (20)–(23), respectively, by using input data on January. For the proposed estimation, we fix the parameters of $c = 3.0 \times 10^{-5}$ and $\varepsilon = 1.0 \times 10^{-8}$ for all (b)–(d), and use the parameter of $\lambda = 5.0 \times 10^{-6}$ in (b), $\lambda = 1.0 \times 10^{-5}$ in (c), and $\lambda = 5.0 \times 10^{-7}$ in (d). The figures show that the backstepping observer gain makes the convergence speed of the estimation to the actual value approximately 5 to 10 times faster at every point in sea ice. As seen in (b)–(d), while the larger value of $\lambda$ makes the convergence speed faster, it causes more overshoot beyond the actual temperature. Hence, the tradeoff between the
convergence speed and overshoot can be handled by tuning the gain parameter $\lambda$ appropriately, thereby the parameters used in (b) achieve the desired performance. The overshoot behavior is noted at the end of Section 3.3 from a theoretical perspective. Consequently, the stability properties stated in Theorem 3 for the simplified model can be observed in numerical results of the proposed estimation applied to the original model (1)–(7). To visualize the convergence of the estimated temperature profile used in (b) more clearly, Fig. 4 illustrates the profiles of both true temperature (black solid) and estimated temperature (red dash) on January 1st–3rd in (a)–(c), respectively. We observe that the estimated temperature profile becomes almost the same as the true temperature profile on January 3rd, which is two days after the estimation algorithm runs. Moreover, Fig. 4(d) depicts the time evolution of $\tilde{H}(t)$, which is an estimation error of the ice’s thickness. We observe that the error is “enlarged” from $\tilde{H}(0) = 0$ due to the error of temperature profile, and returns to zero after the temperature profiles become almost indistinguishable on January 3rd, from which the necessity of the estimator of the ice’s thickness is ensured while the thickness is actually measured.

Finally, we have studied the robustness of the proposed observer by varying the parameters $D_i$, $\beta$, and $F_w$ in the observer (11)–(14) and the gains (20)–(23) to $D_i(1 + \delta_1)$, $\beta(1 + \delta_2)$, and $F_w(1 + \delta_3)$ with setting $\delta_1 = 0.3$, $\delta_2 = -0.3$, and $\delta_3 = 0.4$. Fig. 5(a) shows the contour plots of estimated and true temperature profiles and Fig. 5(b) shows the evolution of $\tilde{H}(t)$. From both figures, we can see that the observer states converge and stay around the true states with a modest error after 5 days, which illustrates robust performance of the proposed observer under the parameters’ uncertainties.

5. Conclusion and future work

In this paper, we develop the estimation algorithm for temperature profile in the Arctic sea ice via backstepping observer design. The observer gains are derived so that the convergence of the state estimate to the actual state is guaranteed theoretically for a simplified model. Numerical simulation is employed to investigate the performance of the observer design with the original thermodynamic model, which illustrates ten times faster convergence of state estimation to the actual temperature than the straightforward open-loop estimation.

While we have assumed the online availability of the measurements, these data acquired by satellites typically accompany a time-delay due to the communication. Such a time-delay can be compensated by extending the method developed in Koga, Bresch-Pietri, and Krstic (2019) for control design under actuator delay to the estimator design under sensor delay following the procedure in Krstic (2009b). In addition, the physical parameters used in this paper are uncertain variables in practice, where the uncertain parameters can be assumed to be constants at each month, and hence it is significant to design a simultaneous state and parameter estimation algorithm such as Moura et al. (2014) using a reduced-order model via Pade-approximation and Benosman (2016) using data-driven extremum seeking as an iterative learning method. Instead of adaptive estimation, interval observers for the state estimation of uncertain parabolic PDEs have been proposed in Kharkovskaia, Efimov, Fridman, Polyakov, and Richard (2017). Moreover, applying the optimal control of the Stefan problem developed in Alessandri, Bagnerini, and Gaggero (2018), Bernauer and Herzog (2011), Hinze and Ziegenbalg (2007) to the estimation of the sea ice model is also an interesting direction. These extensions will be considered as our future work.

Acknowledgments

The authors would like to thank I. Eisenman for suggesting the sea ice model we considered throughout this paper and helpful discussions. The authors would like to thank I. Fenty for enriching our knowledge on recent trend of sea ice state estimation developed in NASA Jet Propulsion Laboratory.

References


Shumon Koga received the B.S. degree in Applied Physics from Keio University (Japan) in 2014, and the M.S. degree in Mechanical and Aerospace Engineering from the University of California, San Diego (USA) in 2016. He is currently pursuing the Ph.D. degree in Mechanical and Aerospace Engineering at the University of California, San Diego. He was an intern at NASA Jet Propulsion Laboratory (USA), and Mitsubishi Electric Research Laboratories (USA), during fall of 2017 and summer of 2018, respectively. He received the O. Hugo Schuck Best Paper Award from American Control Conference in 2019, and the Outstanding Graduate Student Award in Mechanical and Aerospace Engineering from UC San Diego in 2018, respectively. His research interests include distributed parameter systems, optimization by extremum seeking, and their applications to additive manufacturing, battery management, thermal management in buildings, transportation systems, and global climate systems.

Miroslav Krstic is Distinguished Professor of Mechanical and Aerospace Engineering, holds the Aspachen endowed chair, and is the founding director of the Cymer Center for Control Systems and Dynamics at UC San Diego. He also serves as Senior Associate Vice Chancellor for Research at UCSD. As a graduate student, he won the UC Santa Barbara best dissertation award and student best paper awards at CDC and ACC. He has been elected as Fellow of seven scientific societies – IEEE, IFAC, ASME, SIAM, AAAS, IET (UK), and AIAA (Associate Fellow) – and as a foreign member of the Serbian Academy of Sciences and Arts and of the Academy of Engineering of Serbia. He has received the SIAM Reid Prize, ASME Oldenburger Medal, Nyquist Lecture Prize, Paynter Outstanding Investigator Award, Ragazzi Education Award, IFAC Nonlinear Control Systems Award, Chestnut textbook prize, Control Systems Society Distinguished Member Award, the PECASE, NSF Career, and ONR Young Investigator awards, the Schuck ’96 and ’19 and Axelby paper prizes, and the first UCSD Research Award given to an engineer. He has also been awarded the Springer Visiting Professorship at UC Berkeley, the Distinguished Visiting Fellowship of the Royal Academy of Engineering, and the Invitation Fellowship of the Japan Society for the Promotion of Science. He serves as Editor-in-Chief of Systems & Control Letters and has been serving as Senior Editor in Automatica and IEEE Transactions on Automatic Control, as editor of two Springer book series, and has served as Vice President for Technical Activities of the IEEE Control Systems Society and as chair of the IEEE CSS Fellow Committee. He has coauthored thirteen books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.