ELSEVIER

Systems & Control Letters 39 (2000) 173–182

SYSTEMS & CONTROL LETTERS

www.elsevier.com/locate/sysconle

Output-feedback stabilization of stochastic nonlinear systems driven by noise of unknown covariance ☆

Hua Deng, Miroslav Krstić *

Department of AMES, University of California at San Diego, La Jolla, CA 92093-0411, USA

Abstract

We address the class of stochastic output-feedback nonlinear systems driven by noise whose covariance is time varying and bounded but the bound is not known a priori. This problem is analogous to deterministic disturbance attenuation problems. We first design a controller which guarantees that the solutions converge (in probability) to a residual set proportional to the unknown bound on the covariance. Then, for the case of a vanishing noise vector field, we design an adaptive controller which, besides global stability in probability, guarantees regulation of the state of the plant to zero with probability one. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Output-feedback stabilization; Noise-to-state stabilization; Adaptive stabilization

1. Introduction

Despite huge popularity of the LQG control problem, the stabilization problem for *nonlinear stochastic* systems has been receiving relatively little attention until recently. Efforts toward (global) *stabilization* of stochastic nonlinear systems have been initiated in the work of Florchinger [6–8] who, among other things, extended the concept of control Lyapunov functions and Sontag's stabilization formula [15] to the stochastic setting. A breakthrough towards arriving at *constructive* methods for stabilization of broader classes of stochastic nonlinear systems came with the result of Pan and Başar [13] who derived a backstepping design for strict-feedback systems motivated by a risk-sensitive cost criterion [1,10,12,14]. In [2,3], for the same class of systems as in [13], we designed inverse optimal control laws, which, unlike those in [13], can be designed in an automated manner (via symbolic software). The results of [2,3] were extended in [4] to the class of output-feedback systems; this is the first result on global output-feedback stochastic stabilization. Notable advances on input-to- state (rather than equilibrium) stabilization were made by Tsinias [17,18].

The results in [2–4] solve only the equilibrium stabilization problem under the assumption that the noise vector field is vanishing (which preserves the equilibrium) and the assumption that a bound on the noise covariance is known. These assumptions allow some interesting nonlinear systems but exclude linear systems with additive noise! For this reason, in [5] we addressed systems with nonvanishing noise vector field and unknown bound on covariance, and derived both "robust" and "adaptive" controllers for the class of strict-feedback systems. In this paper we extend these results to the class of output feedback systems, and give a second-order simulation example.

☆ This work was supported in part by the National Science Foundation under Grant ECS-9624386 and in part by the Air Force Office of Scientific Research under Grant F08671-9800319.

* Corresponding author. Fax: 619-822-3107.

E-mail addresses: huadeng@mechanics.ucsd.edu (H. Deng), krstic@ucsd.edu (M. Krstić)

PII: S0167-6911(99)00084-5

2. Preliminaries on stochastic stability

In this section, we first state some basic notation and then briefly review some stochastic stability concepts and theorems from [11], which will be used in the following sections.

Definition 2.1. A continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$.

Definition 2.2. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{KL}$ if for each fixed s the mapping $\beta(r,s)$ belongs to class $\mathcal{K}_\infty$ with respect to r, and for each fixed r the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$.

For a matrix $X = [x_1, x_2, \ldots, x_n]$, $\mathrm{Tr}\{\cdot\}$ denotes the trace and

$$|X|_{\mathcal{F}} \triangleq (\mathrm{Tr}\{X^{\mathrm{T}}X\})^{1/2} = (\mathrm{Tr}\{XX^{\mathrm{T}}\})^{1/2} \tag{2.1}$$

denotes the Frobenius norm, and obviously,

$$|X|_{\mathcal{F}} = |\mathrm{col}(X)| \tag{2.2}$$

where $\mathrm{col}(X) = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \ldots, x_n^{\mathrm{T}}]^{\mathrm{T}}$. Consider the nonlinear stochastic system

$$\mathrm{d}x = f(x)\,\mathrm{d}t + g(x)\,\mathrm{d}w, \tag{2.3}$$

where $x \in \mathbb{R}^n$ is the state, w is an r-dimensional independent Wiener process with incremental covariance $\Sigma(t)\Sigma(t)^{\mathrm{T}}\,\mathrm{d}t$, i.e., $E\{\mathrm{d}w\,\mathrm{d}w^{\mathrm{T}}\} = \Sigma(t)\Sigma(t)^{\mathrm{T}}\,\mathrm{d}t$, where $\Sigma(t)$ is a bounded function taking values in the set of nonnegative definite matrices, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are locally Lipschitz, and $f(0) = 0$.

Definition 2.3. The system (2.3) is *noise-to-state stable* (*NSS*) if $\forall \varepsilon > 0$, there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$, such that

$$P\left\{|(x(t)| < \beta(|x_0|, t) + \gamma\left(\sup_{t \geqslant s \geqslant t_0} \left|\Sigma(s)\Sigma(s)^{\mathrm{T}}\right|_{\mathcal{F}}\right)\right\} \geqslant 1 - \varepsilon, \quad \forall t \geqslant 0,\ \forall x_0 \in \mathbb{R}^n \setminus \{0\}. \tag{2.4}$$

NSS is a stochastic analog of input-to-state stability [16].

Theorem 2.1 (Krstić and Deng [11]). *Consider system* (2.3) *and suppose there exists a $\mathcal{C}^2$ function $V(x)$, class $\mathcal{K}_\infty$ functions α_1, α_2 and ρ, and a class $\mathcal{K}$ function α_3, such that*

$$\alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|), \tag{2.5}$$

$$|x| \geqslant \rho(\left|\Sigma\Sigma^{\mathrm{T}}\right|_{\mathcal{F}}) \quad \Rightarrow \quad \mathcal{L}V(x,\Sigma) = \frac{\partial V}{\partial x} f(x) + \tfrac{1}{2}\mathrm{Tr}\left\{\Sigma^{\mathrm{T}} g^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g \Sigma\right\} \leqslant -\alpha_3(|x|). \tag{2.6}$$

Then the system (2.3) *is NSS.*

Consider the nonlinear stochastic system (2.3) with additional assumptions that $g(0) = 0$ and $\Sigma(t) \equiv \mathrm{I}$.

