Stabilization of reaction–diffusions PDE with delayed distributed actuation

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A B S T R A C T

This paper pursues control design for an unstable reaction–diffusion equation with arbitrarily large input delay affecting the in-domain actuator. We introduce a transport PDE which results in an extended spatial domain where the reaction–diffusion PDE and the transport PDE are in cascade. Based on the conception of predictor-feedback, an backstepping integral transformation is introduced to transform the original coupled system to a stable target system. A predictor control which compensates the delay via distributed in-domain actuation is designed. The kernel function weighting the state feedback and the historical control employs a Dirac delta function as its initial condition. Due to singularity of the kernel function, the norm equivalence between the original and the target system is proved before we finally reach the exponential stability in $H^1$ norm for the coupled system with the delay compensated feedback. A numerical simulation illustrates the effectiveness of the control.

1. Introduction

Engineering systems are often subject to transport phenomena which generate the presence of time delay in their actuation path. Typically, input delays arise when modeling a wide class of complex physical systems such as production systems [1], hydraulically actuated systems [2], drill system [3] and even chemical processes. In the last few decades, major contributions intensified the understanding of stability of nonlinear ODEs (Ordinary Differential Equations) contingent on various type of delays that affects their inputs based on predictor-feedback control techniques. The key idea is essentially to define an appropriate time-inversion that enables to derive a nonlinear predictor signal which essentially helps to compensate the effect of the delay when combined with a nominal feedback control law which stabilize the delay-free plant [1,4–10]. More recently, control of PDEs (Partial Differential Equations) with input delays has just attracted attentions recently [11–14]. Based on Lyapunov–Krasovskii functionals method, a stable PDE is controlled by a delayed control operator by applying the Linear Matrix Inequality in [15] and [16] proposes interesting results considering a semilinear case with time-delays. A boundary control to compensate input delay for reaction–diffusion PDE by decomposing the PDE system into one finite-dimensional unstable part and one stable infinite-dimensional part has been developed [17], and [18] designed a controller based on the PDE backstepping transformation which extends the predictor-based method for an unstable reaction–diffusion PDE prototype system with input delay on boundary. Further results, such as [19] who constructed a boundary controller for a reaction–diffusion PDE with input delay and unmatched disturbance, can be found in the existing literature. For a wave PDE with boundary unknown anti-damping, an adaptive boundary controller is developed to trade off the anti-damping term in [20].

However, the aforementioned results on PDE delays mainly address the control design for boundary input delay that only influence the boundary actuator. Input delays of distributed actuation in domain are much more difficult to compensate compared to the boundary input delay. This is since that the delays in domain affect the entire spatially distributed state and bring a new challenge that one can only overcome by designing a controller that compensate the everywhere delays. In this context few results such as [21], which considers the stabilization of reaction diffusion PDEs with state delay in domain can be found. However, [21] only considers the state delay instead of input delay and just applies boundary control. In the sense, the state delay does not need to be compensated given an appropriate target system is defined. Another interesting approach is proposed in [22] where case of delay varying with time is investigated.

In this paper, a controller is designed for an unstable reaction diffusion PDE which subjects to in domain input delays. The
delays are converted into a transport PDE containing a spatial argument which transforms the time delay into a spatial in domain shift [23]. The resulting system is a cascade of reaction–diffusion transport PDE that can be stabilized using backstepping technique. An integral transformation which transforms the coupled system into a stable target system in the frame of the PDE backstepping is introduced. Compared to the boundary control for compensating the input delay in [18] which integrates the past control feedback in the delay time interval, we design a distributed in domain control which integrates the past control feedback both in spatial domain and in the delay time interval. Due to the occurrence of in domain input delay, the resulting kernel equation is characterized by a singular initial condition, i.e., the initial condition is denoted by a Dirac delta function. The explicit form of the kernel function is obtained which is formulated by a series of sine orthogonal basis. We apply the Parseval’s theorem to prove the norm equivalence between the coupled system and the stable target system. It is important because the theorem to prove the norm equivalence between the coupled and the stable target system is the exponential stability of the closed-loopsystem unter the designed controller.

This paper is organized as follows. Section 2 presents the control design for the system with input delay. The H∞ exponential stability of the closed-loop system under the designed controller is proven in Section 3 and supportive simulation results are provided in Section 4. The paper ends with concluding remarks and future works presented in Section 5.

