Adaptive nonlinear control without overparametrization *

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Abstract: A new design procedure for adaptive nonlinear control is proposed in which the number of parameter estimates is minimal, that is, equal to the number of unknown parameters. The adaptive systems designed by this procedure possess stronger stability properties than those using overparametrization.

Keywords: Adaptive control; nonlinear systems; overparametrization; tuning functions; backstepping design.

1. Introduction

In this paper we present a new design procedure for adaptive control of nonlinear systems transformable into the parametric-strict-feedback form

\begin{align}
\dot{x}_i &= x_{i+1} + \theta^T \phi_i(x_1, \ldots, x_i), \quad 1 \leq i \leq n-1, \\
\dot{x}_n &= \phi_0(x) + \theta^T \phi_n(x) + \beta_0(x) u,
\end{align}

where \( \theta \in \mathbb{R}^p \) is the vector of unknown constant parameters, \( \phi_0, \beta_0 \), and the components of \( \phi_i \), \( 1 \leq i \leq n \), are smooth nonlinear functions in \( \mathbb{R}^n \), and \( \beta_0(x) \neq 0 \), for all \( x \in \mathbb{R}^n \).

The global adaptive regulation and tracking problems for systems in this form have recently been solved in [2,4]. This was achieved using a systematic design procedure and without any growth restrictions on nonlinearities. However, the procedure of [2,4] has not removed the need for overparametrization, a significant drawback of earlier adaptive nonlinear schemes [6]. As many as \( np \) estimates of \( p \) unknown parameters had to be continuously updated. Recently, this number was reduced in half [1]. Still, the dynamic order of the resulting adaptive controller is quite high, and is even higher in the case of output-feedback designs [5,3].

The new design procedure eliminates overparametrization while retaining all the advantages of the procedure in [2,4]. It employs exactly \( p \) estimates for \( p \) unknown parameters and significantly reduces the controller’s dynamic order. This enhances the stability properties of the adaptive system and improves parameter convergence.

For clarity, we present the new procedure for the regulation problem. Its extension to the tracking problem is the same as in [2].

Our control objective is to regulate \( x_1 \) to \( x_1^e = 0 \) and to stabilize the corresponding equilibrium \( x^e \):

\begin{align}
x_1^e = 0, \quad x_{i+1}^e = -\theta^T \phi_i^e := -\theta^T \phi_i(0, -\theta^T \phi_1^e, \ldots, -\theta^T \phi_{i-1}^e), \quad i = 1, \ldots, n-1.
\end{align}

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Note that in the special case when \( \phi_i(0) = \cdots = \phi_{n-1}(0) = 0 \), the equilibrium is \( x^e = 0 \) for all values of \( \theta \). However, as we shall see later, a nonzero equilibrium \( x^e \neq 0 \) improves parameter convergence and stability properties.

### 2. Backstepping design with tuning functions

The design procedure is recursive. At its \( i \)-th step and \( i \)-th-order subsystem is stabilized with respect to a Lyapunov function \( V_i \) by the design of a stabilizing function \( \alpha_i \) and a tuning function \( \tau_i \). The update law for the parameter estimate \( \hat{\theta}(t) \) and the feedback control \( u \) are designed at the final step.

**Step 1:** Introducing \( z_1 = x_1 \) and \( z_2 = x_2 - \alpha_1 \), we rewrite \( \dot{x}_1 = x_2 + \theta^T \phi_1(x_1) \) as

\[
\dot{z}_1 = z_2 + \alpha_1 + \theta^T \phi_1(x_1) \tag{2.1}
\]

and use \( \alpha_1 \) as a control to stabilize (2.1) with respect to the Lyapunov function \( V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}(\hat{\theta} - \theta)^T (\hat{\theta} - \theta) \). Then

\[
\dot{V}_1 = z_1(z_2 + \alpha_1 + \hat{\theta}^T \phi_1) + (\hat{\theta} - \theta)^T (\hat{\theta} - \theta) \tag{2.2}
\]

