Deterministic and Stochastic Newton-based extremum seeking for higher derivatives of unknown maps with delays

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We present a Newton-based extremum seeking algorithm for maximizing higher derivatives of unknown maps in the presence of time delays using deterministic perturbations. Different from previous works about extremum seeking for higher derivatives, arbitrarily long input-output delays are allowed. We incorporate a predictor feedback with a perturbation-based estimate for the Hessian's inverse using a differential Riccati equation. As a bonus, the convergence rate of the real-time optimizer can be made user-assignable, rather than being dependent on the unknown Hessian of the higher-derivative map. Averaging method for arbitrary shaped derivatives under delays is presented. Exponential stability and convergence to a small neighbourhood of the unknown extremum point are achieved for locally quadratic derivatives by using a backstepping transformation and averaging theory in infinite dimensions. Furthermore, we give a brief introduction into stochastic Newton-based Extremum Seeking for constant output delays, where we show the differences and similarities with respect to the deterministic case. We also present illustrative numerical examples in order to highlight the effectiveness of the proposed predictor-based extremum seeking for time-delay compensation applying both deterministic and stochastic perturbations.

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1. Introduction

Extremum Seeking (ES) is a non-model based and real-time optimization technique for nonlinear equilibrium maps that impose local optimum, either minimum or maximum. In recent years, there have been lots of publications on ES in theory [5,7,9,13,21,22,30,31] as well as applications [6,26,27,29,33]. In [8], Newton-based ES (free of delays) was deeply studied. A highlight of these works is the approach used to estimate the Hessian's inverse of the nonlinear map, which is generated by means of a Riccati filter. This is applied to remove the dependence of the algorithm's convergence rate on the second derivative (Hessian), making it user-assignable. The results mentioned above dealt only with extremum seeking for the map itself.

However, there are applications where an extremum of the map's higher derivative is sought. In [32] the authors present a refrigeration system where a suitable operating point is located at the maximum negative slope that is subject to change. This point of zero curvature corresponds to a minimum of the first derivative of the input-output map. Hence, being able to track the minimum of the first derivative in real-time would allow the system to operate almost the whole time at the most suitable operating condition.

A Newton-based ES generalization was presented in [19] to maximize arbitrary higher derivatives of an unknown map. Using periodic perturbations, estimation of the gradient and the Hessian of map's $n$th derivative were obtained as well as the local stability proof for the closed-loop system. However, reference [19] does not cope with maps under delays. Time delays are some of the most common phenomena that arise in engineering practice and need to be handled carefully since even small delays may result in a degradation of the system's behaviour or even lead to instability. The first publications that deal with Newton-based ES in the presence of constant and known time delays are [23–25], where only the extremum of a map was sought and an not the maximization-minimization of its derivatives.

In this paper, we extend the applicability and usage of the predictor-based controller with an averaging-based estimate of the Hessian's inverse proposed in [23–25] to maximize higher derivatives of a static map despite the presence of time delays [28]. Our

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generalization for the Newton optimization uses the Hessian estimate of the map’s higher derivative for the purpose of implementing a predictor that compensates the delay and makes the convergence rate independent of the unknown parameters of the nonlinear map. The convergence properties of the ES algorithm for maximizing arbitrary shaped derivatives by using only measurements of the map are outlined first in the spirit of finite spectrum assignment. After that, we rigorously prove the stability for locally quadratic derivatives via backstepping transformation [14] and averaging theory in infinite dimensions [10,16], considering the whole system which is infinite dimensional due to delays.

As an additional contribution, we give a brief introduction into the stochastic generalization of the Newton-based ES algorithm with constant output delays, stating the main differences with respect to the deterministic case in terms of design and analysis. There are clear benefits in investigating the use of stochastic perturbations over the deterministic ES architecture, as discussed in [18]. For instance, limitations of the deterministic ES scheme include the fact that learning using a periodic excitation signal is rather simple-minded and rare in probing-based learning and optimization approaches [17], which may lead to slower convergence rates. In addition, ES algorithms inspired by bio-mimicry [20] and others sensitive to deterministic perturbation signals [3] suggest other perturbation techniques using random motion rather than periodic ones.

At the last, numerical simulations show the applicability of the proposed algorithms in online maximization-minimization problems as well as comparison results for both deterministic and stochastic perturbations.

- Notation and terminology – The 2-norm of a finite-dimensional (ODE) state vector \( x(t) \) is denoted by single bars, \(|x(t)|\). In contrast, norms of functions \( (x) \) are denoted by double bars. By default, \( || \cdot || \) denotes the spatial \( L_2[0, D] \) norm, i.e., \( || \cdot || = || \cdot ||_{L_2[0, D]} \). Since the PDE state variable \( u(x, t) \) is a function of two arguments, we should emphasize that taking a norm in one of the variables makes the norm a function of the other variable. For example, \( L_2[0, D] \) norm of \( u(x, t) \) in \( x \in [0, D] \) is \( ||u(t)|| = \left( \int_0^D u^2(x, t) dx \right)^{1/2} \).

The partial derivatives of \( u(x, t) \) are denoted by \( u_t(x, t) \) and \( u_x(x, t) \) or, occasionally, by \( \partial_t u(x, t) \) and \( \partial_x u(x, t) \) to refer to the operator for its average signal \( u_x(x, t) \). Consider a generic nonlinear system \( \dot{x} = f(t, x, e) \), where \( x \in \mathbb{R}^n, f(t, x, e) \) is periodic in \( t \) with period \( \Pi \), i.e., \( f(t + \Pi, x, e) = f(t, x, e) \). Hence, for \( e > 0 \) sufficiently small, we can obtain its average model given by \( \bar{x}_e = \bar{f}(x, e) \), with \( \bar{f}(x, e) = \frac{1}{\Pi} \int_0^\Pi f(t, x, 0) dt \), where \( x(t) \) denotes the average version of the state \( x(t) \).

As defined in [12], a vector function \( f(x, e) \in \mathbb{R}^n \) is said to be of order \( O(e) \) over an interval \([t_1, t_2]\) if there exist positive constants \( k \) and \( e^* \) such that \( |f(t, e)| \leq ke(e \leq e^*) \) and \( \forall t \in [t_1, t_2] \). In this manuscript we will be referring to \( O(e) \) being an order of magnitude relation, which is valid for \( e \) sufficiently small. Moreover, we define any arbitrary initial time as \( t_0 \geq 0 \).

2. Newton-based extremum seeking of higher derivatives under delays

Scalar ES considers applications in which one wants to maximize (or minimize) the output \( y \in \mathbb{R} \) of an unknown nonlinear static map \( h(\theta) \) by varying the input \( \theta \in \mathbb{R} \) in real time. But like in many technical applications we have to consider that the output may be time-delayed [2], and hence, we additionally assume that there is a constant and known delay \( D \geq 0 \) such that the output is expressed by

\[
y(t) = h(\theta(t - D)).
\]

In this paper, we assume that our system is output-delayed. Since any input delay can be moved to the output of a static map, the results from this paper can be directly extended to the input-delay case. Also the case when input delays \( D_{in} \) and output \( D_{out} \) delays occur simultaneously can be handled by assuming that the total delay is \( D = D_{in} + D_{out} \). With \( D_{in}, D_{out} \geq 0 \). Furthermore, we only consider measurements without noise and/or disturbances that is not an objective of this paper and should be handled separately.

Without loss of generality, let us consider the maximization of \( n^{th} \) derivative of the output in the presence of time delay using Newton-based ES, where the maximizing value of \( \theta \) is denoted by \( \theta^* \). We state our optimization problem as follows:

\[
\max_{\theta \in \mathbb{R}} h^{(n)}(\theta(t - D)),
\]

with nonlinear map \( h(\cdot) \) satisfying the next assumption.

**Assumption 1.** Let \( h^{(n)}(\cdot) \) be the \( n^{th} \) derivative of a smooth function \( h(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \). Now let us define

\[
\Theta_{max} = \{ \theta | h^{(n+1)}(\theta) = 0, \ h^{(n+2)}(\theta) < 0 \}
\]

which is a collection of maxima where \( h^{(n)} \) is locally concave. Now assume that \( \Theta^* \in \Theta_{max} \) and \( \Theta_{max} \neq \emptyset \).

