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On global stabilization of Burgers' equation by boundary control ☆

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Abstract

Although often referred to as a one-dimensional "cartoon" of Navier–Stokes equation because it does not exhibit turbulence, the Burgers equation is a natural first step towards developing methods for control of flows. Recent references include Burns and Kang [Nonlinear Dynamics 2 (1991) 235–262], Choi et al. [J. Fluid Mech. 253 (1993) 509–543], Ito and Kang [SIAM J. Control Optim. 32 (1994) 831–854], Ito and Yan [J. Math. Anal. Appl. 227 (1998) 271–299], Byrnes et al. [J. Dynam. Control Systems 4 (1998) 457–519] and Van Ly et al. [Numer. Funct. Anal. Optim. 18 (1997) 143–188]. While these papers have achieved tremendous progress in local stabilization and global analysis of attractors, the problem of global asymptotic stabilization has remained open. This problem is non-trivial because for large initial conditions the quadratic (convective) term – which is negligible in a linear/local analysis – dominates the dynamics. We derive nonlinear boundary control laws that achieve global asymptotic stability. We consider both the viscous and the inviscid Burgers' equation, using both Neumann and Dirichlet boundary control. We also study the case where the viscosity parameter is uncertain, as well as the case of stochastic Burgers' equation. For some of the control laws that would require the measurement in the interior of the domain, we develop the observer-based versions.

Keywords: Burgers' equation; Boundary control; Stabilization; Nonlinear control

1. Introduction

Although often referred to as the one-dimensional "cartoon" of Navier-Stokes equation because it does not exhibit turbulence, the Burgers equation is a natural first step towards developing methods for control of flows. Recent references include Burns and Kang [1], Choi et al. [3], Ito and Kang [5], Ito and Yan [6], Byrnes et al. including [2], and Van Ly et al. [9]. Byrnes et al. [2] show that a linear boundary controller achieves local exponential stability (the initial condition needs to be small in L_2). Van Ly et al. [9] improve this result (they extend it to L_∞) but remain local. Achieving a global result for the Burgers equation is non-trivial

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because for large initial conditions the quadratic (convective) term – which is negligible in a linear/local analysis – dominates the dynamics.

We derive nonlinear boundary control laws that achieve global asymptotic stability (in a very strong sense). We consider both the viscous and the inviscid Burgers equation, using both Neumann and Dirichlet boundary control. We also study the case where the viscosity parameter is uncertain, as well as the case of stochastic Burgers' equation. For some of the control laws that would require the measurement in the interior of the domain, we develop the observer-based versions.

The study of existence and uniqueness of solutions under boundary feedback laws developed in this paper is much lengthier than the page limitations of this journal permit, and is therefore a topic of another upcoming article.

2. Problem statement

Consider Burgers' equation

$$W_t - \varepsilon W_{xx} + W W_x = 0 \tag{2.1}$$

where $\varepsilon > 0$ is a constant. Our objective is to achieve set point regulation:

$$\lim_{t\to\infty} W(x,t) = W_d, \quad \forall x \in [0,1], \tag{2.2}$$

where W_d is a constant, while keeping $W(x,t)$ bounded for all $(x,t) \in [0,1]\times[0,\infty)$. Without loss of generality we assume that $W_d \geqslant 0$. By defining the regulation error as $w(x,t) = W(x,t) - W_d$, we get the system

$$w_t - \varepsilon w_{xx} + W_d w_x + w w_x = 0. \tag{2.3}$$

We will approach the problem by both Neumann boundary control

$$w_x(0,t) = u_0, \tag{2.4}$$

$$w_x(1,t) = u_1, \tag{2.5}$$

and the Dirichlet boundary control

$$w(0,t) = v_0, \tag{2.6}$$

$$w(1,t) = v_1, \tag{2.7}$$

where u_0, u_1 and v_0, v_1 are scalar control inputs.

It is easy to see that homogeneous Dirichlet boundary conditions $w(0,t) = w(1,t) = 0$ are globally asymptotically L_2-stabilizing for the zero equilibrium of Eq. (2.3). However, the Neumann boundary conditions $wx(0,t) = wx(1,t) = 0$ allow also any nonzero constant to be an equilibrium solution for (2.3) (see [2]). This means that the equilibrium at zero, even though it may be stable, is not *asymptotically* stable. This motivates the development of Neumann boundary controls, which would also be implementable as Dirichlet boundary controls if the physics dictate actuation via $w(0,t)$ and $w(1,t)$, to achieve global asymptotic stability of Eq. (2.3).