If \( x_2 \) were our actual control, we would let \( z_2 = 0 \), that is, \( x_2 = \alpha_1 \). Then, we would eliminate \( \hat{\theta} - \theta \) from \( \dot{V}_1 \) with the update law \( \dot{\theta} = \tau_1 \), where

\[
\tau_1(x_1) = \Gamma z_1 \phi_1(x_1). \tag{2.3}
\]

To make \( \dot{V}_1 = -c_1 z_1^2 \), we would choose

\[
\alpha_1(x_1, \hat{\theta}) = -c_1 z_1 - \hat{\theta}^T \phi_1(x_1). \tag{2.4}
\]

Since \( x_2 \) is not our control, we have \( x_2 \neq 0 \), and we do not use \( \dot{\theta} = \tau_1 \) as an update law. However, we retain \( \tau_1 \) as our first tuning function and \( \alpha_1 \) as our first stabilizing function. We thus postpone the decision about \( \theta \) and tolerate the presence of \( \hat{\theta} - \theta \) in \( \dot{V}_1 \):

\[
\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 + (\hat{\theta} - \theta)^T (\hat{\theta} - \theta). \tag{2.5}
\]

The second term \( z_1 z_2 \) in \( \dot{V}_1 \) will be cancelled at the next step. The closed-loop form of (2.1) with (2.4) is

\[
\dot{z}_1 = -c_1 z_1 + z_2 + (\hat{\theta} - \theta)^T \phi_1(x_1). \tag{2.6}
\]

**Step 2:** Introducing \( z_3 = x_3 - \alpha_2 \), we rewrite \( \dot{x}_2 = x_3 + \theta^T \phi_2(x_1, x_2) \) as

\[
\dot{z}_2 = z_3 + \alpha_2 + \theta^T \phi_2 - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \theta^T \phi_1) - \frac{\partial \alpha_1}{\partial \theta} \hat{\theta}, \tag{2.7}
\]

and use \( \alpha_2 \) as a control to stabilize the \((z_1, z_2)\)-system (2.6)--(2.7) with respect to \( V_2 = V_1 + \frac{1}{2} z_2^2 \). Then

\[
\dot{V}_2 = -c_1 z_1^2 + z_2 \left[ z_1 + z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} x_2 - \frac{\partial \alpha_1}{\partial \theta} \hat{\theta} + \theta^T \left( \phi_2 - \frac{\partial \alpha_1}{\partial x_1} \phi_1 \right) \right] + (\hat{\theta} - \theta)^T (\hat{\theta} - \theta) \left[ z_1 \phi_1 + z_2 \left( \phi_2 - \frac{\partial \alpha_1}{\partial x_1} \phi_1 \right) \right]. \tag{2.8}
\]
If $x_3$ were our actual control, we would let $z_3 \equiv 0$ and eliminate $\hat{\theta} - \theta$ from $\dot{V}_2$ with the update law $\hat{\theta} = \tau_2$, where

$$
\tau_2(x_1, x_2, \hat{\theta}) = \Gamma \left[ z_1 \phi_1 + z_2 \left( \phi_2 - \frac{\partial \alpha_1}{\partial x_1} \phi_1 \right) \right] = \tau_1 + \Gamma z_2 \left( \phi_2 - \frac{\partial \alpha_1}{\partial x_1} \phi_1 \right).
$$

(2.9)

Then, to make $\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2$, we would design $\alpha_2$ such that the bracketed term multiplying $z_2$ equals $-c_2 z_2$, namely

$$
\alpha_2(x_1, x_2, \hat{\theta}) = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \theta} \tau_2 - \hat{\theta}^T \left( \phi_2 - \frac{\partial \alpha_1}{\partial x_1} \phi_1 \right).
$$

(2.10)

It is important to note that in this expression $\tau_2$ replaces $\dot{\hat{\theta}}$. Since $x_3$ is not our control, we have $z_3 \equiv 0$ and we do not use $\hat{\theta} = \tau_2$ as an update law. However, we retain $\tau_2$ as our second tuning function and $\alpha_2$ as our second stabilizing function. The resulting $\dot{V}_2$ is

$$
\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + \left[ z_2 \frac{\partial \alpha_1}{\partial \theta} + (\theta - \hat{\theta})^T \Gamma^{-1} \right] (\tau_2 - \hat{\theta}).
$$

