Systems & Control Letters 19 (1992) 177–185
North-Holland

Adaptive nonlinear control without overparametrization *

M. Krstić, I. Kanellakopoulos and P.V. Kokotović

Department of Electrical and Computer Engineering, University of California, Santa Barbara CA, USA

Received 24 October 1991
Revised 3 April 1992

Abstract: A new design procedure for adaptive nonlinear control is proposed in which the number of parameter estimates is minimal, that is, equal to the number of unknown parameters. The adaptive systems designed by this procedure possess stronger stability properties than those using overparametrization.

Keywords: Adaptive control; nonlinear systems; overparametrization; tuning functions; backstepping design.

1. Introduction

In this paper we present a new design procedure for adaptive control of nonlinear systems transformable into the *parametric-strict-feedback* form

$$\dot{x}_i = x_{i+1} + \theta^{\mathrm{T}}\phi_i(x_1,\dots,x_i), \quad 1 \leqslant i \leqslant n-1, \tag{1.1a}$$

$$\dot{x}_n = \phi_0(x) + \theta^{\mathrm{T}}\phi_n(x) + \beta_0(x)u, \tag{1.1b}$$

where $\theta \in \mathbb{R}^p$ is the vector of unknown constant parameters, ϕ_0, β_0, and the components of ϕ_i, $1 \leqslant i \leqslant n$, are smooth nonlinear functions in $\mathbb{R}^n$, and $\beta_0(x) \neq 0$, for all $x \in \mathbb{R}^n$.

The global adaptive regulation and tracking problems for systems in this form have recently been solved in [2,4]. This was achieved using a systematic design procedure and without any growth restrictions on nonlinearities. However, the procedure of [2,4] has not removed the need for overparametrization, a significant drawback of earlier adaptive nonlinear schemes [6]. As many as np estimates of p unknown parameters had to be continuously updated. Recently, this number was reduced in half [1]. Still, the dynamic order of the resulting adaptive controller is quite high, and is even higher in the case of output-feedback designs [5,3].

The new design procedure eliminates overparametrization while retaining all the advantages of the procedure in [2,4]. It employs exactly p estimates for p unknown parameters and significantly reduces the controller's dynamic order. This enhances the stability properties of the adaptive system and improves parameter convergence.

For clarity, we present the new procedure for the regulation problem. Its extension to the tracking problem is the same as in [2].

Our control objective is to regulate x_1 to $x_1^{\mathrm{e}} = 0$ and to stabilize the corresponding equilibrium x^{e}:

$$x_1^{\mathrm{e}} = 0, \qquad x_{i+1}^{\mathrm{e}} = -\theta^{\mathrm{T}}\phi_i^{\mathrm{e}} := -\theta^{\mathrm{T}}\phi_i\left(0, -\theta^{\mathrm{T}}\phi_1^{\mathrm{e}},\dots,-\theta^{\mathrm{T}}\phi_{i-1}^{\mathrm{e}}\right), \quad i = 1,\dots,n-1. \tag{1.2}$$

Correspondence to: Prof. P. Kokotović, Dept. of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106, USA.

* This work was supported in part by the National Science Foundation under Grant ECS-87-15811 and in part by the Air Force Office of Scientific Research under Grant AFOSR 91-147.

Note that in the special case when $\phi_1(0) = \cdots = \phi_{n-1}(0) = 0$, the equilibrium is $x^e = 0$ for all values of θ. However, as we shall see later, a nonzero equilibrium $x^e \neq 0$ improves parameter convergence and stability properties.

2. Backstepping design with tuning functions

The design procedure is recursive. At its i-th step and i-th-order subsystem is stabilized with respect to a Lyapunov function V_i by the design of a *stabilizing function* α_i and a *tuning function* τ_i. The update law for the parameter estimate $\hat{\theta}(t)$ and the feedback control u are designed at the final step.

Step 1: Introducing $z_1 = x_1$ and $z_2 = x_2 - \alpha_1$, we rewrite $\dot{x}_1 = x_2 + \theta^T\phi_1(x_1)$ as

$$\dot{z}_1 = z_2 + \alpha_1 + \theta^T\phi_1(x_1) \tag{2.1}$$

and use α_1 as a control to stabilize (2.1) with respect to the Lyapunov function $V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}(\hat{\theta} - \theta)^T\Gamma^{-1}(\hat{\theta} - \theta)$. Then

$$\dot{V}_1 = z_1\left(z_2 + \alpha_1 + \hat{\theta}^T\phi_1\right) + (\hat{\theta} - \theta)^T\Gamma^{-1}\left(\dot{\hat{\theta}} - \Gamma z_1\phi_1\right). \tag{2.2}$$

