Distributed adaptive consensus and synchronization in complex networks of dynamical systems

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A B S T R A C T
In this paper we propose novel distributed adaptive controllers for leaderless synchronization in networks of identical discrete-time dynamical systems. Separate algorithms are developed for the cases of known and unknown control directions. Assuming that the directed network graph is strongly connected, it is proved that all agent outputs converge toward an emerging, unknown in advance, synchronization trajectory. This trajectory is not available for use by the agent’s controllers, and its pattern is determined by the internal model built-in in the distributed adaptation mechanism. It is also shown that for each agent the controller parameter estimates are convergent sequences, and the inputs are uniformly bounded signals. The key to obtaining presented results is to construct adaptive algorithms providing l1-boundedness of certain error signals driving the local parameter estimators. The proposed estimators have a non-vanishing step-size.

1. Introduction

Consensus and synchronization phenomena represent a form of cooperative behavior characterized by a global coherent activity and emergence of spontaneous order in a population (system) of interacting members (agents). Synchronization processes play an important role in a variety of natural and man-made systems in biology, physics, chemistry, social environment, and engineering. The so called ‘nearest neighbors’ rule (Vicsek, Czirok, Jacob, Cohen, & Schochet, 1995) and its theoretical analysis (Jadbabaie, Lin, & Morse, 2003) were motivational catalyst for numerous results on distributed control for multi-agent synchronization. This paper considers the case where there is no leader among agents, and the synchronization trajectory emerges spontaneously as a result of inter-agent interactions. As it is pointed out in Ren (2009) leaderless consensus is at play in situations where the specifically predetermined “consensus equilibrium is not what is important, but rather that each system converges to an identical state” (see p.2138 in Ren (2009)). Flocking, rendezvous and attitude stabilization problems are examples of such instances (Ren, 2009). The existing results on adaptive leaderless synchronization prove that the error between the states of any two agents converges to zero. As it is discussed below (see the paragraph ‘Context for the results of this paper’) this assertion does not necessarily implies that the states of all agents simultaneously converge toward the same consensus trajectory. This paper offers novel adaptive controllers for leaderless synchronization, and it demonstrates that outputs of all agents converge to the same, unknown to agents in advance, consensus trajectory. Since the topic of this paper is the problem of adaptive consensus/synchronization based on internal model principle, we do not intend to comment in detail on the work related to non-adaptive, and non-internal model based methods. We make only brief mention of the results involving a fixed gain distributed controllers. A considerable body of work on multi-agent systems is related to the leader-following problem. This topic has been studied in a variety of cases including linear (Cao, Zhang, & Ren, 2015; Grip, Yang, Saberi, & Stoorvogel, 2012; Ni & Cheng, 2010; Su, & Huang, 2012, 2013; Su & Huang, 2012) and nonlinear (Meng, Lin, & Ren, 2013; Su & Huang, 2013, 2014; Xu, Wang, Hong, & Jiang, 2016) agent dynamics, identical (Cao et al., 2015; Ni & Cheng, 2010; Su et al., 2013) and heterogeneous agents (Grip et al., 2012; Meng et al., 2013; Su & Huang, 2012, 2013, 2014), presence of system uncertainties or uncertain systems (Su & Huang, 2013, 2014), as well as fixed and time-varying communication topologies. Similarly, the leaderless consensus problem has been explored for both linear and nonlinear agent dynamics under various scenarios of inter-agent information exchange, and diverse conditions related to graph topologies and system uncertainties (Burbano-L & di Bernardo, 2016; Scardovi & Sepulchre, 2009; Seo,


Shim, & Back, 2009; Su & Huang, 2012; Trentelman, Takaba, & Monshizadeh, 2013; Zhu & Chen, 2014. Conditions sufficient for the existence of a synchronizing control law based on agent output exchange are derived in Ma and Zhang (2010), and Tuna (2009). In addition to the above, a large number of references on non-adaptive consensus can be found for example in the survey paper (Cao, Yu, Ren, & Chen, 2013) and recent monograph (Ren & Cao, 2011).

**Internal model based non-adaptive synchronization:** Important results establishing the connection of synchronization processes with the internal model principle are presented in Kim, Shim, and Seo (2011), Lunze (2012) and Wieland, Sepulchre, and Allgower (2011). It is shown that a synchronization of agents with linear dynamics implies the existence of an observable virtual exosystem (internal model) which defines the output trajectories on the agreement manifold, and is contained within each agent as an internal model. The internal model can be embedded in the controller dynamics without requiring for it to be a part of agent dynamics. Generalization of the results in Wieland et al. (2011) is presented in Grip, Saberi, & Stoorvogel (2013). Extension of the internal model principle to robust output synchronization of a nonlinear heterogeneous agents can be found in a number of interesting papers (Chen, 2014; Isidori, Marconi, & Casadei, 2014; Su & Huang, 2015; Xu et al., 2016; Zhu, Chen, & Middleton, 2016).

**Recent work on adaptive consensus:** The theory of adaptive consensus deals with two distinct issues: (1) the leader-following problem where each agent follows the leader’s state, and (2) the problem of leaderless consensus where the aim is not to follow pre-determined leader, but rather for all agents to achieve agreement on a common, unknown in advance value of their state. Because the subject of our work is a leaderless synchronization, we only briefly comment on the leader-following literature. This problem has been considered for various known and unknown, linear and nonlinear agent dynamics, fixed (undirected and directed) and time-varying graph topologies, identical and heterogeneous agents, as well as leaders with known and unknown inputs (see for example: Bai, Arcak, and Wen, 2009, Das and Lewis, 2010, Ding, 2015, Li, Wen, Duan, and Ren, 2015, Liu, 2015, Su, 2015, Sun, Geng, and Lv, 2016, Tang, Hong, and Wang, 2015, Wang, Wang, and Ji, 2016 Yu and Xia, 2012, Yu, Shen, and Xia, 2013).

Interesting results on leaderless consensus are presented in Li, Ren, Liu, and Fu (2013); Li, Ren, Liu, and Xie (2013) where it is proved that the difference between each agent state and the average of all agent states converges to zero. In Li, Ren, Liu and Xie (2013) the case of known identical linear agent dynamics and undirected graph topology is analyzed, whereas in Li, Ren, Liu and Fu (2013) identical linear and Lipschitz nonlinear dynamics on directed graphs is studied. Similar results are presented in Li & Ding (2015) where it is assumed that all agents have identical linear non-minimum phase SISO dynamics and strongly connected directed graph topology. There it is shown that the error between the outputs of any two agents tends to zero. All of the above mentioned references on adaptive consensus assume that the high-frequency gain or its sign (control direction) is known. Adaptive consensus in case of unknown control directions is treated in Chen, Li, Ren, and Wen (2014), Ding (2015), Junmin and Xudong (2014), Liu (2015), and Su (2015), based on the Nussbaum gain concept. For the first time this problem has been addressed in Chen et al. (2014) and Junmin and Xudong (2014) where the authors consider undirected graph topology, and show that the error between the states of any two agents converges to zero.

**Context for the results of this paper:** In the non-adaptive leaderless case it has been demonstrated that under certain conditions there exist distributed controllers such that all agent outputs, say \(x_i(t)\), reach agreement on a common, bounded reference (synchronization) trajectory, say \(x_r(t)\), that is \(x_i(t) - x_r(t) \to 0\), as \(t \to \infty\), \(i = 1, \ldots, N\), where \(N\) is the number of agents (see for example Isidori et al. (2014), Kim et al. (2011), Lunze (2012), Wieland et al. (2011), Zhu et al. (2016)). To prove this claim in case of adaptive leaderless consensus is a rather challenging proposition and it is still an open problem. The existing results on this topic are limited to showing that the difference between agent outputs, or the error between any agent output and the average of all agent outputs, converges to zero, i.e., \(x_i(t) - x_j(t) \to 0\), or \(x_i(t) - \frac{1}{N} \sum_{j=1}^{N} x_j(t) \to 0\), as \(t \to \infty\) (see for example, Chen et al. (2014), Junmin and Xudong (2014), Li and Ding (2015), Li, Ren, Liu and Fu (2013)). Note that the above assertions do not imply that there exists an \(x_r(t)\) (in case of the previously mentioned work, a constant \(x_r\)) so that \(x_i(t) - x_r \to 0\), as \(t \to \infty\), \(i = 1, \ldots, N\). Take for example \(x_i(t) = \log(t + i)\), or \(x_i(t) = \cos(\log(t + i))\) for \(i = 1, \ldots, N\).