(2.11)

The first two terms in $\dot{V}_2$ are negative definite, the third term will be cancelled at the next step, and the last term is tolerated at this step, as the decision about $\hat{\theta}$ is again postponed. The closed-loop form of (2.7) with (2.10) is

$$
\dot{z}_2 = -z_1 - c_2 z_2 + z_3 + (\theta - \hat{\theta})^T \left( \phi_2 - \frac{\partial \alpha_1}{\partial x_1} \phi_1 \right) + \frac{\partial \alpha_1}{\partial \theta} (\tau_2 - \hat{\theta}).
$$

(2.12)

**Step 3:** Introducing $z_4 = x_4 - \alpha_3$, we rewrite $\dot{x}_3 = x_4 + \theta^T \phi_3(x_1, x_2, x_3)$ as

$$
\dot{z}_3 = z_4 + \alpha_3 + \theta^T \phi_3 - \frac{\partial \alpha_2}{\partial x_1} (x_2 + \theta^T \phi_1) - \frac{\partial \alpha_2}{\partial x_2} (x_3 + \theta^T \phi_2) - \frac{\partial \alpha_2}{\partial \theta} \hat{\theta},
$$

(2.13)

and use $\alpha_3$ as a control to stabilize the $(z_1, z_2, z_3)$-system with respect to $V_3 = V_2 + \frac{1}{2} z_3^2$. Then

$$
\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 + z_2 \frac{\partial \alpha_1}{\partial \theta} (\tau_2 - \hat{\theta})
$$

$$
+ z_3 \left[ z_2 + z_4 + \alpha_3 - \frac{\partial \alpha_2}{\partial x_1} x_2 - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial \theta} \hat{\theta} + \hat{\theta}^T \left( \phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right) \right]
$$

$$
+ (\hat{\theta} - \theta)^T \Gamma^{-1} \left[ \hat{\theta} - \Gamma \left( z_1 \phi_1 + z_2 \left( \phi_2 - \frac{\partial \alpha_1}{\partial x_1} \phi_1 \right) + z_3 \left( \phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right) \right) \right].
$$

(2.14)

If $x_3$ were our actual control, we would let $z_4 = 0$ and eliminate $\hat{\theta} - \theta$ from $\dot{V}_3$ with the update law $\hat{\theta} = \tau_3$, where

$$
\tau_3(x_1, x_2, x_3, \hat{\theta}) = \Gamma \left[ z_1 \phi_1 + z_2 \left( \phi_2 - \frac{\partial \alpha_1}{\partial x_1} \phi_1 \right) + z_3 \left( \phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right) \right]
$$

$$
= \tau_2 + \Gamma z_3 \left( \phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right).
$$

(2.15)

Noting that

$$
\hat{\theta} - \tau_2 = \hat{\theta} - \tau_3 + \tau_3 - \tau_2 = \hat{\theta} - \tau_3 + \Gamma z_3 \left( \phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right),
$$

(2.16)
we rewrite $\dot{V}_3$ as

$$
\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 + z_3 \frac{\partial \alpha_1}{\partial \theta} (\tau_3 - \hat{\theta})
+ z_3 \left[ z_2 + z_4 + \alpha_3 - \frac{\partial \alpha_2}{\partial x_1} x_2 - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial \theta} \hat{\theta} + \left( \hat{\theta}^T - z_2 \frac{\partial \alpha_1}{\partial x_1} \Gamma \right) \left( \phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right) \right]
+ (\hat{\theta} - \theta)^T \Gamma^{-1} (\hat{\theta} - \tau_3). 
$$

(2.17)

Then, to make $\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2$, we would design $\alpha_3$ such that the bracketed term multiplying $z_3$ equals $-c_3 z_3$, namely

$$
\alpha_3(x_1, x_2, x_3, \hat{\theta}) = -z_2 - c_3 z_3 + \frac{\partial \alpha_2}{\partial x_1} x_2 + \frac{\partial \alpha_2}{\partial x_2} x_3 + \frac{\partial \alpha_2}{\partial \theta} \hat{\theta}
+ \left( z_2 \frac{\partial \alpha_1}{\partial \theta} \Gamma - \hat{\theta}^T \right) \left( \phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right),
$$