If x_2 were our actual control, we would let $z_2 \equiv 0$, that is, $x_2 \equiv \alpha_1$. Then, we would eliminate $\hat{\theta} - \theta$ from $\dot{V}_1$ with the update law $\dot{\hat{\theta}} = \tau_1$, where

$$\tau_1(x_1) = \Gamma z_1\phi_1(x_1). \tag{2.3}$$

To make $\dot{V}_1 = -c_1z_1^2$, we would choose

$$\alpha_1\left(x_1, \hat{\theta}\right) = -c_1z_1 - \hat{\theta}^T\phi_1(x_1). \tag{2.4}$$

Since x_2 is not our control, we have $x_2 \not\equiv 0$, and we do *not* use $\dot{\hat{\theta}} = \tau_1$ as an update law. However, we retain τ_1 as our first *tuning function* and α_1 as our first *stabilizing function*. We thus postpone the decision about $\dot{\hat{\theta}}$ and tolerate the presence of $\hat{\theta} - \theta$ in $\dot{V}_1$:

$$\dot{V}_1 = -c_1z_1^2 + z_1z_2 + (\hat{\theta} - \theta)^T\Gamma^{-1}\left(\dot{\hat{\theta}} - \tau_1\right). \tag{2.5}$$

The second term z_1z_2 in $\dot{V}_1$ will be cancelled at the next step. The closed-loop form of (2.1) with (2.4) is

$$\dot{z}_1 = -c_1z_1 + z_2 + (\theta - \hat{\theta})^T\phi_1(x_1). \tag{2.6}$$

Step 2: Introducing $z_3 = x_3 - \alpha_2$, we rewrite $\dot{x}_2 = x_3 + \theta^T\phi_2(x_1, x_2)$ as

$$\dot{z}_2 = z_3 + \alpha_2 + \theta^T\phi_2 - \frac{\partial\alpha_1}{\partial x_1}\left(x_2 + \theta^T\phi_1\right) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}, \tag{2.7}$$

and use α_2 as a control to stabilize the (z_1, z_2)-system (2.6)–(2.7) with respect to $V_2 = V_1 + \frac{1}{2}z_2^2$. Then

$$\begin{aligned}\dot{V}_2 = {} & -c_1z_1^2 + z_2\left[z_1 + z_3 + \alpha_2 - \frac{\partial\alpha_1}{\partial x_1}x_2 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} + \hat{\theta}^T\left(\phi_2 - \frac{\partial\alpha_1}{\partial x_1}\phi_1\right)\right] \\ & + (\hat{\theta} - \theta)^T\Gamma^{-1}\left[\dot{\hat{\theta}} - \Gamma\left(z_1\phi_1 + z_2\left(\phi_2 - \frac{\partial\alpha_1}{\partial x_1}\phi_1\right)\right)\right].\end{aligned} \tag{2.8}$$

If x_3 were our actual control, we would let $z_3 \equiv 0$ and eliminate $\hat{\theta} - \theta$ from $\dot{V}_2$ with the update law $\dot{\hat{\theta}} = \tau_2$, where

$$\tau_2(x_1, x_2, \hat{\theta}) = \Gamma\left[z_1\phi_1 + z_2\left(\phi_2 - \frac{\partial\alpha_1}{\partial x_1}\phi_1\right)\right] = \tau_1 + \Gamma z_2\left(\phi_2 - \frac{\partial\alpha_1}{\partial x_1}\phi_1\right). \tag{2.9}$$

Then, to make $\dot{V}_2 = -c_1z_1^2 - c_2z_2^2$, we would design α_2 such that the bracketed term multiplying z_2 equals $-c_2z_2$, namely

$$\alpha_2(x_1, x_2, \hat{\theta}) = -z_1 - c_2z_2 + \frac{\partial\alpha_1}{\partial x_1}x_2 + \frac{\partial\alpha_1}{\partial\hat{\theta}}\tau_2 - \hat{\theta}^{\mathrm{T}}\left(\phi_2 - \frac{\partial\alpha_1}{\partial x_1}\phi_1\right). \tag{2.10}$$

It is important to note that in this expression τ_2 replaces $\dot{\hat{\theta}}$. Since x_3 is not our control, we have $z_3 \not\equiv 0$ and we do not use $\dot{\hat{\theta}} = \tau_2$ as an update law. However, we retain τ_2 as our second tuning function and α_2 as our second stabilizing function. The resulting $\dot{V}_2$ is

$$\dot{V}_2 = -c_1z_1^2 - c_2z_2^2 + z_2z_3 + \left[z_2\frac{\partial\alpha_1}{\partial\hat{\theta}} + (\theta - \hat{\theta})^{\mathrm{T}}\Gamma^{-1}\right]\left(\tau_2 - \dot{\hat{\theta}}\right). \tag{2.11}$$

