Output-feedback stabilization of stochastic nonlinear systems driven by noise of unknown covariance

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Abstract

We address the class of stochastic output-feedback nonlinear systems driven by noise whose covariance is time varying and bounded but the bound is not known a priori. This problem is analogous to deterministic disturbance attenuation problems. We first design a controller which guarantees that the solutions converge (in probability) to a residual set proportional to the unknown bound on the covariance. Then, for the case of a vanishing noise vector field, we design an adaptive controller which, besides global stability in probability, guarantees regulation of the state of the plant to zero with probability one. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Despite huge popularity of the LQG control problem, the stabilization problem for nonlinear stochastic systems has been receiving relatively little attention until recently. Efforts toward (global) stabilization of stochastic nonlinear systems have been initiated in the work of Florchinger [6–8] who, among other things, extended the concept of control Lyapunov functions and Sontag’s stabilization formula [15] to the stochastic setting. A breakthrough towards arriving at constructive methods for stabilization of broader classes of stochastic nonlinear systems came with the result of Pan and Başar [13] who derived a backstepping design for strict-feedback systems motivated by a risk-sensitive cost criterion [1,10,12,14]. In [2,3], for the same class of systems as in [13], we designed inverse optimal control laws, which, unlike those in [13], can be designed in an automated manner (via symbolic software). The results of [2,3] were extended in [4] to the class of output-feedback systems; this is the first result on global output-feedback stochastic stabilization. Notable advances on input-to-state (rather than equilibrium) stabilization were made by Tsinias [17,18]. The results in [2–4] solve only the equilibrium stabilization problem under the assumption that the noise vector field is vanishing (which preserves the equilibrium) and the assumption that a bound on the noise covariance is known. These assumptions allow some interesting nonlinear systems but exclude linear systems with additive noise! For this reason, in [5] we addressed systems with nonvanishing noise vector field and unknown bound on covariance, and derived both “robust” and “adaptive” controllers for the class of strict-feedback systems. In this paper we extend these results to the class of output feedback systems, and give a second-order simulation example.

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2. Preliminaries on stochastic stability

In this section, we first state some basic notation and then briefly review some stochastic stability concepts and theorems from [11], which will be used in the following sections.

**Definition 2.1.** A continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$.

**Definition 2.2.** A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{KL}$ if for each fixed $s$ the mapping $\beta(r,s)$ belongs to class $\mathcal{K}_\infty$ with respect to $r$, and for each fixed $r$ the mapping $\beta(r,s)$ is decreasing with respect to $s$ and $\beta(r,s) \to 0$ as $s \to \infty$.

For a matrix $X = [x_1, x_2, \ldots, x_n]$, $\text{Tr} \{ \cdot \}$ denotes the trace and

$$
|X|_F = (\text{Tr}(X^T X))^{1/2} = (\text{Tr}(XX^T))^{1/2}
$$

(2.1)

denotes the Frobenius norm, and obviously,

$$
|X|_F = |\text{col}(X)|
$$

(2.2)

where $\text{col}(X) = [x_1^T, x_2^T, \ldots, x_n^T]^T$. Consider the nonlinear stochastic system

$$
dx = f(x) \, dt + g(x) \, dw,
$$

(2.3)

where $x \in \mathbb{R}^n$ is the state, $w$ is an $r$-dimensional independent Wiener process with incremental covariance $\Sigma(t) \Sigma(t)^T \, dt$, i.e., $E(dw \, dw^T) = \Sigma(t) \Sigma(t)^T \, dt$, where $\Sigma(t)$ is a bounded function taking values in the set of nonnegative definite matrices, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times r}$ are locally Lipschitz, and $f(0) = 0$.

**Definition 2.3.** The system (2.3) is noise-to-state stable (NSS) if $\forall \epsilon > 0$, there exists a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$, such that

$$
P \left\{ |x(t)| \leq \beta(|x_0|, t) + \gamma \left( \sup_{t \geq s \geq 0} |\Sigma(s) \Sigma(s)^T|_F \right) \right\} \geq 1 - \epsilon, \quad \forall t \geq 0, \; \forall x_0 \in \mathbb{R}^n \setminus \{0\}.
$$

(2.4)

NSS is a stochastic analog of input-to-state stability [16].

**Theorem 2.1** (Krstić and Deng [11]). Consider system (2.3) and suppose there exists a $\mathcal{C}^2$ function $V(x)$, class $\mathcal{K}_\infty$ functions $\gamma_1$, $\gamma_2$ and $\rho$, and a class $\mathcal{K}$ function $\gamma_3$, such that

$$
|\Sigma| \leq \rho \left( |\Sigma\Sigma^T|_F \right),
$$

(2.5)

$$
\left\{ (x(t), |x|) \right\} \leq V(x) \leq \gamma_2(|x|),
$$

(2.6)

Then the system (2.3) is NSS.

Consider the nonlinear stochastic system (2.3) with additional assumptions that $g(0) = 0$ and $\Sigma(t) \equiv I$.