**Results of the paper:** In this paper we show that it is possible for adaptive leaderless consensus to achieve the same synchronization results as those for a non-adaptive case. It is proved that the proposed distributed controllers guarantee emergence of a synchronization trajectory, say \(x_r(t)\), \(t = 0, 1, \ldots\), such that all agent outputs \(x_i(t)\), \(i = 1, \ldots, N\), satisfy \(x_i(t) - x_r(t) \to 0\), as \(t \to \infty\). Compared to Chen et al. (2014), Junmin and Xudong (2014), Li and Ding (2015) and Li, Ren, Liu and Fu (2013) where \(x_r(t)\) is a constant, we allow arbitrary form for \(x_r(t)\). The trajectory \(x_r(t)\) is not physically measurable nor known in advance to agents, and it is a uniformly bounded sequence whose pattern is determined by the initial conditions and the internal model built-in in the agents’ adaptive controllers. In order to derive the above proposition it is essential to show that the errors between the output of individual agents and certain averages of neighboring agent states are \(l_1\) sequences. For this purpose, novel distributed adaptive controllers have been developed to generate such error signals. To our knowledge this is the first result providing \(l_1\) performance in an adaptive synchronization setting. It is also proved that the parameter estimates of local controllers are convergent sequences. We consider networked discrete-time systems with identical, general order, linear dynamics, and separately analyze the algorithms for the cases of known and unknown control directions. In case of unknown control direction we present a novel algorithm, different than the existing discrete-time counterpart (Lee & Narendra, 1988). There the parameter estimator involves a discontinuous function termed “discrete-Nussbaum gain”, whose values are either \((+1)\) or \((-1)\) depending on the output of an appropriately defined switching algorithm. Our solution for the unknown control directions entails a “smooth” controller without involving any switching logic.

The paper is organized as follows. The problem statement is given in Section 2. Section 3 presents the case of known control directions. The systems with unknown control directions are considered in Section 4.

**Notation:** The superscript \(T\) denotes the transpose of a matrix; \(\|\|\) is the Euclidean norm of vector \(x\); \(\text{sgn}(a)\) is the sign function of a real number \(a\); \(\rho_p\), \(p \in [1, \infty]\) denotes the normed infinite dimensional vector space of sequences \(s(t)\), \(t \geq 0\), satisfying \(\sum_{t=0}^{\infty} |s(t)|^p < \infty\); \(\rho(W)\) is the spectral radius of matrix \(W\); \(\alpha\) is used to denote vector of all ones, i.e., \(\alpha = [1, 1, \ldots, 1]\), and \(I\) is identity matrix. The dimensions of \(\alpha\) and \(I\) are determined by the context. In this paper \(c_i\) \((i = 1, 2, \ldots)\) are used to denote non-negative constants whose specific values are unimportant.

**2. Problem formulation**

Consider a networked system of \(N\) identical agents whose dynamics is given by:

\[
A(q^{-1})x(t+1) = b_0(q^{-1})u(t), \quad b_0 \neq 0
\]
where $t = 0, 1, 2, \ldots$ is a discrete-time index, $i = 1, \ldots, N$, \(x_i(t)\) and \(u_i(t)\), $t \geq 0$ are the $i$th agent state and input sequences, respectively, while $q^{-1}$ represents the unit delay operator, i.e., $q^{-1}x_i(t+1) := x_i(t)$. Polynomials $A(q^{-1})$ and $B(q^{-1})$ are given by:

$$
A(q^{-1}) = 1 + \sum_{k=1}^{M} a_k q^{-k}, \quad B(q^{-1}) = 1 + \sum_{k=1}^{M} b_k q^{-k} \tag{2}
$$

where $a_k$ and $b_k$, $1 \leq k \leq M$, $M \geq 1$, are unknown parameters. The unknown parameter $b_0$ in (1) is often referred to as the control direction, the high-frequency gain, or the instantaneous gain. In this paper we refer to $B(q^{-1})/A(q^{-1})$ as a transfer operator. The corresponding transfer function is denoted by $B(q^{-1})/A(q^{-1})$, where $x$ is complex variable. The communication topology of the above network is characterized by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, N\}$ is the set of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, also called communication links. The node $i$ represents the agent $i$, and the ordered pairs $(i, j)$, $i \neq j$ denote edges. The statement $(j, i) \in \mathcal{E}$ implies that the $i$th agent can receive information from the $j$th agent. The set of neighbors of node $i$ is denoted by $N_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. The following is assumed about the system (1), and the graph topology.

**Assumption 1 (Minimum Phase System).** Polynomial $B(x)$ has zeros strictly outside the unit circle.

**Assumption 2.** The underlying network graph is strongly connected.

Let

$$x(t)^T := [x_1(t), \ldots, x_N(t)]. \tag{3}$$

We say that the multi-agent system achieves consensus/synchronization if all agents reach agreement on a common asymptotic trajectory, i.e., if there exists a uniformly bounded scalar sequence $\{x_i(t)\} \in \mathbb{R}$ so that $\lim_{t \to \infty} (x_i(t) - x_k(t) \omega) = 0$, where $\omega \in \mathbb{R}^M$ is the vector of all ones. Throughout we refer to $[x_i(t)]$ as the consensus/synchronization trajectory, and phrases “consensus” and “synchronization” are used interchangeably. The objective for each agent is to select its own control signal $u_i(t)$, $i \in \mathcal{V}$, so that all outputs $[x_i(t)]$, $i \in \mathcal{V}$ synchronize to a common, unavailable to agents and unknown in advance trajectory $[\bar{x}_i(t)]$.

The consensus trajectory $[x_i(t)]$ can be a constant, a discrete-time cosine signal, or a more complex sequence. For example in flock formation scenarios, $x_i(t)$, $i \in \mathcal{V}$ can represent velocity, and the objective is that all agents agree on a common and constant salling velocity. In the case of harmonically coupled oscillators with identical frequencies, say equal to $\omega_k$, all agents will agree on a common synchronization trajectory $x_i(t) = F_i \cos(\omega_k t + \phi_i)$ for some finite $\phi_i$ and $F_i > 0$. The specific values of $F_i$ and $\phi_i$ in this example, and in general the resulting form of the emerging trajectory is determined by the information shared among agents and the initial conditions. In general the $i$th agent selects own control input $u_i(t)$ so that the error between $x_i(t+1)$ and suitably chosen function of $x_i(k)$ and neighboring states $x_j(k)$, $k \leq t$, $j \in N_i$, where $N_i$ is the set of neighbors of the $i$th agent, is driven to zero. For example this error can be given by

$$e_i(t+1) = x_i(t+1) - \bar{x}_i(t), \quad i \in \mathcal{V}, \tag{4}$$

where

$$\bar{x}_i(t) = \frac{1}{1 + N_i} \sum_{j \in N_i} x_j(t), \quad N_i = N_i \cup \{i\}, \tag{5}$$

with $N_i$ being the cardinality of $N_i$. In order to motivate the derivation of control algorithms, we first provide a heuristic discussion about the connection between the error $e_i(t+1)$ and the induced consensus trajectory $x_i(t)$. Define the matrix $W \in \mathbb{R}^{N \times N}$ as follows

$$W = [w_{ij}], \quad w_{ij} = \begin{cases} 1 & i \in N_i \cup \{i\} \\ 0, \text{ otherwise.} \end{cases} \tag{6}$$