The first two terms in $\dot{V}_2$ are negative definite, the third term will be cancelled at the next step, and the last term is tolerated at this step, as the decision about $\dot{\hat{\theta}}$ is again postponed. The closed-loop form of (2.7) with (2.10) is

$$\dot{z}_2 = -z_1 - c_2z_2 + z_3 + (\theta - \hat{\theta})^{\mathrm{T}}\left(\phi_2 - \frac{\partial\alpha_1}{\partial x_1}\phi_1\right) + \frac{\partial\alpha_1}{\partial\hat{\theta}}\left(\tau_2 - \dot{\hat{\theta}}\right). \tag{2.12}$$

Step 3: Introducing $z_4 = x_4 - \alpha_3$, we rewrite $\dot{x}_3 = x_4 + \theta^{\mathrm{T}}\phi_3(x_1, x_2, x_3)$ as

$$\dot{z}_3 = z_4 + \alpha_3 + \theta^{\mathrm{T}}\phi_3 - \frac{\partial\alpha_2}{\partial x_1}\left(x_2 + \theta^{\mathrm{T}}\phi_1\right) - \frac{\partial\alpha_2}{\partial x_2}\left(x_3 + \theta^{\mathrm{T}}\phi_2\right) - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}}, \tag{2.13}$$

and use α_3 as a control to stabilize the (z_1, z_2, z_3)-system with respect to $V_3 = V_2 + \frac{1}{2}z_3^2$. Then

$$\begin{aligned}\dot{V}_3 = &-c_1z_1^2 - c_2z_2^2 + z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}\left(\tau_2 - \dot{\hat{\theta}}\right)\\ &+ z_3\left[z_2 + z_4 + \alpha_3 - \frac{\partial\alpha_2}{\partial x_1}x_2 - \frac{\partial\alpha_2}{\partial x_2}x_3 - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}} + \hat{\theta}^{\mathrm{T}}\left(\phi_3 - \frac{\partial\alpha_2}{\partial x_1}\phi_1 - \frac{\partial\alpha_2}{\partial x_2}\phi_2\right)\right]\\ &+ (\hat{\theta} - \theta)^{\mathrm{T}}\Gamma^{-1}\left[\dot{\hat{\theta}} - \Gamma\left(z_1\phi_1 + z_2\left(\phi_2 - \frac{\partial\alpha_1}{\partial x_1}\phi_1\right) + z_3\left(\phi_3 - \frac{\partial\alpha_2}{\partial x_1}\phi_1 - \frac{\partial\alpha_2}{\partial x_2}\phi_2\right)\right)\right].\end{aligned} \tag{2.14}$$

If x_3 were our actual control, we would let $z_4 \equiv 0$ and eliminate $\hat{\theta} - \theta$ from $\dot{V}_3$ with the update law $\dot{\hat{\theta}} = \tau_3$, where

$$\begin{aligned}\tau_3(x_1, x_2, x_3, \hat{\theta}) &= \Gamma\left[z_1\phi_1 + z_2\left(\phi_2 - \frac{\partial\alpha_1}{\partial x_1}\phi_1\right) + z_3\left(\phi_3 - \frac{\partial\alpha_2}{\partial x_1}\phi_1 - \frac{\partial\alpha_2}{\partial x_2}\phi_2\right)\right]\\ &= \tau_2 + \Gamma z_3\left(\phi_3 - \frac{\partial\alpha_2}{\partial x_1}\phi_1 - \frac{\partial\alpha_2}{\partial x_2}\phi_2\right).\end{aligned} \tag{2.15}$$

Noting that

$$\dot{\hat{\theta}} - \tau_2 = \dot{\hat{\theta}} - \tau_3 + \tau_3 - \tau_2 = \dot{\hat{\theta}} - \tau_3 + \Gamma z_3\left(\phi_3 - \frac{\partial\alpha_2}{\partial x_1}\phi_1 - \frac{\partial\alpha_2}{\partial x_2}\phi_2\right), \tag{2.16}$$

we rewrite $\dot{V}_3$ as

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} \left(\tau_3 - \dot{\hat{\theta}} \right)$$
$$+ z_3 \left[z_2 + z_4 + \alpha_3 - \frac{\partial \alpha_2}{\partial x_1} x_2 - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} + \left(\hat{\theta}^{\mathrm{T}} - z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \right) \left(\phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right) \right]$$
$$+ (\hat{\theta} - \theta)^{\mathrm{T}} \Gamma^{-1} \left(\dot{\hat{\theta}} - \tau_3 \right). \tag{2.17}$$

