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Brief paper

Prescribed-time output feedback for linear systems in controllable canonical form ☆,☆☆

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ABSTRACT

For linear systems in the controllable canonical form, we introduce a prescribed-time output feedback controller which provides for easy prescription of estimation and stabilization convergence times irrespective of initial conditions and with minimal tuning of the observer and controller parameters. We show that the closed-loop output feedback system is fixed-time globally uniformly asymptotically stable as well as convergent to zero in the prescribed time. Further, we show that a separation principle holds between the prescribed-time controller and the prescribed-time observer provided the scaling power of the time-varying observer gains exceeds the scaling power of the controller gains by twice the order of the system, i.e., provided the observer is fast enough relative to the controller, irrespective of the constant gains of both.

1. Introduction

1.1. Motivation and previous work

This work is motivated by tactical missile guidance (see, e.g., Zarchan (2007)) and other applications in which there exists a short, finite amount of time remaining to achieve state estimation and control objectives, and that time is known to within some small uncertainty. In finite-time applications such as these, state observers and controllers that allow the user to *prescribe* the convergence times a priori and irrespective of initial conditions offer a clear advantage over those that do not.

Existing approaches to estimation and control in finite time are dominated by sliding modes (Angulo, Moreno, & Fridman, 2013; Cruz-Zavala, Moreno, & Fridman, 2011; Levant, 1998, 2003, 2005, 2013) and approaches based on concepts of homogeneity (Andrieu, Praly, & Astolfi, 2008; Du, Qian, Yang, & Li, 2013; Efimov, Levant, Polyakov, & Perruquetti, 2016; Hong, Huang, & Xu, 2001; Levant, 2003, 2005). However, for higher-order systems, these approaches lack constructive methods for selecting observer and controller parameters that allow for prescription of the convergence times a priori. The Implicit Lyapunov Function (ILF) approach used by Lopez-Ramirez, Polyakov, Efimov, and Perruquetti (2016b) and Polyakov, Efimov, and Perruquetti (2015) does offer controllers and observers for higher-order systems that make *fixed-time* convergence possible (i.e., where the convergence time is independent of initial conditions). However, the parameter tuning process is complicated, since it requires numerical schemes to solve systems of linear matrix inequalities (LMIs). There also exist other state observers that provide for prescription of convergence time, but their implementations are relatively complicated also, since they require the use of time delays (Engel & Kreisselmeier, 2002) or a hybrid-systems framework (Li & Sanfelice, 2015; Raff & Allgower, 2007). Interesting results on finite-time output feedback were provided by Amato, Darouach, and De Tommasi (2018), Andrieu et al. (2008), Bernuau, Perruquetti, Efimov, and Moulay (2015), Hong et al. (2001), Levant (2003), Li and Qian (2006), Lopez-Ramirez, Efimov, Polyakov, and Perruquetti (2016a), Shi et al. (2017) and Tian, Zuo, Yan, and Wang (2017), however none of these works offer simple or constructive ways to select controller and observer parameters that allow the user to prescribe the convergence times.

Here, we take an alternative approach to finite-time estimation and control, by leveraging the prescribed-time observer of Holloway and Krstic (2019) and a prescribed-time state feedback controller from Song, Wang, Holloway, and Krstic (2017).

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Namely, we introduce a prescribed-time output feedback controller for linear time-invariant systems in controllable canonical form, which allows for both easy prescription of the convergence times, and minimal tuning of the observer and controller parameters. In addition, we provide stability results of the output-feedback controller, and demonstrate that a separation principle holds between the observer and the state feedback controller provided the scaling power of the time-varying observer gains exceeds the scaling power of the controller gains by twice the order of the system, i.e., provided the observer is fast enough relative to the controller, irrespective of the constant gains of both.

1.2. Problem statement

In this work, we study output feedback for systems whose solutions are only required to exist on a finite-time interval, $t \in [t_0, t_0+t_f)$, where $t_0 \geq 0$ is the initial time, and $t_f > t_0$ is the final or terminal time by which we are required to meet the estimation and control objectives. We restrict our analysis to linear single-input single-output (SISO) systems in the controllable canonical form,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

where

$$A := \begin{bmatrix} 0 & I_{n-1} & & \\ \vdots & & & \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C := \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix}.$$

Here, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^1$ is a known and bounded control input, and $y(t) \in \mathbb{R}^1$ is the measured output. Our objective is to achieve perfect estimation of the state and stabilize it to the origin within a finite time $T : 0 < T \leq t_f$, in a manner in which T is fixed (independent of initial conditions) and freely prescribed by the user a priori.

1.3. Design approach

Our output feedback controller employs a prescribed-time controller of Song et al. (2017) and the prescribed-time observer of Holloway and Krstic (2019). The controller was developed for nonlinear systems in the normal form, which reduces to the controllable canonical form in the case where the nonlinearities vanish. The observer, however, was developed for linear systems in the observer canonical form. Therefore, to use this controller and observer together, we require a state transformation to relate the different canonical realizations. Toward this end, note that there exists a linear time-invariant coordinate transformation $\mathscr{T}$, such that by defining a new state $\xi(t)$ as

$$\xi(t) := \mathscr{T}x(t), \tag{3}$$

the system in controllable canonical form (1), (2) is transformed into the observer canonical form,

$$\dot{\xi}(t) = \mathscr{A}\xi(t) + \mathscr{B}u(t) - \mathscr{D}y(t) \tag{4}$$

$$y(t) = \mathrm{e}_1^T\xi(t), \tag{5}$$

where

$$\mathscr{A} := \begin{bmatrix} 0 & I_{n-1} \\ \vdots & \\ 0 & \dots & 0 \end{bmatrix}, \quad \mathscr{B} := \begin{bmatrix} b_{n-1} \\ \vdots \\ b_0 \end{bmatrix}, \quad \mathscr{D} := \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix},$$

the a_is and b_is are the same as those in (1), (2), and $\mathrm{e}_1 = [1, 0, \dots, 0]^T \in \mathbb{R}^n$ is the first of the n-dimensional unit vectors. Now, we construct a prescribed-time observer for the system (4), (5) according to Holloway and Krstic (2019) as

$$\dot{\hat{\xi}}(t) = \mathscr{A}\hat{\xi}(t) + \mathscr{B}u(t) - \mathscr{D}y(t) + \begin{bmatrix} g_1(t-t_0, T) \\ \vdots \\ g_n(t-t_0, T) \end{bmatrix} \left(y(t) - \hat{\xi}_1(t)\right) \tag{6}$$

where the time-varying observer gains $\{g_i(t-t_0, T)\}_{i=1}^n$ are functions of the prescribed convergence time T, with $0 < T \leq t_f$, and must be designed. Then since the relationship (3) is known, we calculate the state estimate $\hat{x}(t)$ with

$$\hat{x}(t) = \mathscr{T}^{-1}\hat{\xi}(t). \tag{7}$$

The estimate (7) is based on the idea that, if the estimate $\hat{\xi}(t)$ tracks $\xi(t)$ well, then $\hat{x}(t)$ might track $x(t)$ with comparable performance by virtue of (3). Finally, using the state estimate (7), we construct a prescribed-time output feedback controller of the form

$$u(t) = -\hat{L}_0(t) - \hat{L}_1(t) - \hat{L}_2(t) - \hat{L}_3(t), \tag{8}$$

where the expressions $\hat{L}_0(t)$, $\hat{L}_1(t)$, $\hat{L}_2(t)$, $\hat{L}_3(t)$ are linear time-varying functions of the state estimates and are developed in the following.

2. Preliminaries

2.1. Prescribed-time scaling functions

Both the prescribed-time observer from Holloway and Krstic (2019) and the prescribed-time controller from Song et al. (2017) employ the following scaling functions, which are positive monotonic functions of the convergence time, $T > 0$, a parameter which is freely prescribed by the user and independent of initial conditions.