Let

$$e(t+1) = x(t+1) - W\bar{x}(t). \tag{7}$$

Then from (4) we have

$$e(t+1) = x(t+1) - W\bar{x}(t). \tag{8}$$

Assuming that all agents control inputs $u_i(t)$ produce $e_i(t+1) = 0$, $i \in \mathcal{V}$, we can write

$$x(t+1) - W\bar{x}(t) = 0. \tag{9}$$

Observe that (6) implies $W\alpha = \alpha$, where $\alpha = [1, \ldots, 1]$, $\alpha \in \mathbb{R}^N$. In the next section we demonstrate that the solution of (9) satisfies $x(t) = x_i(t)\alpha$, for some $x_i(t) \in \mathbb{R}$. Then (9) yields $x_i(t+1) - x_i(t)\alpha = 0$, $\forall t \geq 0$ implying that $x_i(t)$ is a constant sequence. Consider now a scenario where the $i$th agent is receiving $x_j(t)$ and $\bar{x}_i(t-1)$ from the $j$th neighbor, $j \in N_i$, where $\bar{x}_i(t-1)$ is defined by (5). Assume that in this case the $i$th agent control input $u_i(t)$ drives the following error to zero,

$$e_i(t+1) = x_i(t+1) + p_1\bar{x}_i(t) + \frac{1}{1 + N_i} \sum_{j \in N_i} \bar{x}_j(t-1) \tag{10}$$

where $p_1 \in \mathbb{R}$. Let $\bar{x}_i(t)^T = [\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_N(t)]$. Then (5) implies $\bar{x}_i(t) = W\bar{x}(t)$, where $\bar{x}$ is given by (3) and $W$ is defined by (6). Thus from (10) one can obtain $e(t+1) = x(t+1) + p_1\bar{x}_i(t+1) + W\bar{x}(t-1)$, or

$$e(t+1) = x(t+1) + p_1\bar{x}(t) + W^2\bar{x}(t-1). \tag{11}$$

where $e(t)$ is defined by (7). The next section shows that when $e(t+1) = 0$, then $x(t) = x_i(t)\alpha$ for some $x_i(t)$, where $\alpha \in \mathbb{R}^N$ is a vector of all ones. Consequently (11) gives $(1 + p_1Wq^{-1} + W^2q^{-2})x_i(t) = 0$. Because $W\alpha = \alpha$ (see (6)) the previous equation yields $(1 + p_1q^{-1} + q^{-2})x_i(t) = 0$, implying that if $x_i(0) \neq 0$, agents will synchronize to $x_i(t) = F_i \cos(\omega_k t + \phi_i)$, with $p_1 = -2\cos(\omega_k)$, and some $\phi_i$ and $F_i > 0$. Consider the case of a more complex synchronization sequence $[x_i(t)]$ defined by

$$x_i(t) = x_0 + \sum_{k=1}^{L} F_k \cos(\omega_k t + \phi_k) \tag{12}$$

for some finite $\phi_k$, $x_0 \neq 0$, and $F_k > 0$. $k = 1, \ldots, L$, $L \geq 0$.

It is assumed that $\omega_k \neq \omega_j$ for $k \neq j$, and $\omega_k$ is not a multiple of $\pi$. The above sequence can be generated by the following linear model,

$$H(q^{-1})x_i(t) = 0, \quad H(q^{-1}) = 1 - \sum_{k=1}^{2L+1} h_k q^{-k} \tag{13}$$

where $x_i(0) \neq 0$, and $H(q^{-1})$ is factorable in the form

$$H(q^{-1}) = (1 - q^{-1})^L \prod_{k=1}^{L} (1 + p_k q^{-1} + q^{-2}) \tag{14}$$

where $p_k = -2\cos(\omega_k)$, $1 \leq k \leq L$, with $\omega_k$ being the same as in (12). Coefficients $h_k$, $1 \leq k \leq 2L + 1$ are functions of $p_k$, $1 \leq k \leq L$. Observe that the restrictions on $\omega_k$’s in (12) guarantee that the roots of $H(q^{-1})$ are exactly on the unit circle, and there are no repeated roots. In other words, the solution of (13) is a uniformly bounded sequence of the form given by (12). From the previous
discussion it is intuitively clear that the multi-agent system will synchronize to the above $x_k(t)$ if the following error is driven to zero
\[ e(t) = P(q^{-1})x(t), \quad P(q^{-1}) = I - \sum_{k=1}^{2L+1} h_k W^k q^{-k} \]  
(15)

where $P(q^{-1})$ is factorable as follows
\[ P(q^{-1}) = (I - Wq^{-1}) \prod_{k=1}^{2L+1} (I + p_k Wq^{-1} + W^2 q^{-2}), \]  
(16)

where $W$ is defined by (6), while the coefficients $h_k$ and $p_k$ are the same as in (13) and (14), respectively. This conclusion follows from the fact that when $e(t) = 0$ and $x(t) = x_k(t) x_k$, then (15) implies (13) by virtue of the fact that $W\alpha = \alpha$. Observe that the errors given by (8) and (11) are special cases of (15). We now derive the local errors $e_i(t)$ so that the overall system error $e(t)$ satisfies (15). Let
\[ u_i(t) = [u_{i1}, \ldots, u_{in}], \quad i \in \mathcal{V}, \]  
(17)

where $u_i$ is given by (6). Define the following nested averages $\forall i \in \mathcal{V}$, $k = 2, \ldots, 2L + 1$,
\[ \bar{x}_0(t) = u_i^T \bar{x}_{k-1}(t) = \frac{1}{1 + N_i} \sum_j \bar{x}_j(t-1)(t). \]  
(18)

where
\[ \bar{x}_k(t)^T = [\bar{x}_i(t), \bar{x}_{i2}(t), \ldots, \bar{x}_{iN}(t)] \]  
(19)

with $\bar{x}_i(t)$ given by
\[ \bar{x}_i(t) = u_i^T x(t), \quad i \in \mathcal{V}, \]  
(20)

where $u_i$ and $x(t)$ are defined by (17) and (3), respectively. It is not difficult to show that the overall networked system will have $e(t+1)$ given by (15) if the components $e_i(t+1), \quad i \in \mathcal{V}$ satisfy
\[ e_i(t+1) = x_i(t+1) - \sum_{k=1}^{2L+1} h_k \bar{x}_k(t) - k + 1. \]  
(21)

The following argument justifies this assertion. Observe that from (18)–(20) we can write $\bar{x}_k(t) = Wx_{k-1}(t)$ and $\bar{x}_1(t) = Wx(t)$, where from it follows that $\bar{x}_k(t) = W^k x(t)$, which together with (18) gives
\[ \bar{x}_k(t-k+1) = u_i^T W^k x(t-k+1), \quad k = 1, \ldots, 2L \]  
(22)

where $W$ is defined by (6). Substituting (22) in (21) yields $e_i(t+1) = x_i(t+1) - \sum_{k=1}^{2L+1} h_k u_i^T W^k x(t-k+1)$ from where by (7), (17), and (6) one obtains (15). We are now in a position to derive a distributed adaptive controllers that will drive the errors $e_i(t+1), \quad i \in \mathcal{V}$ to zero. We first consider the case of known sign of the high-frequency gain $b_0$ in (1).

### 3. Known sign of the parameter $b_0$

Similarly as in (18)–(20) let us define the following “nested” control strategy
\[ \bar{u}_k(t) := u_i^T U_{k-1}(t), \quad k = 2, \ldots, 2L + 1 \]  
(23)

where
\[ U_k(t)^T = [\bar{u}_k(t), \bar{u}_{k2}(t), \ldots, \bar{u}_{kN}(t)] \]  
(24)

with
\[ \bar{u}_i(t) = u_i^T u(t), \quad u(t)^T = [u_1(t), \ldots, u_N(t)], \]  
(25)

where $w_i$ is given by (17). We now show that from (1) and (21) one can derive
\[ A(q^{-1}) y_i(t+1) = b_0 B(q^{-1}) (u_1(t) + \beta_i(t-1)) \]  
(26)

where
\[ \beta_i(t-1) := - \sum_{k=1}^{2L+1} h_k \bar{u}_k(t-k) \]  
(27)

with $\bar{u}_k(t)$ defined by (23). Let
\[ y_i(t) := \sum_{k=1}^{2L+1} h_k \bar{x}_k(t-k+1) \]  
(28)

where $\bar{x}_k(t)$ is given by (18). Then
\[ A(q^{-1}) y_i(t) = \sum_{k=1}^{2L+1} h_k \hat{A}_k(t-k + 1). \]  
(29)