**Definition 2.4.** The equilibrium $x = 0$ of the system (2.3) is

- **globally stable in probability** if $\forall \epsilon > 0$ there exists a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$
P \left\{ |x(t)| < \gamma(|x_0|) \right\} \geq 1 - \epsilon, \quad \forall t \geq 0, \; \forall x_0 \in \mathbb{R}^n \setminus \{0\},
$$

(2.7)

- **globally asymptotically stable in probability** if $\forall \epsilon > 0$ there exists a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ such that

$$
P \left\{ |x(t)| < \beta(|x_0|, t) \right\} \geq 1 - \epsilon, \quad \forall t \geq 0, \; \forall x_0 \in \mathbb{R}^n \setminus \{0\}.
$$

(2.8)

**Theorem 2.2** (Krstić and Deng [11]). Consider system (2.3) and suppose there exists a $\mathcal{C}^2$ function $V(x)$ and class $\mathcal{K}_\infty$ functions $\gamma_1$ and $\gamma_2$, such that

$$
\gamma_1(|x|) \leq V(x) \leq \gamma_2(|x|),
$$

(2.9)

$$
\mathcal{L} V(x) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2} \text{Tr} \left\{ g^T \frac{\partial^2 V}{\partial x^2} g \right\} \leq -W(x),
$$

(2.10)
where \( W(x) \) is continuous and nonnegative. Then the equilibrium \( x = 0 \) is globally stable in probability and
\[
P \left\{ \lim_{t \to \infty} W(x) = 0 \right\} = 1. \tag{2.11}
\]
This theorem is a stochastic analog of LaSalle’s theorem.

3. Output-feedback noise-to-state stabilization

In this section we deal with nonlinear output-feedback systems driven by white noise with bounded but unknown covariance. This class of systems is given by the following nonlinear stochastic differential equations:
\[
\begin{align*}
\frac{dx_i}{dt} &= x_{i+1} dt + \phi_i(y) dW, & i = 1, \ldots, n-1 \\
\frac{dx_n}{dt} &= u dt + \phi_n(y) dW, \\
y &= x_1,
\end{align*}
\tag{3.1}
\]
where \( \phi_i(y) \) are \( r \)-vector-valued smooth functions, and \( w \) is an independent \( r \)-dimensional Wiener process with incremental covariance
\[
E[dw dW] = \Sigma(t) \Sigma(t) dW.
\]
Since the states \( x_2, \ldots, x_n \) are not measured, we first design an observer which would provide exponentially convergent estimates of the unmeasured states in the absence of noise. The observer is designed as
\[
\hat{x}_1 = \hat{x}_{i+1} + k_i(y - \hat{x}_1), \quad i = 1, \ldots, n \tag{3.2}
\]
where \( \hat{x}_{n+1} = u \). The observer error \( \hat{x} = x - \hat{x} \) satisfies
\[
\begin{align*}
\dot{\hat{x}} &= \begin{bmatrix} -k_1 \\ \vdots \\ -k_n \end{bmatrix} \hat{x} dt + \phi(y)^T dW = A_0 \hat{x} dt + \phi(y)^T dW, \\
\dot{\hat{x}} &= \begin{bmatrix} x_3 + k_2(y - \hat{x}_1) \end{bmatrix} dt, \\
\vdots \\
\dot{\hat{x}}_n &= \begin{bmatrix} u + k_n(y - \hat{x}_1) \end{bmatrix} dt.
\end{align*}
\tag{3.3}
\]
\[
\text{Our output-feedback design will consist in applying a backstepping procedure to the system } (y, \hat{x}_2, \ldots, \hat{x}_n), \text{ while taking care of the feedback connection through the } \hat{x} \text{ system. In the backstepping method, the error variables } z_i \text{ are given by}
\[
\begin{align*}
z_1 &= y, \\
z_i &= \hat{x}_i - \hat{x}_{i-1}(\hat{x}_1), \quad i = 2, \ldots, n,
\end{align*}
\tag{3.5}
\]
where \( \hat{x}_i = [\hat{x}_1, \ldots, \hat{x}_i]^T \). According to Itô’s differentiation rule, we have
\[
\begin{align*}
\frac{dz_1}{dt} &= (\hat{x}_2 + \hat{x}_2) dt + \phi_1(y)^T dW \\
\frac{dz_i}{dt} &= \left[ \hat{x}_{i+1} + k_1 \hat{x}_1 - \sum_{l=2}^{i-1} \frac{\partial \phi_{i-1}}{\partial \hat{x}_l} (\hat{x}_{l+1} + k_l \hat{x}_1) - \frac{\partial \phi_{i-1}}{\partial y} (\hat{x}_2 + \hat{x}_2) \\
&\quad - \frac{1}{2} \left( \frac{\partial^2 \phi_{i-1}}{\partial y^2} \right) \phi_1(y) \Sigma \Sigma^T \phi_1(y) \right] dt - \frac{\partial \phi_{i-1}}{\partial y} \phi_1(y)^T dW, \quad i = 2, \ldots, n.
\end{align*}
\tag{3.7}
\]
As usual in the stochastic case [11], we employ a quartic Lyapunov function
\[
V(z, \hat{x}) = \frac{1}{4} z^T A z + \frac{1}{4} \sum_{i=2}^{n} z_i^4 + \frac{b}{2} (\hat{x}^T P \hat{x})^2, \tag{3.9}
\]
where $b$ is a positive constant and $P$ satisfies $A_t^TP + PA_0 = -I$. The first two terms in (3.9) constitute a Lyapunov function for the $(y, \hat{x}_2, \ldots, \hat{x}_n)$-system, while the third term in (3.9) is a Lyapunov function for the $\hat{x}$-system.