2. Control design

2.1. Problem description

Consider a reaction–diffusion PDE system as follows

\[ u_t(x, t) = u_{xx}(x, t) + \lambda u(x, t) + U(x, t - D), \quad x \in [0, 1], \quad t > 0 \quad \text{and} \quad D \in \mathbb{R}^+, \]

for \( x \in [0, 1] \), \( t > 0 \) and \( D \in \mathbb{R}^+ \), denoting system (1) delayed by \( D \) time unit. The boundary conditions are

\[ u(0, t) = 0, \quad u(1, t) = 0. \]

For \( D = 0 \), this is a trivial problem, solvable by many different feedback laws, the nominal one being

\[ U(x, t) = -(c + \lambda)u(x, t), \quad c > 0, \]

which stabilizes the system (1)–(3) to a zero equilibrium. However, under the occurrence of time delay acting on the in domain actuator, the system (1) subject to the nominal feedback law (3) becomes unstable and a delay compensator is needed to stabilize the system.

Such a problem has been considered before for a ODE scalar plant with single input

\[ \dot{x}(t) = \lambda x(t) + U(t - D) \]

for \( t > 0 \) and \( D \in \mathbb{R}^+ \). The Predictor feedback for (4) is

\[ U(t) = -(c + \lambda) \int_0^t \gamma(t-\theta)U(\theta)d\theta \]

where

\[ \gamma(s) = e^{\lambda s}, \quad s \in [0, D]. \]

The term in the bracket on the right side of (5) is the predictor which compensates the input delay. And (6) plays the role of state transition.

This example is a special case of the general compensator design in the setting of finite dimensional system.

For reaction–diffusion PDE (1)–(2), we first represent the delay as a transport PDE, which resulting in the following cascade system

\[ u_t(x, t) = u_{xx}(x, t) + \lambda u(x, t) + v(x, 0, t), \quad x \in [0, 1], \quad t \in \mathbb{R}^+. \]

\[ u(t, 0) = u(t, 1) = 0, \]

\[ u(x, 0) = u_0(x), \]

\[ u(x, s, t) = v(x, s, t), \quad x \in [0, 1], \quad s \in [0, D], \]

\[ v(x, D, t) = U(x, t), \]

\[ v(x, s, 0) = u_0(x), \]

Extended domain \((x, s)\) is shown in Fig. 1, \( s \) resulting from time delay is the spatial argument of the first-order hyperbolic PDE (10) and \( x \) relates to spatial derivative in parabolic PDE (7). Note that the solution to \((10)–(12)\) can be expressed as \( v(x, s, t) = U(x, s + t - D) \). More precisely, \( v(x, 0, t) = U(x, t - D) \) implies that there is no explicit delay in the cascade system, but the delay is transformed into a spatial shift denoted by \( s \) and hidden in state \( v(x, 0, t) \). In the next subsection a delay compensated controller is designed for the equivalent cascade system (7)–(12).

2.2. Design for the delay-compensated controller

We apply the well-known PDE backstepping method to design the controller. First, introduce a stable target system as follows

\[ u_t(x, t) = u_{xx}(x, t) - cu(x, t) + z(x, 0, t), \quad x \in [0, 1], \]

\[ u(t, 0) = 0, \quad u(t, 1) = 0, \]

\[ u(x, 0) = u_0(x), \]

\[ z_t(x, s, t) = z_t(x, s, t), \quad x \in [0, 1], \quad s \in [0, D], \]

\[ z(x, D, t) = 0, \]

\[ z(x, s, 0) = z_0(x, s), \]

where the transport system \( z \) has a mild solution

\[ z(x, s, t) = \left\{ \begin{array}{ll} z_0(x, s + t) & 0 < s + t \leq D \\ 0 & s + t > D. \end{array} \right. \]

which means state \( z(x, s, t) \) becomes zero after time \( D \). Inspired by predictor-feedback control (5), we introduce the following integrated transformation

\[ z(x, s, t) = v(x, s, t) + (c + \lambda) \int_0^s \gamma(s, y)u(y, t)dy + (c + \lambda) \int_0^t \gamma(s - r, y)v(y, r, t)dy, \]

where kernel function \( \gamma(s, y) \) is defined on \([0, 1] \times [0, D] \times [0, 1] \). This transformation transforms original system (7)–(12) to target system (13)–(18). Kernel \( \gamma(s, y) \) could be derived
from the equivalence between the original system and the target system. We first calculate the derivative of $z(x, s, t)$ and get