Let $d_{ij}^{(k)}$ be elements of the following vector
\[ w_i^T W^{k-1} = [d_{i1}^{(k)}, d_{i2}^{(k)}, \ldots, d_{in}^{(k)}], \quad i \in \mathcal{V}. \]  
(30)

Then (22) can be written in the form $\bar{x}_k(t-k+1) = \sum_{j=1}^N d_{ij}^{(k)} x_j(t-k+1)$, which together with (29) gives
\[ A(q^{-1}) y_i(t) = b_0 B(q^{-1}) (1 + \sum_{j=1}^{2L+1} h_i d_{ij}^{(k)} W^{k-1} x(t-k)). \]  
(31)

Since by (1) $A(q^{-1}) x(t-k+1) = b_0 B(q^{-1}) x(t-k)$, and by (30) $\sum_{j=1}^N d_{ij}^{(k)} u_i(t-k) = w_i^T W^{k-1} u(t-k)$, from (31) one can derive
\[ A(q^{-1}) y_i(t) = b_0 B(q^{-1}) (1 + \sum_{k=1}^{2L+1} h_i w_i^T W^{k-1} u(t-k)). \]  
(32)

where $u(t)$ is defined by (25). Similarly as in (22), from (23) we can conclude that
\[ \bar{u}_k(t-k) = w_i^T W^{k-1} u(t-k), \quad i \in \mathcal{V}. \]  
(33)

Thus, inserting (33) in (32) yields
\[ A(q^{-1}) y_i(t) = b_0 B(q^{-1}) \left( \sum_{k=1}^{2L+1} h_i \bar{u}_k(t-k) \right), \quad i \in \mathcal{V}. \]  
(34)

Because (21) and (28) imply $e_i(t+1) = x_i(t+1) - y_i(t)$, (26) follows from (1) and (34). In order to derive the control law $u_i(t)$ we write (26) in the form
\[ \frac{1}{b_0} e_i(t+1) = \theta_i^T \phi_i(t) + u_i(t) + \beta_i(t-1), \quad i \in \mathcal{V} \]  
(35)

where
\[ \phi_i(t)^T = [\phi_1(t), \ldots, \phi_i(t-M+1), \phi_i(t-M), \ldots, \phi_i(t-M-1)], \]  
(36)

and
\[ \theta_i^T = [-a_1', \ldots, -a_M', b_1, \ldots, b_M]. \]  
(37)

with $a_k' = a_k/b_0, \quad 1 \leq k \leq M$. Obviously the control law $u_i(t) = -\beta_i(t-1) - \theta_i^T \phi_i(t)$ yields $e_i(t+1) = 0, \quad \forall t \geq M$, provided that (1) is a minimum phase system. In case when the parameter vector $\theta_i$ is unknown, we propose the following controller $\forall t \geq 0, \quad \forall i \in \mathcal{V}$,
\[ u_i(t) = -\beta_i(t-1) - \hat{\theta}_i(t)^T \phi_i(t) \]  
(38)
where \( \hat{\theta}(t) \) is generated by
\[
\hat{\theta}(t+1) = \hat{\theta}(t) + \mu_i \text{sgn } \psi_i \frac{\text{sgn } \psi_i (t+1)}{r_i(t) \log r_i(t)} \leq c_6 < \infty \quad \forall \ n \geq 0, \tag{39}
\]
where \( \mu_i > 0, \ 0 < \epsilon_i < 1 \), and
\[
r_i(t) = r_i(t-1) + \| \psi(t) \|, \quad r_i(0) > 1, \tag{40}
\]
with \( \psi_i(t) \) and \( c_i(t) \) defined by \( \text{(21) and (36)} \). The initial condition \( \hat{\theta}(0) \in \mathbb{R}^{2d} \) is an arbitrary vector.

Note that the parameters \( \hat{h}_k \), \( 1 \leq k \leq 2L \) in \( e_i(t) \) (see \( \text{(21)} \)) are dependent on \( p_k = -2 \cos(\omega_k), \ 1 \leq k \leq L \) (see \( \text{(13) and (14)} \)), where frequencies \( \omega_k \)'s determine the scenario trajectory \( x_i(t) \). Thus the parameters \( p_k \)'s are global information available to all agents. In case of a constant \( x_i(t) = x_0 \neq 0 \), the error \( e_i(t) \) is given by \( \text{(4)} \), and \( H(q^{-1}) \) in \( \text{(13)} \) becomes \( H(q^{-1}) = 1 - q^{-1} \). Hence, in case of a constant consensus trajectory the only global information is the knowledge that the \( i \)-th agent signal \( e_i(t+1) \) in \( \text{(39)} \) is equal to the mismatch between \( x_i(t+1) \) and the delayed average of neighboring states \( x_j(t), j \in N_i \) (see \( \text{(4)} \)).

Remark 1. The global stability analysis of \( \text{(38)-(39)} \) is given in \( \text{Theorem 1} \), and it demonstrates that the choice of the normalizer \( r_i(t) \) in \( \text{(38)} \) is constrained by the following conditions: \( 1) \) \( r_i(t) \) should satisfy \( r_i(t) = 0 \left( \sum_{k=1}^{2L} |e(k)| \right) \); \( 2) \) \( r_i(t) \log r_i(t) \) should not grow faster than \( r_i(t) \log r_i(t) \), which is guaranteed by setting \( \epsilon_i < 1 \); and \( 3) \) \( \sum_{i=1}^{n} \| \psi(t) \|^2 / r_i^2 \leq \infty \).

The next lemma is useful for our analysis. Define
\[
E_i(t) = E_i(t-1) + |e_i(t)|, \quad E_i(0) > 0, \quad i \in \mathcal{V}. \tag{41}
\]

Lemma 1. Let the Assumption A1 hold. Then \( \forall \ i \in \mathcal{V} \) we have
\[
r_i(t) \leq c_i E_i(t), \quad \forall \ n \geq 0, \quad 0 < c_i < \infty, \tag{42}
\]
where \( r_i(t) \) is defined by \( \text{(40)} \).

Proof. The proof is given in the Appendix.

We now prove that \( \{ e_i(t) \} \in l_1 \), and \( \left\{ \hat{\theta}(t) \right\} \) is a convergent sequence, \( \forall \ i \in \mathcal{V} \).

Theorem 1. Let the Assumption A1 hold. Then the algorithm \( \text{(38)-(40)} \) ensures that \( \forall \ i \in \mathcal{V} \),
\[
(1) \quad \sum_{i=0}^{n} |e_i(t)| \leq c_5 < \infty, \quad \forall \ n \geq 0, \tag{43}
\]
(2) \[ \sum_{i=0}^{n} \| \psi(t) \| \leq c_6 < \infty, \quad \forall \ n \geq 0, \tag{44}\]
(3) \[
\hat{\theta}_i = \lim_{t \to \infty} \hat{\theta}(t) \text{ exists}. \tag{45}
\]