We define the function $\mu_1(t-t_0, T) : [t_0, t_0+T) \mapsto \mathbb{R}^+$ as

$$\mu_1(t-t_0, T) := \frac{T}{T+t_0-t}, \tag{9}$$

which starts from 1 at $t = t_0$ and increases monotonically to infinity as $t \to t_0 + T$. We also define the function $\nu(t-t_0, T) : [t_0, t_0+T) \mapsto \mathbb{R}^+$ as

$$\nu(t-t_0, T) := \mu_1(t-t_0, T)^{-1} = \frac{T+t_0-t}{T}, \tag{10}$$

which starts from 1 at $t = t_0$ and *decreases monotonically to zero* as $t \to t_0 + T$. Then also using (9), we define the function $\mu(t-t_0, T) : [t_0, t_0+T) \mapsto \mathbb{R}^+$ as

$$\mu(t-t_0, T) := \mu_1(t-t_0, T)^{n+m} = \frac{T^{n+m}}{(T+t_0-t)^{n+m}}, \tag{11}$$

which also starts from 1 at $t = t_0$ and increases monotonically to infinity as $t \to t_0 + T$, but can be tuned to do so more quickly than $\mu_1(t-t_0, T)$ through the positive integers n (the order of the system) and $m \geq 1$, a design parameter.

Notation: The parameter m in (11) can take on different values in the observer and the controller. Therefore, throughout the paper, we replace m with the subscripted parameters m_O and m_C as applicable for the observer and controller, respectively. To save space, we often omit the explicit $t - t_0$ and T dependence of the functions (9)–(11) and write them simply as μ_1, ν, μ_O, and μ_C, where the "O" and "C" subscripts denote the dependence on the parameters m_O and m_C. Also, for any scalar function $f(t)$ continuous and bounded on $[t_0, t_0+T)$, we use the notation $\|f\|_{[t_0,t]} := \sup_{t_0\le\tau\le t} |f(\tau)|$.

2.2. Fixed-time stability in prescribed time

Our stability analyses employ the following concepts of fixed-time stability which were introduced in Song et al. (2017). Note that in all of these, T is freely prescribed and independent of initial conditions.

Definition 1 (*FT-GUAS*). The system $\dot{x} = f(x,t)$ (of arbitrary dimension of x) is said to be fixed-time globally uniformly asymptotically stable (FT-GUAS) in time T if there exists a class $\mathcal{KL}$ function β such that for all $t \in [t_0, t_0+T)$,

$$|x(t)| \le \beta(|x(t_0)|, \mu_1(t-t_0,T)-1), \tag{12}$$

where the function $\mu_1(t-t_0,T)$ is defined in (9).

Notice that the function $\mu_1(t-t_0,T)-1 = (t-t_0)/(T+t_0-t)$ starts from zero at $t=t_0$ and increases monotonically to infinity as $t \to t_0+T$. Therefore, the function $\beta(|x(t_0)|, \mu_1(t-t_0,T)-1)$ decays to zero as $t \to t_0+T$, i.e., at a time that is prescribed by T.

Definition 2 (*FT-ISS*). The system $\dot{x} = f(x,t,d)$ (of arbitrary dimensions of x and d) is said to be fixed-time input-to-state stable (FT-ISS) with respect to d in time T if there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ, such that for all $t \in [t_0, t_0+T)$,

$$|x(t)| \le \beta(|x(t_0)|, \mu_1(t-t_0,T)-1) + \gamma(\|d\|_{[t_0,t]}), \tag{13}$$

where the function $\mu_1(t-t_0,T)$ is defined in (9).

Note that in the absence of the disturbance $d(t)$, a system that is FT-ISS is FT-GUAS in time T.

Definition 3 (*FT-ISS+C*). The system $\dot{x} = f(x,t,d)$ (of arbitrary dimensions of x and d) is said to be fixed-time input-to-state stable and convergent to zero (FT-ISS+C) in time T if there exist class $\mathcal{KL}$ functions β and β_f, and a class $\mathcal{K}$ function γ, such that for all $t \in [t_0, t_0+T)$,

$$|x(t)| \le \beta_f\left(\beta(|x(t_0)|, t-t_0) + \gamma(\|d\|_{[t_0,t]}), \mu_1(t-t_0,T)-1\right), \tag{14}$$

where the function $\mu_1(t-t_0,T)$ is defined in (9).

So, a system that is FT-ISS+C is also FT-ISS, with the additional property that its state converges to zero in the time T despite the presence of a disturbance.

2.3. Prescribed-time observer

Define the observer error states as

$$\tilde{\xi}_i(t) := \xi_i(t) - \hat{\xi}_i(t), \quad i = 1,\ldots,n. \tag{15}$$

Then with (4), (5), and (6), we obtain the error dynamics

$$\dot{\tilde{\xi}}_i(t) = \tilde{\xi}_{i+1}(t) - g_i(t-t_0,T)\tilde{\xi}_1(t), \quad i = 1,\ldots,n-1 \tag{16}$$

$$\dot{\tilde{\xi}}_n(t) = -g_n(t-t_0,T)\tilde{\xi}_1(t). \tag{17}$$

To facilitate the selection of the observer gains $\{g_i(t-t_0,T)\}_{i=1}^n$, we transform the error system (16), (17) according to the following lemma of Holloway and Krstic (2019), repeated here for reference.

Lemma 1 (*Observer Error Transformation*). *Consider the transformation $\tilde{\xi}_i(t) \mapsto \tilde{\zeta}_i(t)$ defined by*

$$\tilde{\zeta}_i(t) := \mu_O(t-t_0,T)\tilde{\xi}_i(t), \quad i = 1,\ldots,n, \tag{18}$$

and the transformation $\tilde{\zeta}_i(t) \mapsto \tilde{z}_i(t)$ defined by

$$\tilde{z}_i(t) := \sum_{j=1}^{n} p^*_{O_{i,j}}(\mu_1)\tilde{\zeta}_j(t), \quad i = 1,\ldots,n, \tag{19}$$

where the functions $\{p^*_{O_{i,j}}(\mu_1)\}$ are defined by

$$p^*_{O_{i,j}}(\mu_1) := \bar{p}_{O_{i,j}}\mu_1^{i-j}, \quad 1 \le j \le i \le n, \tag{20}$$

and the coefficients $\{\bar{p}_{O_{i,j}}\}$ are constants to be determined. By selecting the $\{\bar{p}_{O_{i,j}}\}$ according to

$$\bar{p}_{O_{i,i}} = 1, \tag{21}$$

$$\bar{p}_{O_{i,j}} = 0, \quad j > i, \tag{22}$$

for $\{\bar{p}_{O_{i,j}}\}$ with $j \ge i$, the recursion relations

$$\bar{p}_{O_{i,j-1}} = -\frac{n+m_O+i-j}{T}\bar{p}_{O_{i,j}} + \bar{p}_{O_{i+1,j}}, \quad n-1 \ge i \ge j \ge 2, \tag{23}$$

$$\bar{p}_{O_{n,j-1}} = -\frac{2n+m_O-j}{T}\bar{p}_{O_{n,j}}, \qquad j = n, n-1, \ldots, 2, \tag{24}$$

for $\{\bar{p}_{O_{i,j}}\}$ with $j < i$, and the observer gains $\{g_i(t-t_0,T)\}_{i=1}^{n-1}$ according to

$$g_i(t-t_0,T) = l_i + \left(\frac{n+m_O+i-1}{T}\bar{p}_{O_{i,1}} - \bar{p}_{O_{i+1,1}}\right)\mu_1^i - \sum_{j=1}^{i-1} g_j(t-t_0,T)\bar{p}_{O_{i,j}}\mu_1^{i-j}, \tag{25}$$