Definition 2.4. The equilibrium $x = 0$ of the system (2.3) is

- *globally stable in probability* if $\forall \varepsilon > 0$ there exists a class $\mathcal{K}$ function $\gamma(\cdot)$ such that
$$P\{|x(t)| < \gamma(|x_0|)\} \geqslant 1 - \varepsilon, \quad \forall t \geqslant 0,\ \forall x_0 \in \mathbb{R}^n \setminus \{0\}, \tag{2.7}$$
- *globally asymptotically stable in probability* if $\forall \varepsilon > 0$ there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that
$$P\{|x(t)| < \beta(|x_0|, t)\} \geqslant 1 - \varepsilon, \quad \forall t \geqslant 0, \quad \forall x_0 \in \mathbb{R}^n \setminus \{0\}. \tag{2.8}$$

Theorem 2.2 (Krstić and Deng [11]). *Consider system* (2.3) *and suppose there exists a $\mathcal{C}^2$ function $V(x)$ and class $\mathcal{K}_\infty$ functions α_1 and α_2, such that*

$$\alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|), \tag{2.9}$$

$$\mathcal{L}V(x) = \frac{\partial V}{\partial x} f(x) + \tfrac{1}{2}\mathrm{Tr}\left\{g^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g\right\} \leqslant -W(x), \tag{2.10}$$

where $W(x)$ is continuous and nonnegative. Then the equilibrium $x = 0$ is globally stable in probability and

$$P\left\{\lim_{t\to\infty} W(x) = 0\right\} = 1. \tag{2.11}$$

This theorem is a stochastic analog of LaSalle's theorem.

3. Output-feedback noise-to-state stabilization

In this section we deal with nonlinear *output-feedback* systems driven by white noise with bounded but unknown covariance. This class of systems is given by the following nonlinear stochastic differential equations:

$$\begin{aligned} \mathrm{d}x_i &= x_{i+1}\,\mathrm{d}t + \varphi_i(y)^{\mathrm{T}}\,\mathrm{d}w, \quad i = 1,\dots,n-1 \\ \mathrm{d}x_n &= u\,\mathrm{d}t + \varphi_n(y)^{\mathrm{T}}\,\mathrm{d}w, \\ y &= x_1, \end{aligned} \tag{3.1}$$

where $\varphi_i(y)$ are r-vector-valued smooth functions, and w is an independent r-dimensional Wiener process with incremental covariance $E\{\mathrm{d}w\,\mathrm{d}w^{\mathrm{T}}\} = \Sigma(t)\Sigma(t)^{\mathrm{T}}\,\mathrm{d}t$.

Since the states $x_2,\dots,x_n$ are not measured, we first design an observer which would provide exponentially convergent estimates of the unmeasured states in the absence of noise. The observer is designed as

$$\dot{\hat{x}}_i = \hat{x}_{i+1} + k_i(y - \hat{x}_1), \quad i = 1,\dots,n \tag{3.2}$$

where $\hat{x}_{n+1} = u$. The observer error $\tilde{x} = x - \hat{x}$ satisfies

$$\mathrm{d}\tilde{x} = \begin{bmatrix} -k_1 & & \\ & & I \\ \vdots & & \\ -k_n & & 0\cdots 0 \end{bmatrix} \tilde{x}\,\mathrm{d}t + \varphi(y)^{\mathrm{T}}\,\mathrm{d}w = A_0\tilde{x}\,\mathrm{d}t + \varphi(y)^{\mathrm{T}}\,\mathrm{d}w, \tag{3.3}$$

where A_0 is designed to be Hurwitz. Now, the entire system can be expressed as

$$\begin{aligned} \mathrm{d}\tilde{x} &= A_0\tilde{x}\,\mathrm{d}t + \varphi(y)^{\mathrm{T}}\,\mathrm{d}w, \\ \mathrm{d}y &= (\hat{x}_2 + \tilde{x}_2)\mathrm{d}t + \varphi_1(y)^{\mathrm{T}}\,\mathrm{d}w, \\ \mathrm{d}\hat{x}_2 &= [\hat{x}_3 + k_2(y - \hat{x}_1)]\,\mathrm{d}t, \\ &\;\;\vdots \\ \mathrm{d}\hat{x}_n &= [u + k_n(y - \hat{x}_1)]\,\mathrm{d}t. \end{aligned} \tag{3.4}$$

Our output-feedback design will consist in applying a backstepping procedure to the system $(y,\hat{x}_2,\dots,\hat{x}_n)$, while taking care of the feedback connection through the $\tilde{x}$ system. In the backstepping method, the error variables z_i are given by

$$z_1 = y, \tag{3.5}$$

$$z_i = \hat{x}_i - \alpha_{i-1}(\bar{\hat{x}}_{i-1}, y), \quad i = 2,\dots,n, \tag{3.6}$$

where $\bar{\hat{x}}_i = [\hat{x}_1,\dots,\hat{x}_i]^{\mathrm{T}}$. According to Itô's differentiation rule, we have

$$\mathrm{d}z_1 = (\hat{x}_2 + \tilde{x}_2)\,\mathrm{d}t + \varphi_1(y)^{\mathrm{T}}\mathrm{d}w \tag{3.7}$$

$$\begin{aligned} \mathrm{d}z_i = \Bigg[&\hat{x}_{i+1} + k_i\tilde{x}_1 - \sum_{l=2}^{i-1} \frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1) - \frac{\partial\alpha_{i-1}}{\partial y}(\hat{x}_2 + \tilde{x}_2) \\ &- \frac{1}{2}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y)\Bigg]\,\mathrm{d}t - \frac{\partial\alpha_{i-1}}{\partial y}\varphi_1(y)^{\mathrm{T}}\mathrm{d}w, \quad i = 2,\dots,n. \end{aligned} \tag{3.8}$$

As usual in the stochastic case [11], we employ a quartic Lyapunov function

$$V(z,\tilde{x}) = \frac{1}{4}y^4 + \frac{1}{4}\sum_{i=2}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^{\mathrm{T}}P\tilde{x})^2, \tag{3.9}$$

where b is a positive constant and P satisfies $A_0^{\mathrm{T}}P + PA_0 = -I$. The first two terms in (3.9) constitute a Lyapunov function for the $(y, \hat{x}_2, \ldots, \hat{x}_n)$-system, while the third term in (3.9) is a Lyapunov function for the $\tilde{x}$-system.

Now we start the process of selecting the functions $\alpha_i(\bar{\hat{x}}_i, y)$ to make $\mathscr{L}V$ in the form

$$\mathscr{L}V \leqslant -\rho(\tilde{x}, y, \hat{x}_2, \ldots, \hat{x}_n) + \gamma(|\Sigma|) \tag{3.10}$$

where ρ is positive definite, radially unbounded, and γ is a class $\mathscr{K}_\infty$ function. Since $\varphi(y)$ is a smooth function, according to mean value theorem, we can write it as

$$\varphi(y) = \varphi(0) + y\psi(y) \tag{3.11}$$

where $\psi(y)$ is a smooth function. Along the solutions of (3.3), (3.7) and (3.8), we have