(2.18)

where $\tau_3$ replaces $\hat{\theta}$. Since $x_3$ is not our control, we have $z_4 \neq 0$ and we do not use $\hat{\theta} = \tau_3$ as an update law. We again postpone the decision about $\hat{\theta}$ and retain $\tau_3$ as our third tuning function and $\alpha_3$ as our third stabilizing function. The resulting $\dot{V}_3$ is

$$
\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 z_4 + \left[ z_2 \frac{\partial \alpha_1}{\partial \theta} + z_3 \frac{\partial \alpha_2}{\partial \theta} + (\theta - \hat{\theta})^T \Gamma^{-1} \right] (\tau_3 - \hat{\theta}).
$$

(2.19)

The closed-loop form of (2.13) with (2.18) is

$$
\dot{z}_3 = -z_2 - c_3 z_3 + z_4 + (\theta - \hat{\theta})^T \left( \phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right)
+ z_2 \frac{\partial \alpha_1}{\partial \theta} \Gamma \left( \phi_3 - \frac{\partial \alpha_2}{\partial x_1} \phi_1 - \frac{\partial \alpha_2}{\partial x_2} \phi_2 \right).
$$

(2.20)

**Step i:** Introducing $z_{i+1} = x_{i+1} - \alpha_i$, we rewrite $\dot{x}_i = x_{i+1} + \theta^T \phi_i(x_1, \ldots, x_i)$ as

$$
\dot{z}_i = z_{i+1} + \alpha_i + \theta^T \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} (x_{k+1} + \theta^T \phi_k) - \frac{\partial \alpha_{i-1}}{\partial \theta} \hat{\theta},
$$

(2.21)

and use $\alpha_i$ as a control to stabilize the $(z_1, \ldots, z_i)$-system with respect to $V_i = V_{i-1} + \frac{1}{2} z_i^2$. Then

$$
\dot{V}_i = - \sum_{k=1}^{i-1} c_k z_k^2 + \left( \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \theta} \right) (\tau_{i-1} - \hat{\theta})
+ z_i \left[ z_{i+1} + \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial \theta} \hat{\theta} + \hat{\theta}^T \left( \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k \right) \right]
+ (\hat{\theta} - \theta)^T \Gamma^{-1} \left[ \hat{\theta} - \Gamma \sum_{l=1}^{i-1} z_l \left( \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k \right) \right].
$$

(2.22)

If $x_{i+1}$ were our actual control, we would let $z_{i+1} = 0$ and eliminate $\hat{\theta} - \theta$ from $\dot{V}_i$ with the update law $\hat{\theta} = \tau_3$, where

$$
\tau_i(x_1, \ldots, x_i, \hat{\theta}) = \Gamma \sum_{l=1}^{i} z_l \left( \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k \right) = \tau_{i-1} + \Gamma z_i \left( \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k \right).
$$