Proof. Substituting \( \text{(38)} \) in \( \text{(35)} \) yields
\[
\frac{1}{b_0} e_i(t+1) = -\theta_i(t)^3 \phi_i(t), \quad \hat{\theta}(t) := \hat{\theta}(t) - \hat{\theta}_i \in \mathcal{V}. \tag{46}
\]
Subtracting \( \hat{\theta} \) from both sides of \( \text{(39)} \), and squaring the obtained equation yields
\[
\| \hat{\theta}(t+1) \|^2 = \| \hat{\theta}(t) \|^2 + 2 \mu_i \text{sgn } b_0 \frac{\hat{\theta}_i(t)^3 \phi_i(t) \text{sgn } e_i(t+1)}{r_i(t) \log r_i(t)^{1+\epsilon_i}/2} + \mu_i \| \psi(t) \|^2 / r_i(t) \log r_i(t)^{1+\epsilon_i}/2, \quad \forall \ i \in \mathcal{V} \tag{47}
\]
where \( \hat{\theta}_i \) is defined with \( \text{(46)} \). After substituting \( \text{(46)} \) in the 2nd term on the RHS of \( \text{(47)} \), and using the fact that \( \| \psi(t) \|^2 / r_i(t) \leq 1 \) (see \( \text{(40)} \)), we obtain
\[
\| \hat{\theta}(t+1) \|^2 \leq \| \hat{\theta}(t) \|^2 - 2 \mu_i \frac{|e_i(t+1)|}{b_0} + \mu_i^2 \frac{\| \psi(t) \|^2 / r_i(t) \log r_i(t)^{1+\epsilon_i}/2}{r_i(t) \log r_i(t)^{1+\epsilon_i}/2}, \quad \forall \ i \in \mathcal{V} \tag{48}
\]
Summing both sides of \( \text{(48)} \) from \( t = 0 \) to \( t = n \) gives
\[
\| \hat{\theta}(n+1) \|^2 + 2 \mu_i \sum_{t=0}^{n} \frac{|e_i(t+1)|}{b_0} \leq \| \hat{\theta}(0) \|^2 + \mu_i \sum_{t=0}^{n} \frac{\| \psi(t) \|^2 / r_i(t) \log r_i(t)^{1+\epsilon_i}/2}{r_i(t) \log r_i(t)^{1+\epsilon_i}/2}, \quad \forall \ i \in \mathcal{V} \tag{49}
\]
for all \( n \geq 0 \). It can be shown that the second term on the RHS of \( \text{(49)} \) is uniformly bounded. Since by \( \text{(40)} \), \( \| \psi(t) \| = r_i(t) - r_{i-1}(t-1) \), monotonicity of \( r_i(t) \) implies
\[
\sum_{t=1}^{\infty} \frac{\| \psi(t) \|^2 / r_i(t) \log r_i(t)^{1+\epsilon_i}/2}{r_i(t) \log r_i(t)^{1+\epsilon_i}/2} < \infty. \tag{50}
\]
Then by virtue of \( \text{(50)} \), from \( \text{(49)} \) one can conclude that \( \forall \ n \geq 0, \ i \in \mathcal{V} \), and some \( c_i > 0 \),
\[
\sum_{t=0}^{n} \frac{|e_i(t+1)|}{b_0} \leq c_i < \infty \tag{51}
\]
where we used the fact that \( \log z^{1+\epsilon_i}/2 \leq \log z \). \( z \geq e \), and \( \epsilon_i \) is the same as in \( \text{(39)} \). By using \( \text{(42)} \) in \( \text{(51)} \) yields
\[
\sum_{t=0}^{n} \frac{|e_i(t+1)|}{b_0} \leq c_i < \infty, \quad \forall \ n \geq 0, \quad i \in \mathcal{V}, \tag{52}
\]
for some \( c_i > 0 \). Since \( |e_i(t+1)| = E_i(t+1) - E_i(t) \) (see \( \text{(41)} \)), previous relation implies
\[
c_i \geq \sum_{t=0}^{n} \frac{E_i(t)}{b_0} \log E_i(t) \int_{E_i(t+1)}^{E_i(t+1)} \frac{dz}{z \log z} = \log(\log E_i(n+1) - \log(\log E_i(0))) \tag{53}
\]
from where by \( \text{(41)} \) it follows that \( E_i(n+1) = E_i(0) + \sum_{t=0}^{n} |e_i(t)| \leq c_i < \infty, \quad \forall \ n \geq 0, \quad i \in \mathcal{V} \). Thus \( \text{(43)} \) is proved. The statement \( \text{(44)} \) follows from \( \text{(40), (42) and (43)} \). We next prove \( \text{(45)} \). After
Remark 2. Because by (44) \( \psi(t) \to 0 \) as \( t \to \infty \), the signal vector \( \psi_i(t) \), \( i \in V \), is not persistently exciting. Therefore it cannot be expected that \( \theta_i(t) \to \theta_i \), as \( t \to \infty \), where \( \theta_i \) is the true parameter vector defined by (37). Also it should be noted that in (39) the algorithm gain (step-size) \( \mu_i/r_i(t) \) is a nonvanishing sequence, which follows from (40), and (44).

The fact that \( e_i(t) \to 0 \) as \( t \to \infty \), implies that the difference between \( x_i(t+1) \) and the corresponding average of the neighboring states (see (21)) converges to zero. It does not imply synchronization to a common consensus trajectory. To clarify this point, assume for a moment that in (21) \( L=0 \) and \( h_1=1 \).

Recall that \( r_i(k) \to 1 \) (see (40)). Then by (44) we have

\[
\sum_{k=0}^{t} \frac{\psi_i(k)\text{sgn}(e_i(k+1))}{r_i(k)\log r_i(k)^{(1+r_1)/2}} \leq c_0 < \infty, \quad \forall t \geq 0,
\]

for \( i \in V \). Hence as \( t \to \infty \) the second term on the RHS of (54) is an absolutely convergent series, implying that \( \theta_i \) in (45) exists. Thus the theorem is proved. \( \square \)

Part 1: Prove that \( [\|\delta(t)\|] \in l_1 \), where

\[
\delta(t) := (I - \sum_{k=1}^{2l+1} h_k q^{-k} x(t))\delta(t)
\]

where \( h_k \), \( 1 \leq k \leq 2l+1 \) are the same as in (13), while \( \alpha \) and \( v \) are vectors defined in (56). Obviously absolute summability of \( [\delta(t)] \) implies \( \delta(t) \to 0 \) as \( t \to \infty \), as a consequence of which we can conclude that asymptotically all elements \( x(t) \) of the vector \( x(t) \) behave as \( \sum_{k=1}^{2l+1} h_k q^{-k} x(t-k) \), thus yielding \( x_i(t) - x_j(t) \to 0 \) as \( t \to \infty \), \( \forall i, j \in V \), \( i \neq j \).

Part 2: Prove that

\[
\lim_{t \to \infty} (v^T x(t) - x_i(t)) = 0
\]

where \( x_i(t) \) is given by (12). Then show that (58) follows from (60) and (61).

Proof of part 1: After substituting (57) in (P(q^{-1})), from (15) we can derive

\[
e(t) = (- \sum_{k=1}^{2l+1} h_k W_i q^{-k} x(t)) + \delta(t)
\]

where \( \delta(t) \) is defined by (60). The above equation can be written in the form

\[
ne(t) = -(\sum_{k=1}^{2l+1} h_k W_i q^{-k}) x(t) - \delta(t)
\]

Since \( W_i \alpha = 0 \) (see the discussion after Eq. (56)), and by (60) \( \delta(t) - x(t) = -\sum_{k=1}^{2l+1} h_k q^{-k} x(t) \), we have \( W_i (\delta(t) - x(t)) = 0 \). Hence, (63) implies

\[
ne(t) = R (q^{-1}) \delta(t), \quad R (q^{-1}) = I - \sum_{k=1}^{2l+1} h_k W_i q^{-k}.
\]

Because by Theorem 1 \( [\|e(t)\|] \in l_1 \), \( [\|\delta(t)\|] \) will be an absolutely summable sequence if \( R (q^{-1}) \) is a stable MIMO transfer operator. We now prove that this is indeed so. Similarly as in (16), operator \( R (q^{-1}) \) can be factored as follows

\[
R (q^{-1}) = (I - W_i q^{-1}) \prod_{k=1}^{L} (I + p_k W_i q^{-k} + W_i q^{-2k}) .
\]

Recall that \( R (z^{-1}) \) is a stable MIMO transfer function iff \( \text{det}(R (z^{-1})) \neq 0 \), \( \forall |z| \geq 1 \). We analyze separately each of the factors in (65). Consider

\[
R_k (z^{-1}) = I + p_k W_i z^{-1} + W_i z^{-2}, \quad k = 1, \ldots, L.
\]

Let \( \eta_i \) and \( \psi_i \), \( i = 1, \ldots, N \), be the eigenvalues (not necessarily distinct), and the corresponding eigenvectors, respectively, of \( W_i \), that is \( W_i \eta_i = \eta_i \psi_i \). Because \( |\psi_i| < 1 \) (see (56)), the eigenvalues \( \eta_i \) satisfy \( |\eta_i| < 1 \), \( i = 1, \ldots, N \). Thus \( R_k (z^{-1}) \psi_i = m_k (z) \eta_i \), where

\[
m_k (z) = 1 + p_k \eta_i z^{-1} + \eta_i^2 z^{-2}, \quad i = 1, \ldots, N.
\]