$$\begin{aligned}
\mathscr{L}V =\ & y^3(\hat{x}_2 + \tilde{x}_2) + \tfrac{3}{2}y^2\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y) + \sum_{i=2}^{n} z_i^3\left[\hat{x}_{i+1} + k_i\tilde{x}_1 - \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1)\right.\\
& \left. -\frac{\partial\alpha_{i-1}}{\partial y}(\hat{x}_2 + \tilde{x}_2) - \frac{1}{2}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y)\right]\\
& + \frac{3}{2}\sum_{i=2}^{n} z_i^2\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y) - b\tilde{x}^{\mathrm{T}}P\tilde{x}|\tilde{x}|^2 + b\,\mathrm{Tr}\left\{\varphi(y)\Sigma^{\mathrm{T}}(2P\tilde{x}\tilde{x}^{\mathrm{T}}P + \tilde{x}^{\mathrm{T}}P\tilde{x}P)\Sigma\varphi(y)^{\mathrm{T}}\right\}\\
\leqslant\ & -b\tilde{x}^{\mathrm{T}}P\tilde{x}|\tilde{x}|^2 + 3bn\sqrt{n}|P|^2\left(\frac{1}{2\varepsilon_2^2}|\varphi(0)^{\mathrm{T}}\varphi(0)| + \frac{1}{\varepsilon_3^2} + \frac{1}{2\varepsilon_4^2}\right)|\tilde{x}|^4 + \frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\varphi(0)^{\mathrm{T}}\varphi(0)||\Sigma|^4\\
& + \frac{3bn\sqrt{n}|P|^2\varepsilon_3^2}{2}|\Sigma|^8 + \frac{bn\sqrt{n}|P|^2\varepsilon_4^2}{4}|\Sigma|^{24} + bn\sqrt{n}|P|^2\left(\frac{3\varepsilon_3^2}{2}|\varphi(0)^{\mathrm{T}}\psi(y)|^4 y + \frac{5\varepsilon_4^2}{4}|\psi(y)|^{24/5}y^{24/5}\right)y^3\\
& + y^3(\alpha_1 + z_2 + \tilde{x}_2) + \frac{3}{2}y^2\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y)\\
& + \sum_{i=2}^{n} z_i^3\left[\alpha_i + z_{i+1} + k_i\tilde{x}_1 - \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1) - \frac{\partial\alpha_{i-1}}{\partial y}(\hat{x}_2 + \tilde{x}_2)\right.\\
& \left. -\frac{1}{2}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y)\right] + \frac{3}{2}\sum_{i=2}^{n} z_i^2\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y)\\
\leqslant\ & -\left[b\lambda - 3bn\sqrt{n}|P|^2\left(\frac{1}{2\varepsilon_2^2}|\varphi(0)^{\mathrm{T}}\varphi(0)| + \frac{1}{\varepsilon_3^2} + \frac{1}{2\varepsilon_4^2}\right) - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\varepsilon_1^4}\right]|\tilde{x}|^4\\
& + y^3\left[\alpha_1 + \frac{3}{4}|\varphi_1(0)|^4 y + \tfrac{3}{2}y^3(\varphi_1(0)^{\mathrm{T}}\psi_1(y))^2 + \tfrac{3}{8}\varepsilon_5^{4/3}y^{7/3}|\psi_1(y)|^{8/3} + \tfrac{3}{4}\delta_1^{4/3}y + \tfrac{3}{4}\varepsilon_1^{4/3}y\right.\\
& \left. + bn\sqrt{n}|P|^2\left(\frac{3\varepsilon_3^2}{2}|\varphi(0)^{\mathrm{T}}\psi(y)|^4 y + \frac{5\varepsilon_4^2}{4}|\psi(y)|^{24/5}y^{9/5}\right)\right]\\
& + \sum_{i=2}^{n} z_i^3\left[\alpha_i + k_i\tilde{x}_1 - \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1) - \frac{\partial\alpha_{i-1}}{\partial y}\hat{x}_2 + \frac{1}{4}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\varphi_1(y)^{\mathrm{T}}\varphi_1(y)\right)^2 z_i^3\right.\\
& \left. + \tfrac{3}{4}\delta_i^{4/3}z_i + \frac{1}{4\delta_{i-1}^4}z_i + \tfrac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i + \frac{3}{4\xi_i^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i(\varphi_1(y)^{\mathrm{T}}\varphi_1(y))^2\right]\\
& + \left(\frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\varphi(0)^{\mathrm{T}}\varphi(0)| + \frac{3}{4}\sum_{i=2}^{n}\xi_i^2 + \tfrac{5}{2}\right)|\Sigma|^4\\
& + \left(\frac{6}{\varepsilon_5^4} + \frac{3bn\sqrt{n}|P|^2\varepsilon_3^2}{2}\right)|\Sigma|^8 + \frac{bn\sqrt{n}|P|^2\varepsilon_4^2}{4}|\Sigma|^{24}
\end{aligned} \tag{3.12}$$

where $\lambda > 0$ is the smallest eigenvalue of P. The inequalities come from substituting $\hat{x}_i = z_i + \alpha_{i-1}$, and Young's inequalities in Appendix A and (A.2), (A.3), (A.4), (A.5) in [4]. At this point, we can see that all the terms can be cancelled by u and α_i. If we choose $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and η_i large enough to satisfy

$$b\lambda - 3bn\sqrt{n}|P|^2\left(\frac{1}{2\varepsilon_2^2}|\varphi(0)^{\mathrm{T}}\varphi(0)| + \frac{1}{\varepsilon_3^2} + \frac{1}{2\varepsilon_4^2}\right) - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\varepsilon_1^4} = p > 0, \tag{3.13}$$

and α_i and u as

$$\alpha_1 = -c_1 y - \tfrac{3}{4}|\varphi_1(0)|^4 y - \tfrac{3}{2} y^3(\varphi_1(0)^{\mathrm{T}}\psi_1(y))^2 - \tfrac{3}{8}\varepsilon_5^{4/3} y^{7/3}|\psi_1(y)|^{8/3} - \tfrac{3}{4}\delta_1^{4/3} y - \tfrac{3}{4}\varepsilon_1^{4/3} y$$
$$-bn\sqrt{n}|P|^2\left(\frac{3\varepsilon_3^2}{2}|\varphi(0)^{\mathrm{T}}\psi(y)|^4 y + \frac{5\varepsilon_4^2}{4}|\psi(y)|^{24/5} y^{9/5}\right), \tag{3.14}$$

$$\alpha_i = -c_i z_i - k_i\tilde{x}_1 + \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1) + \frac{\partial\alpha_{i-1}}{\partial y}\hat{x}_2 - \frac{1}{4}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\varphi_1(y)^{\mathrm{T}}\varphi_1(y)\right)^2 z_i^3$$
$$-\tfrac{3}{4}\delta_i^{4/3} z_i - \frac{1}{4\delta_{i-1}^4} z_i - \tfrac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i - \frac{3}{4\xi_i^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 (\varphi_1(y)^{\mathrm{T}}\varphi_1(y))^2 z_i, \tag{3.15}$$

$$u = \alpha_n, \tag{3.16}$$

where $c_i > 0$ and $\delta_n = 0$, then the infinitesimal generator of the closed-loop system (3.3), (3.7), (3.8) and (3.16) satisfies

$$\mathscr{L}V \leqslant -\sum_{i=1}^{n} c_i z_i^4 - p|\tilde{x}|^4 + \left(\frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\varphi(0)^{\mathrm{T}}\varphi(0)| + \frac{3}{4}\sum_{i=2}^{n}\xi_i^2 + \tfrac{5}{2}\right)|\Sigma|^4$$
$$+\left(\frac{3bn\sqrt{n}|P|^2\varepsilon_3^2}{2} + \frac{6}{\varepsilon_5^4}\right)|\Sigma|^8 + \frac{bn\sqrt{n}|P|^2\varepsilon_4^2}{4}|\Sigma|^{24}. \tag{3.17}$$

With (3.17), according to Theorem 2.1, we have the following stability result.