Now we start the process of selecting the functions $\phi_i(\hat{x}_i, y)$ to make $\mathcal{L}V$ in the form

$$
\mathcal{L}V \leq -\rho(\hat{x}, y, \hat{x}_2, \ldots, \hat{x}_n) + \gamma(\Sigma(t))
$$

where $\rho$ is positive definite, radially unbounded, and $\gamma$ is a class $\mathcal{K}$ function. Since $\phi(y)$ is a smooth function, according to mean value theorem, we can write it as

$$
\phi(y) = \phi(0) + y\phi'(y)
$$

where $\phi'(y)$ is a smooth function. Along the solutions of (3.3), (3.7) and (3.8), we have

$$
\mathcal{L}V = y^3(\hat{x}_2 + \hat{x}_2 + \frac{3}{2}y^2\phi_1(\psi(y)^T\Sigma^T\phi_1(y)) + \sum_{i=2}^n z_i^3 \left( \hat{x}_i + k_i \hat{x}_i - \sum_{i=2}^n \frac{\partial^2 x_{i-1}}{\partial y^2} (\hat{x}_i + k_i \hat{x}_i) \right)
$$

$$
+ \frac{3}{2} \sum_{i=2}^n z_i^2 \left( \frac{\partial^2 x_{i-1}}{\partial y^2} \right)^2 \phi_1(\psi(y)^T\Sigma^T\phi_1(y) - b\hat{x}^TP\hat{x}) + bTR \left\{ \phi(y)^T(2P\hat{x}^TP + \hat{x}^TP\hat{x})\phi(y)^T \right\}
$$

$$
\leq -b\hat{x}^TP\hat{x} + 3bn\sqrt{n}|P|^2 \left( \frac{1}{2c^2_2} |(\phi(0)^T\phi(0)) + \frac{1}{3} \right) + \frac{1}{2c^2_3} |\hat{x}|^4 + \frac{3bn\sqrt{n}|P|^2c^2_3}{2} |\phi(0)^T\phi(0)|\Sigma^4
$$

$$
+ \frac{3bn\sqrt{n}|P|^2c^2_3}{4} |\Sigma|^8 + bn\sqrt{n}|P|^2 \left( \frac{3c^2_3}{2} |(\phi(0)^T\phi(y))|^4y + \frac{5c^2_3}{4} |\phi(y)|^{24/5}y^{24/5} \right)
$$

$$
+ y^3(\hat{x}_2 + \hat{x}_2 + \frac{3}{2}y^2\phi_1(\psi(y)^T\Sigma^T\phi_1(y)) + \sum_{i=2}^n z_i^3 \left( \hat{x}_i + k_i \hat{x}_i - \sum_{i=2}^n \frac{\partial^2 x_{i-1}}{\partial y^2} (\hat{x}_i + k_i \hat{x}_i) \right)
$$

$$
- \frac{1}{2} \left( \frac{\partial^2 x_{i-1}}{\partial y^2} \right)^2 \phi_1(\psi(y)^T\Sigma^T\phi_1(y)) \right) + \frac{3}{2} \sum_{i=2}^n z_i^2 \left( \frac{\partial^2 x_{i-1}}{\partial y^2} \right)^2 \phi_1(\psi(y)^T\Sigma^T\phi_1(y))
$$

$$
\leq -b\lambda - 3bn\sqrt{n}|P|^2 \left( \frac{1}{2c^2_2} |(\phi(0)^T\phi(0)) + \frac{1}{3} \right) + \frac{1}{2c^2_3} |\hat{x}|^4
$$

$$
+ y^3 \left[ x_1 + \frac{3}{4} \phi(0)^T\psi(y) + \frac{3}{4} (\phi(0)^T\psi(y))^2 + \frac{3}{4} (\phi(0)^T\psi(y))^3 |\psi(y)|^{8/3} + \frac{3}{4} |\phi(y)|^{4/3}y + \frac{3}{4} |\phi(y)|^{4/3}y \right]
$$

$$
+ bn\sqrt{n}|P|^2 \left( \frac{3c^2_3}{2} |(\phi(0)^T\phi(y))|^4y + \frac{5c^2_3}{4} |\phi(y)|^{24/5}y^{24/5} \right)
$$