$$z_t(x, s, t) = v_t(x, s, t) + (c + \lambda) \int_0^1 \gamma(x, s, y)u_t(y, t) \, dy$$

$$+ (c + \lambda) \int_0^1 \int_0^1 \gamma(x, s - r, y)u(y, r, t) \, dr \, dy$$

$$= v_t(x, s, t) + (c + \lambda) \gamma_t(x, s, 1)u_t(1, t)$$

$$- (c + \lambda) \gamma_t(x, s, 0)u_t(0, t)$$

$$+ (c + \lambda) \int_0^1 \gamma(x, s, y)u(y, 0, t) \, dy$$

$$+ (c + \lambda) \int_0^1 \gamma(x, 0, y)u(y, s, t) \, dy - (c + \lambda)$$

$$\times \int_0^1 \gamma(x, s, y)v(y, 0, t) \, dy$$

$$- (c + \lambda) \int_0^1 \int_0^1 \gamma(x, s - r, y)v(y, r, t) \, dr \, dy, \quad (21)$$

where we use integration by parts. And then we calculate

$$z_t(x, s, t) = v_t(x, s, t) + (c + \lambda) \int_0^1 \gamma_t(x, s, y)u(y, t) \, dy$$

$$+ (c + \lambda) \int_0^1 \int_0^1 \gamma_t(x, s - r, y)v(y, r, t) \, dr \, dy$$

$$+ (c + \lambda) \int_0^1 \gamma(x, 0, y)u(y, s, t) \, dy. \quad (22)$$

Substituting (21) and (22) into (16) and combining (7), (13) and

$$z(x, 0, t) = u(x, 0, t)$$

$$+ (c + \lambda) \int_0^1 \gamma(x, 0, y)u(y, t) \, dy, \quad (23)$$

we find the kernels equations as follows:

$$\gamma_t(x, s, y) = \gamma_{yy}(x, s, y) + \lambda \gamma(x, s, y). \quad x \in [0, 1],$$

$$y \in (0, 1), \quad s \in (0, D). \quad (24)$$

$$\gamma(x, s, 0) = 0, \quad \gamma(x, s, 1) = 0, \quad (25)$$

$$\gamma(x, 0, y) = \delta(x - y). \quad (26)$$

where $\delta(x)$ is Dirac delta function. The following proposition presents the solution of $\gamma$, which contains singularity.

**Proposition 1.** The singular initial value problem (24)-(26) has a unique solution in the sense of distribution

$$\gamma(x, s, y) = 2e^{2\lambda} \sum_{n=1}^\infty e^{n^2\pi^2\lambda} \sin(n\pi y) \sin(n\pi x), \quad (27)$$

where $\sqrt{2} \sin(n\pi x), \ n \geq 1$ forms an orthogonal basis for $l^2[0, 1]$. 

**Proof.** Substitute the solution (27) back into (24) and (25) and find the equations hold. Knowing that the initial condition (26) are derived from the following equation

$$\int_0^1 \gamma(x, 0, y)u(y, t) \, dy = u(x, t), \quad (28)$$

for any function $f(x) \in L^2[0, 1]$, we have

$$\int_0^1 \gamma(x, 0, y)f(y) \, dy = \sum_{n=1}^\infty 2 \int_0^1 \sin(n\pi x) \sin(n\pi x)f(y) \, dy,$$

$$= \sum_{n=1}^\infty \left(2 \int_0^1 \sin(n\pi x)f(y) \, dy \right) \sin(n\pi x),$$

$$= \sum_{n=1}^\infty a_n \sin(n\pi x) = f(x), \quad (29)$$

where $a_n = 2 \int_0^1 \sin(n\pi x)f(y) \, dy$ for $n = 1, 2, \ldots$ are the Fourier coefficients of $f(x)$, the initial condition holds.