3. Neumann boundary control for viscous Burgers' equation

3.1. Global stabilization

We start from the control Lyapunov function

$$V = \frac{1}{2}\int_0^1 w(x,t)^2 \,\mathrm{d}x \tag{3.1}$$

and compute its time derivative

$$\begin{aligned}
\dot V &= \int_0^1 w(\varepsilon w_{xx} - W_d w_x - w w_x)\,\mathrm{d}x \\
&= \varepsilon \int_0^1 w d(w_x) - W_d \int_0^1 w\,\mathrm{d}w - \frac{1}{3}\int_0^1 \left(w^3\right)_x \mathrm{d}x \\
&= \varepsilon\left(w w_x|_0^1 - \int_0^1 (w_x)^2\,\mathrm{d}x \right) - \frac{W_d}{2} w^2\Big|_0^1 - \frac{1}{3} w^3\Big|_0^1 \\
&= -\varepsilon \int_0^1 (w_x)^2\,\mathrm{d}x \\
&\quad - w(0,t)\left[\varepsilon u_0 - \frac{W_d}{2} w(0,t) - \frac{1}{3} w(0,t)^2\right] \\
&\quad + w(1,t)\left[\varepsilon u_1 - \frac{W_d}{2} w(1,t) - \frac{1}{3} w(1,t)^2\right].
\end{aligned} \tag{3.2}$$

Choosing the control as

$$u_0 = \frac{1}{\varepsilon}\left[\left(c_0 + \frac{W_d}{2}\right) w(0,t) + \frac{1}{3} w(0,t)^2\right], \tag{3.3}$$

$$u_1 = -\frac{1}{\varepsilon}\left[c_1 w(1,t) + \frac{1}{3} w(1,t)^2\right], \tag{3.4}$$

where $c_0, c_1 > 0$, we get

$$\dot V = -\left[\varepsilon \int_0^1 (w_x)^2\,\mathrm{d}x + c_0 w(0,t)^2 + \left(c_1 + \frac{W_d}{2}\right) w(1,t)^2\right].$$

By Lemma A.1, we obtain

$$\dot V \leqslant -\min\left\{\frac{\varepsilon}{2}, c_0, c_1 + \frac{Wd}{2}\right\} V, \tag{3.5}$$

which implies that the equilibrium $w(x,t) \equiv 0$ is globally exponentially stable in $L_2[0,1]$.

Consider now the problem with Dirichlet boundary control, namely, the problem where $W(0,t)$ and $W(1,t)$ are the control inputs and $W_x(0,t)$ and $W_x(1,t)$ are the measured variables. If we want to apply the control laws (3.3) and (3.4), we face the problem that the quadratic functions are not invertible. Thus we seek control laws $w_x(0,t) = u_0(w(0,t))$ and $w_x(1,t) = u_1(w(1,t))$ which are invertible functions and can be applied as both Neumann and Dirichlet boundary controls. To this end, consider (3.2):

$$\begin{aligned}
\dot V &= -\varepsilon \int_0^1 (w_x)^2\,\mathrm{d}x \\
&\quad - w(0,t)\left[\varepsilon w_x(0,t) - \frac{W_d}{2} w(0,t) - \frac{1}{3} w(0,t)^2\right] \\
&\quad - \frac{W_d}{2} w(1,t)^2 + w(1,t)\left[\varepsilon w_x(1,t) - \frac{1}{3} w(1,t)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\varepsilon \int_0^1 (w_x)^2 \,\mathrm{d}x \\
&\quad - w(0,t)\left[\varepsilon w_x(0,t) - \frac{W_d}{2} w(0,t) - \frac{c_0}{2} w(0,t) - \frac{1}{18c_0} w(0,t)^3\right] \\
&\quad - \frac{W_d}{2} w(1,t)^2 + w(1,t)\left[\varepsilon w_x(1,t) + \frac{c_1}{2} w(1,t) + \frac{1}{18c_1} w(1,t)^3\right] \\
&\quad - \frac{c_0}{2}\left[w(0,t) - \frac{1}{3c_0} w(0,t)^2\right]^2 - \frac{c_1}{2}\left[w(1,t) - \frac{1}{3c_1} w(1,t)^2\right]^2 .
\end{aligned} \tag{3.6}
$$