In Fig. 1, we illustrate the proposed scalar version of the Newton-based ES for maximization of higher derivatives based on predictor feedback for delay compensation. The design parameters are \( k, k_8, a, o, c > 0 \) as presented in Fig. 1. According to [19], we switch from maximization to minimization problem by setting \( \text{sgn}(\varphi_0) = \text{sgn}(h^{(n+2)}(\theta^*)) \) with \( \varphi_0 \) as initial value of \( \varphi \).

2.1. System and signals

Let \( \hat{\theta} \) be the estimate of the maximizer and

\[
\hat{\theta}(t) = \hat{\theta}(t) - \theta^*
\]

be the estimation error. From the block diagram in Fig. 1, the error dynamics can be written as

\[
\dot{\hat{\theta}}(t) = \hat{\theta}(t) = U(t).
\]

Moreover, we have

\[
\varphi = k_8(1 - \varphi h^{(n+2)}),
\]

where (6) is a differential Riccati equation. Eq. (6) will be used to generate an estimate of the Hessian’s inverse [8] according to the following error transformation

\[
\dot{\varphi} = \gamma - \frac{1}{h^{(n+2)}(\theta^*)}.
\]

Rearranging the equations given in [19] for the block diagram in Fig. 1 including delays, we can write:

\[
\theta = \hat{\theta} + a \sin(\omega t),
\]

\[
\varphi = C_j \sin \left( j \omega t + \frac{\pi}{4} (1 - (-1)^j) \right),
\]

\[
C_j = \frac{2j}{a^2} (-1)^j.
\]

\[
F = \frac{j! \sin \left( \frac{2|j\pi|}{2} \right)}{2
\]

We have defined the additive dither signal as

\[
S(t) = a \sin(\omega t)
\]

(12)
which is not delayed. However, as shown in Fig. 1, the multiplicative dither signals $\Upsilon_{(n+1)}$ and $\Upsilon_{(n+2)}$ are delayed by $D$ in order to cope with the delayed output $y$ [24] so that the demodulated signals

$$h^{(n+1)}(t) = \Upsilon_{(n+1)}(t - D) y(t - D),$$  
(13)

$$h^{(n+2)}(t) = \Upsilon_{(n+2)}(t - D) y(t - D).$$  
(14)

can be obtained [19]. Basically, $\hat{h}^{(j)}(t) = y \Upsilon_{j}$, are used to estimate the delayed gradient ($j = n + 1$) and the Hessian ($j = n + 2$) of $\hat{h}^{(n)}$. As discussed in [23,25] one could also advance the dither signal (12) with $D$ time units instead of delaying $\Upsilon_{(n+1)}$ and $\Upsilon_{(n+2)}$.

Furthermore, let us define the measurable signal

$$z(t) = y(t) \hat{h}^{(n+1)},$$  
(15)

where $y$ is updated according to (6) and $\hat{h}^{(n+1)}$ being the demodulated signal in (13). We can construct (15) because we only use measurable/available signals $y(t)$ and $\hat{h}^{(n+1)}$.

### 2.2. Averaging analysis

Let $\tilde{\theta}_av$ be the average version of the signal $\hat{\theta}$ defined in (4) and $\tilde{y}_av$ be the average version of $\hat{y}$ in (7), according to the notation and terminology presented in Section 1.

Consider the particular average version of the signal $\eta_j := \Upsilon_j y$ which is given by

$$\eta_j(\tilde{\theta}_av) = \frac{1}{\Pi} \int_0^{\Pi} \eta_j(\tilde{\theta}_av + \theta^* + \sigma \sin(\sigma)) d\sigma,$$  
(16)

where $\Pi = 2\pi/\omega$. Now, one can provide a convenient expression of the demodulated signal $\eta_j(\tilde{\theta}_av)$ for arbitrary $j = n + 1$ as follows [19]

$$\eta_j(\tilde{\theta}_av) = h^{(j)}(\tilde{\theta}_av + \theta^*) + \frac{h^{(j+2)}(\tilde{\theta}_av + \theta^*)}{4(j+1)} \sigma^2 + O(\sigma^4).$$  
(17)

All maps satisfying Assumption 1 can be approximated locally by a quadratic function

$$h^{(n)}(\theta) = Q^* + \frac{H}{2} (\theta - \theta^*)^2,$$  
(18)

for some constants $Q^* > 0$ and $H < 0$, where $H$ is the Hessian of the quadratic approximation [1, p.4]. Noticing that $X = 0$ for locally quadratic assumption of $h^{(n)}$, plugging (18) into (16) we get

$$\eta_{n+1}(\tilde{\theta}_av) = H \hat{\theta}_av(t - D).$$  
(19)

**Lemma 1.** In order to maximize any higher derivative order of the unknown, nonlinear and time-delayed map $h(\theta(t - D))$ satisfying Assumption 1, the average and linearized version of the measurable signal $z(t)$ in (15) can be expressed as:

$$z_{av}(t) = \tilde{\theta}_av(t - D)$$  
(20)

by using the periodic perturbation $S(t)$ in (12), as well as the demodulated signals in (9)–(11), for $j = n + 1$ and $j = n + 2$.

**Proof.** Analogously to [19], the following delayed equations for the average error system can be written

$$\frac{d\hat{\theta}_av(t)}{dt} = -k \gamma \eta_{n+1}(\tilde{\theta}_av(t - D)).$$  
(21)

$$\frac{d\tilde{y}_av(t)}{dt} = k \gamma \eta_{n+2}(\tilde{\theta}_av(t - D)).$$  
(22)

with $\gamma \neq 0$ and $\gamma \neq 0$. Hence, the right-hand side of (21) is zero for $\eta_{n+1}(\tilde{\theta}_av) = 0$ and it follows from (19) that the equilibrium of (21) is $\tilde{\theta}_av = 0$. In order to obtain the equilibrium for (22), we first calculate $\eta_{n+2}(\tilde{\theta}_av)$ which is simply $H$. It follows that (22) is zero for $1 - \gamma \gamma = 0$ which is true for $\gamma H = H^{-1}$. Now, consider Eq. (15) which can be written in its average version as

$$z_{av}(t) = \gamma_{av}(t) \eta_{n+1}(\tilde{\theta}_av(t)).$$  
(23)

We linearize $z_{av}$ in (23) at $\gamma_{av}(t) = 1/h^{(n+2)}(\theta^*) = H^{-1}$ and $\tilde{\theta}_av = 0$, under Assumption 1, such that

$$z_{av}(t) = \gamma_{av}(t) \eta_{n+1}(\tilde{\theta}_av(t))$$

$$= \gamma_{av}(t) h^{(n+1)}(\tilde{\theta}_av(t - D))$$

$$= \frac{\gamma_{av}(t) h^{(n+1)}(\tilde{\theta}_av(t - D))}{\gamma_{av}(t) = 0, \gamma_{av} = H^{-1} = 0}$$

$$+ \frac{\gamma_{av}(t) h^{(n+2)}(\tilde{\theta}_av(t - D))}{\gamma_{av}(t) = 0, \gamma_{av} = H^{-1} = 0}$$

$$+ \frac{\gamma_{av}(t) h^{(n+2)}(\tilde{\theta}_av(t - D))}{\gamma_{av}(t) = 0, \gamma_{av} = H^{-1} = 0}$$

$$= H^{-1} \gamma_{av}(t - D) \hat{\theta}_av(t - D),$$

and hence get for (15) the convenient expression (20). □

### 2.3. Motivation for predictor feedback

From (5), it follows

$$\hat{\theta}(t - D) = U(t - D).$$  
(24)