Theorem 3.1. *The closed-loop stochastic system* (3.1), (3.2), (3.16) *is NSS.*

4. Output-feedback adaptive stabilization

In this section, we deal with output-feedback systems (3.1) with an additional assumption that $\varphi_i(0) = 0$. Since $\varphi_i(0) = 0$, by the mean value theorem, $\varphi(y)$ can be expressed as

$$\varphi(y) = y\psi(y) \tag{4.1}$$

where $\psi(y)$ is a smooth function. As we will see in the sequel, to achieve adaptive stabilization in the presence of unknown Σ, it is not necessary to estimate the entire matrix Σ. Instead, we will estimate only one unknown parameter $\theta = \|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2$ using an estimate $\hat{\theta}$. Employing the same observer (3.2), the entire system is

$$\begin{aligned}
\mathrm{d}\tilde{x} &= A_0\tilde{x}\,\mathrm{d}t + \varphi(y)^{\mathrm{T}}\,\mathrm{d}w\\
\mathrm{d}y &= (\hat{x}_2 + \tilde{x}_2)\,\mathrm{d}t + \varphi_1(y)^{\mathrm{T}}\,\mathrm{d}w\\
\mathrm{d}\hat{x}_2 &= [\hat{x}_3 + k_2(y - \hat{x}_1)]\,\mathrm{d}t\\
&\;\;\vdots\\
\mathrm{d}\hat{x}_n &= [\alpha_n(\hat{x}, y, \hat{\theta}) + k_n(y - \hat{x}_1)]\,\mathrm{d}t\\
\dot{\hat{\theta}} &= \gamma\tau_n(\hat{x}, y, \hat{\theta}),
\end{aligned} \tag{4.2}$$

where α_n and τ_n are functions to be designed. In the backstepping method, the error variables z_i are given by

$$z_1 = y, \tag{4.3}$$

$$z_i = \hat{x}_i - \alpha_{i-1}(\bar{\hat{x}}_{i-1}, y, \hat{\theta}), \quad i = 2, \ldots, n. \tag{4.4}$$

According to Itô's differentiation rule, we have

$$\mathrm{d}z_1 = (\hat{x}_2 + \tilde{x}_2)\,\mathrm{d}t + \varphi_1(y)^{\mathrm{T}}\,\mathrm{d}w \tag{4.5}$$

$$\mathrm{d}z_i = \left[\hat{x}_{i+1} + k_i\tilde{x}_1 - \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1) - \frac{\partial\alpha_{i-1}}{\partial y}(\hat{x}_2 + \tilde{x}_2)\right.$$
$$\left.-\frac{1}{2}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y) - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\right]\mathrm{d}t - \frac{\partial\alpha_{i-1}}{\partial y}\varphi_1(y)^{\mathrm{T}}\,\mathrm{d}w \quad i = 2, \ldots, n. \tag{4.6}$$

As usual in the stochastic case, we employ a Lyapunov function of a quartic form

$$V(z,\tilde{x},\hat{\theta}) = \frac{1}{4}y^4 + \frac{1}{4}\sum_{i=2}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^{\mathrm{T}}P\tilde{x})^2 + \frac{1}{2\gamma}\tilde{\theta}^2, \tag{4.7}$$

where b is a positive constant, P satisfies $A_0^{\mathrm{T}}P + PA_0 = -I$, and $\tilde{\theta} = \|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2 - \hat{\theta}$. Now we start the process of selecting the functions $\alpha_i(\bar{\hat{x}}_i, y, \hat{\theta})$ to make $\mathscr{L}V$ in the form

$$\mathscr{L}V \leqslant -\rho(\tilde{x}, y, \hat{x}_2, \ldots, \hat{x}_n, \hat{\theta}) \tag{4.8}$$

where ρ is a positive semidefinite function. Along the solutions of (3.3), (4.5) and (4.6), we have

$$\begin{aligned}
\mathscr{L}V &= y^3(\alpha_1 + z_2 + \tilde{x}_2) + \tfrac{3}{2}y^2\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y) + \sum_{i=2}^{n} z_i^3\Bigg[\alpha_i + z_{i+1} + k_i\tilde{x}_1 \\
&\quad - \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1) - \frac{\partial\alpha_{i-1}}{\partial y}(\hat{x}_2 + \tilde{x}_2) - \frac{1}{2}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y) - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\Bigg] \\
&\quad + \frac{3}{2}\sum_{i=2}^{n} z_i^2\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y) - b\tilde{x}^{\mathrm{T}}P\tilde{x}|\tilde{x}|^2 \\
&\quad + b\,\mathrm{Tr}\{\varphi(y)\Sigma^{\mathrm{T}}(2P\tilde{x}\tilde{x}^{\mathrm{T}}P + \tilde{x}^{\mathrm{T}}P\tilde{x}P)\Sigma\varphi(y)^{\mathrm{T}}\} - \frac{\tilde{\theta}\dot{\hat{\theta}}}{\gamma} \\
&\leqslant -\left[b\lambda - \frac{3bn\sqrt{n}}{2\varepsilon_2^2}|P|^2 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\varepsilon_1^4}\right]|\tilde{x}|^4 + y^3\Bigg[\alpha_1 + \frac{3\varepsilon_3^2}{4}(\psi_1(y)^{\mathrm{T}}\psi_1(y))^2 y + \tfrac{3}{4}\delta_1^{4/3}y \\
&\quad + \tfrac{3}{4}\varepsilon_1^{4/3}y + \frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\psi(y)|^4 y\|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2 \\
&\quad + \frac{3}{4\varepsilon_3^2}y\|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2 + \frac{\varepsilon_3^2}{4}(\psi_1(y)^{\mathrm{T}}\psi_1(y))^2 y + \frac{3\varepsilon_3^2}{4}(n-1)(\psi_1(y)^{\mathrm{T}}\psi_1(y))^2 y\Bigg] \\
&\quad + \sum_{i=2}^{n} z_i^3\Bigg[\alpha_i + k_i\tilde{x}_1 - \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1) - \frac{\partial\alpha_{i-1}}{\partial y}\hat{x}_2 + \frac{1}{4\varepsilon_3^2}z_i^3\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)^2\|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2 \\
&\quad + \tfrac{3}{4}\delta_i^{4/3}z_i + \frac{1}{4\delta_{i-1}^4}z_i + \tfrac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3}z_i + \frac{3}{4\varepsilon_3^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i\|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2 - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\Bigg] - \frac{\tilde{\theta}\dot{\hat{\theta}}}{\gamma} \\
&= -\left[b\lambda - \frac{3bn\sqrt{n}}{2\varepsilon_2^2}|P|^2 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\varepsilon_1^4}\right]|\tilde{x}|^4 + y^3\Bigg[\alpha_1 + \frac{(3n+1)\varepsilon_3^2}{4}(\psi_1(y)^{\mathrm{T}}\psi_1(y))^2 y + \tfrac{3}{4}\delta_1^{4/3}y \\
&\quad + \tfrac{3}{4}\varepsilon_1^{4/3}y + \frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\psi(y)|^4 y\hat{\theta} + \frac{3}{4\varepsilon_3^2}y\hat{\theta}\Bigg] + \sum_{i=2}^{n} z_i^3\Bigg[\alpha_i + k_i\tilde{x}_1 - \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1) - \frac{\partial\alpha_{i-1}}{\partial y}\hat{x}_2 \\
&\quad + \frac{1}{4\varepsilon_3^2}z_i^3\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)^2\hat{\theta} + \tfrac{3}{4}\delta_i^{4/3}z_i + \frac{1}{4\delta_{i-1}^4}z_i + \tfrac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3}z_i + \frac{3}{4\varepsilon_3^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i\hat{\theta} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\Bigg] \\
&\quad - \tilde{\theta}\left[\frac{\dot{\hat{\theta}}}{\gamma} - \frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\psi(y)|^4 y^4 - \tfrac{3}{4}y^4 - \frac{1}{4\varepsilon_3^2}\sum_{i=2}^{n} z_i^6\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)^2 - \frac{3}{4\varepsilon_3^2}\sum_{i=2}^{n} z_i^4\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4\right]
\end{aligned} \tag{4.9}$$