and $g_n(t-t_0,T)$ according to

$$g_n(t-t_0,T) = l_n + \frac{2n+m_O-1}{T}\bar{p}_{O_{n,1}}\mu_1^n - \sum_{j=1}^{n-1} g_j(t-t_0,T)\bar{p}_{O_{n,j}}\mu_1^{n-j}, \tag{26}$$

where the $\{l_i\}_{i=1}^n$ are constants to be selected, the observer error system (16), (17) is transformed into

$$\dot{\tilde{z}}_i(t) = \tilde{z}_{i+1}(t) - l_i\tilde{z}_1(t), \quad i = 1,\ldots,n-1, \tag{27}$$

$$\dot{\tilde{z}}_n(t) = -l_n\tilde{z}_1(t). \tag{28}$$

The system (27), (28) is stabilized by selecting the constants $\{l_i\}_{i=1}^n$ to make the companion matrix Λ_O Hurwitz, where Λ_O is defined from (27), (28) as

$$\Lambda_O := \begin{bmatrix} -l_1 & I_{n-1} \\ \vdots & \\ -l_n & 0 \end{bmatrix}. \tag{29}$$

Prescribed-time convergence of the observer (4), (5) is stated and proven in Theorem 1 of Holloway and Krstic (2019), restated here as a lemma for reference.

Lemma 2 (*FT-GUAS Observer*). *For the dynamic system (1), (2) defined on the finite time interval $t \in [t_0, t_0+T)$, consider the observer (6) having error dynamics (16), (17) and observer gains $\{g_i(t-t_0,T)\}_{i=1}^n$ given by (25), (26) where the $\{l_i\}_{i=1}^n$ are constants to be selected. If the constants $\{l_i\}_{i=1}^n$ are selected such that the companion matrix (29) is Hurwitz, then the system (16), (17) has* *a FT-GUAS equilibrium at the origin, with a prescribed convergence time T, and there exist positive constants $\tilde{M}$, $\tilde{\delta} > 0$ such that*

$$|\tilde{\xi}(t)| \leq \nu(t-t_0,T)^{m_O+1}\tilde{M}e^{-\tilde{\delta}(t-t_0)}|\tilde{\xi}(t_0)|, \tag{30}$$

for all $t \in [t_0, t_0+T)$, where $\nu(t-t_0,T)$ is defined in (10), and $m_O \geq 1$ is an integer and a design parameter. Furthermore, the output estimation error injection terms $\gamma_i(t-t_0,T) := g_i(t-t_0,T)\tilde{\xi}_1(t)$ for $i = 1,\ldots,n$ remain uniformly bounded over $[t_0, t_0+T)$, and also converge to zero as $t \to t_0+T$.

In summary, the convergence time $T : 0 < T \leq t_f$ is a free parameter selected by the user, the parameters $\{l_i\}_{i=1}^n$ are selected to make the companion matrix Λ_O Hurwitz, and m_O is an integer tuning parameter. The coefficients $\{\bar{p}_{O_{i,j}}\}$ for $j < i$ are explicitly provided by the recursion relations (23), (24), and are calculated using the algorithm of Remark 1 in Holloway and Krstic (2019). Finally, the observer gains $\{g_i(t-t_0,T)\}_{i=1}^n$ provided by (25), (26) are then easily calculated recursively.

2.4. Prescribed-time controller

Prescribed-time stabilization of the state to the origin will be achieved using the controller provided by Theorem 2 of Song et al. (2017). The controller employs a change-of-coordinates transformation of the state,

$$w_1(t) := \mu_C(t-t_0,T)x_1(t), \tag{31}$$

$$w_q(t) := dw_{q-1}(t)/dt, \quad q = 2,\ldots,n+1, \tag{32}$$

and we denote $w_{n+1}(t) := \dot{w}_n(t)$ and $x_{n+1}(t) := \dot{x}_n(t)$. We also define

$$r_1(t) := [w_1(t),\ldots,w_{n-1}(t)]^T = J_1 w(t) \in \mathbb{R}^{n-1} \tag{33}$$

$$r_2(t) := \dot{r}_1(t) = [w_2(t),\ldots,w_n(t)]^T = J_2 w(t) \in \mathbb{R}^{n-1}$$

where $J_1 := [I_{n-1}, 0_{(n-1)\times 1}]$, $J_2 := [0_{(n-1)\times 1}, I_{n-1}]$, and $K_{n-1} := [k_1,\ldots,k_{n-1}]^T \in \mathbb{R}^{n-1}$, where K_{n-1} is an appropriately chosen coefficient vector such that the matrix

$$\Lambda_C := \begin{bmatrix} 0 & & I_{n-2} & \\ -k_1 & -k_2 & \cdots & -k_{n-1} \end{bmatrix} \tag{34}$$

is Hurwitz. The state $w_n(t)$ is replaced by a new variable $\sigma(t)$ defined as

$$\sigma(t) := w_n(t) + K_{n-1}^T r_1(t), \tag{35}$$

which results in the transformed system

$$\dot{r}_1(t) = \Lambda_C r_1(t) + e_{n-1}\sigma(t) \tag{36}$$

$$\dot{\sigma}(t) = \dot{w}_n(t) + K_{n-1}^T J_2 w(t). \tag{37}$$

The latter equation (37) is rewritten using (A.1) of Song et al. (2017) and substitution of the last row of (1), which gives

$$\dot{\sigma}(t) = \mu_C\left(u(t) - a^T x(t) + \sum_{k=1}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{\mu_C^{(k)}}{\mu_C}x_{n+1-k}(t) + \nu^{n+m_C}K_{n-1}^T J_2 w(t)\right) \tag{38}$$

where the vector $a \in \mathbb{R}^n$ is defined from (1) as $a := [a_0, a_1, \ldots, a_{n-1}]^T$. Finally, (38) is rewritten as

$$\dot{\sigma}(t) = \mu_C\left(u(t) + L_0(t) + L_1(t) + L_2(t)\right) \tag{39}$$

where

$$L_0(t) := \sum_{k=1}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{\mu_C^{(k)}}{\mu_C}x_{n+1-k}(t), \tag{40}$$

$$L_1(t) := \nu^{n+m_C}K_{n-1}^T J_2 w(t) \tag{41}$$

$$L_2(t) := -a^T x(t). \tag{42}$$

Using these definitions, we paraphrase Theorem 2 of Song et al. (2017) as another lemma. Note that here, $L_2(t)$ equals the known $f(x,t)$ in that theorem.

Lemma 3 (*FT-GUAS Controller*)**.** *Select $k_n > 0$. Then the system (1) with the controller*

$$u(t) = -\left(L_0(t) + L_1(t) + L_2(t) + k_n\sigma(t)\right) \tag{43}$$

with $L_0(t)$, $L_1(t)$, and $L_2(t)$ as defined in (40)–(42) and $\sigma(t)$ as defined in (35) has a FT-GUAS equilibrium at the origin, with a prescribed convergence time T, and there exist $\tilde{M}$, $\tilde{\delta} > 0$ such that

$$|x(t)| \leq \nu(t-t_0,T)^{m_C+1}\tilde{M}e^{-\tilde{\delta}(t-t_0)}|x(t_0)| \tag{44}$$

for all $t \in [t_0, t_0+T)$. Furthermore, the control $u(t)$ remains uniformly bounded over $[t_0, t_0+T)$, and also converges to zero as $t \to t_0+T$.