Now using (5), the next “shifted” average model can be derived

$$\hat{\theta}_av(t) = U_{av}(t).$$  
(25)
with \( U_{av} \in \mathbb{R} \) being the resulting average control for \( U \in \mathbb{R} \). One can try to feed back the future state \( z_{av}(\sigma + D) \) in the equivalent average system where \( \sigma \) is time. Assuming that \( \sigma = t - D \) we get
\[
\dot{z}_{av}(\sigma + D) = z_{av}(t - D) + \dot{z}_{av}(t)
\]  
and, thus, the time-delay compensated signal which we feed back.
The delayed version of (25) is \( \dot{\theta}_{av}(t - D) = U_{av}(t - D) \). Applying (20) and (25) one has
\[
\dot{z}_{av}(t) = U_{av}(t - D).
\]  
Feeding back the future state \( z_{av}(t + D) \) for the delay compensation motivates the use of predictor feedback design. Applying the variation-of-constants formula [14] to (27), the future state can be calculated as
\[
z_{av}(t + D) = z_{av}(t) + \int_{t-D}^{t} U_{av}(\sigma)d\sigma.
\]  
We derive a controller as follows:
\[
U_{av}(t) = \dot{\theta}_{av}(t) = -kz_{av}(t + D),
\]  
\( \forall t \geq D \) and \( k > 0 \), which yields to an average predictor-based control
\[
U_{av}(t) = -k\left[z_{av}(t) + \int_{t-D}^{t} U_{av}(\sigma)d\sigma\right].
\]  
Furthermore using (23), (29) as well as time shifting we obtain the average error dynamics of the following form
\[
\dot{\theta}_{av}(t) = -kz_{av}(t + D) = -k\theta_{av}(t),
\]  
\( \forall t \geq D \), with eigenvalues that are determined by \(-k\). Thus, the system has an exponentially attractive equilibrium and the convergence rate is independent of the unknown Hessian.

2.4. Predictor feedback

We propose the predictor-based controller which incorporates low-pass filter [23]
\[
U(t) = \frac{c}{s + c}\left[-k\left[z(t) + \int_{t-D}^{t} U(\tau)d\tau\right]\right]
\]  
where \( c > 0 \) is sufficiently large, i.e., the predictor feedback is of the form of a low-pass filtered of the non average version of (30). Note that we mix the time and frequency domain notation in (32) by using the braces \( \{ \cdot \} \) to denote that the transfer function acts as an operator on a time-domain function. The difficulty arises in the existing averaging theorem to infinite dimensions due to delays [10,16]. For the class of functional differential equations (FDEs) studied here, there is no "off the shelf" averaging theorem result oriented for distributed input-output delays. In general, the theory applies only to state-delay systems. This fact leads us to propose a simple modification of the basic predictor-based controller, which employs a low-pass filter, to still achieve our objectives [10,16].

The controller (32) is infinite-dimensional because the integral in (32) involves the control history over the interval \([t - D, t]\), averaging-based (perturbation-based) because \( z \) (15) is updated according to the estimate \( \dot{\gamma} \) for the unknown Hessian’s inverse \( H^{-1} \) given by (6), with \( \tilde{h}^{n+2} \) satisfying the averaging property stated in [16].

3. Convergence analysis for \( t \geq t_0 + D \)

Consider the error transformations stated in (4) and (7). Using Fig. 1 we rewrite (32) to
\[
U(t) = -k - \frac{c}{s + c}z(t + D).
\]  
Now using (4) and (33), one can write the closed-loop dynamics (controller already “kicked in” and compensates the time delay)
\[
\dot{\hat{\theta}} = \hat{\theta} = U,
\]  
\( \hat{U} = -cU - kcz(t + D) \).

We rewrite the system (7), (34), and (35) into a compact form
\[
\frac{d}{dt} \begin{bmatrix} \dot{\theta} \\ U \\ \tilde{\gamma} \end{bmatrix} = \begin{bmatrix} -cU - kcz(t + D) \end{bmatrix},
\]  
with
\[
M = k\left(\tilde{\gamma} + H^{-1}\right)\left(1 - (\tilde{\gamma} + H^{-1})\right)X_{(n+2)}h(\theta + a \sin(\omega t)).
\]  
System (36) is in the form to which averaging method is applicable (\( z \) is the “frozen/slow” variable). The average model of (36) is
\[
\frac{d}{dt} \begin{bmatrix} \bar{\theta} \\ \bar{U} \\ \bar{\tilde{\gamma}} \end{bmatrix} = \begin{bmatrix} \bar{f}_{av} \\ \bar{g}_{av} \\ \bar{\tilde{\gamma}}_{av} \end{bmatrix} = \begin{bmatrix} -cU_{av} - kcz_{av}(t + D) \end{bmatrix},
\]  
with
\[
M_{av} = \left(1 - (\tilde{\gamma}_{av} + H^{-1})\eta_{n+2}(\bar{\theta}_{av})\right)\times k_{l}(\bar{\gamma}_{av} + H^{-1}).
\]  
In the next step we determine the average equilibrium \( (\bar{\theta}_{av}^{E}, \bar{U}_{av}^{E}, \bar{\tilde{\gamma}}_{av}^{E}) \) which satisfies
\[
U_{av} = 0,
\]  
\( z_{av}(t + D) = 0 \),
\( 1 - (\tilde{\gamma}_{av} + H^{-1})\eta_{n+2}(\bar{\theta}_{av}) = 0 \),

since \( \gamma_{av}(t + D) \neq 0 \) [19]. From (23), Eq. (41) is of the form
\[
z_{av}(t + D) = \gamma_{av}(t + D)\eta_{l}(t + D) = 0,
\]  
and it is zero for \( \eta_{l}(t + D) = 0 \). The solutions of (42) and (43) were already stated in [19] as:
\[
\bar{\tilde{\gamma}}_{av}^{E} = \tilde{\gamma}^{*} + O(a^{3}),
\]  
where
\[
\tilde{\gamma}^{*} = - \frac{h^{(n+3)}(\theta^{*})}{4(n + 2)h^{(n+3)}(\theta^{*})}a^{2},
\]  
Applying the results from [19] and some nontrivial calculations, the following lemma can be obtained.

Lemma 2. The Jacobian of the system (38) evaluated at the average equilibrium \( \phi = (\bar{\theta}_{av}^{E}, \bar{U}_{av}^{E}, \bar{\tilde{\gamma}}_{av}^{E}) \) has the following form
\[ A(\phi) = \begin{bmatrix} 0 & 1 & 0 \\ -k_c + O(a^2) & -c & 0 \\ \partial_\theta z & \partial_\theta t & 0 \end{bmatrix}, \]

and is Hurwitz for \( k, c > 0 \) and sufficiently small “\( a \)".

**Proof.** In Appendix A, two methods to prove the lemma are presented: one exploring the cascade connection of two exponentially stable sub-systems and another that is based on Liénard-Chipart's criterion [4]. □

This implies that (38) is locally exponentially attractive about \( \phi = (\theta^E_u, u^E_u, \tilde{\gamma}^E_u) \), for all \( t \geq t_0 + D \), where \( t_0 \) is any arbitrary initial time.

Hence, we can follow that (36) is exponentially convergent. The analysis sketched above is in the spirit of finite spectrum assignment and does not capture the entire system consisting of the ODE in \( \dot{\theta}(t - D) = U(t - D) \), the delayed vector field (5), and the infinite dimensional subsystem of the input delay.

### 4. Stability analysis

In the following, we give the main stability/convergence result for the closed-loop system using steps stated in [23] and [25].