(2.23)
Noting that
\[ \dot{\tau}_{i-1} = \dot{\tau}_i + \tau_i - \tau_{i-1} = \dot{\tau}_i + \Gamma_{i} z_i \left( \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k \right), \]  
we rewrite \( V_i \) as
\[ \dot{V}_i = - \sum_{k=1}^{i-1} c_k z_k^2 + \sum_{k=1}^{i-2} \frac{\partial \alpha_k}{\partial \theta} \left( \tau_i - \dot{\theta} \right) + z_i \left[ z_{i-1} + z_{i+1} + \alpha_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} \right] \]
\[ \frac{\partial \alpha_{i-1}}{\partial \theta} \dot{\theta} \left( \tau_i - \dot{\theta} \right) + \bigg( \dot{\theta}^T - \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \theta} \bigg) \left( \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k \right) \]
\[ + (\dot{\theta} - \theta)^T \Gamma^{-1} (\dot{\theta} - \tau_i). \]  
Then, to make \( \dot{V}_i = - \sum_{k=1}^{i-1} c_k z_k^2 \), we would design \( \alpha_i \) such that the bracketed term multiplying \( z_i \) equals \( - c_i z_i \), namely
\[ \alpha_i(x_1, \ldots, x_i, \dot{\theta}) = -z_{i-1} - c_i z_i + \sum_{k=1}^{i-1} \frac{\partial \alpha_i}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial \theta} \tau_i \]
\[ + \left[ \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \theta} \dot{\theta} \bigg( \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k \bigg) \right], \]  
where \( \tau_i \) replaces \( \dot{\theta} \). Since \( x_{i+1} \) is not our control, we have \( z_i \neq 0 \) and we do not use \( \dot{\theta} = \tau_i \) as an update law. However, we retain \( \tau_i \) as our \( i \)-th tuning function and \( \alpha_i \) as our \( i \)-th stabilizing function. The resulting \( V_i \) is
\[ \dot{V}_i = - \sum_{k=1}^{i} c_k z_k^2 + z_i z_{i+1} + \left[ \sum_{k=1}^{i-1} z_{k+1} \frac{\partial \alpha_k}{\partial \theta} + (\theta - \dot{\theta})^T \Gamma^{-1} \bigg( \tau_i - \dot{\theta} \bigg) \right]. \]  
The closed-loop form of (2.21) with (2.26) is
\[ \dot{z}_i = -z_{i-1} - c_i z_i + z_{i+1} + (\theta - \dot{\theta})^T \left( \phi_i - \sum_{k=1}^{i} \frac{\partial \alpha_i}{\partial x_k} \phi_k \right) \]
\[ + \frac{\partial \alpha_{i-1}}{\partial \theta} \bigg( \tau_i - \dot{\theta} \bigg) + \left( \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \theta} \bigg) \left( \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k \bigg). \]  
Step n: With \( z_n = x_n - \alpha_{n-1} \), we rewrite \( \dot{x}_n = \phi_0(x) + \theta^T \phi_n(x) + \beta_0(x) u \) as
\[ \dot{z}_n = \phi_0 + \theta^T \phi_n + \beta_0 u - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \left( x_{k+1} + \theta^T \phi_k \right) - \frac{\partial \alpha_{n-1}}{\partial \theta} \dot{\theta}. \]  
We now design our actual update law \( \dot{\theta} = \tau_n \) and feedback control \( u \) to stabilize the full z-system with respect to \( V_n = V_{n-1} + \frac{1}{2} z_n^2 \). Our goal is to make \( V_n \) nonpositive:
\[ \dot{V}_n = - \sum_{k=1}^{n-1} c_k z_k^2 + \left( \sum_{k=1}^{n-2} z_{k+1} \frac{\partial \alpha_k}{\partial \theta} \right) \bigg( \tau_{n-1} - \dot{\theta} \bigg) \]
\[ + z_n \left[ z_{n-1} + \beta_0 u + \phi_0 - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} - \frac{\partial \alpha_{n-1}}{\partial \theta} \dot{\theta} + \dot{\theta}^T \left( \phi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \phi_k \right) \right] \]
\[ + (\dot{\theta} - \theta)^T \Gamma^{-1} \left[ \dot{\theta} - \Gamma \sum_{l=1}^{n} z_l \left( \phi_l - \sum_{k=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_k} \phi_k \right) \right]. \]  

(2.30)
To eliminate \( \hat{\theta} - \theta \) from \( \dot{V}_n \) we choose the update law
\[
\dot{\theta} = \tau_n(z, \hat{\theta}) = \Gamma \sum_{l=1}^{n} z_l \left( \phi_l - \sum_{k=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_k} \phi_k \right) = \tau_{n-1} + \Gamma z_n \left( \phi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \phi_k \right).
\] (2.31)