Then, to make $\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2$, we would design α_3 such that the bracketed term multiplying z_3 equals $-c_3 z_3$, namely

$$\alpha_3 \left(x_1, x_2, x_3, \hat{\theta} \right) = -z_2 - c_3 z_3 + \frac{\partial \alpha_2}{\partial x_1} x_2 + \frac{\partial \alpha_2}{\partial x_2} x_3 + \frac{\partial \alpha_2}{\partial \hat{\theta}} \tau_3$$
$$+ \left(z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma - \hat{\theta}^{\mathrm{T}} \right) \left(\phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right), \tag{2.18}$$

where τ_3 replaces $\dot{\hat{\theta}}$. Since x_3 is not our control, we have $z_4 \not\equiv 0$ and we do not use $\dot{\hat{\theta}} = \tau_3$ as an update law. We again postpone the decision about $\dot{\hat{\theta}}$ and retain τ_3 as our third tuning function and α_3 as our third stabilizing function. The resulting $\dot{V}_3$ is

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 z_4 + \left[z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} + z_3 \frac{\partial \alpha_2}{\partial \hat{\theta}} + (\theta - \hat{\theta})^{\mathrm{T}} \Gamma^{-1} \right] \left(\tau_3 - \dot{\hat{\theta}} \right). \tag{2.19}$$

The closed-loop form of (2.13) with (2.18) is

$$\dot{z}_3 = -z_2 - c_3 z_3 + z_4 + (\theta - \hat{\theta})^{\mathrm{T}} \left(\phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right)$$
$$+ \frac{\partial \alpha_2}{\partial \hat{\theta}} \left(\tau_3 - \dot{\hat{\theta}} \right) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \left(\phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right). \tag{2.20}$$

Step i: Introducing $z_{i+1} = x_{i+1} - \alpha_i$, we rewrite $\dot{x}_i = x_{i+1} + \theta^{\mathrm{T}} \phi_i(x_1, \ldots, x_i)$ as

$$\dot{z}_i = z_{i+1} + \alpha_i + \theta^{\mathrm{T}} \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \left(x_{k+1} + \theta^{\mathrm{T}} \phi_k \right) - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}, \tag{2.21}$$

and use α_i as a control to stabilize the $(z_1, \ldots, z_i)$-system with respect to $V_i = V_{i-1} + \frac{1}{2} z_i^2$. Then

$$\dot{V}_i = -\sum_{k=1}^{i-1} c_k z_k^2 + \left(\sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) \left(\tau_{i-1} - \dot{\hat{\theta}} \right)$$
$$+ z_i \left[z_{i-1} + z_{i+1} + \alpha_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \hat{\theta}^{\mathrm{T}} \left(\phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k \right) \right]$$
$$+ (\hat{\theta} - \theta)^{\mathrm{T}} \Gamma^{-1} \left[\dot{\hat{\theta}} - \Gamma \sum_{l=1}^{i} z_l \left(\phi_l - \sum_{k=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_k} \phi_k \right) \right]. \tag{2.22}$$

If x_{i+1} were our actual control, we would let $z_{i+1} \equiv 0$ and eliminate $\hat{\theta} - \theta$ from $\dot{V}_i$ with the update law $\dot{\hat{\theta}} = \tau_i$, where

$$\tau_i \left(x_1, \ldots, x_i, \hat{\theta} \right) = \Gamma \sum_{l=1}^{i} z_l \left(\phi_l - \sum_{k=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_k} \phi_k \right) = \tau_{i-1} + \Gamma z_i \left(\phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k \right). \tag{2.23}$$

Noting that

$$\dot{\hat{\theta}} - \tau_{i-1} = \dot{\hat{\theta}} - \tau_i + \tau_i - \tau_{i-1} = \dot{\hat{\theta}} - \tau_i + \Gamma z_i\left(\phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\phi_k\right), \tag{2.24}$$

we rewrite $\dot{V}_i$ as

$$\begin{aligned}\dot{V}_i = &-\sum_{k=1}^{i-1} c_k z_k^2 + \sum_{k=1}^{i-2} z_{k+1}\frac{\partial \alpha_k}{\partial \hat{\theta}}\left(\tau_i - \dot{\hat{\theta}}\right) + z_i\Bigg[z_{i-1} + z_{i+1} + \alpha_i - \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_k}x_{k+1} \\ &\qquad - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} + \left(\hat{\theta}^{\mathrm{T}} - \sum_{k=1}^{i-2} z_{k+1}\frac{\partial \alpha_k}{\partial \hat{\theta}}\Gamma\right)\left(\phi_i - \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_k}\phi_k\right)\Bigg] \\ &+ (\hat{\theta} - \theta)^{\mathrm{T}}\Gamma^{-1}\left(\dot{\hat{\theta}} - \tau_i\right).\end{aligned} \tag{2.25}$$