$$
+ \sum_{i=2}^n z_i^3 \left[ x_i + k_i \hat{x}_i - \sum_{i=2}^n \frac{\partial^2 x_{i-1}}{\partial y^2} (\hat{x}_i + k_i \hat{x}_i) \right] + \frac{1}{4} \left( \frac{\partial^2 x_{i-1}}{\partial y^2} \phi_1(\psi(y))^T\phi_1(y) \right)^2 z_i^3
$$

$$
+ \frac{3}{4} |z_i|^4 + z_i + \frac{3}{4} (\phi(0)^T\phi(0)) + \frac{3}{4} \sum_{i=2}^n z_i^2 + \frac{3}{4} \frac{\partial^2 x_{i-1}}{\partial y^2} \phi_1(\psi(y))^T\phi_1(y) \right] + \frac{3bn\sqrt{n}|P|^2c^2_3}{2} |\phi(0)^T\phi(0)|
$$

$$
+ \sum_{i=2}^n z_i^3 \left( \hat{x}_i + k_i \hat{x}_i - \sum_{i=2}^n \frac{\partial^2 x_{i-1}}{\partial y^2} (\hat{x}_i + k_i \hat{x}_i) \right)
$$

$$
+ \frac{3}{4} \frac{\partial^2 x_{i-1}}{\partial y^2} \phi_1(\psi(y))^T\phi_1(y) \right] + \frac{3bn\sqrt{n}|P|^2c^2_3}{2} |\phi(0)^T\phi(0)| + \frac{3}{4} |\phi(y)|^{3/4}y + \frac{3}{4} |\phi(y)|^{3/4}y
$$

$$
+ \frac{3}{4} \frac{\partial^2 x_{i-1}}{\partial y^2} \phi_1(\psi(y))^T\phi_1(y) \right] + \frac{3bn\sqrt{n}|P|^2c^2_3}{2} |\phi(0)^T\phi(0)| + \frac{3}{4} |\phi(y)|^{3/4}y + \frac{3}{4} |\phi(y)|^{3/4}y
$$

where $\lambda > 0$ is the smallest eigenvalue of $P$. The inequalities come from substituting $\hat{x}_i = z_i + x_i$, and Young’s inequalities in Appendix A and (A.2), (A.3), (A.4), (A.5) in [4]. At this point, we can see that all the terms can be cancelled by $y$ and $\hat{x}_i$. If we choose $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and $\eta_1$ large enough to satisfy

$$
\frac{b\lambda - 3bn\sqrt{n}|P|^2}{\left( \frac{1}{2c^2_2} |(\phi(0)^T\phi(0)) + \frac{1}{4c^2_3} \right) - \frac{1}{4} \sum_{i=2}^n \frac{1}{\eta_1^2} - \frac{1}{4c^2_3} = p > 0},
$$

$$
(3.13)
$$
and \( x_i \) and \( u \) as
\[
\begin{align*}
x_1 &= -c_1 y - \frac{1}{4} |\varphi(0)|^4 y - \frac{1}{2} y^3 (\varphi(0)^T \psi(y))^2 - \frac{1}{8} c_1^2 y^7/3 |\psi(y)|^{8/3} - \frac{1}{4} c_1^3 y - \frac{1}{2} c_1^4 y \\
- 3 \beta y^2, \quad \beta = \frac{3 \sqrt{n} |\varphi(0)|^2 c_1^2}{n} |\psi(y)|^{2/3}, \quad \psi(y) = \psi(y), \quad \psi(y)^{2/3}, \quad \psi(y)^{4/3} \end{align*}
\]
(3.14)
\[
\begin{align*}
x_i &= -c_i z_i - k_i \hat{x}_1 + \sum_{j=2}^{i-1} \frac{1}{\delta_{i,j}} \delta_{j-1} \hat{x}_j + k_i \hat{x}_i + \frac{1}{4} (\hat{\varphi}_{i-1} \varphi(y)^T \varphi(y))^2 \frac{3}{4} |\psi(y)|^{4/3} z_i \\
- 3 \delta_i^4 z_i - \frac{1}{4 \delta_i^4} z_i - \frac{3}{4} \delta_i^4 (\hat{\varphi}_{i-1} \varphi(y)^T \varphi(y))^4 + \frac{3}{4} \delta_i^2 \varphi(y)^T \varphi(y))^2 z_i, \quad \delta_i = \delta_i, \quad i = 2, \ldots, n
\end{align*}
\]
(3.15)
\[
u = u, \quad \delta_i = \delta_i, \quad i = 0
\]
(3.16)
where \( c_i > 0 \) and \( \delta_i = 0 \), then the infinitesimal generator of the closed-loop system (3.3), (3.7), (3.8) and (3.16) satisfies
\[
\mathcal{L} V \leq - \sum_{i=1}^{n} c_i z_i^4 + \frac{3}{4} b_n \varphi(0)^T \varphi(0) + \frac{3}{4} \frac{n}{c_i^4} + \frac{6}{\delta_i^4} |\Sigma|^8 + \frac{bn \sqrt{n} |\varphi(0)|^2 c_i^2 |\Sigma|^4}.
\]
(3.17)
With (3.17), according to Theorem 2.1, we have the following stability result.