Choosing the control as

$$
w_x(0,t) = \frac{1}{\varepsilon}\left[c_0 + \frac{W_d}{2} + \frac{1}{18c_0} w(0,t)^2\right] w(0,t), \tag{3.7}
$$

$$
w_x(1,t) = -\frac{1}{\varepsilon}\left[c_1 + \frac{1}{18c_1} w(1,t)^2\right] w(1,t), \tag{3.8}
$$

we get

$$
\begin{aligned}
\dot V = &-\left[\varepsilon \int_0^1 (w_x)^2 \,\mathrm{d}x + \frac{c_0}{2} w(0,t)^2 + \frac{1}{2}(c_1 + W_d) w(1,t)^2\right] \\
&- \frac{c_0}{2}\left[w(0,t) - \frac{1}{3c_0} w(0,t)^2\right]^2 - \frac{c_1}{2}\left[w(1,t) - \frac{1}{3c_1} w(1,t)^2\right]^2 .
\end{aligned} \tag{3.9}
$$

By Lemma A.1, we get

$$
\dot V \leqslant -cV, \tag{3.10}
$$

where

$$
c = \frac{1}{2}\min\{\varepsilon, c_0, c_1 + W_d\} > 0. \tag{3.11}
$$

Thus the equilibrium $w(x,t) \equiv 0$ is globally exponentially stable in $L_2[0,1]$.

The Neumann control laws (3.7) and (3.8) are invertible functions, which means that they can be applied also as Dirichlet boundary control:

$$
\begin{aligned}
w(0,t) = &\sqrt[3]{9c_0\varepsilon w_x(0,t) + 3c_0\sqrt{9\varepsilon^2 w_x(0,t)^2 + 24c_0\left(c_0 + \frac{W_d}{2}\right)^3}} \\
&+ \sqrt[3]{9c_0\varepsilon w_x(0,t) - 3c_0\sqrt{9\varepsilon^2 w_x(0,t)^2 + 24c_0\left(c_0 + \frac{W_d}{2}\right)^3}},
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
w(1,t) = &\sqrt[3]{-9c_1\varepsilon w_x(1,t) + 3c_1\sqrt{9\varepsilon^2 w_x(1,t)^2 + 24c_1^4}} \\
&+ \sqrt[3]{-9c_1\varepsilon w_x(1,t) - 3c_1\sqrt{9\varepsilon^2 w_x(1,t)^2 + 24c_1^4}}.
\end{aligned} \tag{3.13}
$$

Even though the homogeneous Dirichlet boundary conditions $w(0,t)=w(1,t)=0$ would be globally asymptotically stabilizing, the boundary controls (3.12), (3.13) are more effective because they enhance the negativity of $\dot{V}$ in Eq. (3.9).

3.2. Boundedness

While the stability analysis guarantees that $\int_0^1 w(x,t)^2\,\mathrm{d}x$ will be bounded and converge to zero, it does not guarantee that $w(x,t)$ is bounded and converges to zero for all $x\in[0,1]$. In particular, we don't know if control takes bounded values or not. This is why we need to prove the boundedness and convergence separately. We do this by showing that $\int_0^1 w_x(x,t)^2\,\mathrm{d}x$, $w(0,t)$, and $w(1,t)$ are bounded and by employing Agmon's inequality (Lemma A.2)

$$\max_{x\in[0,1]}|w(x,t)|\leqslant\min\{|w(0,t)|,|w(1,t)|\}+\sqrt{2}\sqrt[4]{\int_0^1 w(x,t)^2\,\mathrm{d}x}\sqrt[4]{\int_0^1 w_x(x,t)^2\,\mathrm{d}x}\,. \tag{3.14}$$

Let us start by denoting

$$\eta(t)=\int_0^1 w_x(x,t)^2\,\mathrm{d}x+\frac{c_0}{2\varepsilon}w(0,t)^2+\frac{1}{2\varepsilon}(c_1+W_d)\,w(1,t)^2 \tag{3.15}$$

$$\zeta(t)=\int_0^1 w_x(x,t)^2\,\mathrm{d}x+\frac{1}{\varepsilon}\left[\left(c_0+\frac{W_d}{2}\right)w(0,t)^2+c_1w(1,t)^2+\frac{1}{36c_0}w(0,t)^4+\frac{1}{36c_1}w(1,t)^4\right]. \tag{3.16}$$