**Theorem 1.** There exists \( c^* > 0 \) such that, \( \forall c > c^*, \exists \omega(c) > 0 \) such that, \( \forall \omega > \omega^*, \) the delayed closed-loop system (24) and (32), with \( \tilde{z}(t) \) in (15) \( h^{a(t)} \) calculated as in (13), \( y(t) \) in (6) and state \( \tilde{y}(t), \hat{\theta}(t - D), U(\sigma), \sigma \in [t - D, t], \) has a unique locally exponentially stable periodic solution in \( t \) of period \( \Pi = 2\pi / \omega \), denoted by \( \tilde{y}^\Pi(\tau), \hat{\theta}^\Pi(t - D), U^\Pi(\sigma), \forall \sigma \in [t - D, t], \) satisfying, \( \forall t \geq t_0: \)

\[
\left( \left| \tilde{y}^\Pi(t) \right|^2 + \left| \hat{\theta}^\Pi(t - D) \right|^2 + \left[ U^\Pi(t) \right]^2 + \int_{t-D}^{t} \left[ U^\Pi(\tau) \right]^2 d\tau \right)^{1/2} \leq O\left( \frac{1}{t} \right). \]

Furthermore,

\[
\limsup_{t \to +\infty} |\hat{\theta}(t) - \theta^*| = O(\alpha + 1/\omega). \]

**Proof.** The demonstration follows Steps 4.1 to 4.8. □

#### 4.1. Transport PDE for delay representation

According to [14], the delayed version of (5) can be represented using a transport PDE as

\[ \dot{\theta}(t - D) = U(0, t), \]

\[ u(t, x) = u_c(x, t), \quad x \in [0, D], \]

\[ u(D, t) = U(t), \]  

where the solution of (50)–(51) is

\[ u(x, t) = U(t + x - D). \]

#### 4.2. Equations of the closed-loop system

First, we derive an expression for the output in terms of \( \dot{\theta} \) using (8) and plugging it into (1)

\[ y(t) = h(\dot{\theta}(t - D) + \omega(\dot{\theta}(t - D)). \]

By representing the integrand in (32) using transport PDE state it follows

\[ U(t) = \frac{c}{s + c} \left\{ -k \left[ z(t) + \int_0^D u(\sigma, t)d\sigma \right] \right\} \]

with \( z(t) \) as in (15). Now using (53) and (54) one can rewrite (49)–(50) as

\[ \dot{\theta}(t - D) = U(0, t), \]

\[ \partial_t u(t, x) = \partial_t u_c(x, t), \quad x \in [0, D], \]

\[ u(D, t) = \frac{c}{s + c} \left\{ -k \left[ z(t) + \int_0^D u(\sigma, t)d\sigma \right] \right\}. \]

4.3. Average model of the closed-loop system

The average version of system (55)–(57) is:

\[ \dot{\bar{\theta}}(t - D) = \bar{u}(0, t), \]

\[ \partial_t \bar{u}(x, t) = \partial_t \bar{u}_c(x, t), \quad x \in [0, D], \]

\[ \bar{u}(D, t) = \frac{c}{s + c} \left\{ -k \left[ z(t) + \int_0^D u(\sigma, t)d\sigma \right] \right\}, \]

Denoting

\[ \dot{\bar{\theta}}(t) = \bar{\theta}(t - D), \]

\[ \bar{u}(t) = \bar{u}_c(t - D) = \bar{z}(t), \]

which is valid only locally because the quadratic condition invoked to derive (20), one has the following linearized average version of system (55)–(57):

\[ \dot{\bar{\theta}}(t) = \bar{u}(0, t), \]

\[ \partial_t \bar{u}(x, t) = \partial_t \bar{u}_c(x, t), \quad x \in [0, D], \]

\[ \bar{u}(D, t) = \frac{c}{s + c} \left\{ -k \left[ z(t) + \int_0^D u(\sigma, t)d\sigma \right] \right\}, \]

where the filter \( c/s+c \) is also represented in the state-space form. The solution of the transport PDE (64)–(65) is given by \( u_c(x, t) = U_c(t + x - D) \).

From (6) and (18) we know that a local average for the Hessian’s inverse estimation error is \( d\bar{u}(t) = -k \bar{\gamma} \) and its linearized version

\[ \frac{d\bar{\gamma}(t)}{dt} = -k_R \bar{\gamma}(t), \]

which is stable since \( k_R \) is positive.

#### 4.4. Backstepping transformation, its inverse and the target system

Consider the infinite-dimensional backstepping transformation of the delay state

\[ w(x, t) = u_c(x, t) + k \left[ \partial_t \bar{u}(t) + \int_0^t u_c(\sigma, t)d\sigma \right] \]

which maps the system (63)–(65) into the target system

\[ \dot{\bar{\theta}}(t) = -k_R \bar{\gamma}(t) + w(0, t), \]

\[ w(x, t) = w_c(x, t), \quad x \in [0, D], \]

\[ w(D, t) = \frac{1}{c} \partial_t \bar{u}_c(D, t). \]
Using (67) for \( x = D \) and the fact that \( u_{av}(D, t) = U_{av}(t) \), from (70) we get (65), i.e.,

\[
U_{av}(t) = \frac{c}{s+c} \left\{ \int_0^\infty \left[ -k \left[ \frac{\partial}{\partial \sigma} u_{av}(\sigma, t) \right] + \int_0^\sigma u_{av}(\sigma, t) \, d\sigma \right] \right\}
\]

Let us now consider \( w(D, t) \). It is easy to check that

\[
w_t(D, t) = \partial_t u_{av}(D, t) + ku_{av}(D, t),
\]

where \( \partial_t u_{av}(D, t) = \tilde{U}_{av}(t) \). The inverse of (67) is given by

\[
u_{av}(x, t) = w(x, t) - k \left[ e^{kx} \tilde{U}_{av}(t) + \int_0^x e^{-k(x-\sigma)} w(\sigma, t) \, d\sigma \right].
\]

Plugging (70) and (73) into (74), we get

\[
\dot{w}_t(D, t) = -cw(D, t) + kw(D, t)
\]

\[
- k^2 \left[ e^{-kD} \tilde{U}_{av}(t) + \int_0^D e^{-k(D-\sigma)} w(\sigma, t) \, d\sigma \right].
\]

4.5. Lyapunov–Krasovskii functional

Now consider the following Lyapunov functional

\[
V(t) = \frac{\tilde{U}_{av}^2(t)}{2} + \frac{a}{2} \int_0^D (1 + x) w^2(x, t) \, dx + \frac{1}{2} w^2(D, t),
\]

where the parameter \( a > 0 \) is to be chosen later. We have

\[
\dot{V}(t) = -k \tilde{U}_{av}^2(t) + \tilde{U}_{av}(t) w(D, t)
\]

\[
+ a \int_0^D (1 + x) w(x, t) w_t(x, t) \, dx + w(D, t) w_t(D, t)
\]

\[
= -k \tilde{U}_{av}^2(t) + \tilde{U}_{av}(t) w(D, t) + \frac{a(1 + D)}{2} w^2(D, t)
\]

\[
- \frac{1}{2} \dot{w}^2(0, t) + \frac{\tilde{U}_{av}^2(t)}{2a} - \frac{1}{2} \int_0^D w^2(x, t) \, dx + w(D, t) w_t(D, t)
\]

\[
\leq - \frac{k}{2} \tilde{U}_{av}^2(t) + \frac{\tilde{U}_{av}^2(t)}{2a} - \frac{1}{2} \int_0^D w^2(x, t) \, dx + w(D, t) \left[ w_t(D, t) + \frac{a(1 + D)}{2} w(D, t) \right].
\]

Reminding that \( k > 0 \), let us choose \( a = 1/k \). Then,

\[
\dot{V}(t) \leq - \frac{k}{2} \tilde{U}_{av}^2(t) - \frac{1}{2k} \int_0^D w^2(x, t) \, dx
\]

\[
+ w(D, t) \left[ w_t(D, t) + \frac{a(1 + D)}{2} w(D, t) \right]
\]

\[
= - \frac{1}{2a} \tilde{U}_{av}^2(t) - \frac{1}{2} \int_0^D w^2(x, t) \, dx + w(D, t) \left[ w_t(D, t) + \frac{a(1 + D)}{2} w(D, t) \right].
\]

Now consider (75) along with (74). With a completion of squares and remarking that \( \|w(D, t)\| \|e^{kD} - e^{kD-\sigma}\| \|w(D, t)\| \leq \frac{1}{2} \|e^{kD} - e^{kD-\sigma}\| \|w(D, t)\|^2 + \frac{1}{2} \|e^{kD} - e^{kD-\sigma}\| \|w^2(D, t)\| \) by using Cauchy–Schwarz and Young's inequalities, we obtain

\[
\dot{V}(t) \leq - \frac{1}{4a} \tilde{U}_{av}^2(t) + \frac{1}{4} \int_0^D w^2(x, t) \, dx + a \left| k \|e^{kD} - e^{kD-\sigma}\| \|w^2(D, t)\| \right.
\]

\[
+ \frac{a}{2} \left( k \|e^{kD} - e^{kD-\sigma}\| \right) \|w^2(D, t)\| - \left. \right) w^2(D, t) - cw^2(D, t).
\]

From (76), we arrive at

\[
\dot{V}(t) \leq - \frac{1}{4a} \tilde{U}_{av}^2(t) - \frac{a}{4(1 + D)} \int_0^D (1 + x) w^2(x, t) \, dx
\]

\[
- (c - c^*)w^2(D, t),
\]

where \( c^* = a(1 + D)k + a[k^2 e^{-kD}] + \frac{1}{2} \|k^2 e^{-k(D-\sigma)}\|^2\). Hence, from (77), if \( c \) is chosen such that \( c > c^* \), we obtain

\[
\dot{V}(t) \leq -\mu V(t),
\]

for some \( \mu > 0 \). Thus, the closed-loop system is exponentially stable in the sense of the full state norm

\[
\left( \|\tilde{U}_{av}(t)\|^2 + \int_0^t w^2(x, t) \, dx + w^2(D, t) \right)^{1/2},
\]

i.e., in the transformed variable \( (\tilde{U}_{av}, w) \).