Then, to make $\dot{V}_i = -\sum_{k=1}^{i} c_k z_k^2$, we would design α_i such that the bracketed term multiplying z_i equals $-c_i z_i$, namely

$$\begin{aligned}\alpha_i\left(x_1, \dots, x_i, \hat{\theta}\right) = &-z_{i-1} - c_i z_i + \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_k}x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\tau_i \\ &+ \left[\sum_{k=1}^{i-2} z_{k+1}\frac{\partial \alpha_k}{\partial \hat{\theta}}\Gamma - \hat{\theta}^{\mathrm{T}}\right]\left(\phi_i - \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_k}\phi_k\right),\end{aligned} \tag{2.26}$$

where τ_i replaces $\dot{\hat{\theta}}$. Since x_{i+1} is not our control, we have $z_i \not\equiv 0$ and we do not use $\dot{\hat{\theta}} = \tau_i$ as an update law. However, we retain τ_i as our i-th tuning function and α_i as our i-th stabilizing function. The resulting $\dot{V}_i$ is

$$\dot{V}_i = -\sum_{k=1}^{i} c_k z_k^2 + z_i z_{i+1} + \left[\sum_{k=1}^{i-1} z_{k+1}\frac{\partial \alpha_k}{\partial \hat{\theta}} + (\theta - \hat{\theta})^{\mathrm{T}}\Gamma^{-1}\right]\left(\tau_i - \dot{\hat{\theta}}\right). \tag{2.27}$$

The closed-loop form of (2.21) with (2.26) is

$$\begin{aligned}\dot{z}_i = &-z_{i-1} - c_i z_i + z_{i+1} + (\theta - \hat{\theta})^{\mathrm{T}}\left(\phi_i - \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_k}\phi_k\right) \\ &+ \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\left(\tau_i - \dot{\hat{\theta}}\right) + \left(\sum_{k=1}^{i-2} z_{k+1}\frac{\partial \alpha_k}{\partial \hat{\theta}}\Gamma\right)\left(\phi_i - \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_k}\phi_k\right).\end{aligned} \tag{2.28}$$

Step n: With $z_n = x_n - \alpha_{n-1}$, we rewrite $\dot{x}_n = \phi_0(x) + \theta^{\mathrm{T}}\phi_n(x) + \beta_0(x)u$ as

$$\dot{z}_n = \phi_0 + \theta^{\mathrm{T}}\phi_n + \beta_0 u - \sum_{k=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_k}\left(x_{k+1} + \theta^{\mathrm{T}}\phi_k\right) - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}. \tag{2.29}$$

We now design our actual update law $\dot{\hat{\theta}} = \tau_n$ and feedback control u to stabilize the full z-system with respect to $V_n = V_{n-1} + \frac{1}{2}z_n^2$. Our goal is to make $\dot{V}_n$ nonpositive:

$$\begin{aligned}\dot{V}_n = &-\sum_{k=1}^{n-1} c_k z_k^2 + \left(\sum_{k=1}^{n-2} z_{k+1}\frac{\partial \alpha_k}{\partial \hat{\theta}}\right)\left(\tau_{n-1} - \dot{\hat{\theta}}\right) \\ &+ z_n\left[z_{n-1} + \beta_0 u + \phi_0 - \sum_{k=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_k}x_{k+1} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} + \hat{\theta}^{\mathrm{T}}\left(\phi_n - \sum_{k=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_k}\phi_k\right)\right] \\ &+ (\hat{\theta} - \theta)^{\mathrm{T}}\Gamma^{-1}\left[\dot{\hat{\theta}} - \Gamma\sum_{l=1}^{n} z_l\left(\phi_l - \sum_{k=1}^{l-1}\frac{\partial \alpha_{l-1}}{\partial x_k}\phi_k\right)\right].\end{aligned} \tag{2.30}$$

To eliminate $\hat{\theta}-\theta$ from $\dot{V}_n$ we choose the update law

$$\dot{\hat{\theta}}=\tau_n(z,\hat{\theta})=\Gamma\sum_{l=1}^{n} z_l\left(\phi_l-\sum_{k=1}^{l-1}\frac{\partial\alpha_{l-1}}{\partial x_k}\phi_k\right)=\tau_{n-1}+\Gamma z_n\left(\phi_n-\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}\phi_k\right). \tag{2.31}$$

Then, noting that

$$\dot{\hat{\theta}}-\tau_{n-1}=\tau_n-\tau_{n-1}=\Gamma z_n\left(\phi_n-\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}\phi_k\right), \tag{2.32}$$

we rewrite $\dot{V}_n$ as

$$\begin{aligned}\dot{V}_n=-\sum_{k=1}^{n-1}c_k z_k^2+z_n\Bigg[&z_{n-1}+\beta_0 u+\phi_0-\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}x_{k+1}-\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\\&+\left(\hat{\theta}^{\mathrm{T}}-\sum_{k=1}^{n-2}z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\theta}}\Gamma\right)\left(\phi_n-\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}\phi_k\right)\Bigg].\end{aligned} \tag{2.33}$$