Then, noting that
\[
\dot{\theta} - \tau_{n-1} = \tau_n - \tau_{n-1} = \Gamma z_n \left( \phi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \phi_k \right),
\] (2.32)
we rewrite \( \dot{V}_n \) as
\[
\dot{V}_n = - \sum_{k=1}^{n-1} c_k z_k^2 + z_n \left[ z_{n-1} + \beta_0 u + \phi_0 - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} - \frac{\partial \alpha_{n-1}}{\partial \theta} \right.
\]
\[
\left. + \left( \hat{\theta}^T - \sum_{k=1}^{n-2} \frac{\partial \alpha_k}{\partial \theta} \Gamma \right) \left( \phi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \phi_k \right) \right].
\] (2.33)

Finally, we choose the control \( u \) such that the bracketed term multiplying \( z_n \) equals \(- c_n z_n\):
\[
u = \frac{1}{\beta_0} \left[ -z_{n-1} - c_n z_n - \phi_0 + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{n-1}}{\partial \theta} \right.
\]
\[
+ \left( \sum_{k=1}^{n-2} z_{k+1} \frac{\partial \alpha_k}{\partial \theta} \Gamma - \hat{\theta}^T \right) \left( \phi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \phi_k \right). \] (2.34)

We have thus reached our goal:
\[
\dot{V}_n = - \sum_{k=1}^{n} c_k z_k^2.
\] (2.35)

With (2.34) the closed-loop form of (2.29) becomes
\[
\dot{z}_n = -z_{n-1} - c_n z_n + (\theta - \hat{\theta})^T \left( \phi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \phi_k \right) + \left( \sum_{k=1}^{n-2} z_{k+1} \frac{\partial \alpha_k}{\partial \theta} \Gamma \right) \left( \phi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \phi_k \right).
\] (2.36)

In the more compact notation
\[
w_i(x_1, \ldots, x_i, \hat{\theta}) := \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \phi_k,
\] (2.37)
the overall closed-loop system is rewritten as
\[
\dot{z}_1 = -c_1 z_1 + z_2 + (\theta - \hat{\theta})^T w_1,
\] (2.38a)
\[
\dot{z}_2 = -z_1 - c_2 z_2 + z_3 + (\theta - \hat{\theta})^T w_2 - \sum_{k=3}^{n} \frac{\partial \alpha_1}{\partial \theta} \Gamma z_k w_k,
\] (2.38b)
\[
\dot{z}_3 = -z_2 - c_3 z_3 + z_4 + (\theta - \hat{\theta})^T w_3 - \sum_{k=3}^{n} \frac{\partial \alpha_2}{\partial \theta} \Gamma z_k w_k + z_2 \frac{\partial \alpha_1}{\partial \theta} \Gamma w_3,
\] (2.38c)
\[
\vdots
\]
\[
\begin{align*}
\dot{x}_i &= -c_i x_i + \theta \phi_i(x_1, \ldots, x_i, x_{i+1}) + \theta \phi_i(x_1, \ldots, x_i, x_{i+1}) + \sum_{k=1}^{i-2} \frac{\partial \alpha_k}{\partial \theta} \Gamma w_k + \sum_{k=i+1}^{n} \frac{\partial \alpha_k}{\partial \theta} \Gamma w_k, \\
\dot{x}_n &= -c_n x_n + \theta \phi_n(x_1, \ldots, x_n) + \sum_{k=1}^{n-2} \frac{\partial \alpha_k}{\partial \theta} \Gamma w_n, \\
\dot{\theta} &= \Gamma \sum_{i=1}^{n} z_i w_i.
\end{align*}
\] (2.38d, 2.38e, 2.38f)

**Remark 1.** Even though the new procedure is presented for strict-feedback systems (1.1), its first \( n - 1 \) steps remain the same for parametric-pure-feedback systems:

\[
\begin{align*}
\dot{x}_i &= \phi_0(x) + \theta \phi_0(x) + \beta_0(x) + \theta \beta(x) u, \\
\dot{x}_n &= \phi_0(x) + \theta \phi_n(x) + \beta_0(x) + \theta \beta(x) u.
\end{align*}
\] (2.39a, 2.39b)

where \( \phi_0, \beta_0 \), and the components of \( \beta \in \mathbb{R}^p \) and \( \phi_i \in \mathbb{R}^p \), \( 1 \leq i \leq n \), are smooth nonlinear functions in \( B_x \), a neighborhood of the origin \( x = 0 \), \( \beta_0(x) \neq 0 \), for all \( x \in B_x \). The completion of the \( n \)-th step requires that feasibility conditions similar to those in [2] be satisfied. A verifiable geometric characterization of this class of systems is given in [2].

### 3. Stability and convergence

To prove stability of the closed-loop system (2.38), we express \( \phi_i, \alpha_i, \tau_i, w_i \) in the \( z \)-coordinates. Then the global stability of the equilibrium \( z = 0 \), \( \dot{\theta} = 0 \) follows from the fact that the derivative of \( V_n = \frac{1}{2} z^T z + \frac{1}{2} (\theta - \hat{\theta})^T \Gamma^{-1} (\theta - \hat{\theta}) \) for (2.38) is given by (2.35). From LaSalle's invariance theorem, it further follows that the state \( (z(t), \hat{\theta}(t)) \) converges to the largest invariant set \( E \) of (2.38) contained in \( \mathbb{R}^{n+\rho} \setminus \mathbb{R}^{n+\rho} | z = 0 \), that is, in the set where \( \dot{V}_n = 0 \).

We now set out to determine \( M \). On this invariant set, we have \( z = 0 \) and \( \dot{z} = 0 \). Setting \( z = 0 \), \( \dot{z} = 0 \) in (2.38) we obtain \( \dot{\theta} = 0 \) and

\[
(\theta - \hat{\theta})^T w_i = 0, \quad i = 1, \ldots, n, \quad \forall (z, \hat{\theta}) \in E.
\] (3.1)

Using (2.37) and (3.1) for \( i = 1 \) we get \( (\theta - \hat{\theta})^T \phi_1(0) = (\theta - \hat{\theta})^T \phi_1^e = 0 \) on \( M \). Recall from (2.4) that \( \alpha_1 = -c_1 z_1 - \hat{\theta}^T \phi_1 \). Therefore, on \( M \) we have \( \alpha_i = -\hat{\theta}^T \phi_i^e = -\theta^T \phi_i^e \). Combining this with \( z_2 = 0 \), we get \( x_2 = x_2^e \) on \( M \). Using (2.37) and (3.1) for \( i = 2 \), we obtain \( (\theta - \hat{\theta})^T (\phi_2 - \alpha_1 \phi_1) = 0 \). Since on \( M \) we have \( (\theta - \hat{\theta})^T \phi_1^e = 0 \) and \( \phi_2(x_1, x_2) = \phi_2(0, -\theta^T \phi_1^e) = \phi_2^e \), this implies that \( (\theta - \hat{\theta})^T \phi_2^e = 0 \). Continuing in the same fashion, we prove that \( x_i = x_i^e \) and \( (\theta - \hat{\theta})^T \phi_i^e = 0 \) on \( M \), \( i = 1, \ldots, n \).

Thus, the largest invariant set \( M \) in \( E \) is

\[
M = \left\{ (z, \hat{\theta}) \in \mathbb{R}^{n+\rho}: z = 0, \ (\theta - \hat{\theta})^T \phi_i^e = 0, \ i = 1, \ldots, n \right\}.
\] (3.2)

These two equivalent expressions for \( M \) and the convergence of \( (z(t), \hat{\theta}(t)) \) to \( M \) prove that \( x(t) \rightarrow x^e \) as \( t \rightarrow \infty \).

Another important property of \( M \) is its dimension \( p - r \), where \( r = \text{rank}[\phi_1^e, \ldots, \phi_n^e] \). It is well known from the adaptive control literature that as the dimension of \( M \) is reduced, the robustness properties of
the adaptive system are improved. When \( r = p \), the dimension of \( M \) is zero, that is, \( M \) becomes the equilibrium point \( x = x^e, \dot{\theta} = \theta \). This means that the parameter estimates converge to their true values, so that the equilibrium \( x = x^e, \dot{\theta} = \theta \) is globally asymptotically stable.

The above facts prove the following result:

**Theorem 1.** Suppose that the design procedure is applied to the parametric-strict-feedback system (1.1). Then, the equilibrium \( x = x^e, \dot{\theta} = \theta \) of the resulting adaptive system is globally stable. Furthermore, its state \((x(t), \dot{\theta}(t))\) converges to the \((p - r)\)-dimensional manifold \( M \) given by (3.2). The equilibrium \( x = x^e, \dot{\theta} = \theta \) is globally asymptotically stable if and only if \( r = p \). \[QED\]

The new procedure can achieve global asymptotic stability with as many as \( p = n \) unknown parameters, while in earlier procedures that number was at most \( p = 2 \).

4. A design example

Let us illustrate the difference between the new procedure and the procedure of [2,4] on the 'benchmark' example:

\[
\begin{align*}
\dot{x}_1 &= x_2 + \theta \phi(x_1), \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= u.
\end{align*}
\]

The controller of [2,4] employs three estimates \( \theta_1, \theta_2, \theta_3 \):

\[
\begin{align*}
\dot{\theta}_1 &= z_1 \phi, \\
\dot{\theta}_2 &= -z_2 \frac{\partial \alpha_1}{\partial x_1} \phi, \\
\dot{\theta}_3 &= -z_3 \frac{\partial \alpha_2}{\partial x_1} \phi.
\end{align*}
\]

The corresponding stabilizing functions and the feedback control \( u \) are

\[
\begin{align*}
\alpha_1 &= -z_1 - \dot{\theta}_1 \phi, \\
\alpha_2 &= -z_2 - z_1 + \frac{\partial \alpha_1}{\partial x_1} (x_2 + \dot{\theta}_2 \phi) + \frac{\partial \alpha_1}{\partial \theta_1} z_1 \phi, \\
u &= -z_3 - z_2 + \frac{\partial \alpha_2}{\partial x_1} (x_2 + \dot{\theta}_3 \phi) + \frac{\partial \alpha_2}{\partial x_2} x_3 + \frac{\partial \alpha_2}{\partial \theta_2} z_1 \phi - \frac{\partial \alpha_2}{\partial \theta_2} z_2 \frac{\partial \alpha_1}{\partial x_1} \phi.
\end{align*}
\]

In contrast to the three estimates in (4.2), the new design procedure needs only one estimate \( \dot{\theta} \). The tuning functions are \( \tau_1 = z_1 \phi, \tau_2 = \tau_1 - z_2 (\partial \alpha_1 / \partial x_1) \phi, \tau_3 = \tau_2 - z_3 (\partial \alpha_2 / \partial x_1) \phi \), and the update law for \( \dot{\theta} \) is

\[
\dot{\theta} = \tau_3 = z_1 \phi - z_2 \frac{\partial \alpha_1}{\partial x_1} \phi - z_3 \frac{\partial \alpha_2}{\partial x_1} \phi.
\]
while the stabilizing functions and the feedback control $u$ are

$$
\alpha_1 = -z_1 - \dot{\phi}, \quad (4.5a)
$$

$$
\alpha_2 = -z_2 - z_1 + \frac{\partial \alpha_1}{\partial x_1} (x_2 + \dot{\phi} \phi) + \frac{\partial \alpha_1}{\partial \theta} \tau_2, \quad (4.5b)
$$

$$
u = -z_3 - z_2 + \frac{\partial \alpha_2}{\partial x_1} (x_2 + \dot{\phi} \phi) + \frac{\partial \alpha_2}{\partial x_2} x_3 + \frac{\partial \alpha_2}{\partial \theta} \tau_3 - z_2 \frac{\partial \alpha_1}{\partial \theta} \frac{\partial \alpha_2}{\partial x_1} \phi. \quad (4.5c)
$$

Comparing (4.2)-(4.3) with (4.4)-(4.5), we see that the new design procedure reduces the dynamic order of the adaptive controller.

5. Concluding remarks

The new design procedure eliminates the need for more than the minimum number of parameter estimates. Among the advantages of having exactly $p$ estimates for $p$ parameters are a reduction in the controller’s dynamic order, enhanced stability properties, and improved parameter convergence. In particular, if there are at most $n$ unknown parameters and $r = p$, then the adaptive system is globally asymptotically stable, and, hence, more robust to disturbances.

References