4.6. Exponential stability estimate (in \( L_2 \) norm) for the average system (63)–(65)

To obtain exponential stability in the sense of the norm \( (\|\tilde{U}_{av}(t)\|^2 + \int_0^t u_{av}^2(t) \, dt) \), we need to show there exist positive numbers \( \alpha_1 \) and \( \alpha_2 \) such that

\[
\alpha_1 \Psi(t) \leq V(t) \leq \alpha_2 \Psi(t),
\]

where \( \Psi(t) \equiv \|\tilde{U}_{av}(t)\|^2 + \int_0^t u_{av}^2(t) \, dt + u_{av}^2(D, t) \). or equivalently

\[
\Psi(t) \leq \frac{\alpha_2}{\alpha_1} e^{-\alpha t} \Psi(0),
\]

which completes the proof of exponential stability.

4.7. Invoking averaging theorem

Using (6), (4) and (7), the dynamics of the estimation error of the Hessian can be written as

\[
\tilde{y} = \omega_I \left[ \tilde{y} + H^{-1} \right].
\]

Now, note that the closed-loop system (52), (38) and (83) can be rewritten as:

\[
\dot{\tilde{H}} = U(t - D),
\]

\[
\dot{U}(t) = -cU(t) - cK \left[ z(t) + \int_0^t U(\tau) \, d\tau \right],
\]

\[
\tilde{y}(t) = \omega_I \left[ \tilde{y} + H^{-1} \right]
\]

\[
\times [1 - h^{(\pi/2)} \tilde{y}(t) + H^{-1}].
\]

where \( \tilde{z}(t) = \tilde{y}(t) + H^{-1} \) is the state vector. Noting that \( \int_0^t U(\tau) \, d\tau = \int_0^t U(t + \tau) \, d\tau \) and recalling (6), (13), and (15) and (15),

\[
\dot{\xi}(t) = \alpha(t, \xi(t)),
\]

where \( \xi(t) = \xi(t + \Theta) \) for \( -\Theta \leq \Theta \leq 0 \) and \( f \) is an appropriate continuous functional, such that the averaging theorem by Hale–Nelson and Lehm [10,16] in Appendix B can be applied considering \( T = 2\pi/\omega \) and \( \omega = 1/\epsilon \).

From (82), the origin of the average closed-loop system (63)–(65) with transport PDE for delay representation is exponentially stable. In addition, we conclude the equilibrium \( \tilde{y}_{av}(t) = 0 \) of the linearized error system (66) is also exponentially stable since \( k_2 > 0 \). Thus, there exist positive constants \( \alpha \) and \( \beta \) such that all solutions
satisfy \( \hat{\Psi}(t) \leq \alpha e^{-\beta t} \hat{\Psi}(0) \), for all \( t \geq 0 \), where \( \hat{\Psi}(t) \triangleq |\hat{\gamma}_m(t)|^2 + \left| \hat{\theta}_\nu(t - D) \right|^2 + \int_{t-D}^t U^2_\nu(\tau) d\tau \).

Then, according to the averaging theorem [10,16], for \( \omega \) sufficiently large, (84)–(86) has a unique locally exponentially stable periodic solution around the origin satisfying (47).

4.8. Asymptotic convergence to a neighborhood of the extremum

By using the change of variables (61) and then integrating both sides of (49) within the interval \([t, t + D]\), we have

\[
\tilde{\theta}(\sigma + D) = \hat{\theta}(t) + \int_t^{t+D} U(0, s) ds. \tag{88}
\]

From (52), we can rewrite (88) in terms of \( U \), namely

\[
\tilde{\theta}(\sigma + D) = \hat{\theta}(t) + \int_{t-D}^{\sigma} U(\tau) d\tau. \tag{89}
\]

Now, note that

\[
\tilde{\theta}(\sigma) = \tilde{\theta}(\sigma + D), \quad \forall \sigma \in [t - D, t]. \tag{90}
\]

Hence,

\[
\tilde{\theta}(\sigma) = \hat{\theta}(t - D) + \int_{t-D}^{\sigma} U(\tau) d\tau, \quad \forall \sigma \in [t - D, t]. \tag{91}
\]

By applying the supremum norm in both sides of (91), we have

\[
\sup_{t-D \leq \sigma \leq t} |\tilde{\theta}(\sigma)| \leq \sup_{t-D \leq \sigma \leq t} |\hat{\theta}(t - D)| + \sup_{t-D \leq \sigma \leq t} \left| \int_{t-D}^{\sigma} U(\tau) d\tau \right|
\]

\[
\leq \sup_{t-D \leq \sigma \leq t} |\hat{\theta}(t - D)| + \sup_{t-D \leq \sigma \leq t} \int_{t-D}^\sigma |U(\tau)| d\tau
\]

\[
\leq |\hat{\theta}(t - D)| + \int_{t-D}^t |U(\tau)| d\tau \tag{92}
\]

By applying the square in the right-hand side of (92), we can write

\[
|\hat{\theta}(t - D)| \leq \left( |\hat{\theta}(t - D)|^2 + \int_{t-D}^t U^2(\tau) d\tau \right)^{1/2}. \tag{93}
\]

\[
\left( \int_{t-D}^t U^2(\tau) d\tau \right)^{1/2} \leq \left( |\hat{\theta}(t - D)|^2 + \int_{t-D}^t U^2(\tau) d\tau \right)^{1/2}. \tag{94}
\]

By using (93) and (94), one has

\[
|\hat{\theta}(t - D)| + \sqrt{D} \left( \int_{t-D}^t U^2(\tau) d\tau \right)^{1/2}
\]

\[
\leq (1 + \sqrt{D}) \left( |\hat{\theta}(t - D)|^2 + \int_{t-D}^t U^2(\tau) d\tau \right)^{1/2}. \tag{95}
\]

From (92), it is straightforward to conclude that

\[
\sup_{t-D \leq \sigma \leq t} |\tilde{\theta}(\sigma)| \leq (1 + \sqrt{D}) \left( |\hat{\theta}(t - D)|^2 + \int_{t-D}^t U^2(\tau) d\tau \right)^{1/2}. \tag{96}
\]

and, consequently,

\[
|\hat{\theta}(t)| \leq (1 + \sqrt{D}) \left( |\hat{\theta}(t - D)|^2 + \int_{t-D}^t U^2(\tau) d\tau \right)^{1/2}. \tag{97}
\]

Inequality (97) can be given in terms of the periodic solution \( \tilde{\theta}^*(t - D), U^\Pi(t), \forall t \in [t - D, t] \) as follows:

\[
|\hat{\theta}(t)| \leq (1 + \sqrt{D}) \left( |\hat{\theta}(t - D) - \tilde{\theta}^*(t - D) + \tilde{\theta}^*(t - D)|^2 + \int_{t-D}^t \left[ U(t) - U^\Pi(t) + U^\Pi(t) \right]^2 d\tau \right)^{1/2}. \tag{98}
\]

By applying Young’s inequality and some algebra, the right-hand side of (98) and \(|\hat{\theta}(t)|\) can be majorized by

\[
|\hat{\theta}(t)| \leq \sqrt{2} \left( 1 + \sqrt{D} \right) \left( |\hat{\theta}(t - D) - \tilde{\theta}^*(t - D)|^2 + \int_{t-D}^t [U(t) - U^\Pi(t) + U^\Pi(t)]^2 d\tau + \int_{t-D}^t [U^\Pi(t)]^2 d\tau \right)^{1/2}. \tag{99}
\]

According to the averaging theorem [10,16], we can conclude that

\[
|\hat{\theta}(t - D) - \tilde{\theta}^*(t - D)| \to 0, \tag{100}
\]

\[
\int_{t-D}^t \left[ U(t) - U^\Pi(t) \right]^2 d\tau \to 0, \tag{101}
\]

exponentially. Hence,

\[
\limsup_{t \to +\infty} |\hat{\theta}(t)| = \sqrt{2} \left( 1 + \sqrt{D} \right) \times \left( |\tilde{\theta}^*(t - D)|^2 + \int_{t-D}^t [U^\Pi(t)]^2 d\tau \right)^{1/2}. \tag{102}
\]

From (47) and (102), we can write

\[
\limsup_{t \to +\infty} |\hat{\theta}(t)| = O(1/\omega). \tag{103}
\]

From (4) and reminding that \( \theta(t) = \tilde{\theta}(t) + S(t) \), one has that \( \theta(t) - \theta^* = \tilde{\theta}(t) + S(t) \). \tag{104}

Since the first term in the right-hand side of (104) is ultimately of order \( O(1/\omega) \) and the second term is of order \( O(\omega) \), then (48) is verified. \( \Box \)

5. Stochastic Newton-based extremum seeking for constant delays

In the previous sections, we have presented a basic Newton-based extremum seeking using deterministic (periodic) perturbations, which are used to probe the higher derivatives of the nonlinear map and estimate both its gradient and Hessian.

Basically, all problem formulation constructed before repeats itself for the purpose of obtaining a stochastic version of the proposed algorithm. The principal change is the employment of stochastic perturbations via the sinusoid of a Wiener process about the boundary of a circle redefining the signal \( S(t) \):

\[
S(t) = a \sin(\eta(t)). \tag{105}
\]

for \( a > 0 \) sufficiently small, such that \( \theta(t) = \tilde{\theta}(t) + a \sin(\eta(t)) \). \tag{106}

where

\[
\eta(t) = \omega \tau (1 + \|W_{\omega t}\|). \tag{107}
\]

This stochastic perturbation process (107) represents a homogeneous ergodic Markov process with invariant distribution, where \( W_{\omega t} \) is a standard Brownian motion process (also referred to as the Wiener process) [18]. To satisfy the Markov property, there is no difference in future predictions based on just the current process state or with its full history. We then refer to a Markov process as a stochastic process which satisfies the Markov property with respect to its natural filtration.
By inducing a slight movement of the output through (106), it allows a sufficiently precise estimate of the gradient and Hessian in an average sense through the special demodulation signals \( \Upsilon_{(n+1)} \) and \( \Upsilon_{(n+2)} \), which are also dependent on the same stochastic process (107). Newton’s Method is then applied utilizing these perturbation-based derivatives of the unknown dynamic map.

Reminding \( \hat{\theta} \) as the best estimate of the maximizing value \( \theta^* \), we can write again
\[
\frac{d}{dt} \hat{\theta}(t - D) = U(t - D),
\]
(108)
where the control signal \( U(t) \) is generated through a predictor using stochastic variables rather than deterministic.

The Riccati filter with stochastic input is defined by
\[
\frac{d}{dt} \gamma(t) = k_k \gamma(t) \left( 1 - \gamma(t) \hat{h}^{(n+2)}(t) \right)
\]
(109)
to dynamically estimate the unknown Hessian’s inverse, i.e., the inverse second of the derivative for \( h^{(n)}(\theta) \).

As already stated, the demodulated signals
\[
\hat{h}^{(k)}(t) = \Upsilon_k(t-D)\gamma(t-D)
\]
(110)
have the purpose of estimating the gradient \( k = n + 1 \) and the Hessian \( k = n + 2 \) of the n-th derivative. The demodulation signal (9) and the corresponding normalizing gain (10) are now changed to:
\[
\Upsilon_k(t) = C_k \sin \left( \frac{k \eta(t)}{4} + \frac{\pi}{2} (1 - (-1)^k) \right),
\]
(111)
\[
C_k = \frac{2^{k!}}{\alpha^k (-1)^{\frac{k-1}{2}}},
\]
(112)
with the stochastic perturbation \( \eta(t) \) described in (107).

In order to cope with the delayed output, again one has to delay the demodulation signal \( \Upsilon_k(t) \) by \( D \) time units. Moreover, advancing the perturbation signal (105) by \( D \) time units would lead to the same solution.

Similar to (15), we define
\[
z(t) = \gamma(t) \hat{h}^{(n+1)}(t),
\]
(113)
where \( \gamma(t) \), the inverse Hessian estimate, is updated according to (109), and \( \hat{h}^{(n+1)}(t) \) represents the gradient estimate (110) for \( k = n + 1 \).

By means of a similar averaging analysis performed in Section 2.2 but now exploring the ergodicity and invariant distribution of the stochastic process (107) [11] for the same error metrics given in (4) and (7), repeated below:
\[
\hat{\theta}(t) = \hat{\theta}(t) - \theta^*,
\]
(114a)
\[
\hat{\gamma}(t) = \gamma(t) - H^{-1},
\]
(114b)
it is possible to show that (113) is locally represented by
\[
z_{\text{per}}(t) = \hat{\theta}_{\text{per}}(t-D).
\]
(115)
In other words, regardless of choosing the perturbation frequency to be periodic/deterministic or stochastic, the average linearized version of the measurable signal \( z_{\text{per}} \) remains the same. Hence, the generalized version of the predictor feedback introduced in (32) for the stochastic case is given by
\[
\hat{U}(t) = -c_z U(t) - c_\kappa \left[ z(t) + \int_{t-D}^t U(\tau) d\tau \right],
\]
(116)
where the positive constants \( c_z \) and \( c_\kappa \) have the same role of \( c \) and \( k \) in the deterministic case, respectively. Beside such different constants, the filtered predictor (116) employs a stochastic signal \( z(t) \) in (113) constructed with the demodulated signals \( \hat{h}^{(n+1)}(t) = \Upsilon_{(n+1)}(t-D)\gamma(t-D) \) and \( \hat{h}^{(n+2)}(t) = \Upsilon_{(n+2)}(t-D)\gamma(t-D) \) from (109) and (108)−(12), based on \( \eta(t) \) given in (107).

The stability proof for the closed-loop system described by (108), (109) and (116) follows exactly the same steps as done in Section 4. However, instead of using the averaging theory for deterministic FDEs stated in [10,16], one has to use the averaging theorem for stochastic FDEs by Katsafiotis–Tsaykov presented in [11] in order to prove the stability of the non-average system.

The full-error system is given by:
\[
\frac{d}{dt} \left[ \begin{array}{c}
\hat{\theta}(t-D) \\
U(t) \\
\hat{\gamma}(t) \\
\end{array} \right] = \left[ \\
U(t-D) \\
\hat{U}(t-D) \\
\hat{\gamma}(t-D) \\
\end{array} \right] = \\
-\left[ \\
c_z U(t) - c_\kappa (\gamma(t) + H^{-1}) \hat{h}^{(n+1)}(t) + \int_0^D U(t) d\delta(t) \\
-\left[ \\
c_z U(t) - c_\kappa (\gamma(t) + H^{-1}) \hat{h}^{(n+1)}(t) + \int_0^D U(t) d\delta(t) \\
\end{array} \right] \right].
\]
(117)

By applying the time scale \( \tau = \omega t \) and the stochastic chain rule to (107), one obtains
\[
d\eta = -\frac{\pi}{2} \sin(W_t) dt + \pi \cos(W_t) dW_t,
\]
(118)
and we can rewrite (117) in the new time scale in the form of a three-dimensional stochastic FDE:
\[
\frac{d}{d\tau} u^\tau(t) = G(u^\tau(t)) + \epsilon F(t, u^\tau(t), \eta(t), \varepsilon),
\]
(119)
with \( \varepsilon = 1/\omega \), state vector defined by \( u^\tau = [\hat{\theta} U \hat{\gamma}]^T \) and \( u^\tau(\delta) = u^\tau(t+\delta) \) for \(-D < \delta \leq 0\).

Therefore, since \( \eta(t) \) is a homogenous ergodic Markov process with values in the phase space \( Y \) and unique invariant measure \( \mu(d\eta) \) satisfying the property of exponential ergodicity, and \( G : \mathbb{C}_1([-D,0]) \to \mathbb{R}^3 \) as well as the Lipschitz perturbing term \( F : \mathbb{R}_+ \times \mathbb{C}_1([-D,0]) \times \mathbb{R} \times [0,1] \to \mathbb{R}^3 \) with \( F(t, 0, \eta, \varepsilon = 0) \) are appropriate three-dimensional continuous mappings verifying the conditions required in [11], we can apply Katsafiotis-Tsaykov’s averaging theorem to conclude local exponential stability for initial conditions arbitrarily close to the equilibrium of (117) from the exponential stability estimate of its average system, assuming \( \varepsilon = 1/\omega \) is sufficiently small \( \varepsilon \to 0 \).

6. Simulation results

In the next, we present three numerical examples in order to illustrate the applicability of the proposed real-time algorithms for optimization subject to time delays. The first one shows a maximization problem using a globally quadratic map, whereas the second one handles a locally quadratic function, both scenarios under the action of deterministic (periodic) excitation signals. After that, the third example brings comparison results between deterministic and stochastic extremum seeking schemes, highlighting the convergence speed of each one in a minimization setup.

6.1. Globally quadratic functions

Consider the following static cubic nonlinear map
\[
h(\theta) = -(\theta - 0.5)^3 + \theta,
\]
(120)
since the output delay \( D = 5 \) s, where we want to maximize \( h^{(1)}(\theta) \) according to Eq. (2). Recall that \( y = h(\theta(\theta(t-D)) = h(\hat{\theta}(\theta(t-D) + a \sin(\omega(t+D)) \rangle \), and the variable \( \theta \) is described as the best estimate of \( \theta \). If no prediction-based controller is used, the closed-loop system is unstable due to the large value of the considered delay. Fig. 2 shows (120) and its first derivative.
Fig. 2. The derivatives of $h^{(n)}(\theta)$ for $n=0, 1$ where $h^{(0)}(\theta) = h(\theta) = \text{black (continuous), } h^{(1)}(\theta) = \text{red (semicolon). The blue dashed line is the target } (\theta, y, y')$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 3. Maximization of $h^{(3)}(\theta)$ where $h(\theta) = -((\theta - 0.5)^2 + \theta$. The plot on top displays $y(t)$, in the middle one has $\theta(t)$; and $\gamma$ in the bottom the curve. The horizontal axis presents the time variable in seconds. For the simulation, we have used $\omega = 25 \text{ rad/s}$, $a = 0.1$. $D = 5 \text{ s}, k = 0.1$ and $k_{\theta} = 0.6$. The initial conditions were $\theta_0 = 0$ and $y_0 = -0.01$. The blue dashed lines represent the targets $\gamma = 0.5$, $\theta' = 0.5$ and $y' = -0.16$, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In our simulations we use low-pass and washout filters with corner frequencies ($\omega_0$ and $\omega_1$) to improve the controller performance as usual in extremum seeking designs [15], see [8, Figure 4]. We present simulations of the predictor (32), where $c = 20$. $z$ is given by (15) with $h^{(1)}$ as in (13)-(14) and $\gamma$ in (6).

The tests are performed with the $\omega_0 = \omega_1 = 1$. Fig. 3 shows maximization of the first derivative of (120) so that after some transient the Riccati equation converges close enough to the actual value of the Hessian’s inverse. The initial transient of $\gamma$ (after the controller “kicks in”) at $D = 5 \text{ sec}$ is driven by $h^{(2)} < 0$ and makes $\gamma$ increasingly positive.

From Fig. 3, it follows that $y(t)$ converges to the value where the desired derivative has its extremum. Note that, by increasing the order of the derivative causes decreasing of the convergence rate for the Hessian estimate, which can be improved with smaller values of $\alpha$. However, this also implies larger gains of $C_z$ as well as larger time constants [19]. Thus, a trade-off between performance and tuning for the design parameters is needed.

6.2. Locally quadratic functions

Consider the following function

$$l(\theta) = \tan(\theta),$$

whereas second derivative satisfies Assumption 1 only locally such that its quadratic approximation is given by (18).

In Fig. 4, Eq. (121) and its first two derivatives are depicted. Our objective this time is to maximize the second derivative of the nonlinear map (121). The maximization of $I^{(2)}(\theta)$ is again carried out with Newton-based ES using the predictor feedback (32) for delay compensation. Fig. 5 shows the simulation results. Unsurprisingly, $\theta$ converges to its maximum at $\theta^* \approx -0.58$. As already stated in [19], higher-order derivatives decreases the speed of convergence of the Hessian’s inverse estimate, which forced us to increase perturbation frequency $\omega$.

6.3. Comparison of the periodic and stochastic dithers – minimization case

For the demonstration of the results for stochastic perturbations, we consider the following globally cubic map

$$\nu(\theta) = \theta^3 - \frac{17}{4} \theta,$$

where we are interested into minimization of $\nu^{(1)}(\theta)$ given in Fig. 6.

In order to show the effectiveness of the stochastic perturbation technique, we compare it with the periodic perturbations case and show the results in Fig. 7.

Again, we assume $D = 5s$. For the periodic case, we use (32) with $c = 5$ and $k_0 = 0.1$. For the stochastic case, we apply predictor (116) and $\eta(t)$ defined as (107) with $c_\zeta = c$, $k_\zeta = 0.1$, $\omega = 1$.

As can be seen in Fig. 7, all signals converge to the expected values despite the delay. However, we notice that the stochastic perturbation case outperforms the periodic one when using similar
The convergence constant.

respectively.

simulation parameters, since the convergence rate of the stochastic algorithm is faster than of the deterministic approach.

**7. Conclusions**

In this article, we presented and proved local stability of a Newton-based extremum seeking algorithm in presence of input-output delays, which maximizes arbitrary $n$th derivative of an unknown static map using deterministic periodic perturbations.

The only available measurement is from the map’s output itself and not of its derivatives. The delay is assumed to be known and constant.

The resulting approach preserves exponential stability and convergence of the system input to a small neighborhood around the point where the desired derivative has an extremum point, despite of the presence of delays.

A rigorous proof is given in terms of backstepping transformation and averaging analysis in infinite dimensions for locally quadratic objective functions. Using a Riccati filter, the convergence rate is made user-assignable, rather than being dependent on the unknown Hessian of the map.

Moreover, we present a short introduction to Newton-based extremum seeking of higher derivatives using stochastic perturbations in the presence of time delays in the output signal. We show the main differences with respect to the periodic case, detailing a similar predictor feedback for delay compensation as well as stating the application of stochastic averaging theorem for the stability proof. Simulation shows that the stochastic version outperforms the periodic one in terms of faster convergence rates.

Generalization to include unknown and time-varying delays as well as a complete proof for stochastic perturbations case will be published in future research.

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**Appendix A**

**Proof of Lemma 2.**

**Method 1:** Cascade of exponentially stable systems

The linearized average system with Jacobian matrix (46) is simply

\[
\dot{\theta}_{av} = U_{av},
\]

\[
U_{av} = -cU_{av} + \left[-kc + \mathcal{O}(a^2)\right]\theta_{av}.
\]

\[
\dot{\gamma}_{av} = -k\gamma_{av} + \frac{\partial h_{av}}{\partial \theta_{av}} \theta_{av}.
\]
From subsystems (123) and (124), we write the characteristic polynomial
\[
det(sI - A1) = s^3 + cs + kc - O(a) = 0. \tag{126}
\]
which has negative roots if “a” is chosen sufficiently small, with \(A_1\) being a sub-matrix of the Jacobian \(A(\phi)\) given by
\[
A_1 = \begin{bmatrix} 0 & 1 \\ -kc + O(a) & -c \end{bmatrix}. \tag{127}
\]
Thus, we can conclude that \(\tilde{\gamma}_w(t)\) and \(U_w(t)\) in (123)-(124) tend to zero exponentially. Since (125) is exponentially input-to-state stable with respect to the input \(\tilde{\gamma}_w\), we can also conclude that \(\gamma_w(t) \to 0\) exponentially as \(t \to +\infty\). completing the proof of Lemma 2.

**Method 2:** Complete analysis

The Jacobian of the average system (38) is written as
\[
J(\tilde{\gamma}_w; U_{aw}; \tilde{\gamma}_w) = \begin{bmatrix} \frac{\partial f_w}{\partial \tilde{\gamma}_w} & \frac{\partial f_w}{\partial U_{aw}} & \frac{\partial f_w}{\partial \gamma_w} \\ \frac{\partial g_w}{\partial \tilde{\gamma}_w} & \frac{\partial g_w}{\partial U_{aw}} & \frac{\partial g_w}{\partial \gamma_w} \\ \frac{\partial h_w}{\partial \tilde{\gamma}_w} & \frac{\partial h_w}{\partial U_{aw}} & \frac{\partial h_w}{\partial \gamma_w} \end{bmatrix}. \tag{128}
\]
Hence, for (128) we get\(^1\)
\[
\begin{align*}
(a) & \quad \frac{\partial f_w}{\partial \tilde{\gamma}_w} \bigg|_{\tilde{\gamma}_w; U_{aw}; \tilde{\gamma}_w} = 0 \\
(b) & \quad \frac{\partial f_w}{\partial U_{aw}} \bigg|_{\tilde{\gamma}_w; U_{aw}; \tilde{\gamma}_w} = 1 \\
(c) & \quad \frac{\partial g_w}{\partial \tilde{\gamma}_w} \bigg|_{\tilde{\gamma}_w; U_{aw}; \tilde{\gamma}_w} = -kc + O(a^2) \\
(d) & \quad \frac{\partial g_w}{\partial U_{aw}} \bigg|_{\tilde{\gamma}_w; U_{aw}; \tilde{\gamma}_w} = -c \\
(e) & \quad \frac{\partial h_w}{\partial \tilde{\gamma}_w} \bigg|_{\tilde{\gamma}_w; U_{aw}; \tilde{\gamma}_w} = 0 \\
(f) & \quad \frac{\partial h_w}{\partial U_{aw}} \bigg|_{\tilde{\gamma}_w; U_{aw}; \tilde{\gamma}_w} = -kc \\
\end{align*}
\]
Since (46) is a 3x3 matrix, in order to prove that it is Hurwitz, and hence the system is locally exponentially convergent, first we have to calculate the characteristic polynomial \([4]\) as follows:
\[
det(sI - J) = a_3s^3 + a_2s^2 + a_1s + a_0 = 0. \tag{129}
\]
The characteristic polynomial can easily be calculated using (46) and is given by
\[
s^3 + s^2(k_c + c) + s(k_c - O(a^2)) + ckc - k_c(k_c + O(a^2)) = 0. \tag{130}
\]
According to \([4]\), the Hurwitz matrix of (130) has the following form
\[
H(\phi) = \begin{bmatrix} kc - O(a^2) & 1 & 0 \\ k_c(k_c + O(a^2)) & k_c + c & 0 \\ 0 & kc - O(a^2) & 1 \end{bmatrix}, \tag{131}
\]
where \(\phi = (\tilde{\gamma}_w; U_{aw}; \tilde{\gamma}_w).\)
Now, applying Liénard–Chipart Criterion \([4]\):
1. All coefficients of the matrix \(H\) have to be positive, hence:
\[
(a) kc - O(a^2) > 0, \text{ which is true for } k, c > 0 \text{ and "a" sufficiently small};
\]
\[
(b) k_c(k_c + O(a^2)) > 0, \text{ which is true for } k_c, c > 0 \text{ and "a" sufficiently small};
\]
\[
(c) k_c + c > 0, \text{ which is true for } k_c, c > 0;
\]
\[
(d) kc - O(a^2) > 0, \text{ which is true for } k, c > 0 \text{ and "a" sufficiently small}.
\]
Therefore, all coefficients of (131) are positive for sufficiently small “a” and \(k, k_c, c > 0\).
All minors of the Matrix \(H\) have to be positive.
In order to prove this condition, we rewrite (131) to
\[
H = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_5 \end{bmatrix}, \tag{132}
\]
where the minors of (131) are defined as
\[
D_1 = a_1, \tag{133}
\]
\[
D_2 = \det \begin{bmatrix} a_1 & a_3 \\ a_0 & a_2 \end{bmatrix}, \tag{134}
\]
\[
D_3 = \det H = \det \begin{bmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{bmatrix}, \tag{135}
\]
while the coefficients \(a_1\) to \(a_5\) are substitutions for terms in (131). For the Hurwitz matrix (131) we have to check:
\[
(a) D_1 = kc - O(a^2) > 0, \text{ which is again valid for } k, c > 0 \text{ and "a" sufficiently small};
\]
\[
(b) D_2, D_3 > 0. \text{ Since } a_4 \text{ and } a_5 \text{ are zero, it follows } D_2 = D_3. \text{ Now calculating } D_3 \text{ we get that the criterion } kc > k_c + O(a^2) \text{ is only true if } c \text{ is sufficiently large since } k > 0 \text{ is user-assigned.}
\]
Thus, we can conclude that the matrix \(J\) is Hurwitz for \(k, k_c, c > 0\), sufficiently large \(c\) and “a” sufficiently small. Hence, the average system (36) is locally exponentially attractive around its equilibrium. This concludes the proof of Lemma 2. \(\square\)

**Appendix B**

**Theorem 2.** Consider the delay system
\[
\dot{x}(t) = f(t/\epsilon, x_\epsilon), \quad t > 0, \tag{136}
\]
where \(\epsilon\) is real parameter, \(x_\epsilon(\theta) = x(t + \theta)\) for \(-r \leq \theta \leq 0\), and \(f : \mathbb{R} \times \Omega \to \mathbb{R}^n\) is a continuous functional from a neighborhood \(\Omega\) of 0 of the supremum-normed Banach space \(X = C([-r, 0]; \mathbb{R}^n)\) of continuous functions from \([-r, 0]\) to \(\mathbb{R}^n\). Assume that \(f(t, \varphi)\) is periodic in \(t\) uniformly with respect to \(\varphi\) in compact subsets of \(\Omega\) and that \(f\) has a continuous Fréchet derivative \(\frac{df(t, \varphi)}{d\varphi}\) in \(\varphi\) on \(\mathbb{R} \times \Omega\). If \(y = y_0 \in \Omega\) is an exponentially stable equilibrium for the average system
\[
\dot{y}(t) = f_0(y(t)), \quad t > 0, \tag{137}
\]
where
\[
f_0(\varphi) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(s, \varphi)ds, \tag{138}
\]
then, from some \(\epsilon_0 > 0\) and \(0 < \epsilon \leq \epsilon_0\), there is a unique periodic solution \(t \to x^*(t, \epsilon)\) of (136) with the properties of being continuous in \(t\) and \(\epsilon\), satisfying \(|x^*(t, \epsilon) - y_0| \leq \tilde{O}(\epsilon)\), for \(t \in \mathbb{R}\), and such that there is \(\rho > 0\) so that if \(x(\cdot, \varphi)\) is a solution of (136) with \(x(s) = \varphi\) and \(|\varphi - y_0| < \rho\), then \(|x(t) - x^*(t, \epsilon)| \leq Ce^{-\gamma(t-s)}\), \(\tag{139}\)
for \(C > 0\) and \(\gamma > 0\).

**Proof.** See \([10, 16]\). \(\square\)
References