Finally, we choose the control u such that the bracketed term multiplying z_n equals $-c_n z_n$:

$$\begin{aligned}u=\frac{1}{\beta_0}\Bigg[&-z_{n-1}-c_n z_n-\phi_0+\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}x_{k+1}+\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\tau_n\\&+\left(\sum_{k=1}^{n-2}z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\theta}}\Gamma-\hat{\theta}^{\mathrm{T}}\right)\left(\phi_n-\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}\phi_k\right)\Bigg].\end{aligned} \tag{2.34}$$

We have thus reached our goal:

$$\dot{V}_n=-\sum_{k=1}^{n}c_k z_k^2. \tag{2.35}$$

With (2.34) the closed-loop form of (2.29) becomes

$$\dot{z}_n=-z_{n-1}-c_n z_n+(\theta-\hat{\theta})^{\mathrm{T}}\left(\phi_n-\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}\phi_k\right)+\left(\sum_{k=1}^{n-2}z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\theta}}\Gamma\right)\left(\phi_n-\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}\phi_k\right). \tag{2.36}$$

In the more compact notation

$$w_i(x_1,\ldots,x_i,\hat{\theta}):=\phi_i-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}\phi_k, \tag{2.37}$$

the overall closed-loop system is rewritten as

$$\dot{z}_1=-c_1 z_1+z_2+(\theta-\hat{\theta})^{\mathrm{T}}w_1, \tag{2.38a}$$

$$\dot{z}_2=-z_1-c_2 z_2+z_3+(\theta-\hat{\theta})^{\mathrm{T}}w_2-\sum_{k=3}^{n}\frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma z_k w_k, \tag{2.38b}$$

$$\dot{z}_3=-z_2-c_3 z_3+z_4+(\theta-\hat{\theta})^{\mathrm{T}}w_3-\sum_{k=4}^{n}\frac{\partial\alpha_2}{\partial\hat{\theta}}\Gamma z_k w_k+z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma w_3, \tag{2.38c}$$

$$\vdots$$

$$\dot{z}_i = -z_{i-1} - c_i z_i + z_{i+1} + (\theta - \hat{\theta})^{\mathrm{T}} w_i - \sum_{k=i+1}^{n} \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma z_k w_k + \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \Gamma w_i, \tag{2.38d}$$

$$\vdots$$

$$\dot{z}_n = -z_{n-1} - c_n z_n + (\theta - \hat{\theta})^{\mathrm{T}} w_n + \sum_{k=1}^{n-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \Gamma w_n, \tag{2.38e}$$

$$\dot{\hat{\theta}} = \Gamma \sum_{l=1}^{n} z_l w_l. \tag{2.38f}$$

Remark 1. Even though the new procedure is presented for strict-feedback systems (1.1), its first $n-1$ steps remain the same for *parametric-pure-feedback* systems:

$$\dot{x}_i = x_{i+1} + \theta^{\mathrm{T}} \phi_i(x_1, \ldots, x_{i+1}), \quad 1 \leqslant i \leqslant n-1, \tag{2.39a}$$

$$\dot{x}_n = \phi_0(x) + \theta^{\mathrm{T}} \phi_n(x) + \left[\beta_0(x) + \theta^{\mathrm{T}} \beta(x)\right] u, \tag{2.39b}$$

where ϕ_0, β_0, and the components of $\beta \in \mathbb{R}^p$ and $\phi_i \in \mathbb{R}^p$, $1 \leqslant i \leqslant n$, are smooth nonlinear functions in B_x, a neighborhood of the origin $x = 0$, $\beta_0(x) \neq 0$, for all $x \in B_x$. The completion of the n-th step requires that feasibility conditions similar to those in [2] be satisfied. A verifiable geometric characterization of this class of systems is given in [2].

3. Stability and convergence

To prove stability of the closed-loop system (2.38), we express ϕ_i, α_i, τ_i, w_i in the z-coordinates. Then the global stability of the equilibrium $z = 0$, $\hat{\theta} = \theta$ follows from the fact that the derivative of $V_n = \frac{1}{2} z^{\mathrm{T}} z + \frac{1}{2}(\hat{\theta} - \theta)^{\mathrm{T}} \Gamma^{-1} (\hat{\theta} - \theta)$ for (2.38) is given by (2.35). From LaSalle's invariance theorem, it further follows that the state $(z(t), \hat{\theta}(t))$ converges to the largest invariant set M of (2.38) contained in $E = \{(z, \hat{\theta}) \in \mathbb{R}^{n+p} \mid z = 0\}$, that is, in the set where $\dot{V}_n = 0$.