Therefore for a fixed \( z \), \( m_k (z) \) is an eigenvalue of \( R_k (z^{-1}) \). It is not difficult to see that \( m_k (z) \neq 0 \), \( \forall |z| \geq 1 \). Note (67) can be written as \( m_k (z) = 1 + s_1 \eta_i z^{-1} + s_2 \eta_i^2 z^{-2} \), \( s_1 = z \eta_i, \quad 1 \leq i \leq N \). Since the zeros \( z^* \) of \( m_k (z) \) satisfy \( |s_1^*| = 1 \) we have \( |z^*| = |\eta_i| < 1 \). Hence \( m_k (z) \) is analytic for all \( |z| \geq 1 \). Thus the determinant of \( R_k (z^{-1}) \) (see (66)) satisfies

\[
\text{det}(R_k (z^{-1})) = \prod_{i=1}^{N} m_k (z) \neq 0, \quad \forall |z| \geq 1.
\]
for $k = 1, \ldots, L$. Similarly it can be shown that the remaining factor in (65), $I - W_i z^{-1}$ satisfies $det(I - W_i z^{-1}) \neq 0$, $\forall \ |z| > 1$ by virtue of the fact that $\rho(W_i) < 1$. Based on the above consideration we conclude that $det(R(z^{-1})) \neq 0$, $\forall \ |z| > 1$, i.e., $R(z^{-1})^{-1}$ is a stable MIMO transfer function. Thus by (43) and (64) we have

$$\sum_{i=0}^{n} \left| \delta(t) \right| \leq c_{11} < \infty, \forall \ n \geq 0. \tag{69}$$

**Proof of part 2:** Multiplying both sides of (60) by $v^T$ gives

$$v^T \delta(t) = H(q^{-1})v^T X(t) \tag{70}$$

where $H(q^{-1})$ is defined by (13) (or (14)), and where we used the fact that $v^T \alpha = 1$ (see (56)). Let $g(t), \ t \geq 0$ be the unit impulse response of $H(z^{-1})^{-1}$. Then from (70) it follows that

$$v^T X(t) = g(t) + \gamma(t), \ \gamma(t) := v^T \delta(t) \tag{71}$$

where $\gamma$ denotes the convolution operator. From (14) we have

$$g(t) = g_0 + \sum_{k=1}^{L} G_k \cos(\omega_k t + \chi_k), \ t \geq 0. \tag{72}$$

for some $g_0$, $\chi_k$, and $G_k > 0$. After substituting (72) in (71) one can derive

$$v^T X(t) = g_0 \sum_{i=0}^{t} \gamma(i) + \frac{1}{2} \sum_{i=0}^{t} \sum_{k=1}^{L} G_k \left[ \exp[i(\omega_k t - i + \chi_k)] + \exp[-i(\omega_k t - i + \chi_k)]\right] \gamma(i)$$

$$= g_0 \sum_{i=0}^{t} \gamma(i) + \frac{1}{2} \sum_{k=1}^{L} G_k \left[ \exp[i(\omega_k t + \chi_k)] \sum_{i=0}^{t} e^{-j\omega_k i} \gamma(i) + \exp[-i(\omega_k t + \chi_k)] \sum_{i=0}^{t} e^{j\omega_k i} \gamma(i) \right]$$

where $\gamma$ is defined as follows

$$\Gamma_i(j\omega) := \sum_{i=0}^{t} e^{-j\omega i} \gamma(i), \ \omega \in [0, 2\pi). \tag{74}$$

Since $\Gamma_i(j\omega) = \left| \Gamma_i(j\omega) \right| \exp[j\phi_i(t)]$, $f_i(t) = \arg \Gamma_i(j\omega)$ for $k = 1, \ldots, L$, from (73) we can obtain

$$v^T X(t) = g_0 \sum_{i=0}^{t} \gamma(i) + \sum_{i=1}^{L} G_k \left| \Gamma_i(j\omega) \right| \cos(\omega_k t + \chi_k + f_k) \tag{75}$$

where we used the fact that $|\Gamma_i(j\omega)|$ is an even function of argument $\omega$, and $\gamma(-j\omega) = \Gamma_i(-j\omega) \exp[-j\phi_k(t)]$. Observe that by (69) and the definition of $\gamma(t)$ (see (71)), we have $\gamma(t) \in L^1$. This implies that $\Gamma_i(j\omega)$ (see (74)) uniformly converges as $t \to \infty$. Hence the Fourier transform of $\gamma(t)$ exists and it is equal to

$$\tilde{\Gamma}(j\omega) := \lim_{t \to \infty} \Gamma_i(j\omega), \ \omega \in [0, 2\pi). \tag{76}$$

Define

$$y_i(t) = g_0 \tilde{T}(0) + \sum_{k=1}^{L} G_k |\tilde{T}(j\omega)| \cos(\omega_k t + \chi_k + f_k) \tag{77}$$

where $\tilde{T}_k := \arg \tilde{T}(j\omega)$. Then from (75) and (77) one can conclude that $\lim_{t \to \infty} (v^T x(t) - y_i(t)) = 0$, where we took into account that $f_k(t) \to \tilde{f}_k$, $k = 1, \ldots, L$ and $|\Gamma_i(j\omega)| \to |\tilde{T}(j\omega)|, \ \forall \omega$, as $t \to \infty$ (see (76)). Thus (61) is proved provided that in (12) we set $x_{\delta} = g_0 \tilde{T}(0)$, $F_k = G_k |\tilde{T}(j\omega)|$, and $\delta_k = \chi_k + f_k$, $k = 1, \ldots, L$. Since by (13) $\alpha(1 - \sum_{k=1}^{L} \tilde{h}_k q^{-k})x(t) = 0$, from (60) one can derive $\delta(t) = x(t) - \alpha x(t) - \left( \sum_{k=1}^{L} \tilde{h}_k q^{-k} \alpha \right) (v^T x(t) - x(t))$, from where by (61) and the fact that $|\left| \tilde{\delta}(t) \right| \| \leq c_{11}$ we obtain (58). Because by (58) $\tilde{x}(t)$ is bounded, and by Assumption A1, $1/B q^{-1}$ is a stable transfer function, from (1) it follows that $|u_i(t)| = |\tilde{A}(q^{-1}) - b_i B q^{-1}|x_i(t + 1)| \leq c_{21} \max_{0 \leq t \leq q} |x_i(t + 1)| \leq c_{20} < \infty, \forall \ t \geq 0, 0 < c_{21}, c_{20} < \infty \ i \in \mathbb{V}$. Thus the theorem is proved. \(\square\)

Remark 3. The above analysis shows that the control law defined by (38) and (39) forces overall network dynamics to satisfy (70) where $\delta(t)$ is the sequence (see (69)). In other words, this controller embeds the internal model specified by (14) into the closed loop agent dynamics. It should be noted that $|\chi_i(t)|$ is not known or available to the local controllers. It rather emerges as time progresses as a result of the adaptation process and the inter-agent information flow. The resulting form of $|\chi_i(t)|$ depends on the network graph topology and the conditions $x_i(0)$, $u_i(0)$, and $\tilde{\theta}_i(0)$. In this paper $H(q^{-1})$ (see (14)) has been restricted so that $|\chi_i(t)|$ is bounded. If $H(q^{-1})$ has repeated roots on the unit circle, then $|\chi_i(t)|$ will diverge as polynomial function of time $t$. In this case agent outputs $x_i(t)$ and consequently $\tilde{u}_i(t)$ (see Assumption A1) will diverge with the same rate as $x_i(t)$. In Radenković and Tadić (2016) (58) is obtained for the special case of integrator dynamics and a constant $x_i$. There the algorithm provides only $L_1$ performance, as a consequence of which the obtained result cannot be generalized to higher order systems or more complex patterns of $x_i(t)$. When in (13) $L = 0$ and $h_1 = 1$, the error $e_i(t + 1)$ is given by (4). It is not difficult to see that then synchronization trajectory converges to a constant dependent on the average of the initial states $x_i(0), - (M - 1) \leq n \leq 0, \forall i \in \mathbb{V}$.