**Theorem 3.1.** The closed-loop stochastic system (3.1), (3.2), (3.16) is NSS.

### 4. Output-feedback adaptive stabilization

In this section, we deal with output-feedback systems (3.1) with an additional assumption that \( \varphi(0) = 0 \). Since \( \varphi(0) = 0 \), by the mean value theorem, \( \varphi(y) \) can be expressed as
\[
\varphi(y) = y \psi(y)
\]
(4.1)
where \( \psi(y) \) is a smooth function. As we will see in the sequel, to achieve adaptive stabilization in the presence of unknown \( \Sigma \), it is not necessary to estimate the entire matrix \( \Sigma \). Instead, we will estimate only one unknown parameter \( \theta = \| \Sigma \Sigma^T \|_\infty \) using an estimate \( \hat{\theta} \). Employing the same observer (3.2), the entire system is
\[
\begin{align*}
d\hat{x} &= A \hat{x} dt + \varphi(y)^T dw \\
\hat{y} &= (\hat{x}_1 + \hat{x}_2) dt + \varphi(y)^T dw \\
\hat{\varphi}_1 &= [\hat{x}_1 + k_1 (y - \hat{x}_1)] dt \\
\vdots \\
\hat{\varphi}_n &= [\varphi_{n}(\hat{x}, y, \hat{\theta}) + k_n (y - \hat{x}_1)] dt \\
\hat{\theta} &= \gamma \tau_\theta(\hat{x}, y, \hat{\theta}),
\end{align*}
\]
(4.2)
where \( x_n \) and \( \tau_\theta \) are functions to be designed. In the backstepping method, the error variables \( z_i \) are given by
\[
z_1 = y, \\
z_i = z_i - z_{i-1}(\hat{x}_{i-1}, y, \hat{\theta}), \quad i = 2, \ldots, n.
\]
(4.3)
According to Itô’s differentiation rule, we have
\[
\begin{align*}
dz_1 &= (\hat{x}_2 + \hat{x}_2) dt + \varphi_1(y)^T dw \\
dz_i &= \left[ \hat{x}_{i+1} + k_i \hat{x}_1 - \sum_{j=2}^{i-1} \frac{1}{\delta_{i,j}} \delta_{j-1} \hat{x}_j + k_i \hat{x}_i \right] dt + \frac{1}{2} \frac{\partial^2 \varphi_1}{\partial y^2} \varphi_1(y)^T \Sigma \Sigma^T \varphi_1(y) - \frac{\partial \varphi_1}{\partial \theta} \hat{\theta} dt \\
&- \frac{1}{2} \left[ \frac{\partial^2 \varphi_1}{\partial y^2} \varphi_1(y)^T \Sigma \Sigma^T \varphi_1(y) - \frac{\partial \varphi_1}{\partial \theta} \hat{\theta} ight] dt + \frac{\partial \varphi_1}{\partial \theta} \varphi_1(y)^T dw \quad i = 2, \ldots, n.
\end{align*}
\]
(4.6)
As usual in the stochastic case, we employ a Lyapunov function of a quartic form
\[
V(z, x, \hat{\theta}) = \frac{1}{4} x^4 + \frac{1}{4} \sum_{i=1}^{n} z_i^4 + \frac{b}{2} (x^T P x)^2 + \frac{1}{2 \gamma^2} \hat{\theta}^2,
\]  
(4.7)
where \(b\) is a positive constant, \(P\) satisfies \(A_0^T P + P A_0 = -I\), and \(\hat{\theta} = \|\Sigma \Sigma^T\|_\infty - \hat{\theta}\). Now we start the process of selecting the functions \(z_i(x, y, \hat{\theta})\) to make \(\mathcal{L}V\) in the form
\[
\mathcal{L}V \leq -\rho(x, y, \hat{x}, \ldots, \hat{x}, \hat{\theta})
\]  
(4.8)
where \(\rho\) is a positive semidefinite function. Along the solutions of (3.3), (4.5) and (4.6), we have
\[
\mathcal{L}V = y^3 (z_1 + z_2 + \hat{x}_2) + \frac{3}{4} y^2 \phi_1(y)^T \Sigma \Sigma^T \phi_1(y) + \sum_{i=2}^{3} \left[ z_i + z_{i+1} + k_i \hat{x}_1 
\right. \]
\[