By taking the L_2-inner product of Eq. (2.3) with w_{xx}, substituting $\dot{\zeta}$, and using Eqs. (3.7) and (3.8), we get

$$\begin{aligned}
\dot{\zeta}(t)&=-2\varepsilon\int_0^1 w_{xx}^2\,\mathrm{d}x+\int_0^1 2(W_d+w)w_xw_{xx}\,\mathrm{d}x\\
&\leqslant-2\varepsilon\int_0^1 w_{xx}^2\,\mathrm{d}x+\int_0^1\left[2\varepsilon w_{xx}^2+\frac{1}{2\varepsilon}(W_d+w)^2w_x^2\right]\mathrm{d}x\\
&=\frac{1}{2\varepsilon}\int_0^1(W_d+w)^2w_x^2\,\mathrm{d}x\\
&\leq\frac{1}{\varepsilon}\left(W_d^2+\max_{x\in[0,1]}w(x,t)^2\right)\int_0^1 w_x^2\,\mathrm{d}x\,.
\end{aligned} \tag{3.17}$$

By combining Lemmas A.2 and A.1, it is easy to see that there exists a positive constant q such that

$$\max_{x\in[0,1]}w(x,t)^2\leqslant q\zeta(t)\,. \tag{3.18}$$

Thus we can write Eq. (3.17) as

$$\varepsilon\dot{\zeta}(t)\leqslant q\eta(t)\zeta(t)+W_d^2\eta(t)\,. \tag{3.19}$$

By Lemma B.6 in [8], we get

$$\zeta(t)\leqslant\left(\zeta(0)+\frac{W_d^2}{\varepsilon}\int_0^\infty\eta(t)\,\mathrm{d}t\right)\exp\left[\frac{q}{\varepsilon}\int_0^\infty\eta(t)\,\mathrm{d}t\right]. \tag{3.20}$$

From Eq. (3.9) it follows that

$$\int_0^\infty\eta(t)\,\mathrm{d}t\leqslant\frac{1}{2\varepsilon}\int_0^1 w(x,0)^2\,\mathrm{d}x\,. \tag{3.21}$$

Thus we get

$$\zeta(t) \leqslant \left[\int_0^1 w_x(x,0)^2\,\mathrm{d}x + \left(c_0 + \frac{W_d}{2}\right) w(0,0)^2 + c_1 w(1,0)^2 \right.$$

$$\left. + \frac{1}{36c_0} w(0,0)^4 + \frac{1}{36c_1} w(1,0)^4 + \frac{W_d}{2\varepsilon^2}\int_0^1 w(x,0)^2\,\mathrm{d}x\right] \exp\left[\frac{q}{2\varepsilon^2}\int_0^1 w(x,0)^2\,\mathrm{d}x\right]. \tag{3.22}$$

From Lemma A.1, (3.22), and (3.18), it follows that if $\int_0^1 1 w_x(x,0)^2\,\mathrm{d}x$, $w(0,0)$, and $w(1,0)$ are finite, then $w(x,t)$ is bounded on $(x,t) \in [0,1] \times [0,\infty)$. To prove that $\lim_{t\to\infty} w(x,t)$ for all $x \in [0,1]$, we note that the proof of Lemma A.2 can be adapted to show that

$$\max_{x\in[0,1]} |w(x,t)| \leqslant \min_{x\in[0,1]} |w(x,t)| + \sqrt{2}\sqrt[4]{\int_0^1 w(x,t)^2\,\mathrm{d}x}\sqrt[4]{\int_0^1 w_x(x,t)^2\,\mathrm{d}x}. \tag{3.23}$$

Since we have shown that $\int_0^1 w(x,t)^2\,\mathrm{d}x$ converges to zero (which also means that $\min_{x\in[0,1]}|w(x,t)|$ converges to zero) and that $\int_0^1 w_x(x,t)^2\,\mathrm{d}x$ is bounded, it follows that

$$\lim_{t\to\infty} \max_{x\in[0,1]} |w(x,t)| = 0. \tag{3.24}$$

Thus we have proved the following global result.

Theorem 3.1. *Consider the system* (2.3) *with boundary control* (3.7), (3.8). *If the quantities* $\int_0^1 w_x(x,0)^2\,\mathrm{d}x$, $w(0,0)$, *and* $w(1,0)$ *are finite, then*

1. $\sup_{(x,t)\in[0,1]\times[0,\infty)} |w(x,t)| < \infty$,
2. $\lim_{t\to\infty} \max_{x\in[0,1]} |w(x,t)| = 0$,
3. $\int_0^1 w(x,t)^2\,\mathrm{d}x$ *decays to zero exponentially.*

This implies, in particular, that the control input is bounded and converges to zero in both Neumann and Dirichlet implementations.