where $\lambda > 0$ is the smallest eigenvalue of P, $x_{n+1} = u$, $z_{n+1} = 0$, $\alpha_n = u$. The inequalities come from substituting $\hat{x}_i = z_i + \alpha_{i-1}$, and Young's inequalities in Appendix B and (A.2), (A.3), (A.4), (A.5) in [4]. Let

$$\tau_1 = \frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\psi(y)|^4 y^4 + \tfrac{3}{4}y^4, \tag{4.10}$$

$$\tau_i = \tau_{i-1} + z_i^3\omega_i, \quad i = 2, \ldots, n, \tag{4.11}$$

$$\dot{\hat{\theta}} = \gamma\tau_n, \tag{4.12}$$

where

$$\omega_i = \frac{1}{4\varepsilon_3^2}z_i^3\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)^2 + \frac{3}{4\varepsilon_3^2}z_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4. \tag{4.13}$$

Then

$$\mathscr{L}V \leqslant -\left[b\lambda - \frac{3bn\sqrt{n}}{2\varepsilon_2^2}|P|^2 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\varepsilon_1^4}\right]|\tilde{x}|^4 + y^3\left[\alpha_1 + \frac{(3n+1)\varepsilon_3^2}{4}(\psi_1(y)^{\mathrm{T}}\psi_1(y))^2 y + \tfrac{3}{4}\delta_1^{4/3}y\right.$$
$$\left. + \tfrac{3}{4}\varepsilon_1^{4/3}y + \frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\psi(y)|^4 y\hat{\theta} + \frac{3}{4\varepsilon_3^2}y\hat{\theta}\right] + \sum_{i=2}^{n} z_i^3\left[\alpha_i + k_i\tilde{x}_1 - \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1)\right.$$
$$-\frac{\partial\alpha_{i-1}}{\partial y}\hat{x}_2 + \tfrac{3}{4}\delta_i^{4/3}z_i + \frac{1}{4\delta_{i-1}^4}z_i$$
$$\left. + \tfrac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i - \omega_i\hat{\theta} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\sum_{j=2}^{i}\gamma z_j^3\omega_j - \sum_{j=2}^{i-1}\gamma z_j^3\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\omega_i\right]. \tag{4.14}$$

If we choose ε_1, ε_2 and η_i large enough to satisfy

$$b\lambda - \frac{3bn\sqrt{n}}{2\varepsilon_2^2}|P|^2 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\varepsilon_1^4} = p > 0, \tag{4.15}$$

and α_i and u as

$$\alpha_1 = -c_1 y - \frac{(3n+1)\varepsilon_3^2}{4}(\psi_1(y)^{\mathrm{T}}\psi_1(y))^2 y - \tfrac{3}{4}\delta_1^{4/3}y - \tfrac{3}{4}\varepsilon_1^{4/3}y - \frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\psi(y)|^4 y\hat{\theta} - \frac{3}{4\varepsilon_3^2}y\hat{\theta}, \tag{4.16}$$

$$\alpha_i = -c_i z_i - k_i\tilde{x}_1 + \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_l\tilde{x}_1) + \frac{\partial\alpha_{i-1}}{\partial y}\hat{x}_2 - \tfrac{3}{4}\delta_i^{4/3}z_i$$
$$-\frac{1}{4\delta_{i-1}^4}z_i - \tfrac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i + \omega_i\hat{\theta} + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\sum_{j=2}^{i}\gamma z_j^3\omega_j + \sum_{j=2}^{i-1}\gamma z_j^3\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\omega_i, \tag{4.17}$$

$$u = \alpha_n, \tag{4.18}$$

where $c_i > 0$ and $\delta_n = 0$, then the infinitesimal generator of the closed-loop system (3.3), (4.5), (4.6) and (4.18) satisfies

$$\mathscr{L}V \leqslant -\sum_{i=1}^{n} c_i z_i^4 - p|\tilde{x}|^4. \tag{4.19}$$

Since $z=0$ and $\tilde{x}=0$ implies $x=0$, by Theorem 2.2, we have the following result.

Theorem 4.1. *The equilibrium* $x=0$, $\hat{\theta} = \|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2$ *of the closed-loop system* (3.1), (3.2), (4.12) *and* (4.18) *is globally stable in probability and*

$$P\left\{\lim_{t\to\infty} x(t) = 0 \text{ and } \lim_{t\to\infty}\hat{x}(t) = 0\right\} = 1. \tag{4.20}$$

Remark 4.1. Since $\mathscr{L}V$ is nonpositive, $EV(t)$ is nonincreasing. Since V is also bounded from below by zero, $EV(t)$ has a limit. Since $z(t)$ and $\tilde{x}(t)$ converge to zero with probability one, thus $E\{\tilde{\theta}(t)^2\}$ has a limit.