- \sum_{i=2}^{n} \frac{\partial z_{i-1}}{\partial x_i} (\hat{x}_{i-1} + k_i \hat{x}_1) - \frac{\partial z_{i-1}}{\partial y} (\hat{x}_2 + \zeta_2) - \frac{1}{2} \left( \frac{\partial^2 z_{i-1}}{\partial y^2} \right) \phi_1(y)^T \Sigma \Sigma^T \phi_1(y) - \frac{\hat{\theta}_i}{\delta_1} \hat{\theta}_i
\]
\[
+ \sum_{i=2}^{3} \left[ z_i + k_i \hat{x}_1 - \sum_{i=2}^{n} \frac{\partial z_{i-1}}{\partial x_i} (\hat{x}_{i-1} + k_i \hat{x}_1) - \frac{\partial z_{i-1}}{\partial y} \hat{x}_2 + \frac{1}{4 z_i^3} \left( \frac{\partial^2 z_{i-1}}{\partial y^2} \right)^2 \|\Sigma \Sigma^T\|_\infty^2
\]
\[
+ \frac{3}{4 z_i^4} y + \frac{3}{4 z_i^3} \phi(\Sigma \Sigma^T)^2 \|\Sigma \Sigma^T\|^2_\infty
\]
\[
+ \frac{3}{4 z_i^3} (\phi(\Sigma \Sigma^T)^2 \|\Sigma \Sigma^T\|^2_\infty + \phi(\Sigma \Sigma^T)^2 \|\Sigma \Sigma^T\|_\infty - \phi(\Sigma \Sigma^T)^2 \|\Sigma \Sigma^T\|_\infty - \phi(\Sigma \Sigma^T)^2 \|\Sigma \Sigma^T\|_\infty - \phi(\Sigma \Sigma^T)^2 \|\Sigma \Sigma^T\|_\infty - \phi(\Sigma \Sigma^T)^2 \|\Sigma \Sigma^T\|_\infty - \phi(\Sigma \Sigma^T)^2 \|\Sigma \Sigma^T\|_\infty - \phi(\Sigma \Sigma^T)^2 \|\Sigma \Sigma^T\|_\infty - \phi(\Sigma \Sigma^T)^2 \|\Sigma \Sigma^T\|_\infty - \phi(\Sigma \Sigma^T)^2 \]  
(4.9)
where \(\hat{\theta} > 0\) is the smallest eigenvalue of \(P\), \(z_{n-1} = u, z_n = 0, \phi = \phi\). The inequalities come from substituting \(\hat{x}_i = z_i + \hat{z}_i\), and Young’s inequalities in Appendix B and (A.2), (A.3), (A.4), (A.5) in [4]. Let
\[
\tau_1 = \frac{3 b n \sqrt{n} P [z_i^3]}{4 z_i^3} |\phi(y)|^4 y^4 + \frac{1}{3} y^4,
\]  
(4.10)
\[
\tau_i = \tau_{i-1} + z_i^3 \omega_i, \quad i = 2, \ldots, n,
\]  
(4.11)
\[
\hat{\theta}_i = \gamma \tau_i,
\]  
(4.12)
where
\[
\omega_i = \frac{1}{4 z_i^3} \left( \frac{\partial^2 z_{i-1}}{\partial y^2} \right)^2 + \frac{3}{4 z_i^3} z_i^3 \left( \frac{\partial z_{i-1}}{\partial y} \right)^4.
\]  
(4.13)
Then
\[
\mathcal{L}V \leq - \left[ b\lambda - \frac{3b\sqrt{n}}{2\epsilon_2^2} |p|^2 - \frac{1}{4} \sum_{i=2}^n \frac{1}{\epsilon_i^4} |\dot{x}|^4 + y^3 \left[ x_1 + \frac{(3n + 1)\alpha_3^2}{4} (\psi_1(y))^T \psi_1(y))^2 y + \frac{3}{4} \delta_1^4 y \right] + \frac{3}{4} \epsilon_{n+1}^4 y + \frac{3b\sqrt{n}}{2\epsilon_2^2} |\psi(y)|^4 y \dot{\theta} + \frac{3}{4} \delta_1^4 y \right] \right] - \frac{1}{4} \epsilon_1^4 |p|^2 - \frac{1}{4} \epsilon_1^4 \leq p > 0, \tag{4.15}
\]
and \( x_i \) and \( u \) as
\[
x_i = -c_i y - \frac{(3n + 1)\alpha_3^2}{4} (\psi_1(y))^T \psi_1(y))^2 y - \frac{1}{4} \delta_1^4 y - \frac{3b\sqrt{n}}{2\epsilon_2^2} |\psi(y)|^4 y \dot{\theta} - \frac{3}{4} \delta_1^4 y \dot{\theta}, \tag{4.16}
\]
\[
x_i = -c_i z_i - k_i \dot{x}_i + \frac{1}{4} \delta_1^4 z_i - \frac{3}{4} \epsilon_{n+1}^4 z_i - \frac{1}{4} \delta_1^4 z_i - \frac{3}{4} \epsilon_{n+1}^4 z_i,
\]
\[
u = u, \tag{4.18}
\]
where \( c_i > 0 \) and \( \delta_n = 0 \), then the infinitesimal generator of the closed-loop system (3.3), (4.5), (4.6) and (4.18) satisfies
\[
\mathcal{L}V \leq - \sum_{i=1}^n c_i z_i^4 - p|\dot{x}|^4, \tag{4.19}
\]
Since \( z = 0 \) and \( \dot{x} = 0 \) implies \( x = 0 \), by Theorem 2.2, we have the following result.