Simple inspection of (1), (27), (35), (38), and (39) shows that if in (13) $L \geq 1$, and the initial conditions satisfy $u_i(n - 1) = 0, x_i(0) = 0$ (and therefore $\tilde{\phi}(0) = 0, \tilde{\phi}(n - 1) = 0$) for $n \geq 0, \forall i \in \mathbb{V}$, all states $x_i(t)$ will be equal to zero $\forall t \geq 0$, implying that $\gamma(t)$ (see (77) and (75)) degenerates to zero. Generally when $L \geq 1$, irrespective of the initial values $x_i(0), n \leq 0, i \in \mathbb{V}$, synchronization to $y_i(t)$ will occur if at least one agent selects a nonzero initial value of its control input (see (38)).

4. The case of unknown sign of parameter $h_0$

In this section we assume that the sign of parameter $h_0$ in (1) is unknown. In this case the local control inputs $u_i(t), \ i \in \mathbb{V}$ are generated by (38) with $\hat{\psi}_i(t)$ obtained by using the following adaptive estimator $\forall \ t \geq 0$,

$$\hat{\psi}_i(t + 1) = \hat{\psi}_i(t) + \mu_i(t + 1) \frac{\psi_i(t)sgn(e_i(t + 1))}{T_i(t)D_i(t + 1)} \tag{78}$$

where

$$D_i(t + 1) = D_i(t) + \frac{|e_i(t + 1)|}{T_i(t)}, \ D_i(0) > 0, \tag{79}$$

$$T_i(t) := 1 + \max_{0 \leq t \leq \infty} \| \tilde{\phi}_i(t) \|. \tag{80}$$

while $e_i(t + 1)$ and $\psi_i(t)$ are given by (21) and (36), respectively. The gain $\mu_i(t + 1)$ is defined as follows

$$\mu_i(t + 1) := \frac{L_i(t + 1) - L_i(t)}{\Delta_i(t + 1) - \Delta_i(t)}, \tag{81}$$

with

$$L_i(t) = \Delta_i(t)^2 \cos(\Delta_i(t)), \tag{82}$$

$$\Delta_i(t + 1) = \Delta_i(t) + \frac{|e_i(t + 1)|}{T_i(t)D_i(t + 1)}, \ \Delta_i(0) > 0. \tag{83}$$

In the above algorithm $\hat{\theta}_i(0)$ is an arbitrary vector.
Comment about \( \mu_i(t) \) in (78). Note that by (82) we have \( L(t + 1) - L_i(t) = (\Delta(t + 1)^2 - \Delta_i(t)^2) \cos(\Delta_i(t) + 1) - \Delta_i(t)^2 \cos(\Delta_i(t) + 1) - \cos(\Delta_i(t))) \) where from by using simple trigonometry, (81) can be transformed into

\[
\mu_i(t + 1) = (\Delta_i(t + 1) + \Delta_i(t)) \cos(\Delta_i(t + 1)) - \Delta_i(t)^2.
\]

\[
\sin\left(\frac{\Delta_i(t + 1) + \Delta_i(t)}{2}\right) \sin\left(\frac{\Delta_i(t + 1) - \Delta_i(t)}{2}\right)
\]

where \( \sin(z) := \sin(z)/z \). Clearly \( \mu_i(t) \) is a continuous function of \( \Delta_i(t) \). The presence of \( \cos(\cdot) \) and \( \sin(\cdot) \) terms in (84) ensures that the gain \( \mu_i(t) \) can be positive or negative, depending on the size of \( \Delta_i(t) \). It is expected that throughout the adaptation process \( \mu_i(t) \) will change its sign until a stabilizing \( \hat{\theta}_i(t) \) in (38) is found.

**Remark 4.** The global convergence of the algorithm (78)–(83) is presented in Theorem 3. The proof of this theorem reveals that the choice of \( \mu_i(t), T_i(t) \) and \( D_i(t) \) in (78)–(83) is dictated by the following constraints: (1) \( T_i(t) \) and \( D_i(t) \) are non-decreasing positive sequences; (2) \( \mu_i(t) \) is selected so that \( \sum_{t=0}^{\infty} \mu_i(t) |\epsilon_i(t)| / T_i(t - 1) \) is bounded, where \( L_i(t) = L_i(t - 1) \), where \( L_i(t) \) changes its sign if \( L_i(t) \rightarrow \infty \) as \( n \rightarrow \infty \); (3) \( T_i(t) \) and \( D_i(t) \) are selected so that: (a) the 2nd term on the RHS of (78) satisfies \( \sum_{t=0}^{\infty} \mu_i(t + 1)^2 |\epsilon_i(t)|^2 / T_i(t + 1)^2 < \infty \), and (b) boundedness of \( \Delta_i(t) \) implies \( |\epsilon_i(t)| \) in (1). One possible choice of \( D_i(t) \) is found. The algorithm gain is a discontinuous function of its argument, (78) a smooth parameter estimation. By “smooth” we mean that the algorithm gain \( \mu_i(\cdot) \) and consequently \( \hat{\theta}_i(\cdot) \) are continuous functions of the argument \( \Delta_i(t) \) (see (78) and (84)).

The following lemma examines some properties of \( \Delta_i(t), \mu_i(t), T_i(t) \) and \( D_i(t), i \in \mathbb{V} \).

**Lemma 2.** Suppose the Assumption A1 holds. Then \( \forall i \in \mathbb{V} \),

\[
\Delta_i(t) \leq \Delta_i(0) + \log D_i(t), \forall t \geq 0
\]

(85)

\[
|\mu_i(t)| \leq c_{13}(\Delta_i(t) + \log D_i(t))^2, \forall t \geq 0
\]

(86)

\[
I_i(n) := \sum_{t=1}^{n} \mu_i(t + 1)^2 \frac{|\epsilon_i(t)|}{T_i(t + 1)^2} \leq c_{14} < \infty,
\]

(87)

\[
T_i(t) \leq 1 + c_{15} \max_{0 \leq t \leq T_i(0)} |\epsilon_i(t)|,
\]

(88)

\[
\forall n \geq 1, \text{ and some positive constants } c_{13}, c_{14} \text{ and } c_{15}.
\]

**Proof.** The proof is given in the Appendix. \( \Box \)

The next proposition proves that \( \{\epsilon_i(t)\} \in \mathbb{I}_1 \), and \( \{\hat{\theta}_i(t)\} \) is a convergent sequence for all \( i \in \mathbb{V} \).

**Theorem 3.** Let the Assumption A1 hold. Then the algorithm (38), (78)–(83) ensures that for \( i \in \mathbb{V} \),

\[
\Delta_i := \lim_{t \to \infty} \Delta_i(t) \text{ exists},
\]

(89)

\[
D_i := \lim_{t \to \infty} D_i(t) \text{ exists},
\]

(90)

\[
n \sum_{t=0}^{n} |\epsilon_i(t)| \leq c_{25} < \infty, \forall n \geq 0,
\]

(91)

\[
\hat{\theta}_i := \lim_{t \to \infty} \hat{\theta}_i(t) \text{ exists},
\]

(92)
On the other hand by Lemma 1 (see (40), (42) and (80)) we have that for some $c_{26} > 0$,\footnote{\[|\epsilon_i(t)| \leq c_{26} T_i(n-1), \quad \forall i \geq 1, \text{ thus implying } \lim_{n \rightarrow \infty} \sup_{i \geq 1} |\epsilon_i(t)| \leq c_{26} T_i / n \rightarrow 0, \text{ which contradicts (102). Hence (26) must be a bounded sequence. Consequently, (90) and (101) imply (91). Let us prove (92). From (78) we have}$$\hat{\theta}_i(t + 1) = \hat{\theta}_i(0) + S(t), \quad i \in \mathcal{V} \quad (103)$$\] where
\[S(t) = \sum_{k=1}^{t} \mu_i(k+1) \frac{\psi_i(k) \text{sgn}(\epsilon_i(k+1))}{T_i(k) D_i(k+1)}.
\]
Observe that by (86) and (90) $\mu_i(k)$ is a bounded sequence. Then by virtue of the fact that (40), (42) and (91) imply $\sum_{k=1}^{t} |\psi_i(k)| \leq c_{29} \leq \infty, \forall t \geq 0$, one can derive $|S(t)| \leq c_{29} \sum_{k=1}^{t} |\psi_i(k)| \leq c_{29} \leq \infty, \forall t \geq 0, i \in \mathcal{V},$ where we took into account that $D_i(k) > 0$ and $T_i(k) \geq 1, \forall k \geq 0$. Hence as $t \rightarrow \infty$, the 2nd term on the RHS of (103) is an absolutely convergent series, implying that (92) holds. In order to derive the statement (93) we should note that $\sum_{i} |\epsilon_i(t+1) - \Delta_i(t)| / 2 \rightarrow 0$ as $\Delta_i(t) \rightarrow \Delta_i$. Then from (84) and (89) we can easily arrive at $\lim_{t \rightarrow \infty} \mu_i(t + 1) = 2 \Delta_i \cos(\Delta_i) - 2 \Delta_i \sin(\Delta_i), \quad i \in \mathcal{V}$. Thus the theorem is proved. \hfill \Box

The following result is identical to Theorem 2.

**Theorem 4.** Let the Assumptions A1 and A2 hold. Then the algorithm (38), (78)–(83) guarantees (58) and (59).

**Proof.** See the proof of Theorem 2. \hfill \Box

**5. Simulation example**

In this example we consider the algorithm (78)–(83), and a network of six agents whose directed graph topology is described by the following adjacency matrix
\[
\mathbf{A}_d = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\quad (104)
\]
where $\mathbf{A}_d(i, j) = 1$ indicates that the agent $i$ can directly receive information from the $j$th agent. Otherwise $\mathbf{A}_d(i, j) = 0$. It is assumed that the agent dynamics is given by $x_i(t + 1) = 1.5 x_i(t) + 1.4 x_i(t - 1) - 2 u_i(t) + 0.8 u_i(t - 1)$, with $x_i(0) = i(1)$, $x_i(1) = 2i(1)^{1/2}$, where $i(1)$ is defined by $H(\delta^{-1}) = (1 - \delta^{-1})^1 + p_1 \delta^{-1} + \delta^{-2}$, which $p_1 = -2 \cos(\alpha_0)$ with $\alpha_0 = \pi / 90$. In this case the emerging synchronization trajectory is $x_i(t) = x_0 + F_1 \cos(\omega_0 t + \phi_i)$, where $x_0, F_1 > 0$, and $\phi_i$ depend on the initial conditions. Fig. 1 shows that all states $x_i(t), i = 1, \ldots, 6$, converge toward the same $x_i(t), t \geq 0$. Fig. 2 indicates that the parameter estimates $\hat{\theta}_i(t)$ (for $i = 1$) are convergent sequences. As it is commented in Remark 2, $\hat{\theta}_i(t), i \in \mathcal{V}$ do not necessarily converge toward the true parameters $\theta_i$.

**Appendix**

**Proof of Lemma 1.** Because by the Assumption A1, $\mathbf{A}(q^{-1}) / B(q^{-1})$ is a stable operator, from (26) we have
\[
|u_i(t - 1) + \beta_i(t - 2)| \leq c_2 \lambda^{g_2} + c_3 \sum_{j=0}^{t} \lambda^{g_2} |\epsilon_i(t - j)|
\quad (A.1)
\]

**Fig. 1.** Agent states $x_i(t), i = 1, \ldots, 6$.

**Fig. 2.** Convergence of parameter estimates $\hat{\theta}_i(t)$.

for some $0 < \lambda < 1$, and $0 < c_2, c_3 < \infty$. The role of $c_2 \lambda^{g_2}$ is to capture the effect of nonzero initial conditions. On the other hand, definition (40) implies $r_i(t) \leq r_i(0) + M \sum_{i=0}^{t} |\epsilon_i(t)| + |u_i(t - 1) + \beta_i(t - 2)|$, which together with (A.1) gives (42).

**Proof of Lemma 2.** (1) From (79), (80) and (83) we can obtain $\Delta_i(t + 1) = \Delta_i(t) + \sum_{k=1}^{t} \int_{T_i(k)}^{T_i(k+1)} |\epsilon_i(t+1) - \Delta_i(t)| / 2 \leq \Delta_i(t) + \log(D_i(t + 1))$. (2) Because $|\sin(\Delta_i)| \leq 1, \forall \Delta_i$, (86) follows from (84) and (85).

(3) Note that $(\log D_i(t + 1)^2 / D_i(t + 1)^2) \leq c_{18} \leq c_{18}$, for $0 < c_{2} < 1$. Then by the fact that $\|\psi_i(t)\| / T_i(t) \leq 1$ (see (80)) and (86) and the definition of $I_2(t)$ (see (87)) we can derive
\[
I_2(t) \leq c_{18} \sum_{i=1}^{n} \left( \frac{\Delta_i(0) + \log D_i(t + 1)}{T_i(t) D_i(t + 1)^2} \right)^4 \frac{|\psi_i(t)|}{T_i(t) D_i(t + 1)^2}
\quad (A.2)
\]
for some $\varepsilon_2$, $0 < \varepsilon_2 < 1$. We now examine $\|\psi(t)\|$. Definition (36) gives $\|\psi(t)\| \leq \sum_{p=0}^{M-1} |e_p(t-p)| + \sum_{p=1}^{M-1} |u_p(t-p) + \beta(t-p-1)|$, by which from (A.2) one can write

$$I_3(n) \leq c_{19}I_3(n) + I_4(n), \quad (A.3)$$

where $I_3(n) := \sum_{p=1}^{n} \sum_{t=p}^{n} \frac{1}{|l(t-p)|} \sum_{p=0}^{M-1} |e_p(t-p)| + |u_p(t-p)| + \beta(t-p-1)|$, and $I_4(n) := \sum_{t=0}^{n} \sum_{p=0}^{M-1} |e_p(t)| + |u_p(t)| + \beta(t-p)|$. Let us remark that $\tau(t) = 0$, $u_i(t) = 0$ and $\beta(t) = 0$, $\forall \tau < 0$. This implies that the summation indices in $I_3(n)$ and $I_4(n)$ start with $t = p$. By exploiting monotonicity of $D_i(t)$ and $T_i(t)$ (see (79) and (80)) we can obtain

$$I_3(n) \leq \sum_{t=0}^{n} \sum_{p=0}^{M-1} \frac{|e_p(t-p)|}{|l(t-p)|} \leq c_{19} \sum_{t=1}^{n} \int_{[t-l(t-1)]}^{l(t)} |e_i(t)| dt$$

$$\leq c_{19} + M \sum_{t=1}^{n} \int_{[t-l(t-1)]}^{l(t)} |e_i(t)| dt \leq c_{20} < \infty, \quad (A.4)$$

for some $0 < c_{19} < \infty$, and where we used the fact that $0 < \varepsilon_2 < 1$. Similarly one can write

$$I_4(n) \leq \sum_{p=1}^{n} \sum_{t=p}^{n} \frac{|u_i(t-p) + \beta(t-p-1)|}{|l(t-p)|} \leq c_1 + c_2 \sum_{t=1}^{n} \frac{|e(t+1)|}{|l(t)|} \leq c_1 + c_2 \sum_{t=1}^{n} \frac{|e(t+1)|}{|l(t)|} \leq c_1 \sum_{t=0}^{n} \frac{|e(t) + 1|}{|l(t)|} \leq c_1 \sum_{t=0}^{n} \frac{|e(t) + 1|}{|l(t)|} \leq c_1 \sum_{t=0}^{n} \frac{|e(t+1)|}{|l(t)|} \leq c_2 \sum_{t=0}^{n} \frac{|e(t+1)|}{|l(t)|} \leq c_2 < \infty. \quad (A.6)$$

Similarly as in (A.4), the “discrete index summation” in (A.6) can be converted into “continuous integration” to conclude that $I_4(n) \leq c_{22} < \infty$. In all $n \geq 0$. Thus both $I_3(n)$ and $I_4(n)$ are uniformly bounded sequences. Hence, (A.3) implies that (87) holds.

(4) Statement (88) follows from (80), (36), and (A.1). Thus the lemma is proved.

References