We now set out to determine M. On this invariant set, we have $z = 0$ and $\dot{z} = 0$. Setting $z = 0$, $\dot{z} = 0$ in (2.38) we obtain $\dot{\hat{\theta}} = 0$ and

$$(\theta - \hat{\theta})^{\mathrm{T}} w_i = 0, \quad i = 1, \ldots, n, \forall (z, \hat{\theta}) \in M. \tag{3.1}$$

Using (2.37) and (3.1) for $i = 1$ we get $(\theta - \hat{\theta})^{\mathrm{T}} \phi_1(0) = (\theta - \hat{\theta})^{\mathrm{T}} \phi_1^{\mathrm{e}} = 0$ on M. Recall from (2.4) that $\alpha_1 = -c_1 z_1 - \hat{\theta}^{\mathrm{T}} \phi_1$. Therefore, on M we have $\alpha_1 = -\hat{\theta}^{\mathrm{T}} \phi_1^{\mathrm{e}} = -\theta^{\mathrm{T}} \phi_1^{\mathrm{e}}$. Combining this with $z_2 = 0$, we get $x_2 = x_2^{\mathrm{e}}$ on M. Using (2.37) and (3.1) for $i = 2$, we obtain $(\theta - \hat{\theta})^{\mathrm{T}} (\phi_2 - (\partial \alpha_1 / \partial x_1) \phi_1) = 0$. Since on M we have $(\theta - \hat{\theta})^{\mathrm{T}} \phi_1^{\mathrm{e}} = 0$ and $\phi_2(x_1, x_2) = \phi_2(0, -\theta^{\mathrm{T}} \phi_1^{\mathrm{e}}) = \phi_2^{\mathrm{e}}$, this implies that $(\theta - \hat{\theta})^{\mathrm{T}} \phi_2^{\mathrm{e}} = 0$. Continuing in the same fashion, we prove that $x_i = x_i^{\mathrm{e}}$ and $(\theta - \hat{\theta})^{\mathrm{T}} \phi_i^{\mathrm{e}} = 0$ on M, $i = 1, \ldots, n$.

Thus, the largest invariant set M in E is

$$\begin{aligned} M &= \left\{(z, \hat{\theta}) \in \mathbb{R}^{n+p} \colon z = 0, (\theta - \hat{\theta})^{\mathrm{T}} \phi_i^{\mathrm{e}} = 0, i = 1, \ldots, n\right\} \\ &= \left\{(x, \hat{\theta}) \in \mathbb{R}^{n+p} \colon x = x^{\mathrm{e}}, (\theta - \hat{\theta})^{\mathrm{T}} \phi_i^{\mathrm{e}} = 0, i = 1, \ldots, n\right\}. \end{aligned} \tag{3.2}$$

These two equivalent expressions for M and the convergence of $(z(t), \hat{\theta}(t))$ to M prove that $x(t) \to x^{\mathrm{e}}$ as $t \to \infty$.

Another important property of M is its dimension $p - r$, where $r = \operatorname{rank}[\phi_1^{\mathrm{e}}, \ldots, \phi_n^{\mathrm{e}}]$. It is well known from the adaptive control literature that as the dimension of M is reduced, the robustness properties of the adaptive system are improved. When $r = p$, the dimension of M is zero, that is, M becomes the equilibrium point $x = x^e$, $\hat{\theta} = \theta$. This means that the parameter estimates converge to their true values, so that the equilibrium $x = x^e$, $\hat{\theta} = \theta$ is globally *asymptotically* stable.

The above facts prove the following result:

Theorem 1. *Suppose that the design procedure is applied to the parametric-strict-feedback system* (1.1). *Then, the equilibrium $x = x^e$, $\hat{\theta} = \theta$ of the resulting adaptive system is globally stable. Furthermore, its state $(x(t), \hat{\theta}(t))$ converges to the $(p - r)$-dimensional manifold M given by* (3.2). *The equilibrium $x = x^e$, $\hat{\theta} = \theta$ is globally asymptotically stable if and only if $r = p$.* □

The new procedure can achieve global asymptotic stability with as many as $p = n$ unknown parameters, while in earlier procedures that number was at most $p = 2$.