Because of Lemma 3, we define the linear feedback

$$L_3(t) := k_n\sigma(t) \tag{45}$$

$$= k_n\left(e_n^T + K_{n-1}^T J_1\right)w(t) \tag{46}$$

after using (35) and (33). Then using (45), the controller (43) becomes

$$u(t) = -\left(L_0(t) + L_1(t) + L_2(t) + L_3(t)\right). \tag{47}$$

The quantities $L_0(t)$, $L_1(t)$, $L_2(t)$, and $L_3(t)$ are linear feedbacks of the state, and can be written in terms of either the original state variables or the transformed ones. The former are more useful for implementation, while the latter facilitate stability analyses. In terms of the transformed variables, we have

$$L_0(t) = \nu^{m_C}l_0(\nu)w(t) \tag{48}$$

$$L_1(t) = \nu^{n+m_C}K_{n-1}^T J_2 w(t) \tag{49}$$

$$L_2(t) = -\nu^{m_C+1}a^T Q_C(\nu)w(t) \tag{50}$$

$$L_3(t) = k_n\left(e_n^T + K_{n-1}^T J_1\right)w(t) \tag{51}$$

where (48) is true by Lemma 4 in Song et al. (2017), and (50) results from using Lemma 3 of the same reference in (42). Then by using Lemma 2 of the same reference to replace $w(t)$, the feedbacks (48)–(51) are rewritten in terms of the original state variables as

$$L_0(t) = \mu_1 l_0(\nu)P_C(\mu_1)x(t) \tag{52}$$

$$L_1(t) = \mu_1^{1-n}K_{n-1}^T J_2 P_C(\mu_1)x(t) \tag{53}$$

$$L_2(t) = -a^T x(t) \tag{54}$$

$$L_3(t) = \mu_1^{m_C+1}k_n\left(e_n^T + K_{n-1}^T J_1\right)P_C(\mu_1)x(t). \tag{55}$$

Lastly, note that as with the observer, $T : 0 < T \leq t_f$ is a free parameter selected by the user, the parameters $\{k_i\}_{i=1}^{n-1}$ are selected to make the companion matrix (34) Hurwitz, and m_C is an integer tuning parameter. The user can select any $k_n > 0$, and then the remaining coefficients needed to build the controller are provided by Lemma 2 and Lemma 4 in Song et al. (2017).

3. Prescribed-time output feedback

3.1. Synthesis

Having only the measured output and state estimate available for feedback, we replace the controller (47) with the output feedback

$$u(t) = -(\hat{L}_0(t) + \hat{L}_1(t) + \hat{L}_2(t) + \hat{L}_3(t)) \quad (56)$$

$$= -\Big(\mu_1 l_0(\nu) P_C(\mu_1) + \mu_1^{1-n} K_{n-1}^T J_2 P_C(\mu_1) - a^T + \mu_1^{m_C+1} k_n \left(e_n^T + K_{n-1}^T J_1\right) P_C(\mu_1)\Big)\hat{x}(t), \quad (57)$$

where $\hat{L}_0(t)$, $\hat{L}_1(t)$, $\hat{L}_2(t)$, and $\hat{L}_3(t)$ denote the quantities (52)–(55) but with $x(t)$ replaced by $\hat{x}(t)$. With (7), (15), and (3), we obtain

$$\hat{x}(t) = x(t) - \mathscr{T}^{-1}\tilde{\xi}(t). \quad (58)$$

Then substituting (58) into (57) gives

$$u(t) = -\Big(\mu_1 l_0(\nu) P_C(\mu_1) + \mu_1^{1-n} K_{n-1}^T J_2 P_C(\mu_1) - a^T + \mu_1^{m_C+1} k_n \left(e_n^T + K_{n-1}^T J_1\right) P_C(\mu_1)\Big)\left(x(t) - \mathscr{T}^{-1}\tilde{\xi}(t)\right)$$
$$= -(L_0(t) + L_1(t) + L_2(t) + L_3(t)) + (\tilde{L}_0(t) + \tilde{L}_1(t) + \tilde{L}_2(t) + \tilde{L}_3(t)) \quad (59)$$

where we have defined

$$\tilde{L}_0(t) := \mu_1 l_0(\nu) P_C(\mu_1)\mathscr{T}^{-1}\tilde{\xi}(t) \quad (60)$$
$$\tilde{L}_1(t) := \mu_1^{1-n} K_{n-1}^T J_2 P_C(\mu_1)\mathscr{T}^{-1}\tilde{\xi}(t) \quad (61)$$
$$\tilde{L}_2(t) := -a^T\mathscr{T}^{-1}\tilde{\xi}(t) \quad (62)$$
$$\tilde{L}_3(t) := \mu_1^{m_C+1} k_n \left(e_n^T + K_{n-1}^T J_1\right) P_C(\mu_1)\mathscr{T}^{-1}\tilde{\xi}(t). \quad (63)$$

The second group of terms in parentheses on the right-hand side of (59) can be viewed as an "input error" that acts as a disturbance against the controller, which we define as the quantity

$$\tilde{u}(t) := \tilde{L}_0(t) + \tilde{L}_1(t) + \tilde{L}_2(t) + \tilde{L}_3(t). \quad (64)$$

Then with (64), (59) becomes

$$u(t) = -(L_0(t) + L_1(t) + L_2(t) + L_3(t)) + \tilde{u}(t). \quad (65)$$

In summary, (56) is equivalent to (65).

3.2. Stability results

In this section, we state new results which characterize the stability of the output feedback controller (65). The proofs are provided later in the next section. Notice from (65) that the stability of the output feedback system depends on the behavior of the input error, $\tilde{u}(t)$. The following lemma and remark show how desirable $\tilde{u}(t)$ dynamics can be obtained.

Lemma 4. *By selecting the constants $\{l_i\}_{i=1}^n$ to make the companion matrix Λ_O Hurwitz, and the integers $m_O \geq 1$ and $m_C \geq 1$ according to*

$$m_O \geq m_C + 2n - 1, \quad (66)$$

the function $|\tilde{u}(t)|$ remains uniformly bounded over $t \in [t_0, t_0 + T)$, and also goes to zero as $t \to t_0 + T$. Furthermore, there exists a positive integer α and positive constants M_α, δ_α such that

$$|\tilde{u}(t)| \leq \nu(t - t_0, T)^\alpha M_\alpha e^{-\delta_\alpha(t-t_0)}|\tilde{\xi}(t_0)|. \quad (67)$$

Remark 1. Suppose (1) is replaced by an nth-order chain of integrators. Then by selecting the constants $\{l_i\}_{i=1}^n$ to make the companion matrix Λ_O Hurwitz, and the integers $m_O \geq 1$ and $m_C \geq 1$ such that

$$m_O \geq m_C + n,$$

the function $|\tilde{u}(t)|$ remains uniformly bounded over $t \in [t_0, t_0 + T)$, and also goes to zero as $t \to t_0 + T$.

Lemma 4 provides conditions to obtain the following stability result for the closed-loop system.

Lemma 5 (*FT-ISS+C Output Feedback*)**.** *Consider the system (1), (2) with observer (6) and the controller feedback (65). If the constants $\{k_i\}_{i=1}^{n-1}$ and $\{l_i\}_{i=1}^n$ are selected such that the matrices Λ_C and Λ_O are Hurwitz, k_n is selected positive, and if the integers $m_O \geq 1$ and $m_C \geq 1$ are selected according to (66), then the closed-loop output feedback system is FT-ISS+C, and there exist $\check{M}, \check{\delta}, \check{\gamma} > 0$ such that for all $t \in [t_0, t_0 + T)$,*

$$|x(t)| \leq \nu(t - t_0, T)^{m_C+1}\left(\check{M}e^{-\check{\delta}(t-t_0)}|x(t_0)| + \check{\gamma}\|\tilde{u}\|_{[t_0,t]}\right). \quad (68)$$

Furthermore, the control $u(t)$ remains uniformly bounded over $[t_0, t_0 + T)$.

The previous results facilitate the proof of our main result, which is the separation principle between the prescribed-time observer (6) and the prescribed-time controller (65).