Due to $|e^{-n^2\pi^2t}| \sin(n\pi x)| \leq |e^{-n^2\pi^2t}| \leq \frac{1}{n^2\pi^2}$ for $\forall s \in (0, D]$ and $n \geq 1$, we deduce that (27) is bounded for $0 < s \leq D$. □

**Remark 1.** As $s = 0$, it is known $\gamma(x, 0, y) = \sum_{n=1}^\infty e^{n^2\pi^2\lambda} \sin(n\pi x) \sin(n\pi x)$ from (27). It is well known [24] that the shifting Dirac Delta function where $0 \leq x, y \leq 1$ can be expanded in Fourier series as

$$\delta(x - y) = \sum_{n=1}^\infty 2 \sin(n\pi x) \sin(n\pi y). \quad (30)$$

There are singular points in kernel $\gamma(x, s, t)$. Fig. 2 shows that sum blows up for $s = 0, y = x$, where parameters are chosen as $\lambda = 10, D = 4$ and $c = 0$.

The delay-compensated controller $U(x, t)$ is derived substituting $s = D$ into the transformation (20) and considering the boundary condition (11), (17), which gives

$$U(x, t) = -(c + \lambda) \int_0^1 \gamma(x, D, y)u(y, t) \, dy$$

$$- (c + \lambda) \int_0^1 \int_0^t \gamma(x, t - \tau, y)U(y, \tau) \, d\tau \, dy. \quad (31)$$

Substituting (27) into (31), which gives

$$U(x, t)$$

$$= -2(c + \lambda) \left[e^{2\lambda} \sum_{n=1}^\infty e^{-n^2\pi^2\lambda} \sin(n\pi x) \int_0^1 \sin(n\pi x)u(y, t) \, dy$$

$$\int_0^1 \int_{t-D}^{t-D} e^{-n^2\pi^2\lambda(t-\tau)} \sin(n\pi x)U(y, \tau) \, d\tau \, dy \right]. \quad (32)$$
Although control term $U(y, t)$ appears on the right hand side of the above equation, the control (31) can be calculated in an iteration algorithm as $t > D$. Control (32) with distributed actuation for constant distributed delays takes some kind of similar form of the control of the ODE system with distributed delay for a discrete input in [6]. But they have two main differences. First, the second term of control (32) which compensates delay integrates both in domain and in delay time interval, while the control in [6] only integrates in the delay time interval. Second, control (32) involves infinite spatial spectrum due to the infinite dimension of PDE, while the control in [6] contains finite inputs.

2.3. Inverse transformation

Consider the inverse transformation from the target system to the original system

$$
\nu(x, s, t) = z(x, s, t) - (c + \lambda) \int_0^1 \eta(x, s, y)u(y, t)dy - (c + \lambda) \int_0^1 \int_0^1 \eta(x, s - r, y)z(y, r, t)drdy,
$$

where $\eta(x, s, y)$ defined on $[0, 1] \times [0, D] \times [0, 1]$ are scalar kernel functions. From the equivalence between the target and the original systems, the kernel functions satisfy

$$
\eta_t(x, s, y) = \eta_x(x, s, y) - \gamma \eta(x, s, y), \quad x, y \in [0, 1], \quad s \in [0, D],
$$

(34)

$$
\eta(x, s, 0) = 0, \quad \eta(x, 0, s) = \delta(x - y).
$$

(35)

$$
\eta(x, 0, 0) = 0.
$$

(36)

The solution of kernel equation (34)–(36) is expressed as

$$
\eta(x, s, y) = \sum_{n=1}^{\infty} 2e^{-c + \pi^2n^2} \sin(np\pi y) \sin(np\pi x).
$$

(37)

By using the inverse transformation (33) and combining with (11) and (17), (31) also can be rewritten as

$$
U(x, t) = -(c + \lambda) \int_0^1 \eta(x, D, y)u(y, t)dy - (c + \lambda) \int_0^1 \int_0^1 \eta(x, D - r, y)z(y, r, t)drdy.
$$

(38)

Control expressed by (38) also can be regarded as a good realization due to $z(y, r, t) = z_0(y, r + t)$ and $z_0(y, r)$ can be easily obtained from $\nu_0(y, r)$ through (20).