**Theorem 4.1.** The equilibrium \( x = 0, \dot{\theta} = \|\Sigma\Sigma^T\|_{\infty} \) of the closed-loop system (3.1), (3.2), (4.12) and (4.18) is globally stable in probability and
\[
P \left\{ \lim_{t \to \infty} x(t) = 0 \text{ and } \lim_{t \to \infty} \dot{x}(t) = 0 \right\} = 1. \tag{4.20}
\]

**Remark 4.1.** Since \( \mathcal{L}V \) is nonpositive, \( EV(t) \) is nonincreasing. Since \( V \) is also bounded from below by zero, \( EV(t) \) has a limit. Since \( z(t) \) and \( \dot{x}(t) \) converge to zero with probability one, thus \( E\{\dot{\theta}(t)^2\} \) has a limit.

5. Example

We give a second-order example to illustrate the two methods given in above sections. Consider the system
\[
dx_1 = x_2 \, dt + \frac{3\alpha_3^4}{2} y \, dw,
\]
\[
dx_2 = u \, dt,
\]
\[y = x_1. \tag{5.1}\]
For this system, the estimator is
\[\hat{x}_1 = \hat{x}_2 + k_1(y - \hat{x}_1), \quad \hat{x}_2 = u + k_2(y - \hat{x}_1). \tag{5.2}\]
According to the NSS design, the virtual control \( x_1 \) and control \( u \) are
\[
x_1 = -c_1 y - \frac{3\alpha_3^4}{32} y^2 - \frac{3\alpha_3^4}{4} y - \frac{3\alpha_3^4}{8} y - \frac{5\sqrt{2b}|p|^2\epsilon_i^2}{64} y^{3/5}, \tag{5.3}\]
Fig. 1. The states and control effort of the NSS design.

\[ u = -c_2 z_2 - k_2 \dot{x}_1 + \frac{\hat{c}_1}{\hat{y}} \dot{y}_2 - \frac{1}{64} \left( \frac{\hat{c}_1}{\hat{y}} \right)^2 y^8 z_2^3 - \frac{3}{4} \delta_2^3 z_2 - \frac{1}{40} \delta_2^2 \]

\[ - \frac{3}{4} \delta_2 \left( \frac{\hat{c}_1}{\hat{y}} \right)^4 y^4 z_2. \]

(5.4)

We choose \( k_1 = 3 \), \( k_2 = 4.5 \), \( c_1 = e_1 = e_4 = e_5 = \eta_2 = 0.01 \), \( c_2 = b = 0.1 \), \( \delta_1 = \delta_2 = 0.4 \), \( \xi_2 = 50 \), and set the initial condition at \( x_1(0) = 1.3 \), \( x_2(0) = 0 \), \( \dot{x}_1(0) = 0 \), \( \dot{x}_2(0) = x_1(0) \), the states and control of the system are shown in Fig. 1 for \( \Sigma = 1 \).

According to the adaptive design, the virtual control \( z_1 \) and control \( u \) are:

\[ x_1 = -c_1 y - \frac{7 \eta_2^3}{64} y^5 - \frac{1}{4} (\delta_1^4 + \delta_2^4) y - \frac{3 \sqrt{2 b} |P|^2 \eta_2^3}{16} y^5 \dot{\hat{\theta}} - \frac{3}{4 \eta_2^3} y \dot{\hat{\theta}}, \]

(5.5)

\[ u = -c_2 z_2 - k_2 \dot{x}_1 + \frac{\hat{c}_1}{\hat{y}} \dot{y}_2 - \frac{3}{4} \delta_2^3 z_2 - \frac{1}{40} \delta_2^2 \]

\[ - \frac{1}{4 \eta_2^3} z_2 - \frac{3}{4} \delta_2 \left( \frac{\hat{c}_1}{\hat{y}} \right)^4 z_2^2 + \omega_2 \dot{\hat{\theta}} + \frac{\hat{c}_1}{\hat{y}} \dot{\hat{\theta}} \omega_2^2, \]

(5.6)

and the adaptive law is

\[ \dot{\hat{\theta}} = \gamma \tau_2 = \gamma (\tau_1 + z_2 \omega_2), \]

(5.7)

where

\[ \tau_1 = \frac{3 \sqrt{2 b} |P|^2 \eta_2^3}{16} y^8 + \frac{3}{4} y^4, \]

(5.8)

\[ \omega_2 = \frac{1}{4 \eta_2^3} \left( \frac{\hat{c}_1}{\hat{y}} \right)^2 + \frac{3}{4 \eta_2^3} z_2 \left( \frac{\hat{c}_1}{\hat{y}} \right)^4. \]

(5.9)

We choose the same estimator as in NSS design, set parameters as \( c_1 = e_1 = e_4 = e_5 = b = 0.01 \), \( c_2 = \eta_2 = 0.1 \), \( \delta_1 = \delta_2 = 0.4 \), \( \gamma = 10^{-5} \), and the initial condition at \( x_1(0) = 1.3 \), \( x_2(0) = 0 \), \( \dot{x}_1(0) = 0 \), \( \dot{x}_2(0) = x_1(0) \), the states and control of the system are shown in Fig. 2 for \( \Sigma = 1 \).