Theorem 1 (*FT-GUAS Separation Principle*)**.** *Consider the system (1), (2) with observer (6) and the controller feedback (65). If the constants $\{k_i\}_{i=1}^{n-1}$ and $\{l_i\}_{i=1}^n$ are selected such that the matrices Λ_C and Λ_O are Hurwitz, k_n is selected positive, and the integers $m_O \geq 1$ and $m_C \geq 1$ are selected according to (66), then the quantity $|x(t)| + |\hat{\xi}(t)|$ has a FT-GUAS equilibrium at the origin, and there exists a positive constant $\bar{M}$ such that for all $t \in [t_0, t_0 + T)$,*

$$|x(t)| + |\hat{\xi}(t)| \leq \nu(t - t_0, T)^{m_C+1}\bar{M}\left(|x(t_0)| + |\hat{\xi}(t_0)|\right). \quad (69)$$

3.3. Proofs of results

Proof of Lemma 4. From (64), the triangle inequality, and (60)–(63), and since $\mu_1 > 0$, we obtain

$$|\tilde{u}(t)| \leq \Big(\mu_1|l_0(\nu)| + \mu_1^{1-n}|K_{n-1}^T||J_2| + \mu_1^{m_C+1}|k_n(e_n^T + K_{n-1}^T J_1)|\Big)|P_C(\mu_1)||\mathscr{T}^{-1}||\tilde{\xi}(t)| + |a||\mathscr{T}^{-1}||\tilde{\xi}(t)|. \quad (70)$$

Now, since $\mu_1 = \nu^{-1}$, from Lemma 2 in Song et al. (2017) we have

$$p_{C_{i,j}}(\mu_1) = \bar{p}_{C_{i,j}}\nu^{-n-i+j+1}, \quad (71)$$

and multiplying and dividing by ν^{2n-2} gives

$$p_{C_{i,j}}(\mu_1) = \frac{1}{\nu^{2n-2}}\bar{p}_{C_{i,j}}\nu^{n+j-i-1}. \quad (72)$$

Now define the matrix $P_C^*(\nu)$ having elements

$$p^*_{C_{i,j}}(\nu) = \bar{p}_{C_{i,j}}\nu^{n+j-i-1}. \quad (73)$$

Since i can be at most equal to n, and j is at least equal to one, for all i, j, the exponent on ν in (73) is nonnegative. This means that $|P_C^*(\nu)|$ is a continuous function of the bounded argument $\nu \in (0, 1]$, so it remains bounded for all $t \in [t_0, t_0 + T)$. Now, from (72) and (73) we obtain

$$p_{C_{i,j}}(\mu_1) = \frac{1}{\nu^{2n-2}}p^*_{C_{i,j}}(\nu), \quad (74)$$

and therefore

$$P_C(\mu_1) = \mu_1^{2n-2}P_C^*(\nu). \quad (75)$$

Then from (75) we have for all $t \in [t_0, t_0 + T)$

$$|P_C(\mu_1)| \leq \mu_1^{2n-2}|P_C^*(\nu)|. \quad (76)$$

Substituting (76) into (70) obtains

$$|\tilde{u}(t)| \le \Big(\nu^{-2n+1}|l_0(\nu)| + \nu^{-n+1}|K_{n-1}^T||J_2| + \nu^{-2n-m_C+1}|k_n(\mathrm{e}_n^T + K_{n-1}^T J_1)| \Big)|P_C^*(\nu)||\mathscr{T}^{-1}||\tilde{\xi}(t)| + |a||\mathscr{T}^{-1}||\tilde{\xi}(t)|, \tag{77}$$

and then substituting (30) into (77) gives

$$|\tilde{u}(t)| \le \Big\{ \Big(\nu^{m_O-2n+2}|l_0(\nu)| + \nu^{m_O-n+2}|K_{n-1}^T||J_2| + \nu^{m_O-m_C-2n+2}|k_n(\mathrm{e}_n^T + K_{n-1}^T J_1)| \Big)|P_C^*(\nu)| + \nu^{m_O+1}|a| \Big\} |\mathscr{T}^{-1}|\tilde{M}\mathrm{e}^{-\tilde{\delta}(t-t_0)}|\tilde{\xi}(t_0)|. \tag{78}$$

Since $\nu \in (0, 1]$, for positive integers i, j with $i < j$, it is true that $\nu^j \le \nu^i$. Similarly, $\nu^{m_O-n+2} \le \nu^{m_O-2n+2} \le \nu^{m_O-m_C-2n+2}$, and $\nu^{m_O+1} \le \nu^{m_O-m_C-2n+2}$. Then from (78) we obtain

$$|\tilde{u}(t)| \le \nu^{m_O-m_C-2n+2} \Big\{ \Big(|l_0(\nu)| + |K_{n-1}^T||J_2| + |k_n(\mathrm{e}_n^T + K_{n-1}^T J_1)| \Big)|P_C^*(\nu)| + |a| \Big\} \times |\mathscr{T}^{-1}|\tilde{M}\mathrm{e}^{-\tilde{\delta}(t-t_0)}|\tilde{\xi}(t_0)|. \tag{79}$$

Define the function

$$\phi(\nu) := \Big(|l_0(\nu)| + |K_{n-1}^T||J_2| + |k_n(\mathrm{e}_n^T + K_{n-1}^T J_1)| \Big)|P_C^*(\nu)| + |a|, \tag{80}$$

and the quantity

$$\bar{\phi} := \sup_{0<\nu\le 1} |\phi(\nu)|. \tag{81}$$

By (73) and the statements that immediately followed, and the last statement of Lemma 4 in Song et al. (2017), it is clear that $|P_C^*(\nu)|$ and $|l_0(\nu)|$ remain bounded for all $t \in [t_0, t_0 + T)$. Therefore, from (80) and (81), it is clear that for all $t \in [t_0, t_0 + T)$, $\bar{\phi}$ is bounded also.

And so, with (79), (80), and (81), we obtain

$$|\tilde{u}(t)| \le \nu^{m_O-m_C-2n+2}\bar{\phi}|\mathscr{T}^{-1}|\tilde{M}\mathrm{e}^{-\tilde{\delta}(t-t_0)}|\tilde{\xi}(t_0)|, \tag{82}$$

where $\bar{\phi}$ is a positive constant. Then by selecting $m_O \ge m_C + 2n - 1$, the exponent on ν in (82) is positive, and we have proven the claim with $\alpha = m_O - m_C - 2n + 2$, $M_\alpha = \bar{\phi}|\mathscr{T}^{-1}|\tilde{M}$, and $\delta_\alpha = \tilde{\delta}$. ■

For brevity, the proof of Remark 1 is omitted.

Proof of Lemma 5. Substituting the controller (65) into the transformed system (36), (39) yields

$$\dot{r}_1(t) = \Lambda_C r_1(t) + \mathrm{e}_{n-1}\sigma(t) \tag{83}$$

$$\dot{\sigma}(t) = -k_n\mu_C\sigma(t) + \mu_C\tilde{u}(t). \tag{84}$$

Differentiating the Lyapunov function candidate $V(t) := \frac{1}{2}\sigma(t)^2$ and using (84) gives

$$\dot{V}(t) = -k_n\mu_C\sigma(t)^2 + \mu_C\sigma(t)\tilde{u}(t). \tag{85}$$

Using Young's inequality and Lemma 1 of Song et al. (2017), we obtain for all $t \in [t_0, t_0 + T)$,

$$V(t) \le \zeta_C(t - t_0, T)^{k_n} V(t_0) + \frac{\|\tilde{u}\|_{[t_0,t]}^2}{2k_n^2}, \tag{86}$$

where $\zeta_C(t - t_0, T)$ is the monotonically decreasing function

$$\zeta_C(t - t_0, T) := \mathrm{e}^{\frac{T}{m_C+n-1}\left(1-\mu_1(t-t_0,T)^{m_C+n-1}\right)},$$

which has the properties $\zeta_C(0, T) = 1$ and $\zeta_C(T, T) = 0$. From (86) we obtain

$$\sigma(t)^2 \le \zeta_C(t - t_0, T)^{k_n}\sigma(t_0)^2 + \frac{\|\tilde{u}\|_{[t_0,t]}^2}{k_n^2}, \tag{87}$$

such that for all $t \in [t_0, t_0 + T)$,

$$|\sigma(t)| \le \zeta_C(t - t_0, T)^{\frac{k_n}{2}}|\sigma(t_0)| + \gamma_\sigma\|\tilde{u}\|_{[t_0,t]}, \tag{88}$$

where $\gamma_\sigma := 1/k_n$. Now, by Lemma 4, by selecting $m_O \ge m_C + 2n - 1$, we guarantee that $\tilde{u}(t)$ not only remains bounded, but also goes to zero as $t \to t_0 + T$. The former fact combined with (88) shows that the $\sigma(t)$ system is FT-ISS with respect to $\tilde{u}(t)$.

The $r_1(t)$ and $\sigma(t)$ systems (83) and (84) combine to form a cascade system whose stability depends on the stability of each subsystem. Define

$$\bar{w}(t) := \begin{bmatrix} r_1(t) \\ \sigma(t) \end{bmatrix}. \tag{89}$$

Then it is true that

$$|\bar{w}(t)| \le |r_1(t)| + |\sigma(t)| \tag{90}$$

$$|r_1(t_0)| \le |\bar{w}(t_0)| \tag{91}$$

$$|\sigma(t_0)| \le |\bar{w}(t_0)|. \tag{92}$$

Recall that the $r_1(t)$ system is a linear system that is ISS w.r.t. $\sigma(t)$. This means there exist positive constants M_1, δ_1, γ_1 such that for all $t \in [t_0, t_0 + T)$,

$$|r_1(t)| \le M_1\mathrm{e}^{-\delta_1(t-t_0)}|r_1(t_0)| + \gamma_1\|\sigma\|_{[t_0,t]}. \tag{93}$$

We obtain a "fading memory" stability estimate for $|r_1(t)|$ as follows. For any time $t \in [t_0, t_0 + T)$, (93) provides

$$|r_1(t)| \le M_1\mathrm{e}^{-\delta_1\left(t-\frac{t+t_0}{2}\right)}\left|r_1\left(\frac{t+t_0}{2}\right)\right| + \gamma_1 \sup_{\frac{t+t_0}{2}\le\tau\le t}|\sigma(\tau)|. \tag{94}$$

Now, from (88) we obtain

$$\sup_{\frac{t+t_0}{2}\le\tau\le t}|\sigma(\tau)| \le \zeta_C\left(\frac{t+t_0}{2} - t_0, T\right)^{\frac{k_n}{2}}|\sigma(t_0)| + \gamma_\sigma\|\tilde{u}\|_{[t_0,t]}, \tag{95}$$

and then from (93), (88), and since $\zeta_C(0, T) = 1$, we obtain

$$\left|r_1\left(\frac{t+t_0}{2}\right)\right| \le M_1\mathrm{e}^{-\delta_1\left(\frac{t+t_0}{2}-t_0\right)}|r_1(t_0)| + \gamma_1|\sigma(t_0)| + \gamma_1\gamma_\sigma\|\tilde{u}\|_{[t_0,\frac{t+t_0}{2}]}. \tag{96}$$

Now, inserting (96) and (95) into (94) gives

$$|r_1(t)| \le M_1^2\mathrm{e}^{-\delta_1(t-t_0)}|r_1(t_0)| + \gamma_1\left(M_1\mathrm{e}^{-\delta_1\left(t-\frac{t+t_0}{2}\right)} + \zeta_C\left(\frac{t+t_0}{2} - t_0, T\right)^{\frac{k_n}{2}}\right)|\sigma(t_0)| + \gamma_1\gamma_\sigma(M_1 + 1)\|\tilde{u}\|_{[t_0,t]}. \tag{97}$$

Next, using (97) and (88) in (90), we obtain for all $t \in [t_0, t_0 + T)$

$$|\bar{w}(t)| \le M_1^2\mathrm{e}^{-\delta_1(t-t_0)}|r_1(t_0)| + \left(\gamma_1 M_1\mathrm{e}^{-\delta_1\left(t-\frac{t+t_0}{2}\right)} + \gamma_1\zeta_C\left(\frac{t+t_0}{2} - t_0, T\right)^{\frac{k_n}{2}} + \zeta_C(t - t_0, T)^{\frac{k_n}{2}}\right)|\sigma(t_0)| + (\gamma_1\gamma_\sigma(M_1 + 1) + \gamma_\sigma)\|\tilde{u}\|_{[t_0,t]}. \tag{98}$$

Then using (91) and (92), (98) gives

$$|\bar{w}(t)| \leq \left(M_1^2 e^{-\delta_1(t-t_0)} + \gamma_1 M_1 e^{-\delta_1\left(t-\frac{t+t_0}{2}\right)} + \gamma_1 \zeta_C\left(\frac{t+t_0}{2} - t_0, T\right)^{\frac{k_n}{2}} + \zeta_C(t-t_0,T)^{\frac{k_n}{2}}\right)|\bar{w}(t_0)| + (\gamma_1\gamma_\sigma (M_1+1) + \gamma_\sigma)\|\tilde{u}\|_{[t_0,t]}. \quad (99)$$

For all $t \in [t_0, t_0+T)$, every term in the parentheses has decayed from its initial value at t_0. So, there exist some positive constants $M_{\bar{w}}$, $\delta_{\bar{w}}$, and $\gamma_{\bar{w}}$ such that

$$|\bar{w}(t)| \leq M_{\bar{w}} e^{-\delta_{\bar{w}}(t-t_0)}|\bar{w}(t_0)| + \gamma_{\bar{w}}\|\tilde{u}\|_{[t_0,t]} \quad (100)$$

where $\gamma_{\bar{w}} := (\gamma_1\gamma_\sigma(M_1+1)+\gamma_\sigma)$. Therefore, the $\bar{w}(t)$ cascade system is ISS w.r.t. the disturbance $\tilde{u}(t)$.

It remains to invert the dynamics of the scaled state to obtain those of the original state. Define the matrix $\mathscr{R}$ and its inverse from the equation

$$\bar{w}(t) = \left(J_1^T + e_n K_{n-1}^T\right) r_1(t) + e_n w_n(t) = \mathscr{R} w(t), \quad (101)$$

such that $\mathscr{R} := I + e_n K_{n-1}^T J_1$ and $\mathscr{R}^{-1} = I - e_n K_{n-1}^T J_1$. Using Lemma 3 in Song et al. (2017) and (101), we obtain

$$x(t) = \nu^{m_C+1} Q_C(\nu)\mathscr{R}^{-1}\bar{w}(t), \quad (102)$$

and then from (101), Lemma 2 in Song et al. (2017), and $\mu_1(0,T) = 1$, we have

$$\bar{w}(t_0) = \mathscr{R}P_C(1)x(t_0). \quad (103)$$

Then with (102), the last statement of Lemma 3 in Holloway and Krstic (2019), and substituting in (100), we obtain

$$|x(t)| \leq \nu^{m_C+1}\bar{q}_C|\mathscr{R}^{-1}|\left(M_{\bar{w}} e^{-\delta_{\bar{w}}(t-t_0)}|\bar{w}(t_0)| + \gamma_{\bar{w}}\|\tilde{u}\|_{[t_0,t]}\right). \quad (104)$$

Then from (103), we obtain

$$|\bar{w}(t_0)| \leq |\mathscr{R}P_C(1)||x(t_0)|, \quad (105)$$

which after substituting into (104) obtains

$$|x(t)| \leq \nu^{m_C+1}\left(\check{M} e^{-\check{\delta}(t-t_0)}|x(t_0)| + \check{\gamma}\|\tilde{u}\|_{[t_0,t]}\right) \quad (106)$$

with $\check{M} := \bar{q}_C|\mathscr{R}^{-1}||\mathscr{R}P_C(1)|M_{\bar{w}}$, $\check{\delta} := \delta_{\bar{w}}$, and $\check{\gamma} := \bar{q}_C|\mathscr{R}^{-1}|\gamma_{\bar{w}}$. Therefore, the output feedback system is FT-ISS+C.

As for the claim regarding $u(t)$, from (65), (48), (49), and (45) we obtain

$$u(t) = -\nu^{m_C}\left(l_0(\nu) + \nu^n K_{n-1}^T J_2 - \nu a^T Q_C(\nu)\right) w(t) - k_n\sigma(t) + \tilde{u}(t). \quad (107)$$

The terms involving $w(t)$ are uniformly bounded for $t \in [t_0, t_0+T)$ and go to zero as $t \to t_0+T$. By Lemma 4 and our choice of the observer and controller parameters, the same is true for $\tilde{u}(t)$. Finally, by Lemma 4 and (88), $\sigma(t)$ remains uniformly bounded for $t \in [t_0, t_0+T)$. The claim on $u(t)$ is proven. ■

Proof of Theorem 1. From (15), the triangle inequality, and (7) we obtain

$$|\tilde{\xi}(t_0)| \leq |\mathscr{T}|\left(|x(t_0)| + |\hat{\xi}(t_0)|\right). \quad (108)$$

Inserting (108) into (30) then gives

$$|\tilde{\xi}(t)| \leq \nu^{m_O+1}\tilde{M}|\mathscr{T}|e^{-\tilde{\delta}(t-t_0)}\left(|x(t_0)| + |\hat{\xi}(t_0)|\right). \quad (109)$$

Using (68) and (109), we obtain

$$|x(t)| + |\hat{\xi}(t)| \leq \nu^{m_C+1}\left(\check{M}e^{-\check{\delta}(t-t_0)} + \tilde{M}|\mathscr{T}|e^{-\tilde{\delta}(t-t_0)}\right) \times \left(|x(t_0)| + |\hat{\xi}(t_0)|\right) + \nu^{m_C+1}\check{\gamma}\|\tilde{u}\|_{[t_0,t]}, \quad (110)$$

and then from (82) and (108), we obtain

$$\sup_{t_0 \leq \tau \leq t} |\tilde{u}(\tau)| \leq \bar{\phi}|\mathscr{T}^{-1}|\tilde{M}|\mathscr{T}|\left(|x(t_0)| + |\hat{\xi}(t_0)|\right).$$

Using this result in (110) gives

$$|x(t)| + |\hat{\xi}(t)| \leq \nu^{m_C+1}\bar{M}\left(|x(t_0)| + |\hat{\xi}(t_0)|\right) \quad (111)$$

for some positive constant $\bar{M}$. Define $r := |x(t_0)| + |\hat{\xi}(t_0)|$, and $s := \mu_1 - 1$. Then ν can be expressed as $\nu = 1/(s+1)$, and it is clear that the right-hand side of (111) is a class $\mathscr{KL}$ function of the form $\beta(r,s) = \bar{M}r/(s+1)^{m_C+1}$. Thus we have proven the claim. ■

3.4. Implementation

The main benefit of this approach to finite-time stabilization is it offers simple parameter tuning which prescribes convergence times. The parameters $\{l_i\}_{i=1}^n$ and $\{k_i\}_{i=1}^n$ are selected to make the matrices (29) and (34) Hurwitz, and the integer tuning parameters $m_O \geq 1$, $m_C \geq 1$ are selected simply according to (66). But most importantly, the time of convergence, T, is freely prescribed by the user, completely independent of initial conditions and other parameters.

On the other hand, the main drawback of our approach is clear: the output feedback controller (65) was developed for linear systems having no plant or measurement uncertainties, which is limiting in practice. Because of this, the presence of unmodeled plant or measurement disturbances will degrade the stabilization of the state to a nonzero neighborhood of the origin.

Lastly, as discussed in Holloway and Krstic (2019) and Song et al. (2017), the use of the observer (6) and controller (43) in practice will exhibit numerical instability in the final instants as $t \to t_0+T$. However, these works also offer simple solutions to circumvent this problem, such as increasing T to be slightly larger than t_f. These solutions decrease the accuracy of the observer and controller convergence, but in manageable ways that can be tuned easily if needed.

4. Numerical example

Consider the double integrator with only the first state measured as output, $\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = u(t)$, and $y(t) = x_1(t) + \eta(t)$, where $\eta(t)$ is zero-mean, white Gaussian measurement noise. Using (52)–(55) and Lemma 2 and Lemma 4 from Song et al. (2017), we find the state feedback controller (47) to be $u(t) = -\kappa(t-t_0,T)^T x(t)$, where $\kappa(t-t_0,T) := [\kappa_1(t-t_0,T), \kappa_2(t-t_0,T)]^T$, and $\kappa_1(t-t_0,T) = k_1\frac{m_C+2}{T}\mu_1 + \frac{(m_C+3)(m_C+2)}{T^2}\mu_1^2 + k_1k_2\mu_1^{m_C+2} + k_2\frac{m_C+2}{T}\mu_1^{m_C+3}$, $\kappa_2(t-t_0,T) = k_1 + 2\frac{m_C+2}{T}\mu_1 + k_2\mu_1^{m_C+2}$. We replace $x(t)$ with $\hat{x}(t)$ and use (7) to implement the control $u(t) = -\kappa(t-t_0,T)^T\hat{\xi}(t)$, since $\mathscr{T}^{-1}$ equals identity for the chain of integrators. The state estimate $\hat{\xi}(t)$ is obtained from the observer (6), which for this example becomes $\dot{\hat{\xi}}_1(t) = \hat{\xi}_2(t) + g_1(t-t_0,T)\left(y(t) - \hat{\xi}_1(t)\right)$, $\dot{\hat{\xi}}_2(t) = u(t) + g_2(t-t_0,T)\left(y(t) - \hat{\xi}_1(t)\right)$. As in the example of Holloway and Krstic (2019), we obtain the constants $\bar{p}_{O_{1,1}} = \bar{p}_{O_{2,2}} = 1$, $\bar{p}_{O_{1,2}} = 0$, and $\bar{p}_{O_{2,1}} = -\frac{m_O+2}{T}$, and the gains $g_1(t-t_0,T) = l_1 + 2\frac{m_O+2}{T}\mu_1$, $g_2(t-t_0,T) = l_2 + l_1\frac{m_O+2}{T}\mu_1 + \frac{(m_O+1)(m_O+2)}{T^2}\mu_1^2$. Fig. 1–3 show simulation results of this example with $t_f = 10$, $m_C = 1$ and $m_O = 3$ (see Remark 1), and $l_1 = l_2 = k_1 = k_2 = 1$, for three sets of initial conditions, and with measurement noise standard deviation of $\sigma_\eta = 2$. To ensure stabilization at $t = t_0 + t_f$ without numerical instability (see Section 3.4), we set $T = 13$. Fig. 1 shows the

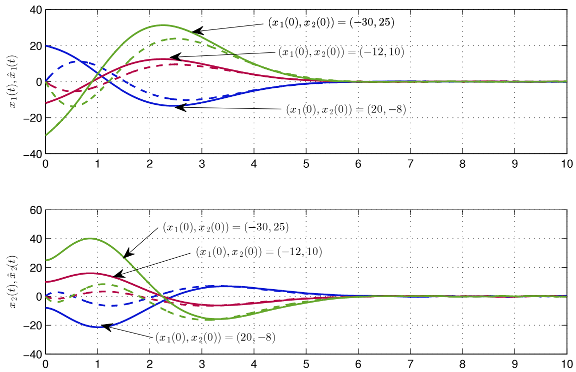

Fig. 1. Time histories of states (solid) and state estimates (dashed) for the double integrator example.

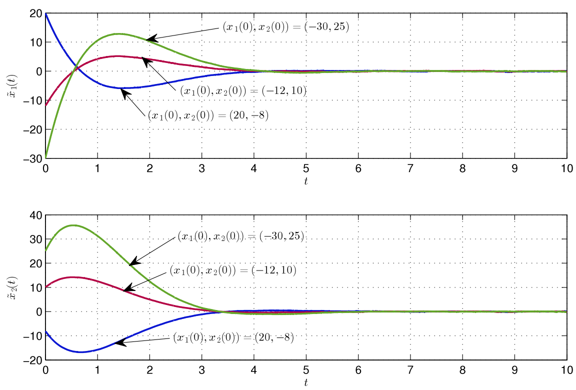

Fig. 2. Time histories of state estimation errors for the double integrator example.

dynamics of the state and state estimate, and Fig. 2 shows the dynamics of the estimation error. The observer gains, controller gains, and control input are shown in Fig. 3. Despite the presence of the measurement noise, stabilization of both the state and state estimate are achieved to within a small neighborhood of the origin in the prescribed time. Also, despite the gains going to infinity as $t \to t_0 + T$, the control input remains bounded.

5. Conclusions

The output feedback controller presented allows the user to easily prescribe convergence times a priori, irrespective of initial conditions. A separation principle holds between the prescribed-time observer of Holloway and Krstic (2019) and the uncertainty-free prescribed-time controller of Song et al. (2017) provided the scaling power of the time-varying observer gains exceeds the scaling power of the controller gains by twice the order of the system, i.e., provided the observer is fast enough relative to the controller, irrespective of the constant gains of both. In practice, the presence of significant plant or measurement disturbances will degrade stabilization of the state to a nonzero neighborhood of the origin. For problems in which such disturbances impede the performance of (65) unacceptably, the output feedback controllers of Lopez-Ramirez et al. (2016a) and Tian et al. (2017) may offer viable solutions, however these will require additional tuning of parameters to prescribe the convergence times.

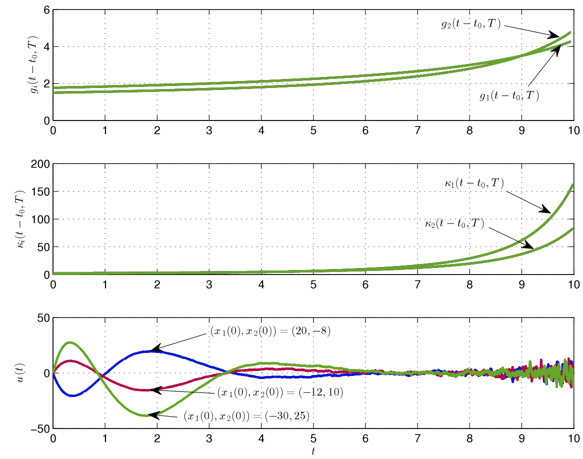

Fig. 3. Time histories of observer gains, controller gains, and control input for the double integrator example.

References

Amato, F., Darouach, M., & De Tommasi, G. (2018). Finite-time stabilizability, detectability and dynamic output feedback finite-time stabilization of linear systems. *IEEE Transactions on Automatic Control*, (in press).

Andrieu, V., Praly, L., & Astolfi, A. (2008). Homogeneous approximation, recursive observer design, and output feedback. *SIAM Journal of Optimization and Control*, *47*(4), 1814–1850.

Angulo, M., Moreno, J., & Fridman, L. (2013). Robust exact uniformly convergent arbitrary order differentiator. *Automatica*, *49*, 2489–2495.

Bernuau, E., Perruquetti, W., Efimov, D., & Moulay, E. (2015). Robust finite-time output feedback stabilisation of the double integrator. *International Journal of Control*, *88*(3), 451–460.

Cruz-Zavala, E., Moreno, J., & Fridman, L. (2011). Uniform robust exact differentiator. *IEEE Transactions on Automatic Control*, *56*(11), 2727–2733.

Du, H., Qian, C., Yang, S., & Li, S. (2013). Recursive design of finite-time convergent observers for a class of time-varying nonlinear systems. *Automatica*, *49*, 601–609.

Efimov, D., Levant, A., Polyakov, A., & Perruquetti, W. (2016). Supervisory acceleration of convergence for homogeneous systems. *International Journal of Control*, http://dx.doi.org/10.1080/00207179.2016.1269949.

Engel, R., & Kreisselmeier, G. (2002). A continuous-time observer which converges in finite time. *IEEE Transactions on Automatic Control*, *47*(7), 1202–1204.

Holloway, J., & Krstic, M. (2019). Prescribed-time observers for linear systems in observer canonical form. *IEEE Transactions on Automatic Control*, (submitted for publication).

Hong, Y., Huang, J., & Xu, Y. (2001). On an output feedback finite-time stabilization problem. *IEEE Transactions on Automatic Control*, *46*(2), 305–309.

Levant, A. (1998). Robust exact differentiation via sliding mode technique. *Automatica*, *34*(3), 379–384.

Levant, A. (2003). Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control*, *76*(9/10), 924–941.

Levant, A. (2005). Homogeneity approach to high-order sliding mode design. *Automatica*, *41*(5), 823–830.

Levant, A. (2013). On fixed and finite time stability in sliding mode control. In *Proc. 52nd IEEE conference on decision and control* (pp. 4260–4265).

Li, J., & Qian, C. (2006). Global finite-time stabilization by dynamic output feedback for a class of continuous nonlinear systems. *IEEE Transactions on Automatic Control*, *51*(5), 879–884.

Li, Y., & Sanfelice, R. (2015). A finite-time convergent observer with robustness to piecewise-constant measurement noise. *Automatica*, *57*, 222–230.

Lopez-Ramirez, F., Efimov, D., Polyakov, A., & Perruquetti, W. (2016). Fixed-time output stabilization of a chain of integrators. In *Proc. IEEE 55th conference on decision and control* (pp. 3886–3891).

Lopez-Ramirez, F., Polyakov, A., Efimov, D., & Perruquetti, W. (2016). Finite-time and fixed-time observers design via implicit Lyapunov function. In *Proc. european control conference* (pp. 289–294).

Polyakov, A., Efimov, D., & Perruquetti, W. (2015). Finite-time and fixed-time stabilization: implicit lyapunov function approach. *Automatica*, *51*, 332–340.

Raff, T., & Allgower, F. (2007). An impulsive observer that estimates the exact state of a linear continuous-time system in predetermined finite time. In *Proc. mediterranean conference on control and automation* (pp. 1–3).

Shi, S., Xu, S., Yu, X., Lu, J., Chen, W., & Zhang, Z. (2017). Robust output-feedback finite-time regulator of systems with mismatched uncertainties bounded by positive functions. *IET Control Theory & Applications*, *11*(17), 3107–3114.

Song, Y., Wang, Y., Holloway, J., & Krstic, M. (2017). Time-varying feedback for robust regulation of normal-form nonlinear systems in prescribed finite time. *Automatica*, *83*, 243–251.

Tian, B., Zuo, Z., Yan, X., & Wang, H. (2017). A fixed-time output feedback control scheme for double integrator systems. *Automatica*, *80*, 17–24.

Zarchan, P. (2007). Tactical and strategic missile guidance. In *Progress in astronautics and aeronautics* (5th ed.).

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