Remark 2. The control $U(x, t)$ (38) whose another form of expression is (31) can be proved to be bounded if the target system (13)–(18) is stabil. We also know $\eta(x, s, y)$ is bounded as $0 < s \leq D$ and $\eta(x, 0, y) = -\delta(x - y)$ from the proof of Proposition 1. Let $\sup_{0<s\leq D} \eta(x, s, y) = M_\eta$, then

$$
\int_0^1 \eta(x, D, y)u(y, t)dy \leq |M_\eta| \int_0^1 |u(y, t)|dy.
$$

(39)

Also, in order to state the boundedness of the control, we consider the second term of (38)

$$
\int_0^1 \int_0^D \eta(x, D - r, y)z(y, r, t)drd (x, D - r, y)z(y, r, t)drd (x, D - r, y)z(y, r, t)drd
$$

$$
= \sum_{n=1}^{\infty} e^{-c + \pi^2n^2} \sin(np\pi x) \int_0^t z_0(r, t)dr,
$$

(40)

where $z_0(r, t) = \int_0^t z(y, r, t) \sin(n\pi y)dy$ is the coefficient of $z(y, r, t)$. According to Abel’s series convergence test, (40) is convergent since $\{e^{-c + \pi^2n^2} \sin(n\pi x)\}$ is monotone sequence and uniformly bounded on $[0, D]$ and $\sum_{n=1}^{\infty} \sin(n\pi x)z_0(r, t) = z(x, r, t)$. Hence (38) is bounded so does (31).

3. Stability

Define the $L^2$ norm for function $f(x) \in L^2[0, 1]$ and function $g(x, s) \in L^2([0, 1] \times [0, D])$ as follows:

$$
\|f\|^2_2 = \int_0^1 f^2(x)dx, \quad \|g\|^2_2 = \int_0^1 \int_0^D g^2(x, s)dsdx.
$$

In order to analyze the stability of the system, we define the Lyapunov function

$$
V_1 = \|u\|^2_2 + \|u_x\|^2_2 + \|v\|^2_2 + \|v_x\|^2_2 + \|v(\cdot, 0, t)\|_2^2.
$$

(41)

Correspondingly, the Lyapunov function of the target system is

$$
V_2 = \|u\|^2_2 + \|u_x\|^2_2 + A\|z\|^2_2 + \|z(\cdot, 0, t)\|^2_2,
$$

(42)

where $A > 0$ is a weighting constant to be chosen in the following analysis.

Our main result is the following theorem.

Theorem 1. Consider the system consisting of the plant (7)–(12) and the control law (32). If the initial conditions $u_0 \in H^2[0, 1], v_0 \in L^2[0, 1] \times H^1[0, D]$ are compatible such that $u_0(0) = u_0(1) = 0, v_0(x, D) = U(x, 0)$, then the system is exponential stable, i.e.,

$$
V_1(t) \leq Me^{-\alpha t} V_1(0),
$$

(43)

where $M, \alpha$ are positive constants.

To prove Theorem 1, we first present the stability of the target system in the following Proposition 2. And then we state the norm equivalence between the original system and the target system in the following Proposition 3.

Proposition 2. Consider the system (13)–(18), if the initial conditions $u_0 \in H^2[0, 1], z_0 \in L^2[0, 1] \times H^1[0, D]$ are compatible such that $u_0(0) = u_0(1) = 0$, and $z_0(x, D) = 0$, then the system is exponential stable, i.e.,

$$
V_2(t) \leq Me^{-\alpha t} V_2(0),
$$

(44)

where $M, \alpha$ are positive constants.

Proof. Taking time derivative of $\int_0^1 u^2(x, t)dx$ and using the Cauchy and Schwarz Inequality, we get

$$
\frac{d}{dt} \left( \int_0^1 u^2(x, t)dx \right) \leq -2 \int_0^1 u_x^2dx - (2c - \gamma) \int_0^1 u^2dx
$$

$$
+ \frac{1}{\gamma} \int_0^1 z^2(x, 0, t)dx,
$$

(45)

where letting $0 < \gamma < 2c$. Take time derivative of $\int_0^1 u_x^2(x, t)dx$ and use the Cauchy and Schwarz Inequality, which gives

$$
\frac{d}{dt} \left( \int_0^1 u_x^2(x, t)dx \right) \leq -2(1 - \gamma \gamma) \int_0^1 u^2dx - 2c \int_0^1 u^2dx
$$

$$
+ \frac{1}{\gamma} \int_0^1 z^2(x, 0, t)dx,
$$

(46)

where letting $0 < \gamma - 1 < 2$.

Introduce a new Lyapunov function $V_3 = \int_0^1 \int_0^D e^{b}(z^2(x, s, t) + z^2(x, s, t))dsdx$ with constant $b > 0$, and let

$$
V_4 = \|u\|^2_2 + \|u_x\|^2_2 + AV_3 + \|z(\cdot, 0, t)\|^2_2.
$$

(47)
\( V_4 \) is equivalent to \( V_2 \) in norm, since
\[
\int_0^1 \int_0^D (z^2 + z^2)dsdx \leq \int_0^1 \int_0^D e^{bt}(z^2 + z^2)dsdx \\
\leq e^{bD} \int_0^1 \int_0^D (z^2 + z^2)dsdx.
\]
(47)

The time derivative of \( V_3 \), satisfies the following estimate
\[
\dot{V}_3 = \int_0^1 \int_0^D 2e^{b\gamma}(z^2_1 + z_2)dsdx \\
\leq -\int_0^1 \int_0^D \left[ z_1^2 + (x, 0, t) + z^2(x, 0, t) \right] dx \\
-\ b\int_0^1 \int_0^D e^{b\gamma}(z_1^2 + z_2)dsdx.
\]
(48)

and, moreover
\[
\frac{d}{dt} \left( \int_0^1 z^2(x, 0, t) dx \right) \leq \gamma_2 \int_0^1 z_1^2(x, 0, t) dx \\
+ \frac{1}{\gamma_2} \int_0^1 z^2(x, 0, t) dx, \text{ for } 0 < \gamma_2.
\]
(49)

From (45), (46), (48) and (49), we obtain the following inequality
\[
\dot{V}_4 \leq -(2c - \gamma) \| u \|^2_2 - (2c + 2) \| u_x \|^2_2 - 2(1 - \frac{\gamma_1}{2}) \| u_x \|^2_2 \\
- Ab \| z_1 \|^2_2 - Ab \| z_1 \|^2_2 - (A - \gamma_2) \| z_2 \|^2_2 \\
- (A - \frac{1}{\gamma_2} - \frac{1}{\gamma_2}) \| z_1(0, 0, t) \|^2_2, \\
\]
where we chose \( \gamma_1 = \frac{1}{3}, \gamma_2 = \frac{1}{2} \text{ and } A > \frac{1}{2} + 1 \text{ with } 0 < \gamma < 2c, \)
therefore
\[
\dot{V}_4 \leq -(2c - \gamma) \| u \|^2_2 - (2c + 2) \| u_x \|^2_2 - Ab \| z_1 \|^2_2 - Ab \| z_2 \|^2_2 \\
- (A - \frac{1}{\gamma_2} - 1) \| z_1(0, 0, t) \|^2_2 \\
\leq -\alpha_0 V_2, \\
\]
(51)

where \( \alpha_0 = \min[2c - \gamma, b, A - \frac{1}{2} - 1] \). From the fact that \( V_4 \) is equivalent to \( V_2 \) in norm, we derive the stability result (44) with \( M_0 = e^{bD}. \) \( \square \)

Before presenting the norm equivalence between the original system and the target system, we first introduce Lemma 1-Lemma 4 as follows.

Lemma 1. Let \( \eta(x, s, y) \) be given as (37). The following inequalities are true:
\[
\int_0^1 \int_0^D \left( \int_0^1 \int_0^D \eta(x, s, y)f(y)dy \right)^2 dsdx \leq \| f \|^2_{L^2}, \\
\int_0^1 \int_0^D \int_0^1 \int_0^1 \eta(x, s, r, y)g(y, r)drdy \right)^2 dsdx \leq \| g \|_{L^2}^2.
\]
(52) (53)

Proof. From the Parseval’s theorem and (37), it holds that
\[
\int_0^1 \left( \int_0^1 \int_0^1 \eta(x, s, y)f(y)dy \right)^2 dsdx \\
= \sum_{n=1}^{\infty} e^{-2(c+n^2\pi^2)y} \left( \int_0^1 f(y) \sin(n\pi y)dy \right)^2.
\]
(54)

Substituting (54) into the left side of (52), we have
\[
\int_0^1 \int_0^D \int_0^1 \int_0^D \eta(x, s, y)f(y)dy \right)^2 dsdx \\
= 4 \int_{s=1}^{\infty} \int_0^0 \int_0^1 \int_0^1 f(y) \sin(n\pi y)dy \right)^2 \\
= 4 \int_{s=1}^{\infty} \int_0^0 \int_0^1 \int_0^1 f(y) \sin(n\pi y)dy \right)^2 \\
\leq 4 \int_{s=1}^{\infty} \int_0^0 \int_0^1 \int_0^1 f(y) \sin(n\pi y)dy \right)^2 = \| f \|^2_{L^2},
\]
(55)

where we use the Parseval’s theorem again for \( f(x) = \sum_{n=0}^{\infty} b_n \sin(n\pi x) \) and \( b_n = 2 \int_0^1 f(y) \sin(n\pi y)dy \).

For the second inequality, we can use a similar fashion to prove it. Again, we use the Parseval’s theorem, which gives,
\[
\int_0^1 \int_0^1 \int_0^1 \int_0^1 \eta(x, s, r, y)g(y, r)drdy \right)^2 dsdx \\
= 4 \int_{s=1}^{\infty} \int_0^0 \int_0^1 \int_0^1 f(y) \sin(n\pi y)dy \right)^2 \\
= 4 \int_{s=1}^{\infty} \int_0^0 \int_0^1 \int_0^1 f(y) \sin(n\pi y)dy \right)^2 \\
\leq 4 \int_{s=1}^{\infty} \int_0^0 \int_0^1 \int_0^1 f(y) \sin(n\pi y)dy \right)^2 = \| g \|_{L^2}^2.
\]
(56)

To obtain the last line, we use Parseval’s theorem for \( g(y, r) = \sum_{s=0}^{\infty} g_s(r) \sin(n\pi y) \) and \( g_s(r) = 2 \int_0^1 g(r) \sin(n\pi y)dy \). Therefore, the inequality (53) holds. \( \square \)

Lemma 2. Let \( \eta(x, s, y) \) be given as (37), the following inequalities are true:
\[
\int_0^1 \int_0^D \int_0^1 \int_0^1 \eta(x, s, y)f(y)dy \right)^2 dsdx \leq C_1 \int_0^1 f(y)^2 dy, \\
\forall f \in L^2[0, 1], \ H^2[0, 1] = \{ f \in H^2[0, 1], f(0) = f'(1) = 0 \}.
\]
\[
\int_0^1 \int_0^D \int_0^1 \int_0^1 \eta(x, s, r, y)g(y, r)drdy \right)^2 dsdx \\
\leq C_4 \int_0^1 \int_0^1 \eta(x, s, r, y)g(y, r)drdy \right)^2 dsdx \\
\leq C_4 \| g \|^2_{L^2} + C_5 \| g \|^2_{H^2} + C_6 \| g \|^2_{H^2}. \\
\forall g \in L^2[0, 1] \times H^2[0, D].
\]
(57) (58)

where \( C_3, C_4, C_5 \) and \( C_6 \) are positive constants.

Proof. First consider \( \forall f \in H^2[0, 1] \), it holds that
\[
\int_0^1 \eta(x, s, y)f(y)dy \\
= \int_0^1 \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} (c + n^2\pi^2)e^{-2(c+n^2\pi^2)y}2 \sin(n\pi x) \sin(n\pi y)f(y)dy
\]
In the last line we use the Parseval’s theorem by considering \( \int_0^1 \int_0^1 \left( \int_0^1 \eta(x, s, r) f(y)dy \right)^2 dsdx \)
\[
\leq \sum_{n=1}^{\infty} \left( \frac{c + n^2 \pi^2}{2n^2 \pi^2} \right)^2 \left( \frac{c + n^2 \pi^2}{n^2 \pi^2} \right) \left( \frac{2 - e^{-2(c + n^2 \pi^2)0}}{2(c + n^2 \pi^2)} \right) \left( \int_0^1 \cos(n\pi y) f'(y)dy \right)^2 
\leq \frac{c + \pi^2}{2n^2 \pi^2} \sum_{n=1}^{\infty} \left( \int_0^1 \cos(n\pi y) f'(y)dy \right)^2 
\leq \frac{c + \pi^2}{2n^2 \pi^2} \int_0^1 \left( f'(y) \right)^2 dy.
\]
where we use the Parseval’s theorem. Third,
\[
\int_0^1 \int_0^1 \left( \int_0^1 \eta(x, s, r) f(y)dy \right)^2 dsdx 
\leq \sum_{n=1}^{\infty} \left( \frac{c + n^2 \pi^2}{2n^2 \pi^2} \right)^2 \left( \frac{c + n^2 \pi^2}{n^2 \pi^2} \right) \left( \frac{2 - e^{-2(c + n^2 \pi^2)0}}{2(c + n^2 \pi^2)} \right) \left( \int_0^1 \cos(n\pi y) f'(y)dy \right)^2 
\leq \frac{c + \pi^2}{2n^2 \pi^2} \sum_{n=1}^{\infty} \left( \int_0^1 \cos(n\pi y) f'(y)dy \right)^2 
\leq \frac{c + \pi^2}{2n^2 \pi^2} \int_0^1 \left( f'(y) \right)^2 dy.
\]
where the Parseval’s theorem and Cauchy–Schwarz inequality are used. Combine (62), (63) and (64), we get (58). □

In a same way, we obtain the following two Lemmas.

**Lemma 3.** Let \( y(x, s, y) \) be given as (27). The following inequalities hold
\[
\int_0^1 \int_0^1 \left( \int_0^1 y(x, s, y) f(y)dy \right)^2 dsdx \leq \|f\|^2_C,
\]
\[
\int_0^1 \int_0^1 \left( \int_0^1 y(x, s, y) g(y, r)dy \right)^2 dsdx \leq \|g\|^2_2.
\]

**Lemma 4.** Let \( y(x, s, y) \) be given as (27), the following inequalities are true:
\[
\int_0^1 \int_0^1 \left( \int_0^1 y(x, s, y) f(y)dy \right)^2 dsdx \leq G_3 \int_0^1 \left( f'(y) \right)^2 dy,
\]
\[
\forall f \in H^2_1[0, 1], \ H^2_1[0, 1] = \{f \in H^2[0, 1], f(0) = 0\}.
\]
\[
\int_0^1 \int_0^1 \left( \int_0^1 y(x, s, r, y) g(y, r)dy \right)^2 dsdx 
\leq G_4 \|g\|^2_2 + G_5 \|g(x, 0)\|^2_2 + G_6 \|g(x, s)\|^2_2,
\]
\[\forall g \in L^2[0, 1] \times H^1[0, 1].\]

where \( G_3, G_4, G_5 \) and \( G_6 \) are positive constants.

The following proposition states the equivalence between the two Lyapunov functions (41) and (42).

**Proposition 3.** The Lyapunov function \( V_1 \) and \( V_2 \) are equivalence in norm, i.e.,
\[
V_1 \leq \alpha_1 V_2,
\]
\[
V_2 \leq \alpha_2 V_1,
\]

where \( \alpha_1 \) and \( \alpha_2 \) are positive constants.

**Proof.** First consider the \( L^2 \) norm of \( v \):
\[
\int_0^1 \int_0^1 v^2(x, s, t) dsdx 
\leq \int_0^1 \int_0^1 \left( \int_0^1 2(x, s, t) + \int_0^1 \eta(x, s, y) u(x, t) dy \right) dsdx 
+ \int_0^1 \int_0^1 \left( \int_0^1 \eta(x, s, r, y) z(x, r, t) drdy \right)^2 dsdx 
\leq 3\|x\|^2_2 + 3 \int_0^1 \int_0^1 \left( \int_0^1 \eta(x, s, y) u(x, t) dy \right)^2 dsdx
the controller for simulation and express it in a semi-discrete form for clear illustration,

\[
U(x, t) = -(c + \lambda) \left[ \int_{0}^{1} \gamma(x, D, y)u(y, t) \, dy + U(x, t) \right] \\
- \sum_{i=0}^{n-1} \int_{0}^{1} \gamma(x, D, y)u(y, t) \, dy, \\
- \sum_{i=0}^{n-1} \int_{0}^{1} \gamma(x, D, y)u(y, t - D(n-i)) \, dy, 
\]

where \( n = \frac{D \pi}{2} \), and in Fig. 3(b), the controller (32) cannot be used in simulation directly.

The state diverges in the case without delay compensated control, while the state converges to zero with delay compensated control in Fig. 3(b). The norms of state \( u(x, t) \) are also shown in Fig. 3(c), which illustrate the convergence rate of the state.

5. Conclusions

This paper proposes an extension of the backstepping method for the boundary actuation with delay to the in-domain control with delay. First, a transport PDE also containing the spatial argument is introduced to formulate the time delay, which results the considered system is coupled with the transport PDE. Then, control of the coupled system is designed by the PDE backstepping method. The solution of the kernel equations is expressed by a series in sine basis. \( H^1 \) norm stability is proved in the paper. The main difficulty is to prove the norm equivalence between the target and the compensated controlled system which contains singularity. Future research would consider the control design problem with non-constant input delay, such as the delay varying with spatial argument or varying with time.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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