From Figs. 1 and 2, we can see that (5.4) achieves noise to state stability (i.e., regulation to within a residual set proportional, in appropriate sense, to \( \Sigma = 1 \)) and (5.6) achieves stability of \( x = \hat{x} = \dot{\hat{\theta}} = 0 \) and regulation of \( x \) and \( \dot{x} \) (in probability). A closer inspection of Fig. 1 (left) indicates the possibility that controller (5.4) may be achieving not only NSS but also regulation to zero, however, we do not have hard simulation evidence that \( x_1 \) continues approaching zero over a longer time window, thus we only claim NSS.

Appendix A

In this and the following appendix, we use Young’s inequality [9, Theorem 156]:

\[ x y \leq \frac{e^p}{p} |x|^p + \frac{1}{q e^q} |y|^q, \]

(A.1)
where $\epsilon > 0$, the constants $p > 1$ and $q > 1$ satisfy $(p - 1)(q - 1) = 1$, and $(x, y) \in \mathbb{R}^2$. Applying these inequalities leads to

$$\frac{3}{2} \sum_{i=2}^{n} \left( \frac{\partial \xi_{i-1}}{\partial y} \right)^2 \phi_1(y)^T \Sigma \Sigma^T \phi_1(y) \leq \frac{3}{4} \sum_{i=2}^{n} \frac{1}{\epsilon_i^2} \left( \frac{\partial \xi_{i-1}}{\partial y} \right)^4 \phi_1(y)^T \phi_1(y) + \frac{3}{4} \sum_{i=2}^{n} \epsilon_i^2 |\Sigma \Sigma^T|^2, \quad (A.2)$$

$$b \text{Tr} \{ \phi(y)^T \Sigma^T \left( 2P \Sigma P^T + \dot{x}^T P \dot{x} \right) \Sigma \phi(y)^T \} \leq 3bn \sqrt{n} |\phi(y)^T \phi(y)| |P|^2 |\dot{x}|^2 |\Sigma|^2$$

$$= 3bn \sqrt{n} |P|^2 |\phi(0)^T \phi(0)| |\dot{x}|^2 |\Sigma|^2 + 6bn \sqrt{n} |P|^2 |y \phi(0)^T \psi(y) - \dot{x}^T \phi(1) \phi(1) + 3bn \sqrt{n} |P|^2 |\psi(y)^T \psi(y)| |\dot{x}|^2 |\Sigma|^2$$

$$\leq \frac{3bn \sqrt{n} |P|^2}{2\epsilon_3^2} |\phi(0)^T \phi(0)||\dot{x}|^4 + \frac{3bn \sqrt{n} |P|^2 \epsilon_2^2}{2} |\phi(0)^T \phi(0)||\Sigma|^4 + 3bn \sqrt{n} |P|^2 \epsilon_3^2 |\psi(y)^T \psi(y)||\dot{x}|^2 |\Sigma|^2$$

$$+ \frac{3bn \sqrt{n} |P|^2}{\epsilon_3^2} |\dot{x}|^4 + \frac{3bn \sqrt{n} |P|^2}{2\epsilon_2^4} |\dot{x}|^4 + \frac{3bn \sqrt{n} |P|^2 \epsilon_3^2}{2} |\psi(y)^T \psi(y)||\dot{x}|^2 |\Sigma|^4$$

$$\leq \frac{3bn \sqrt{n} |P|^2}{2\epsilon_2^2} |\phi(0)^T \phi(0)| |\dot{x}|^4 + \frac{1}{\epsilon_3^2} + \frac{1}{2\epsilon_2^4} |\dot{x}|^4 + \frac{3bn \sqrt{n} |P|^2 \epsilon_3^2}{2} |\phi(0)^T \phi(0)| |\Sigma|^4 + \frac{3bn \sqrt{n} |P|^2 \epsilon_3^2}{2} |\Sigma|^8$$

$$+ \frac{3bn \sqrt{n} |P|^2 \epsilon_3^2}{2} |\psi(y)^T \psi(y)| |\dot{x}|^4 + \frac{bn \sqrt{n} |P|^2 \epsilon_3^2}{4} |\Sigma|^4 + \frac{5bn \sqrt{n} |P|^2 \epsilon_3^2}{4} |\psi(y)^T \psi(y)| |\dot{x}|^4 |\Sigma|^4, \quad (A.3)$$
The references are as follows: