Brief paper

Sampled boundary observer for strict-feedback nonlinear ODE systems with parabolic PDE sensor

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ABSTRACT

We design an observer for ODE-PDE cascades where the ODE is nonlinear of strict-feedback structure and the PDE is a linear and of parabolic type. The observer provides online estimates of the (finite-dimensional) ODE state vector and the (infinite-dimensional) state of the PDE, based only on sampled boundary measurements. A design that simultaneously addresses nonlinear ODEs and boundary measurement sampling is the paper’s key contribution. Our observer design combines the backstepping design approach and the high-gain observer methodology and our analysis employs a Lyapunov–Krasovskii type functional to establish exponential convergence. Our sufficient conditions for convergence involve the maximum sampling interval and the PDE domain length.

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1. Introduction

Designing boundary observers for cascade systems involving PDE sensing and/or actuation has recently become a highly active research topic. Linear ODE-PDE cascades with first-order hyperbolic PDEs have been dealt with in Krstic and Smyshlyaev (2008). The proposed boundary observer is a copy of the system with additional innovation terms involving domain-dependent gains which are obtained through the solution of a first order ODE. The innovation terms provide the observer with a feedback structure and introduce a coupling between its ODE and PDE parts. Exponential stability of the resulting error system has been analysed using an appropriate error coordinate backstepping transformation and a quadratic Lyapunov functional. This study has been extended in Ahmed-Ali, Giri, and Krstic (2017) to the case where the system ODE part is nonlinear of globally Lipschitz type and with a triangular structure. The ODE nonlinearity is accounted for in the observer design by using the high-gain formalism, especially in the part devoted to the ODE state observation. The system nonlinearity gives rise to a limitation on the admissible length $D$ of the PDE domain: the larger the Lipschitz constant, the smaller the admissible values of $D$. This limitation has been coped with in Ahmed-Ali, Giri, Krstic, and Kahelras (2018) by using a cascade observer involving a number, say $m$, of partial observers each one providing the estimates of the PDE state on a subdomain of length $D/m$. Accordingly, the observer (error system) exponential stability can be guaranteed, however large the Lipschitz constant of the ODE nonlinearity, provided the number $m$ is taken sufficiently large.

Observer design for linear ODE-PDE cascades that involve parabolic PDEs has been introduced in Krstic (2009). The observer involves domain position dependent gains that are governed by second order ODEs. Again, exponential stability of the corresponding observation error system is established using a quadratic Lyapunov functional. In Ahmed-Ali, Giri, Krstic, and Lamnabhi-Lagarrigue (2015), we have extended the result of Krstic (2009) to ODE-PDE systems where the ODE part is nonlinear Lipschitz with triangular structures. Observer design for nonlinear multivariable systems with parabolic PDE sensor dynamics and application to output feedback control was studied in Wu and Wang (2013).

Boundary observer design for linear ODE-ODE cascade systems with first-order hyperbolic PDEs has been studied in e.g. Hasan, Aamo, and Krstic (2016). The observer is constructed in a collocated setup, which means that both sensing and actuation are located at the same boundary. The observer gains are computed analytically by solving Goursat-type PDEs in terms of Bessel function of the first
2. Observer problem statement

2.1. Class of observed systems

We are considering continuous-time nonlinear systems that can be modelled by the following ODE-PDE cascade:

\[ \dot{X}(t) = AX(t) + f(X(t), v(t)), \quad f(x, t) = au_0(x, t) - cu(x, t), \]

for \( x, t \in (0, D) \times (0, +\infty) \)

\[ u_0(0, t) = qu(0, t), \quad \forall t \geq 0 \]

\[ u(D, t) = CX(t), \quad \forall t \geq 0 \]

with,

\[ A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{k \times n} \]

\[ C = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{1 \times n} \]

\[ f(X) = \begin{pmatrix} f_1(X_1, v) \\ f_2(X_1, X_2, v) \\ \vdots \\ f_n(X_1, \ldots, X_n, v) \end{pmatrix} \]

and \( f_i: \mathbb{R}^{i+m} \rightarrow \mathbb{R} \)

where \( X(t) \in \mathbb{R}^n \) denotes the state vector of the finite-dimensional subsystem, described by (2) with any initial condition \( X(0) = X_0 \in \mathbb{R}^n; v(t) \in \mathbb{R}^m \) (for some known \( m \geq 1 \)) is an external known bounded signal of class \( C^2(\mathbb{R}_+: \mathbb{R}_m^m); u(x, t) \in \mathbb{R} \) designates the state of the infinite-dimensional subsystem described by the parabolic type PDE (3) with boundary conditions (4) and any initial condition \( u(x, 0) = (u(0)|x) \in \mathbb{R} \) for \( x \in (0, D); a > 0, c \in \mathbb{R} \) and \( q \geq 0 \) are known real parameters. Letting \( f_i(X, v) \) denote the Jacobian matrix of \( f(\cdot) \) w.r.t. \( X \), it is supposed that:

\[ \exists \beta_f > 0, \forall X \in \mathbb{R}^n: |f_k(X, v)| \leq \beta_f \]

where \( \beta_f \) might depend on the external signal \( v \). The pair \( (A, C) \) is observable and the whole system is observed through a ZOH sampling of the signal \( u(0, t) \), i.e. the system output is:

\[ y(t) = u(0, t), \quad \forall t \in [t_k, t_{k+1}] \text{ and } k = 0, 1, 2, \ldots \]

where \( \{t_k\}_{k=0}^\infty \) denotes the sampling time sequence. This is a partition of \( \mathbb{R}_+: \{t_k\}_{k=0}^\infty \) is increasing with \( t_0 = 0 \) and \( t_k \rightarrow \infty \) as \( k \rightarrow \infty \). It is also supposed that \( 0 < h < +\infty \) with

\[ h := \sup_{k \in \mathbb{N}} (t_k - t_{k-1}) \]

We seek an observer that provides accurate online estimates of both the (finite-dimensional) state vector \( X(t) \) and the distributed state \( u(x, t), 0 \leq x \leq D \), based only on the measurements of the external signals \( v(t) \) and \( y(t) \).

Remark 1. Sampled-output observers have recently been developed in Ahmed-Ali and Karafyllis et al. (2016, 2017) for subclasses of (2)–(4) corresponding to \( f(\cdot) = 0 \). Then, Luenberger type

kind. Finally, boundary observer design for linear coupled PDE-ODE, with hyperbolic PDE, has been considered in e.g. Tang and Xie (2010, 2011).

A common characteristic of the above mentioned observers is that they all require the system boundary outputs to be continuously accessible to measurements. As only sampled measurements are available in practice, continuous-time measurement based observers may not meet their theoretical performances. This explains the great deal of interest that has been paid to sampled-data based observer design over the last few years. However, quite a few sampled-measurement observers have been developed for systems described by PDEs, see e.g. Ahmed-Ali, Fridman, Giri, Burlion, and Lamnabhi-Lagarrigue (2016) and Fridman and Blighovsky (2012) for parabolic PDEs. The problem of sampled-data observer design for linear ODE-PDE cascades has only recently been investigated in Ahmed-Ali, Karafyllis, Giri, Krstic, and Lamnabhi-Lagarrigue (2016) and Ahmed-Ali, Karafyllis, Giri, Krstic and Lamnabhi-Lagarrigue (2017) considering a specific class of parabolic PDEs and boundary conditions. The observer design developed there can be viewed as a sampled version of that in Krstic (2009) as it relied on a coordinate transformation of the ODE and PDE states. However, the error system exponential stability has been investigated using a small-gain (input–output) stability analysis, rather than Lyapunov functional analysis in Krstic (2009).
linear observers were used, while high-gain observer is necessary to deal with the present problem. The considered strict-feedback structure of the system and the globally Lipschitz assumption on \(f(\cdot)\) are precisely considered to make the high-gain observer design applicable, see e.g. Khalil (2015, p. 271). In this respect, note that all uniformly observable SISO systems are diffeomorphic to the strict-feedback form. Accordingly, considering this form entails covering all uniformly observable SISO systems. Owing to the global Lipschitz assumption made on \(f(\cdot)\), although this is quite usual, it actually entails limitation of the set of admissible nonlinearities.

3. Observer design and analysis

Consider the following backstepping transformation:

\[
p(x, t) = u(x, t) - CM^{-1}(D)M(x)X(t),
\]

where \(M(x) \in \mathbb{R}^{n \times n}\) is defined by the following ODE and boundary conditions:

\[
d^2M\over dx^2(x) = a^{-1}M(x)(A + cl)
\]

\[
M(0) = I, \frac{dM}{dx}(0) = qI
\]

where \(I \in \mathbb{R}^{n \times n}\) denotes the identity matrix. The solution of the problem (9)–(10) is analytic and expressed by the following globally convergent series

\[
M(x) = (1 + qx)x + \sum_{i=1}^{\infty} a^{-i}(A + cl)^i \left( \frac{x^{2i}}{(2i)!} + q \frac{x^{2i+1}}{(2i+1)!} \right)
\]

The transformation (8) also involves the inverse of \(M(D)\) which implicitly entails an additional assumption on \(A\). This issue is no longer investigated in this paper. Using (9), (2) and (3), it follows from (8) that \(p(x, t)\) is governed by the following PDE, for all \((x, t) \in (0, D) \times (0, +\infty)\):

\[
p(x, t) = ap_{rx}(x, t) - cp(x, t) - CM^{-1}(D)M(x)f(X)
\]

For convenience, the new system representation in terms of the states \((X(t), p(x, t))\) is rewritten:

\[
\hat{X}(t) = AX(t) + f(\hat{X}(t), \nu(t)), \text{ for all } t \geq 0
\]

\[
p_t(x, t) = ap_{rx}(x, t) - cp(x, t) - CM^{-1}(D)M(x)f(X, v), \text{ for all } (x, t) \in (0, D) \times (0, +\infty)
\]

\[
p_{x}(0, t) = qp_{r}(0, t), \text{ for all } t \geq 0
\]

\[
p(D, t) = 0, \text{ for all } t \geq 0
\]

\[
u(x, t) = p(x, t) + CM^{-1}(D)M(x)X(t)
\]

where (17) and the boundary conditions (15)–(16) are derived from (8) using (4) and (10). The \((X, p)-\) system representation in (13)–(17) mainly differs from the initial \((X, u)-\) representation in that the boundary conditions (15)–(16) of the former are \(X\)-free leading to homogeneous boundary conditions (unlike those of the initial representation, see (2)–(4)). Homogeneous boundary conditions are required in the existence/uniqueness theorems for PDEs. For convenience, the well posedness issue will be discussed latter (see Remark 4).

To get online estimates \(\hat{X}(t)\) and \(\hat{p}(x, t)\) of the unmeasurable states \(X(t)\) and \(p(x, t)\), we propose the following sampled-output observer:

\[
\hat{X}(t) = \Delta \hat{X}(t) + f(\hat{X}(t), \nu(t)) - \theta M(D)A^{-1}L\hat{y}(t_k), \text{ for all } t \in [t_k, t_{k+1}) \text{ and } k = 0, 1, 2, \ldots
\]

\[
\hat{p}(x, t) = ap_{rx}(x, t) - cp(x, t) - CM^{-1}(D)M(x)f(\hat{X}(t), \nu(t)), \text{ for all } (x, t) \in (0, D) \times (0, +\infty)
\]

\[
\hat{p}_{x}(0, t) - qp_{r}(0, t) = \hat{p}(D, t) = 0, \text{ for all } t \geq 0
\]

\[
\hat{u}(x, t) = \hat{p}(x, t) + CM^{-1}(D)M(x)\hat{X}(t), \text{ for all } (x, t) \in (0, D) \times (0, +\infty)
\]

\[
\hat{y}(t) = \hat{y}(t) - y(t), \text{ with } y(t) = \hat{u}(0, t)
\]

with

\[
\Delta = \text{diag}\left\{ \frac{1}{\theta}, \ldots, \frac{1}{\theta^{n-1}} \right\} \in \mathbb{R}^{n \times n},
\]

where the real scalars \(K\) and \(\theta > 1\) are design parameters and \(L \in \mathbb{R}\) is any vector such that \(A - LC\) is Hurwitz (this is not an issue since the pair \((A, C)\) is observable).

Remark 2. In the case of linear systems (i.e. if \(f(\cdot) = 0\) then) the use of a high-gain observer is not necessary. In such a case, one can consider simpler sampled-data observers like those in Ahmed-Ali and Karafyllis et al. (2016, 2017) and Karafyllis et al. (2017). Presently, this amounts to letting \(\theta = 1\) and \(\Delta = I\) in the ODE part of the observer (given par (18)). In all mentioned observers, the ‘measurement delay’ caused by the heat PDE sensor dynamics is compensated for in the observer ODE part by the matrix \(M(D)\) (Ahmed-Ali and Karafyllis et al., 2016, 2017; Karafyllis et al., 2017; Kristic, 2009). In a sense, \(M(D)\) acts as a ‘transition’ matrix between the measurement point, located at \(x = 0\), and the output of the ODE part, located at \(x = D\). In the case of no PDE sensor dynamics (i.e. if \(D = 0\), \(M(D) = M(0) = I\)) and the observer part (18) boils down to a standard high-gain observer. To analyse the proposed observer, we introduce the estimation errors:

\[
\hat{X}(t) = \hat{X}(t) - X(t),
\]

\[
\hat{p}(x, t) = \hat{p}(x, t) - p(x, t), \text{ where } \hat{u}(x, t) = \hat{u}(x, t) - u(x, t)
\]

Using (10), (22) and (24a), one immediately gets by subtracting (17) to (21) and letting there \(x = 0, t = t_k\):

\[
\hat{y}(t_k) = \hat{p}(0, t_k) + CM^{-1}(D)\hat{X}(t_k)
\]

Using the observer Eqs. (18)–(22), the system equations (13)–(17), and definitions (24a)–(24b), we get the following error equations, for all \(x \in [0, 1], t \in [t_k, t_{k+1})\) and \(k \in \mathbb{N}\):

\[
\hat{X}(t) = \Delta \hat{X}(t) + f(\hat{X}(t), \nu(t)) - f(X(t), u(t)) - \theta M(D)A^{-1}L\hat{y}(t_k)
\]

\[
\hat{p}(x, t) = ap_{rx}(x, t) - cp(x, t) - CM^{-1}(D)M(x)f(\hat{X}(t), \nu(t))
\]

\[
\hat{p}(0, t) - qp_{r}(0, t) = \hat{p}(D, t) = 0, \text{ for all } t \geq 0
\]
\[ \tilde{u}(x, t) = \tilde{p}(x, t) + CM^{-1}(D)M(x)\tilde{X}(t), \]

(29)

The exponential stability of the error system (26)–(29) is stated in the following theorem:

**Theorem 1 (Main Result).** Consider the class of systems defined by Eqs. (2)–(4) with parameters \( a > 0, c \in \mathbb{R} \) and \( q \geq 0 \). Consider the output-sampled observer equations (18)–(22), with the scalar parameter \( \theta > 0 \) arbitrary, the gain \( L \in \mathbb{R}^q \) such that the matrix \( A - LC \) is Hurwitz, and the matrix function \( M(x) \) defined by (9)–(11). Then, there exists \( \theta^* > 0 \) such that for any \( \theta > \theta^* \), there exist \( 0 < D^* < 1/\sqrt{\theta^*} \) and \( h^* > 0 \) so that if (the domain length \( D \) of the PDE (3) and the diameter \( h \) of the time partition \( \{ t_k \}_{k=0}^\infty \) are such that) \( 0 < D < D^* \) and \( 0 < h < h^* \) then (1) The observer (18)–(22) is exponentially convergent in the sense that,

\[ \|\tilde{X}(t)\| \leq \int_0^t \|\tilde{u}(x, t')\| dx \]

(30)

exponentially converge to zero, wherever the initial conditions \( X(0), \tilde{X}(0) \in \mathbb{R}^n, \tilde{u}(0), \tilde{p}(0) \in C^1([0, D]; \mathbb{R}) \), with \( \tilde{p}(0)(0) = \tilde{p}(0)(D) = 0, \tilde{u}(0)(0) = \tilde{u}(0)(D) = 0, \) and \( u(0)(0) = C\tilde{X}(0) \).

(2) In the case of linear systems, i.e., if the Lipschitz coefficient \( \beta_f \) in (6) is zero then, \( D \) can be arbitrarily large.

**Proof.** Introduce the following notation:

\[ Z = \Delta M^{-1}(D)\tilde{X} \]

(31)

\[ \chi(D, t) = \int_0^1 f_k(X(t) + \alpha X(t)) \rho(t) dt \]

(32)

Then, differentiating both sides of (31), one obtains using (26) and applying the mean-value theorem:

\[ \dot{Z}(t) = \Delta M^{-1}(D)\dot{\tilde{X}}(t) + \Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \]

\[ = \Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \]

\[ = \Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t), \]

for all \( x \in [0, D], t \in [t_k, t_{k+1}], k \in \mathbb{N} \)

where the second equality is obtained using the following equalities which are direct consequences of the following properties:

\[ C \Delta = C \]

(34)

\[ \Delta A = \theta A \Delta \]

(35)

\[ AM(x) = M(x)A, \quad M^{-1}(x) = M^{-1}(x)A, \]

(36)

for all \( x \in [0, D] \).

In turn, (27) is rewritten as follows:

\[ \tilde{p}_r(x, t) = \tilde{p}_r(x, t) \]

(37)

where the last term on the right side is obtained using (34), Eq. (37) is completed with the boundary conditions (28)–(29). To analyse the (\( Z, \tilde{p} \))-system, we consider the following Lyapunov–Krasovskii functional:

\[ V(Z, \tilde{p}, t) = Z^T(t)PZ(t) + \frac{\alpha_0}{2} \int_0^t \tilde{p}_r^2(x, t) dt \]

\[ + \frac{\alpha_1}{2} \int_0^t \tilde{p}_r^2(x, t) + \int_0^{\alpha_2} \tilde{p}_r^2(0, t) dt \]

\[ + \alpha_2 \int_0^t (s - t + h) \tilde{Z}(s)^2 ds \]

(38)

for \( t_k \leq t < t_{k+1} \), \( k \in \mathbb{N} \), where \( \alpha_0 > 0, \alpha_1 > 0 \) and \( \alpha_2 > 0 \) are real scalars such that:

\[ \alpha_0 + \alpha_1 > 0 \]

(39a)

and \( P \) any positive definite matrix satisfying the inequality,

\[ P(A - LC) + (A - LC)^T P \leq -\mu I \]

(39b)

for some \( \mu > 0 \), where \( I \) denotes the identity matrix of \( \mathbb{R}^{n \times n} \). Note that \( P \) exists (whatever \( \mu > 0 \)) because \( A - LC \) is Hurwitz. Also, from (28) one has the following relations which will prove to be useful:

\[ Z(t_k) = Z(t) - \int_{t_k}^t \dot{Z}(s) ds, \tilde{p}(0, t) = -\int_0^t \tilde{p}_r(x, t) dx \]

(40)

Using (33), (37) and (39b), it follows differentiating \( V(Z, \tilde{p}, t) \) with respect to time:

\[ \dot{V}(Z, \tilde{p}, t) = 2Z^T(t) (P(A - LC) + (A - LC)^T P) Z(t) \]

\[ + 2\theta Z^T(t)PL \int_0^t \tilde{Z}(s) ds \]

\[ + 2\theta Z^T(t) PL \int_0^t \tilde{p}_r(x, t) dx \]

\[ + 2Z^T(t)P \Delta M^{-1}(D) \chi(D, t) M(D) \Delta^{-1} \]

\[ - \alpha_0 \int_0^t \tilde{p}_r(x, t) (\alpha \tilde{p}_r(x, t) - c \tilde{p}(x, t)) dx \]

\[ - \alpha_0 \int_0^t \tilde{p}_r(x, t) (\alpha \tilde{p}_r(x, t) - c \tilde{p}(x, t)) dx \]

\[ + \alpha_1 \int_0^t \tilde{p}_r(x, t) \tilde{p}_r(x, t) dx + q \alpha_2 \tilde{p}(0, t) \tilde{p}(0, t) \]

\[ + h \alpha_2 \| \tilde{Z}(t) \|^2 - \alpha_2 \int_0^t \| \tilde{Z}(s) \|^2 ds \]

(42)

Existence and continuity of \( \tilde{p}_r \) on the right side of (42) can be proved as follows: replacing \( x \) by \( 1 - x \) and using the regularity properties of all functions, it follows from Proposition 2.12 in Karafyllis and Krstic (2017b) that \( \tilde{p}_n \) exists and is continuous for all \( x \in [0, D] \) and all \( t > 0 \), except for the \( t_k \)'s. Now, the various terms on the right side of (42) are examined in order and, some of them, bounded from above. Using (39b), the first term is bounded as follows:

\[ \theta Z^T(t) (P(A - LC) + (A - LC)^T P) Z(t) \leq -\theta \mu \| Z(t) \|^2 \]

(43)
The second term on the right side of (42) is developed as follows, using Young’s inequality:

\[ 2\theta Z^T(t)PLC \int_{t_0}^{t} \dot{Z}(s)ds \leq \theta \xi \|Z(t)\|^2 + \frac{\theta}{\xi} \|PL\|^2 \int_{t_0}^{t} \dot{Z}(s)ds \]

\[ \leq \theta \xi \|Z(t)\|^2 + \frac{h\theta}{\xi} \|PL\|^2 \int_{t_0}^{t} \dot{Z}(s)ds \]

(44)

for any \( \xi > 0 \), where the last inequality is obtained using Jensen’s inequality. The third term on the right side of (42) is bounded from above as follows, using Young’s inequality:

\[ 2\theta Z^T(t)PLC \int_{0}^{t} \tilde{p}_k(x, t)dx \]

\[ \leq \theta \xi \|Z(t)\|^2 + \frac{\theta}{\xi} \|\tilde{p}_k\|^2 \int_{0}^{t} \tilde{p}_k^2(x, t)dx \]

(45)

for any \( \xi > 0 \). Owing to the fourth term on the right side of (42), this can be rewritten as follows:

\[ 2Z^T(t)P \Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \]

\[ = 2Z^T(t)P \Delta M^{-1}(D)\Delta^{-1} \Delta \chi(D, t) \Delta^{-1} \Delta M(D)\Delta^{-1}Z(t) \]

In view of (5c) and (23) that, for all \( \theta > 1 \), one has:

\[ \|\Delta \chi(D, t)\Delta^{-1}\| = \Delta \left( \int_{0}^{\infty} f_\chi(X(t) + s\tilde{x}(t), \nu(t)) \right) \Delta^{-1} \leq \beta_f, \]

(46)

using (6). Also, using (35) and (6), it readily follows from (11) that, for all \( x \in [0, D] \):

\[ \|\Delta M(x)\Delta^{-1}\| = \left\| (1 + qx)I + \sum_{l=1}^{\infty} a^{-l} (\theta A + clI)^l \right\| \frac{\chi^2l}{(2l)!} + \frac{\chi^{2l+1}}{(2l+1)!} \]

\[ \leq (1 + qD) + \sum_{l=1}^{\infty} a^{-l} \left\| (\theta A + cI)^l \right\| \frac{(2l)!}{2l!} + \frac{(2l+1)!}{2l!} \]

(47)

Assuming \( \theta D^2 < 1 \), one immediately gets from (47) that:

\[ \|\Delta M(x)\Delta^{-1}\| \leq \beta_M, \]

for all \( x \in [0, D] \)

(48)

with

\[ \beta_M = 1 + qD + \max(1, q) \left\| \left[ \exp(a^{-1}(A + clI)) - I \right] \right\| \]

\[ \times \sum_{l=1}^{\infty} a^{-l} \left\| (A + clI)^l \right\| \frac{l!}{l!} \]

is independent of \( \theta \). Using similar argument, it is shown that if \( \theta D^2 < 1 \) then:

\[ \|\Delta M^{-1}(x)\Delta^{-1}\| \leq \gamma_M, \]

for some real constant \( \gamma_M \) independent of \( \theta \).

Using (46), (48) and (49), one gets:

\[ 2Z^T(t)P \Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \leq 2\beta_f \beta_M \|P\| \|Z(t)\|^2 \]

(50)

The fifth term on the right side of (42) is bounded as follows:

\[ \alpha_0 \int_{0}^{t} \tilde{p}(x, t)(\alpha \tilde{p}_\alpha(x, t) - \tilde{c}\tilde{p}(x, t))dx \]

\[ \leq -\alpha_0 \alpha \tilde{p}_\alpha^2(0, t) - \alpha_0 \int_{0}^{t} \tilde{p}_\alpha^2(x, t)dx - \alpha_0 \int_{t_0}^{t} \tilde{p}_\alpha^2(x, t)dx \]

(51)

where we have used an integration by parts and the boundary conditions (28). The sixth term on the right side of (42) is first bounded as follows:

\[ \alpha_0 \int_{0}^{t} \tilde{p}(x, t)(CM^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t))dx \]

\[ \leq \frac{\alpha_0}{\sigma} \int_{0}^{t} \tilde{p}(x, t)dx \]

\[ + \alpha_0 \sigma \int_{0}^{t} (CM^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t))^2 dx \]

(52)

for any \( \sigma > 0 \). As \( C\Delta = C \) the squared term in the last integral, on the right side of (50), develops as follows:

\[ CM^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \]

\[ = C \Delta M^{-1}(D)\Delta^{-1} \Delta \chi(D, t) \Delta^{-1} \Delta M(D)\Delta^{-1}Z(t) \]

Then, it follows using (45), (47) and (48), one gets:

\[ \left\| CM^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \right\| \leq \beta_f \beta_M \gamma_M \|Z(t)\| \]

(53)

Using (53), it follows from (52) that, for \( t_k \leq t < t_{k+1} \), \( k \in \mathbb{N} \):

\[ \alpha_0 \int_{0}^{t} \tilde{p}(x, t)(CM^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t))dx \]

\[ \leq \frac{\alpha_0}{\sigma} \int_{0}^{t} \tilde{p}(x, t)dx + \alpha_0 \sigma \left( \beta_f \beta_M \gamma_M \right)^2 \|Z(t)\|^2 \]

(54)

The seventh and eighth terms on the right side of (42) are dealt with as follows, for \( t_k \leq t < t_{k+1} \), \( k \in \mathbb{N} \):

\[ \alpha_1 \int_{0}^{t} \tilde{p}(x, t)\tilde{p}_\alpha(x, t)dx + \alpha_1 \int_{t_0}^{t} \tilde{p}(x, t)\tilde{p}_\alpha(x, t)dx \]

\[ = \alpha_1 \int_{0}^{t} \tilde{p}(x, t)\tilde{p}_\alpha(x, t)dx - \alpha_1 \int_{0}^{t} \tilde{p}(x, t)\tilde{p}_\alpha(x, t)dx \]

\[ + \alpha_1 \int_{t_0}^{t} \tilde{p}(x, t)\tilde{p}_\alpha(x, t)dx \]

\[ = -\alpha_1 \int_{0}^{t} \tilde{p}(x, t)\left( \alpha \tilde{p}_\alpha(x, t) - \tilde{c}\tilde{p}(x, t) \right)dx \]

\[ + \alpha_1 \int_{t_0}^{t} \tilde{p}(x, t)\left( CM^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \right)dx \]

\[ \leq -\alpha_1 \int_{0}^{t} \tilde{p}_\alpha^2(x, t)dx - \alpha_1 \int_{t_0}^{t} \tilde{p}_\alpha^2(x, t)dx \]

\[ + \frac{\alpha_1}{\gamma} \int_{t_0}^{t} \tilde{p}_\alpha^2(x, t)dx \]

(55)

for any real \( \eta > 0 \), where the penultimate equality is obtained using (28), while the last inequality is obtained using an integration by part, applying Young’s inequality and using (53). Using (33) and (40), the penultimate term on the right side of (42) is bounded from above as follows:

\[ h\alpha_2 \|Z(t)\|^2 = h\alpha_2 \left\| \theta AZ(t) - \theta LCZ(t) - \theta \tilde{c}\tilde{p}(0, t) \right\|^2 \]

\[ + \Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \]

\[ = h\alpha_2 \left[ \theta AZ(t) - \theta LCZ(t) \right] \int_{t_0}^{t} \tilde{Z}(s)ds + \theta L \int_{t_0}^{t} \tilde{p}(x, t)dx \]

\[ + \Delta M^{-1}(D)\Delta^{-1} \Delta \chi(D, t) \Delta^{-1} \Delta M(D)\Delta^{-1}Z(t) \]

\[ \leq 4h\alpha_2 \|A - LC\|^2 \|Z(t)\|^2 + 4\alpha_2 \|L\|^2 \|\tilde{c}\|^2 \|LC\|^2 \int_{t_0}^{t} \|\tilde{Z}(s)\|^2ds \]

\[ + 4h\alpha_2 \|L\|^2 \|\tilde{c}\|^2 \int_{t_0}^{t} \|\tilde{p}(x, t)\|^2 \]

\[ + 4h\alpha_2 \left( \beta_f \beta_M \gamma_M \right)^2 \|Z(t)\|^2 , \text{ for } t_k \leq t < t_{k+1}, \ k \in \mathbb{N} \]

(56)
Then, we let the maximum sampling period \( h \) so that the following inequalities hold:

\[
\begin{align*}
4\alpha_0\sigma D \left( \beta_1^2 \| \beta_M \| M \right)^2 &< \frac{\theta \mu - 2\beta_1^2 \| P \|}{6} \\
4\alpha_0\sigma D \theta^2 \| A - LC \|^2 &< \frac{\theta \mu - 2\beta_1^2 \| P \|}{6}
\end{align*}
\]

Finally, we let the domain length \( D \) small enough so that:

\[
\frac{\alpha_1 \eta D}{2} \left( \beta_1^2 \| \beta_M \| M \right)^2 < \frac{\theta \mu - 2\beta_1^2 \| P \|}{6}
\]

The still free parameters \( a_0 \) and \( \eta \) are let to be large enough so that the following additional couple of conditions hold:

\[
a_0 = \frac{\theta D \| P \|}{\varsigma} > 0, \quad a = \frac{1}{2\eta} > 0
\]

Then, applying Wirtinger's inequality (1), one has:

\[
\begin{align*}
\dot{V} &\leq -\rho_1 \| Z(t) \|^2 - \rho_2 \int_0^t \int_0^s \| \dot{Z}(s) \|^2 ds ds
\end{align*}
\]

with

\[
\begin{align*}
\rho_1 &= \theta \mu - \theta \varsigma \theta \varsigma - 2\beta_1 \beta_0 \| P \| - 4\alpha_0 \sigma D \left( \beta_1^2 \| \beta_M \| M \right)^2 \\
&= \alpha_0 \sigma D \left( \beta_1^2 \| \beta_M \| M \right)^2 - 4\alpha_0 \sigma D \theta^2 \| A - LC \|^2
\end{align*}
\]

We need the following condition to be satisfied:

\[
\rho_1 > 0
\]

First, let the parameter \( \theta \) be selected such that:

\[
\theta < \frac{2\beta_1 \beta_0}{\| P \|}
\]

Then, we let the maximum sampling period \( h \), the domain length \( D \), and the free positive real parameters \( \{ \sigma, \varsigma, \zeta \} \) be small enough so that (59) holds. A possible choice is to let \( \{ \sigma, \varsigma, \zeta \} \) be small enough so that:

\[
\begin{align*}
\sigma &< \frac{\theta \varsigma - 2\beta_1 \beta_0}{6} \\
\varsigma &< \frac{\theta \zeta - 2\beta_1 \beta_0}{\| P \|}
\end{align*}
\]

Since \( \rho_1 \) to \( \rho_6 \) are positive, it follows from (70) that:

\[
\begin{align*}
\dot{V} &\leq -\rho_1 \| Z(t) \|^2 - \rho_2 \int_0^t \int_0^s \| \dot{Z}(s) \|^2 ds ds
\end{align*}
\]

with

\[
\begin{align*}
\rho_1 &= \frac{\pi^2}{4D^2} \left( \alpha_0 - \frac{\theta D \| P \|}{\varsigma} \right) + \alpha_0 \left( c - \frac{1}{\sigma} \right)
\end{align*}
\]

By letting \( D \) be small enough, the following properties hold:

\[
\rho_i > 0, \quad (i = 2 \ldots 6)
\]

Using (77), we get

\[
\begin{align*}
\dot{V} &\leq -\rho_1 \| Z(t) \|^2 - \rho_2 \int_0^t \int_0^s \| \dot{Z}(s) \|^2 ds ds
\end{align*}
\]

with

\[
\begin{align*}
\rho_1 &= \frac{\pi^2}{4D^2} \left( \alpha_0 - \frac{\theta D \| P \|}{\varsigma} \right) + \alpha_0 \left( c - \frac{1}{\sigma} \right)
\end{align*}
\]

(60)

(61)

(62)

(63)

(64)

(65)

(66)

(67)

(68)

(69)

(70)

(71)

(72)

(73)

(74)

(75)

(76)

(77)

(78)
Now, if
\[
\frac{2\rho_6}{\alpha_1} < \rho
\]  
(79)
then it follows applying Halanay’s inequality to (77) (see e.g. Fridman & Blizhkovsky, 2010, Lemma 3):
\[
V(Z, \bar{p}, t) \leq e^{-2\gamma(t-t_0)} \sup_{-\delta_5 \leq \tau \leq 0} V(Z, \bar{p}, t_0 + \tau); \ t \geq t_0
\]  
(80)
where \( \gamma \) is the unique solution of the algebraic equation:
\[
2\gamma = \frac{-2\rho_6}{\alpha_1} e^{2\gamma y}
\]  
(81)
Note that the condition \( \frac{2\rho_6}{\alpha_1} < \rho \) holds if \( D \) is small enough, because \( \rho_6 \to 0 \) as \( D \to 0 \), due to (75). It readily follows from (80) that both \( \|Z(t)\| \) and \( \int_0^D \tilde{g}^2(x, t) \) converge to zero. In view of (31) and (29), so do \( \|\bar{X}(t)\| \) and \( \int_0^D \tilde{g}^2(x, t) \) dx. Finally, recall that the domain length \( D \) is required (in the above analysis) to be small only to make sure that the inequalities (66) and (79) do hold. It is readily seen that, in the case of a zero Lipschitz coefficient \( \beta_j \), those inequalities hold irrespective of the value \( D \). This proves
\[ \text{Theorem 1}. \]

**Remark 3.** (1) From (60) it follows that the minimal observer gain \( \sigma^* \) in Theorem 1 is such that \( \sigma^* > 2\beta_1 \beta_M^2 \|P\| / \mu \). The larger the nonlinearity coefficient \( \beta_j \), the larger \( \sigma^* \).
(2) Inequalities (61)–(67), (69), (76), and (79) are sufficient conditions, on the maximum sampling interval \( h \) and the domain length \( D \). It is readily checked that those conditions define nonempty sets of admissible values. Let us check it for \( h \). To meet (61)–(63) and (the second part of) (67), let the free scalars \( (\sigma, \delta, \gamma) \) be set as follows:
\[
\delta = \frac{\theta \mu - 2\beta_1 \beta_M^2 \|P\|}{12\sigma}, \quad \gamma = \frac{\theta \mu - 2\beta_1 \beta_M^2 \|P\|}{12\sigma}
\]  
(82)
\[
\eta = \frac{1}{\sigma}
\]  
(83)
To meet (64) and (65), the sampling interval is set so that:
\[
h < h_0
\]  
(84)
with
\[
h_0 = \min \{h_1, h_2, h_3\}
\]  
(85)
\[
h_1 = \frac{\theta \mu - 2\beta_1 \beta_M^2 \|P\|}{24\alpha_2 (\beta_1 \beta_M^2)}, \quad h_2 = \frac{\theta \mu - 2\beta_1 \beta_M^2 \|P\|}{24\alpha_2 \|x - \bar{x}\|}
\]  
(86)
and \( h_3 \) the unique value for which the left side of (69) is zero i.e.
\[
h_3 = \frac{\alpha^2}{\sigma^2} \left( \frac{12 \|\bar{x} - \bar{X}\|^2}{\theta \mu - 2\beta_1 \beta_M^2 \|P\|} + 4\alpha_2 \|\bar{x} - \bar{X}\|^2 \right)^{-1}
\]  
(87)
where (83) has been used to get (87).
(3) In addition to the results of Theorem 1, it is noticed that all conditions on the sampling interval, namely inequalities (64), (65) and (69), are useless if \( h \to 0 \) i.e. in the continuous-measurement case. In such a case, only the condition on the domain length \( D \) stands. If further the Lipschitz constant is zero then, by Part 2 of Theorem 1, it follows that also the condition on \( D \) disappears, retrieving thus a similar result as in (Krstic, 2009).
(4) In practical applications, the domain length \( D \) might not be small enough to meet the related smallness requirement of Theorem 1. Then, a practical solution is to implement several sensors (say \( N \)) providing the measurements of \( u(\bar{X}, t_0)/(i = 0 \cdots N - 1) \), with \( N \) selected large enough so that the ratio \( D/N \) meets the requirement on \( D \) in Theorem 1. Doing so, the observer (18)–(22) provides the estimates of the state \( X(t) \) and the distributed state \( \bar{u}(x, t) \), for \( \frac{i}{N} D \leq x \leq D \), based only on the measurements of \( \bar{u}(t) \) and \( u(\bar{X}, t_0) \). Then, the state \( \bar{u}(t) \) of the rest of the subdomains \( iD/N \), \( i = 0 \cdots N - 2 \) can be simultaneously estimated using the measurements \( u(\bar{X}, t_0) \) for \( i = 0 \cdots N - 2 \) and existing observers for parabolic PDEs like (3), see e.g. Fridman and Blizhkovsky (2012) and Schaum, Alvarez, Meurer, and Moreno (2016).

### 4. Observer design extension

In this section, we extend the observer design method of Section 3 to the following wider class of systems:
\[
\dot{X}(t) = AX(t) + f(X(t), v(t)) + g_3(v(t)), \quad t \geq 0
\]  
(88)
\[
u_i(x, t) = a u_{xx}(x, t) + bu_i(x, t) - cw(x, t) + g_2(x, v(t)), \quad \text{for } (x, t) \in (0, D) \times (0, +\infty) \text{ a.e.}
\]  
(89)
\[
w_i(0, t) = qu(0, t) + p_0(v(t)), \quad \text{for } t \geq 0
\]  
(90)
\[
u(D, t) = CX(t) + p_1(v(t)), \quad \text{for } t \geq 0
\]  
(91)
for some functions \( g_3 \) and \( g_2 \) of class \( C^1(\mathbb{R}^2; \mathbb{R}) \) and \( C^1([0, D] \times \mathbb{R}^2; \mathbb{R}) \), respectively, and some functions \( p_0, p_1 \) of class \( C^2(\mathbb{R}^2; \mathbb{R}) \); the exogenous signal \( v(t) \in \mathbb{R}^d \) (for some integer \( m \geq 1 \)) is accessible to measurements and of class \( C^1(R_+; \mathbb{R}) \). All other quantities remain unchanged with respect to (2)–(4). Compared to the initial class of systems defined by (2)–(4), the new PDE equation (47) includes the convection term \( bu_i(x, t) \) (with \( b \in \mathbb{R} \)).

We seek an exponentially convergent observer for the new system (88)–(91). To this end, introduce the state transformation,
\[
u(x, t) = e^{(x-d)/2} u(x, t), \quad \text{for } (x, t) \in [0, D] \times [0, +\infty)
\]  
(92)
with
\[
r = \frac{b}{2a}
\]  
(93)
Differentiating \( u(x, t) \), one gets using (40):
\[
u_1(x, t) = a e^{(x-d)/2} u_{xx}(x, t) + b e^{(x-d)/2} u_x(x, t)
\]  
(94)
\[
u_3(x, t) = r e^{(x-d)/2} u(X(t), v(t)) + e^{(x-d)/2} u_{xx}(x, t)
\]  
(95)
\[
u_{xx}(x, t) = r^2 e^{(x-d)/2} u(x, t) + 2r e^{(x-d)/2} u_x(x, t)
\]  
(96)
\[
u(x, t) = (x + b) u + e^{(x-d)/2} g_2(x, v(t))
\]  
(97)
\[
u(x, t) = (x + b) u + e^{(x-d)/2} g_2(x, v(t))
\]  
(98)
\[
u_x(0, t) = (q + \frac{b}{2a}) u(0, t) + e^{-t\rho_0} p_0(v(t)), \quad \text{for } t \geq 0
\]  
(99)
\[
u(D, t) = CX(t) + p_1(v(t)), \quad \text{for } t \geq 0
\]  
(100)
\[
u_i(x, t) = a u_{xx}(x, t) - (c + \frac{b^2}{4a}) u + e^{(x-d)/2} g_2(x, v(t))
\]  
(101)
Similarly, the following boundary conditions are readily obtained from (92) and (95), using (90) and (91):
\[
u(D, t) = u(D, t) = CX(t) + p_1(v(t)), \quad \text{for } t \geq 0
\]  
(98)
\[
u_x(0, t) = (q + \frac{b}{2a}) u(0, t) + e^{-t\rho_0} p_0(v(t)), \quad \text{for } t \geq 0
\]  
(99)
The transformed system modelled by Eqs. (88) and (97)–(99), is rewritten here for convenience:
\[
X(t) = AX(t) + f(X(t), v(t)) + g_3(v(t)), \quad t \geq 0
\]  
(100)
for \((x, t) \in (0, D) \times (0, +\infty)\) a.e.

\[
\hat{u}(D, t) = CX(t) + p_t(v(t)), \text{ for all } t \geq 0
\]  

\[
u_0(x, t) = (q + \frac{b}{2a})u(x, 0) + e^{-\theta D}p_0(v(t)), \text{ for } t \geq 0
\]  

\[
w(x, t) = e^{-r(x-D)}u(x, t)
\]

where the last equation is obtained from (92) to make the link between the new system model (100)-(103) and its initial model (88)-(91). Except for the terms involving the external signal \(v(t)\), Eqs. (100)-(103) fit the model structure (2)-(4). In the latter the parameter \(c\) (resp. \(q\)) is replaced by \(c + \frac{b^2}{4a}\) (resp. \(q + \frac{b}{2a}\)). Accordingly, it is supposed that:

\[
a > 0, \quad q + \frac{b}{2a} \geq 0
\]

(105)

The system (100)-(104) is observed through the sampled measurements \(y(t) = u(0, t_k)\) for \(t \in [t_k, t_k+1)\) and \(k = 0, 1, 2, \ldots\) (this amounts to assuming \(u(0, t_k) = e^t u(0, t_k)\) accessible to measurements). Again, consider the transformation (8), with \(X(x) \in \mathbb{R}^{\alpha \times \alpha}\) defined by the ODE (9) and initial conditions (10) replacing there \(c\) (resp. \(q\)) by \(c + \frac{b^2}{4a}\) (resp. \(q + \frac{b}{2a}\)). It turns out that \(M(x)\) is defined by:

\[
d^2M \frac{dx^2}{dx^2} = a^{-1} M(x) \left( A + (c + \frac{b^2}{4a})I \right)
\]

(106)

\[
M(0) = I, \quad \frac{dM}{dx}(0) = (q + \frac{b}{2a})I
\]

(107)

Then, the system (100)-(103) rewrites as follows in terms of the \((X, p)\)-coordinates:

\[
\dot{X}(t) = AX(t) + f(X(t), v(t)) + g_1(v(t)), \text{ for } t \geq 0
\]

(108)

\[
p_l(x, t) = ap_0(x, t) - (c + \frac{b^2}{4a})p_l(x, t) + e^{\epsilon(x-D)}g_2(x, v(t)) \quad CM^{-1}(D)M(x) f(X(t), v(t)) + g_1(v(t))
\]

(109)

\[
p_l(0, t) = (q + \frac{b}{2a})p_l(0, t) + e^{-\theta D}p_0(v(t)), \text{ for } t \geq 0
\]

(110)

\[
p(D, t) = p_t(v(t)), \text{ for all } t \geq 0
\]

(111)

\[
u(x, t) = p(x, t) + CM^{-1}(D)M(x)X(t)
\]

(112)

\[
w(x, t) = e^{-r(x-D)}u(x, t)
\]

(113)

Then, inspired by the observer (65)-(70), we propose the following observer for the system (108)-(113):

\[
\hat{X}(t) = AX(t) + f(X(t), v(t)) + g_1(v(t)) - L(\dot{y}(t_k) - y(t_k)), \text{ for } t \in [t_k, t_{k+1}) , \quad k \geq 0
\]

(114)

\[
\hat{p}_l(x, t) = ap_0(x, t) - (c + \frac{b^2}{4a})\hat{p}_l(x, t) + e^{\epsilon(x-D)}g_2(x, v(t)) - CM^{-1}(D)M(x) f(X(t), v(t)) + g_1(v(t))
\]

(115)

\[
\hat{p}_0(0) - \left( q + \frac{b}{2a} \right) \hat{p}_0(0) - \exp(-r)\hat{p}_0(v(0)) = 0
\]

\[
\hat{p}_0(1) - p_1(v(0)) = 0
\]

and for every locally bounded function \(y: \mathbb{R}^+ \rightarrow \mathbb{R}\), there exist unique mappings \(\hat{X} \in C^0(\mathbb{R}_+) \cap C^1(I)\), where \(l = \left[ \frac{b}{2a} \right]_0^+ \) \(\hat{p}_l \in C^0(\mathbb{R}_+ × [0, 1]) \cap C^1(I × [0, 1])\) with \(\hat{p}[I] \in C^2([0, 1])\), for all \(t \geq 0, \hat{X}(0) = \hat{X}_0, \hat{p}(0) = \hat{p}_0\) and \(\hat{X}_0 \in C^0(\mathbb{R}_+) \cap C^1(I)\) being right-differentiable on \(\mathbb{R}_+\), so that (114) holds for all \(t \in [t_k, t_{k+1})\) and \(k = 0, 1, 2, \ldots\) (115) holds for all \((x, t) \in [0, 1] \times I\) and (116), (117) hold for \(t \geq 0\). A similar result can be stated for system (18)-(20).
5. Simulation

To illustrate the observer design, we consider the ODE-PDE cascade (2)--(4) with
\begin{align}
n &= 2, \quad a = D = 1, \quad c = -1, \quad q = 0 \tag{121} \\
f(X, v) &= \begin{bmatrix} 0 & -g(X, v) \end{bmatrix}^T \tag{122}
\end{align}
where \( g(X, v) = v + 5X_2 + X_1 + (1 + X_1^2) \). The ODE part (2) represents a mass–spring system with viscous force and nonlinear restoring force. The mass–spring system is subject to the effect of an external force. For this case, the matrix \( M(x) \) defined by (9), (10) is given by
\begin{align}
M(x) &= \begin{bmatrix} \cos(x) & x \sin(x)/2 \\ 0 & \cos(x) \end{bmatrix}, \quad \text{for} \ x \in [0, 1] \tag{123}
\end{align}
Using \( L = [2 \ 1]^T \), \( \theta = 2 \), the observer (18), (19), (20), (21), (22), (23) is expressed by the equations
\begin{align}
\frac{d}{dt} \hat{X}_1 &= \hat{X}_2 - 2[2 \cos(1) + \sin(1)](\hat{u}(0, t_k) - y(t_k)) \\
\frac{d}{dt} \hat{X}_2 &= -g(\hat{X}, v) - 4 \cos(1)(\hat{u}(0, t_k) - y(t_k)) \\
\hat{u}(0, t) &= \hat{p}(0, t) + \frac{1}{\cos(1)} \hat{X}_1(t) - \frac{\sin(2)}{2 \cos^2(1)} \hat{X}_2(t) \\
\hat{p}_1(x, t) &= \hat{p}_2(x, t) + \hat{p}(x, t) + \kappa(x)g(\hat{X}(t), v(t)) \\
\hat{p}_2(0, t) &= \hat{p}(1, t) = 0 \tag{125}
\end{align}
where
\begin{align}
\kappa(x) &= \frac{\cos(1)x \sin(x) - \cos(x) \sin(1)}{2 \cos^2(1)}, \quad \text{for} \ x \in [0, 1] \tag{126}
\end{align}
In order to simulate numerically system (2), (14), (15), (16), (17), (124), (125) we used an eigenfunction expansion of the solution of (125) up to 30 terms, i.e., we simulated the ODE system for \( n = 0, 1, \ldots, 30 \)
\begin{align}
\frac{d \tilde{a}_n}{dt}(t) &= -\left(\frac{\pi}{2} + \frac{n \pi}{2}\right)^2 - 1 \tilde{a}_n(t) \\
&= 8(-1)^n (2n + 1) \pi \sqrt{2} \tag{127}
\end{align}
and approximated the solution of (125) by the formula
\[ \hat{p}(x, t) = \sqrt{2} \sum_{n=0}^{30} \tilde{a}_n(t) \cos \left(\frac{n \pi}{2} + \frac{\pi}{2}\right) \]
A similar approximation was applied for the solution of (14), (15), (16), (17).
We also have considered the presence of noise, i.e., we set
\begin{align}
y(t) &= u(0, t_k) + \xi(t) \tag{128}
\end{align}
where \( \xi(t) \) is the noise. We have simulated the system described by (2), (14)--(17) and (121)--(122), on the one hand, and the observer represented by (124)--(125), on the other, with initial conditions \( X_1(0) = X_2(0) = 0 \), \( X_1(0) = X_2(0) = 1 \), \( \hat{p}(x, 0) = \sqrt{2} \sum_{n=0}^{\infty} \cos \left(\frac{n \pi}{2} + \frac{\pi}{2}\right) x \), \( \hat{p}(x, 0) = 0 \), input \( v(t) = \cos(t) \). The simulation is performed considering noise-free output and noisy output. In the last case, the sequence \( \xi(t) \) is a zero-mean Gaussian white noise with variance 0.03. We have used a uniform sampling schedule \( t_k = kh \), for all integers \( k \geq 0 \), with two values of the sampling interval: \( h = 0.1 \) and \( h = 0.2 \). The obtained results are illustrated by Figs. 1 to 5. First, Fig. 1 shows the effect of the injected noise on the output. Figs. 2 and 3 illustrate the observer performances in the free-noise case (\( \xi = 0 \)), while Figs. 4 and 5 show the performances in the noisy case. Clearly, the observer errors converge to zero in the noise-free case (Figs. 2 and 3), with greater convergence rate when the smaller sampling period \( h = 0.1 \) is used. Furthermore, it is readily seen in Figs. 4 and 5 that the noise has only a weak effect on the observer convergence quality. This can be explained by the fact the observer gain \( \theta = 2 \) is presently not too large. Finally, we have checked that the good convergence quality is preserved for sampling periods up to \( h = 0.35 \).
6. Conclusion

We have developed a sampled boundary observer design method that applies to the class of ODE-PDE cascades modelled by (2)–(4) and (88)–(91). The main characteristics of the new design method are: (i) the coordinate transformations (8) and (31); (ii) the matrix-valued function M(x) (defined by (10)–(11)) and the high-gain matrix Δ used in the observer gains; (iii) the Lyapunov–Krasovskii type functional (38). Theorem 1 shows that exponential stability of the observation error system (26)–(29) entails conditions on the maximum (time) sampling interval and the PDE domain length. Accordingly, the maximum domain length is a decreasing function of the Lipschitz constant βs and tends to infinity when βs → ∞, retrieving thus the linear case result of Ahmed-Ali and Karafyllis et al. (2016, 2017) and Karafyllis et al. (2017). To the authors’ knowledge, it is the first time that a sampled boundary observer design is developed for ODE-PDE cascades that involve nonlinear ODEs. The present work can be pursued in many senses including the extension to more general classes of ODEs, involving non-globally Lipschitz nonlinearities.

References


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