4. A design example

Let us illustrate the difference between the new procedure and the procedure of [2,4] on the 'benchmark' example:

$$\dot{x}_1 = x_2 + \theta\phi(x_1), \tag{4.1a}$$

$$\dot{x}_2 = x_3, \tag{4.1b}$$

$$\dot{x}_3 = u. \tag{4.1c}$$

The controller of [2,4] employs *three* estimates ϑ_1, ϑ_2, ϑ_3:

$$\dot{\vartheta}_1 = z_1\phi, \tag{4.2a}$$

$$\dot{\vartheta}_2 = -z_2\frac{\partial\alpha_1}{\partial x_1}\phi, \tag{4.2b}$$

$$\dot{\vartheta}_3 = -z_3\frac{\partial\alpha_2}{\partial x_1}\phi. \tag{4.2c}$$

The corresponding stabilizing functions and the feedback control u are

$$\alpha_1 = -z_1 - \vartheta_1\phi, \tag{4.3a}$$

$$\alpha_2 = -z_2 - z_1 + \frac{\partial\alpha_1}{\partial x_1}(x_2 + \vartheta_2\phi) + \frac{\partial\alpha_1}{\partial\vartheta_1}z_1\phi, \tag{4.3b}$$

$$u = -z_3 - z_2 + \frac{\partial\alpha_2}{\partial x_1}(x_2 + \vartheta_3\phi) + \frac{\partial\alpha_2}{\partial x_2}x_3 + \frac{\partial\alpha_2}{\partial\vartheta_1}z_1\phi - \frac{\partial\alpha_2}{\partial\vartheta_2}z_2\frac{\partial\alpha_1}{\partial x_1}\phi. \tag{4.3c}$$

In contrast to the three estimates in (4.2), the new design procedure needs *only one* estimate $\hat{\theta}$. The tuning functions are $\tau_1 = z_1\phi$, $\tau_2 = \tau_1 - z_2(\partial\alpha_1/\partial x_1)\phi$, $\tau_3 = \tau_2 - z_3(\partial\alpha_2/\partial x_1)\phi$, and the update law for $\hat{\theta}$ is

$$\dot{\hat{\theta}} = \tau_3 = z_1\phi - z_2\frac{\partial\alpha_1}{\partial x_1}\phi - z_3\frac{\partial\alpha_2}{\partial x_1}\phi, \tag{4.4}$$

while the stabilizing functions and the feedback control u are

$$\alpha_1 = -z_1 - \hat{\theta}\phi, \tag{4.5a}$$

$$\alpha_2 = -z_2 - z_1 + \frac{\partial \alpha_1}{\partial x_1}\left(x_2 + \hat{\theta}\phi\right) + \frac{\partial \alpha_1}{\partial \hat{\theta}}\tau_2, \tag{4.5b}$$

$$u = -z_3 - z_2 + \frac{\partial \alpha_2}{\partial x_1}\left(x_2 + \hat{\theta}\phi\right) + \frac{\partial \alpha_2}{\partial x_2}x_3 + \frac{\partial \alpha_2}{\partial \hat{\theta}}\tau_3 - z_2\frac{\partial \alpha_1}{\partial \hat{\theta}}\frac{\partial \alpha_2}{\partial x_1}\phi. \tag{4.5c}$$

Comparing (4.2)–(4.3) with (4.4)–(4.5), we see that the new design procedure reduces the dynamic order of the adaptive controller.

5. Concluding remarks

The new design procedure eliminates the need for more than the minimum number of parameter estimates. Among the advantages of having exactly p estimates for p parameters are a reduction in the controller's dynamic order, enhanced stability properties, and improved parameter convergence. In particular, if there are at most n unknown parameters and $r = p$, then the adaptive system is globally asymptotically stable, and, hence, more robust to disturbances.

References

[1] Z.P. Jiang and L. Praly, Iterative designs of adaptive controllers for systems with nonlinear integrators, *Proc. 30th IEEE Conf. Decision Control* (Brighton, UK, 1991) 2482–2487.

[2] I. Kanellakopoulos, P.V. Kokotović and A.S. Morse, Systematic design of adaptive controllers for feedback linearizable systems, *IEEE Trans. Automat. Control* **36** (1991) 1241–1253.

[3] I. Kanellakopoulos, P.V. Kokotović and A.S. Morse, Adaptive output-feedback control of a class of nonlinear systems, *Proc. 30th IEEE Conf. Decision Control* (Brighton, UK, 1991) 1082–1087.

[4] P.V. Kokotović, I. Kanellakopoulos and A.S. Morse, Adaptive feedback linearization of nonlinear systems, in: P.V. Kokotović, Ed., *Foundations of Adaptive Control* (Springer-Verlag, Berlin, 1991) 311–346.

[5] R. Marino and P. Tomei, Global adaptive observers and output-feedback stabilization for a class of nonlinear systems, in: P.V. Kokotović, Ed., *Foundations of Adaptive Control* (Springer-Verlag, Berlin, 1991) 455–493.

[6] S.S. Sastry and A. Isidori, Adaptive control of linearizable systems, *IEEE Trans. Automat. Control* **34** (1989) 1123–1131.