5. Example

We give a second-order example to illustrate the two methods given in above sections. Consider the system

$$\begin{aligned} \mathrm{d}x_1 &= x_2\,\mathrm{d}t + \tfrac{1}{2}x_1^2\,\mathrm{d}w, \\ \mathrm{d}x_2 &= u\,\mathrm{d}t, \\ y &= x_1. \end{aligned} \tag{5.1}$$

For this system, the estimator is

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = u + k_2(y - \hat{x}_1). \tag{5.2}$$

According to the NSS design, the virtual control α_1 and control u are

$$\alpha_1 = -c_1 y - \frac{3\varepsilon_5^{4/3}}{32}y^5 - \tfrac{3}{4}\delta_1^{4/3}y - \tfrac{3}{4}\varepsilon_1^{4/3}y - \frac{5\sqrt{2}b|P|^2\varepsilon_4^2}{64}y^{33/5}, \tag{5.3}$$

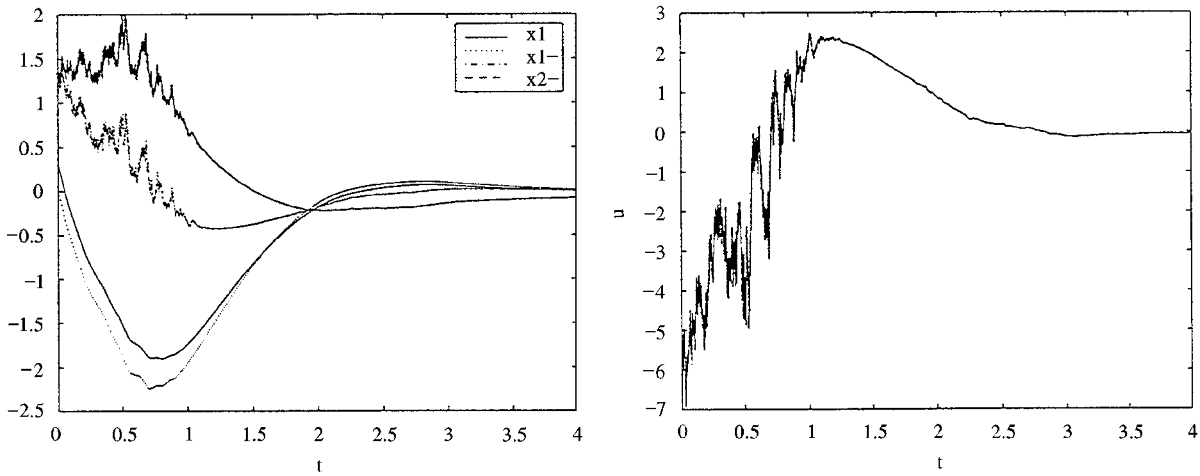

Fig. 1. The states and control effort of the NSS design.

$$u = -c_2 z_2 - k_2 \tilde{x}_1 + \frac{\partial \alpha_1}{\partial y}\hat{x}_2 - \frac{1}{64}\left(\frac{\partial^2 \alpha_1}{\partial y^2}\right)^2 y^8 z_2^3 - \tfrac{3}{4}\delta_2^{4/3} z_2 - \frac{1}{4\delta_1^4} z_2$$
$$- \tfrac{3}{4}\eta_2^{4/3}\left(\frac{\partial \alpha_1}{\partial y}\right)^{4/3} z_2 - \frac{3}{64\xi_2^2}\left(\frac{\partial \alpha_1}{\partial y}\right)^4 y^8 z_2. \tag{5.4}$$

We choose $k_1 = 3$, $k_2 = 4.5$, $c_1 = \varepsilon_1 = \varepsilon_4 = \varepsilon_5 = \eta_2 = 0.01$, $c_2 = b = 0.1$, $\delta_1 = \delta_2 = 0.4$, $\xi_2 = 50$, and set the initial condition at $x_1(0) = 1.3$, $x_2(0) = 0$, $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = \alpha_1(0)$, the states and control of the system are shown in Fig. 1 for $\Sigma = 1$.

According to the adaptive design, the virtual control α_1 and control u are:

$$\alpha_1 = -c_1 y - \frac{7\varepsilon_3^2}{64} y^5 - \tfrac{3}{4}(\delta_1^{4/3} + \varepsilon_1^{4/3}) y - \frac{3\sqrt{2} b |P|^2 \varepsilon_2^2}{16} y^5 \hat{\theta} - \frac{3}{4\varepsilon_3^2} y \hat{\theta}, \tag{5.5}$$

$$u = -c_2 z_2 - k_2 \tilde{x}_1 + \frac{\partial \alpha_1}{\partial y}\hat{x}_2 - \tfrac{3}{4}\delta_2^{4/3} z_2 - \frac{1}{4\delta_1^4} z_2 - \tfrac{3}{4}\eta_2^{4/3}\left(\frac{\partial \alpha_1}{\partial y}\right)^{4/3} z_2 + \omega_2 \hat{\theta} + \frac{\partial \alpha_1}{\partial \hat{\theta}}\gamma z_2^3 \omega_2, \tag{5.6}$$

and the adaptive law is

$$\dot{\hat{\theta}} = \gamma \tau_2 = \gamma(\tau_1 + z_2^3 \omega_2), \tag{5.7}$$

where

$$\tau_1 = \frac{3\sqrt{2} b |P|^2 \varepsilon_2^2}{16} y^8 + \tfrac{3}{4} y^4, \tag{5.8}$$

$$\omega_2 = \frac{1}{4\varepsilon_3^2} z_2^3 \left(\frac{\partial^2 \alpha_1}{\partial y^2}\right)^2 + \frac{3}{4\varepsilon_3^2} z_2 \left(\frac{\partial \alpha_1}{\partial y}\right)^4. \tag{5.9}$$

We choose the same estimator as in NSS design, set parameters as $c_1 = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = b = 0.01$, $c_2 = \eta_2 = 0.1$, $\delta_1 = \delta_2 = 0.4$, $\gamma = 10^{-5}$, and the initial condition at $x_1(0) = 1.3$, $x_2(0) = 0$, $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = \alpha_1(0)$, the states and control of the system are shown in Fig. 2 for $\Sigma = 1$.

From Figs. 1 and 2, we can see that (5.4) achieves noise to state stability (i.e., regulation to within a residual set proportional, in appropriate sense, to $\Sigma = 1$) and (5.6) achieves stability of $x = \hat{x} = \hat{\theta} = 0$ and regulation of x and $\hat{x}$ (in probability). A closer inspection of Fig. 1 (left) indicates the possibility that controller (5.4) may be achieving not only NSS but also regulation to zero, however, we do not have hard simulation evidence that x_1 continues approaching zero over a longer time window, thus we only claim NSS.