3.3. Optimality

A striking feature of the control law (3.7), (3.8) is that it can be used to design optimal controllers. For example, the Neumann control law

$$w_x(0,t) = \frac{1}{\varepsilon}\left[2c_0 + W_d + \frac{1}{9c_0} w(0,t)^2\right] w(0,t), \tag{3.25}$$

$$w_x(1,t) = -\frac{1}{\varepsilon}\left[2c_1 + \frac{1}{9c_1} w(1,t)^2\right] w(1,t) \tag{3.26}$$

can be shown to be the minimizer of the cost functional

$$J = \int_0^\infty \left\{ 2\varepsilon \int_0^1 w_x(x,t)^2\,\mathrm{d}x + c_0 w(0,t)^2 + (c_1 + W_d) w(1,t)^2 \right.$$

$$+c_0\left[w(0,t)-\frac{1}{3c_0}w(0,t)^2\right]^2+c_1\left[w(1,t)-\frac{1}{3c_1}w(1,t)^2\right]^2$$

$$\left.+\frac{\varepsilon}{2c_0+W_d+\frac{1}{9c_0}w(0,t)^2}w_x(0,t)^2+\frac{\varepsilon}{2c_1+\frac{1}{9c_1}w(1,t)^2}w_x(1,t)^2\right\}\mathrm{d}t\,. \tag{3.27}$$

The Dirichlet version of the optimal control law (3.25), (3.26) is also possible to derive in a fashion analogous to Eqs. (3.12) and (3.13).

3.4. Robust and adaptive control

Suppose now that the viscosity ε is not known but a lower bound $\underline{\varepsilon}$ is known. Choosing the Neumann control as

$$w_x(0,t)=\frac{1}{\underline{\varepsilon}}\left[c_0+\frac{W_d}{2}+\frac{1}{18c_0}w(0,t)^2\right]w(0,t), \tag{3.28}$$

$$w_x(1,t)=-\frac{1}{\underline{\varepsilon}}\left[c_1+\frac{1}{18c_1}w(1,t)^2\right]w(1,t), \tag{3.29}$$

we get

$$\dot V\leqslant-\left[\varepsilon\int_0^1(w_x)^2\,\mathrm{d}x+\frac{c_0}{2}w(0,t)^2+\frac{1}{2}(c_1+W_d)\,w(1,t)^2\right]. \tag{3.30}$$

The Dirichlet version of this control achieves, of course, the same.

Since a control using $\underline{\varepsilon}$ involves high gain, we are motivated to look for an adaptive controller which employs an on-line estimate $\hat\varepsilon$ of ε:

$$w_x(0,t)=\frac{1}{\hat\varepsilon}\left[c_0+\frac{W_d}{2}+\frac{1}{18c_0}w(0,t)^2\right]w(0,t) \tag{3.31}$$

$$w_x(1,t)=-\frac{1}{\hat\varepsilon}\left[c_1+\frac{1}{18c_1}w(1,t)^2\right]w(1,t). \tag{3.32}$$

Consider the Lyapunov function

$$U=\frac{1}{2}\int_0^1 w(x,t)^2\,\mathrm{d}x+\frac{1}{2\gamma}(\varepsilon-\hat\varepsilon)^2$$

with $\gamma>0$. Its derivative along the solutions of the Burgers equation with controls (3.31) and (3.32) is given by

$$\dot U\leqslant-\left[\varepsilon\int_0^1(w_x)^2\,\mathrm{d}x+\frac{c_0}{2}w(0,t)^2+\frac{1}{2}(c_1+W_d)\,w(1,t)^2\right]$$

$$+(\varepsilon-\hat\varepsilon)\left[-w(0,t)w_x(0,t)+w(1,t)w_x(1,t)-\frac{1}{\gamma}\dot{\hat\varepsilon}\right]. \tag{3.33}$$