Appendix A

In this and the following appendix, we use Young's inequality [9, Theorem 156]:

$$xy \leqslant \frac{\varepsilon^p}{p}|x|^p + \frac{1}{q\varepsilon^q}|y|^q, \tag{A.1}$$

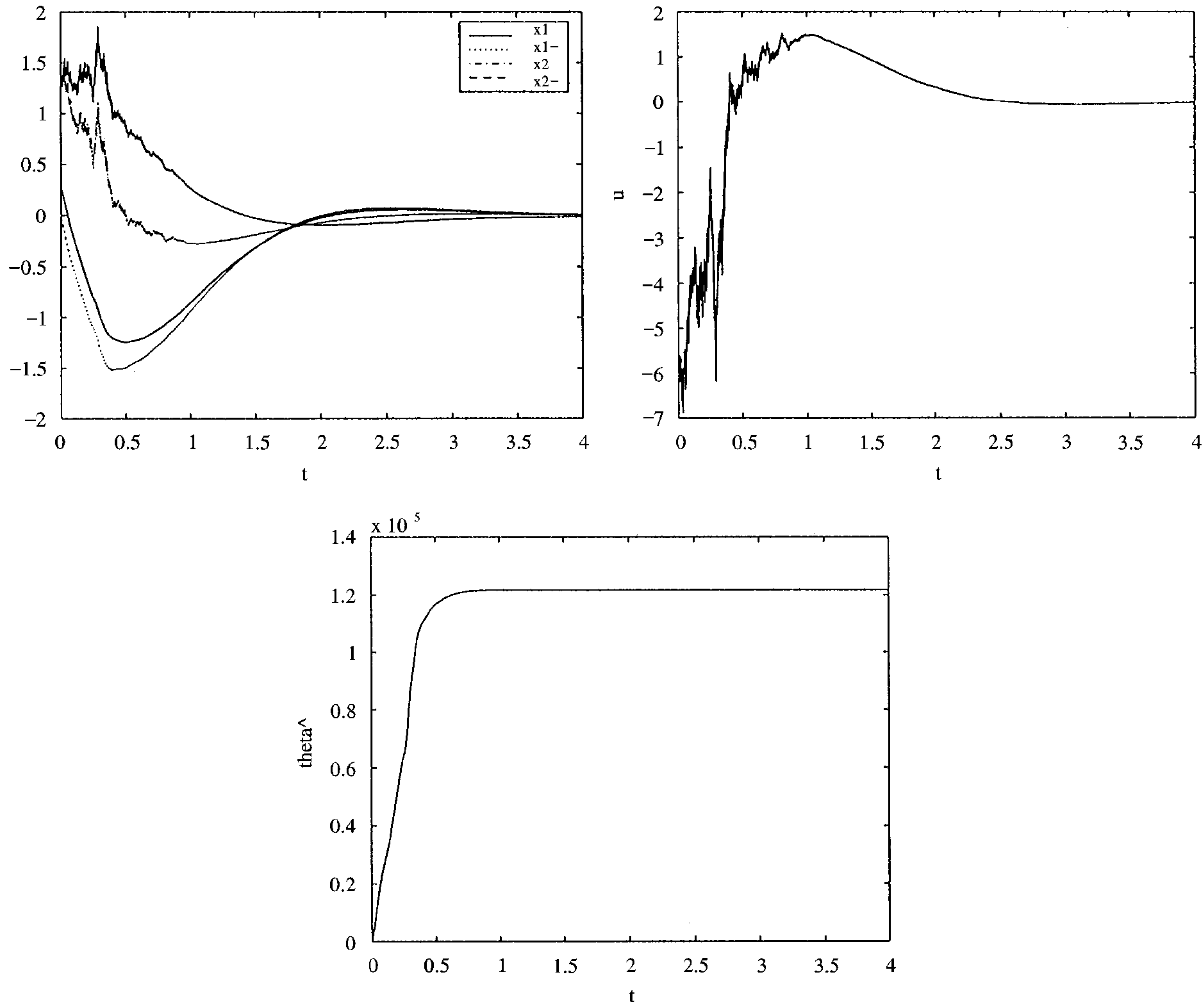

Fig. 2. The states, estimate and control effort of the adaptive design.

where $\varepsilon > 0$, the constants $p > 1$ and $q > 1$ satisfy $(p-1)(q-1) = 1$, and $(x, y) \in \mathbb{R}^2$. Applying these inequalities leads to

$$\frac{3}{2}\sum_{i=2}^{n} z_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y) \leqslant \frac{3}{4}\sum_{i=2}^{n} \frac{1}{\xi_i^2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^4 z_i^4(\varphi_1(y)^{\mathrm{T}}\varphi_1(y))^2 + \frac{3}{4}\sum_{i=2}^{n} \xi_i^2|\Sigma\Sigma^{\mathrm{T}}|^2, \quad \text{(A.2)}$$

$$\begin{aligned}
& b\,\mathrm{Tr}\{\varphi(y)\Sigma^{\mathrm{T}}(2P\tilde{x}\tilde{x}^{\mathrm{T}}P + \tilde{x}^{\mathrm{T}}P\tilde{x}P)\Sigma\varphi(y)^{\mathrm{T}}\} \\
&\quad \leqslant 3bn\sqrt{n}|\varphi(y)^{\mathrm{T}}\varphi(y)|\,|P|^2|\tilde{x}|^2|\Sigma|^2 \\
&\quad = 3bn\sqrt{n}|P|^2|\varphi(0)^{\mathrm{T}}\varphi(0)|\,|\tilde{x}|^2|\Sigma|^2 + 6bn\sqrt{n}|P|^2|y\varphi(0)^{\mathrm{T}}\psi(y)|\,|\tilde{x}|^2|\Sigma|^2 + 3bn\sqrt{n}|P|^2y^2|\psi(y)^{\mathrm{T}}\psi(y)|\,|\tilde{x}|^2|\Sigma|^2 \\
&\quad \leqslant \frac{3bn\sqrt{n}|P|^2}{2\varepsilon_2^2}|\varphi(0)^{\mathrm{T}}\varphi(0)||\tilde{x}|^4 + \frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\varphi(0)^{\mathrm{T}}\varphi(0)||\Sigma|^4 + 3bn\sqrt{n}|P|^2\varepsilon_3^2|\varphi(0)^{\mathrm{T}}\psi(y)|^2y^2|\Sigma|^4 \\
&\qquad + \frac{3bn\sqrt{n}|P|^2}{\varepsilon_3^2}|\tilde{x}|^4 + \frac{3bn\sqrt{n}|P|^2}{2\varepsilon_4^2}|\tilde{x}|^4 + \frac{3bn\sqrt{n}|P|^2\varepsilon_4^2}{2}y^4|\psi(y)^{\mathrm{T}}\psi(y)|^2|\Sigma|^4 \\
&\quad \leqslant 3bn\sqrt{n}|P|^2\left(\frac{1}{2\varepsilon_2^2}|\varphi(0)^{\mathrm{T}}\varphi(0)| + \frac{1}{\varepsilon_3^2} + \frac{1}{2\varepsilon_4^2}\right)|\tilde{x}|^4 + \frac{3bn\sqrt{n}|P|^2\varepsilon_2^2}{2}|\varphi(0)^{\mathrm{T}}\varphi(0)|\,|\Sigma|^4 + \frac{3bn\sqrt{n}|P|^2\varepsilon_3^2}{2}|\Sigma|^8 \\
&\qquad + \frac{3bn\sqrt{n}|P|^2\varepsilon_3^2}{2}|\varphi(0)^{\mathrm{T}}\psi(y)|^4y^4 + \frac{bn\sqrt{n}|P|^2\varepsilon_4^2}{4}|\Sigma|^{24} + \frac{5bn\sqrt{n}|P|^2\varepsilon_4^2}{4}|\psi(y)|^{24/5}y^{24/5}, \qquad \text{(A.3)}
\end{aligned}$$

$$
\begin{aligned}
&\tfrac{3}{2}y^2\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y)\\
&\quad\leqslant \tfrac{3}{2}y^2|\varphi_1(y)^{\mathrm{T}}\varphi_1(y)||\Sigma|^2\\
&\quad= \tfrac{3}{2}y^2(\varphi_1(0)^{\mathrm{T}}\varphi_1(0)+2y\varphi_1(0)^{\mathrm{T}}\psi_1(y)+\ y^2\psi_1(y)^{\mathrm{T}}\psi_1(y))|\Sigma|^2\\
&\quad\leqslant \tfrac{3}{4}|\varphi_1(0)|^4y^4+\tfrac{3}{2}y^6(\varphi_1(0)^{\mathrm{T}}\psi_1(y))^2+\ \tfrac{3}{8}\varepsilon_5^{4/3}y^{16/3}|\psi_1(y)|^{8/3}+\tfrac{9}{4}|\Sigma|^4+\frac{6}{\varepsilon_5^4}|\Sigma|^8 \qquad \text{(A.4)}\\
&\qquad -\frac{1}{2}\frac{\partial^2\alpha_{i-1}}{\partial y^2}z_i^3\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y)\leqslant \tfrac{1}{4}z_i^6\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\varphi_1(y)^{\mathrm{T}}\varphi_1(y)\right)^2+\tfrac{1}{4}|\Sigma|^4. \qquad \text{(A.5)}
\end{aligned}
$$

Appendix B

Similar to Appendix A, in the following inequalities, ε_2 is a constant to be chosen.

$$
\begin{aligned}
&\frac{3}{2}\sum_{i=2}^{n}z_i^2\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y)\\
&\quad\leqslant \frac{3}{4\varepsilon_3^2}\sum_{i=2}^{n}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4z_i^4\|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2+\frac{3\varepsilon_3^2}{4}\sum_{i=2}^{n}y^4(\psi_1(y)^{\mathrm{T}}\psi_1(y))^2, \qquad \text{(B.1)}\\
&b\,\mathrm{Tr}\{\varphi(y)(2P\tilde{x}\tilde{x}^{\mathrm{T}}P+\tilde{x}^{\mathrm{T}}P\tilde{x}P)\varphi(y)^{\mathrm{T}}\}\\
&\quad\leqslant 3bn\sqrt{n}y^2|\psi(y)|^2|P|^2|\tilde{x}|^2\|\Sigma\Sigma^{\mathrm{T}}\|_\infty\\
&\quad\leqslant \frac{3bn\sqrt{n}\varepsilon_2^2|P|^2}{2}y^4|\psi(y)|^4\|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2+\frac{3bn\sqrt{n}|P|^2}{2\varepsilon_2^2}|\tilde{x}|^4, \qquad \text{(B.2)}\\
&\tfrac{3}{2}y^2\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y)\\
&\quad\leqslant \frac{3\varepsilon_3^2}{4}(\varphi_1(y)^{\mathrm{T}}\varphi_1(y))^2+\frac{3}{4\varepsilon_3^2}y^4\|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2=\frac{3\varepsilon_3^2}{4}y^4(\psi_1(y)^{\mathrm{T}}\psi_1(y))^2+\frac{3}{4\varepsilon_3^2}y^4\|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2, \qquad \text{(B.3)}\\
&\quad -\frac{1}{2}\frac{\partial^2\alpha_{i-1}}{\partial y^2}z_i^3\varphi_1(y)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_1(y)\\
&\quad\leqslant \frac{1}{4\varepsilon_3^2}z_i^6\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)^2\|\Sigma\Sigma^{\mathrm{T}}\|_\infty^2+\frac{\varepsilon_3^2}{4}y^4(\psi_1(y)^{\mathrm{T}}\psi_1(y))^2. \qquad \text{(B.4)}
\end{aligned}
$$

References

[1] T. Başar, P. Bernhard, H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach, 2nd Edition, Boston, MA, Birkhäuser, 1995.

[2] H. Deng, M. Krstić, Stochastic nonlinear stabilization — Part I: A backstepping design, Systems Control Lett. 32 (1997) 143–150.

[3] H. Deng, M. Krstić, Stochastic nonlinear stabilization — Part II: Inverse optimality, Systems Control Lett. 32 (1997) 151–159.

[4] H. Deng, M. Krstić, Output-feedback stochastic nonlinear stabilization, IEEE Trans. Automat. Control 44 (1999) 328–333.

[5] H. Deng, M. Krstić, Stabilization of stochastic nonlinear systems driven by noise of unknown covariance, Proceedings of ACC, Philadelphia, 1998.

[6] P. Florchinger, A universal formula for the stabilization of control stochastic differential equations, Stochastic Anal. Appl. 11 (1993) 155–162.

[7] P. Florchinger, Lyapunov-like techniques for stochastic stability, SIAM J. Control Optim. 33 (1995) 1151–1169.

[8] P. Florchinger, Global stabilization of cascade stochastic systems, Proceedings of the 34th Conference on Decision and Control, New Orleans, LA, 1995, pp. 2185–2186.

[9] G. Hardy, J.E. Littlewood, G. Polya, Inequalities, 2nd Edition, Cambridge University Press, Cambridge, 1989.

[10] M.R. James, J. Baras, R.J. Elliott, Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems, IEEE Trans. Automat. Control 39 (1994) 780–792.

[11] M. Krstić, H. Deng, Stabilization of Nonlinear Uncertain Systems, Springer, Berlin, 1998.

[12] H. Nagai, Bellman equations of risk-sensitive control, SIAM J. Control Optim. 34 (1996) 74–101.

[13] Z. Pan, T. Başar, Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criterion, SIAM J. Control Optim. 37(3) (1999) 957–995.

[14] T. Runolfsson, The equivalence between infinite horizon control of stochastic systems with exponential-of-integral performance index and stochastic differential games, IEEE Trans. Automat. Control 39 (1994) 1551–1563.

[15] E.D. Sontag, A 'universal' construction of Artstein's theorem on nonlinear stabilization, Systems Control Lett. 13 (1989) 117–123.

[16] E.D. Sontag, Smooth stabilization implies coprime factorization, IEEE Trans. Automat. Control 34 (1989) 435–443.

[17] J. Tsinias, The concept of 'exponential ISS' for stochastic systems and applications to feedback stabilization, preprint, 1997.

[18] J. Tsinias, Stochastic input-to-state stability and applications to global feedback stabilization, (special issue on Breakthrough in the Control of Nonlinear Systems) Int. J. Control, 1998 71(5) (1998) 907–930.