Denote

$$\begin{aligned}\tau(w,\hat\varepsilon) &= -w(0,t)w_x(0,t)+w(1,t)w_x(1,t)\\ &= -\frac{1}{\hat\varepsilon}\left\{\left[c_0+\frac{W_d}{2}+\frac{1}{18c_0}w(0,t)^2\right]w(0,t)^2+\left[c_1+\frac{1}{18c_1}w(1,t)^2\right]w(1,t)^2\right\}\\ &\leqslant 0 \end{aligned}\tag{3.34}$$

and take the update law with "projection"

$$\dot{\hat\varepsilon}=\gamma\tau(w,\hat\varepsilon)\begin{cases}1, & \hat\varepsilon>\underline{\varepsilon},\\ 0, & \hat\varepsilon\leqslant\underline{\varepsilon},\end{cases}\tag{3.35}$$

where $\hat\varepsilon(0)\geqslant\underline{\varepsilon}$. The discontinuity in the update law (3.35) can be easily removed (see Appendix E in [8]). It is easy to see that the substitution of (3.35) yields

$$\dot U\leqslant-\left[\varepsilon\int_0^1(w_x)^2\,\mathrm{d}x+c_0w(0,t)^2+\left(c_1+\frac{W_d}{2}\right)w(1,t)^2\right].\tag{3.36}$$

By the Invariance Principle (Walker [10], Theorem IV.4.2), the equilibrium solution $w(x,t)\equiv 0,\hat\varepsilon(t)\equiv\varepsilon$ is stable and

$$\lim_{t\to\infty}\int_0^1 w(x,t)^2\,\mathrm{d}x=0.$$

From (3.35) and (3.34) we see that $\hat\varepsilon(t)$ is non-increasing, which means that the controller gain increases. The estimate $\hat\varepsilon(t)$ may or may not reach $\underline{\varepsilon}$ (depending on the behavior of $w(x,t)$), at which point adaptation stops. This means that the adaptive controller may (or may not) be less conservative than the robust (fixed-gain) controller.

The controller (3.31), (3.32) can also be implemented in the Dirichlet form (3.12), (3.13) with ε replaced by $\hat\varepsilon$.

3.5. Control of stochastic viscous Burgers' equation

While the Burgers equation, as a one-dimensional convection-diffusion equation, cannot exhibit "turbulent" behavior (but only "shock" type behavior), a Burgers equation with stochastic forcing is sometimes regarded as a model of turbulance. The stochastic Burgers' equation was studied by Choi et al. [3]. We consider it in the Ito form

$$\mathrm{d}w=(\varepsilon w_{xx}-W_dw_x-ww_x)\,\mathrm{d}t+\mathrm{d}n(x,t),\tag{3.37}$$

where $n(x,t)$ is an independent Wiener process with $E\left\{\int_{x=0}^{x=1}\mathrm{d}n(x,t)^2\right\}=\sigma(t)^2\,\mathrm{d}t$. Taking either the control law (3.3), (3.4) or (3.7), (3.8) and the same Lyapunov function as in (3.1), using the Ito differentiation rule, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}EV\leqslant-E\left[\varepsilon\int_0^1(w_x)^2\,\mathrm{d}x+c_0w(0,t)^2+\left(c_1+\frac{W_d}{2}\right)w(1,t)^2\right]+\frac{1}{2}\sigma(t)^2.\tag{3.38}$$

By Lemma A.1, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}EV\leqslant-cEV+\frac{1}{2}\sigma(t)^2,\tag{3.39}$$

where c is defined in (3.11). From (3.39) it follows that

$$E\left\{\int_0^1 w(x,t)^2\,\mathrm{d}x\right\}\leqslant\mathrm{e}^{-ct}E\left\{\int_0^1 w(x,0)^2\,\mathrm{d}x\right\}+\frac{1}{c}\sup_{0\leqslant\tau\leqslant t}\sigma(\tau)^2.\tag{3.40}$$

4. Inviscid Burgers' equation

4.1. Global stabilization

Consider now the inviscid problem ($\varepsilon = 0$):

$$w_t + W_d w_x + w w_x = 0\,. \tag{4.1}$$

With the Lyapunov function (3.1) we get

$$\dot V = w(0,t)\left[\frac{W_d}{2}w(0,t) + \frac{1}{3}w(0,t)^2\right] - w(1,t)\left[\frac{W_d}{2}w(1,t) + \frac{1}{3}w(1,t)^2\right]. \tag{4.2}$$

Recalling that $W_d > 0$ (no loss of generality), we select the Dirichlet boundary control law as

$$w(0,t) = 0, \tag{4.3}$$

$$w(1,t) = \sqrt[3]{3c\left[\int_0^1 |w_x(x,t)|^3\,\mathrm{d}x + W_d w_x(0,t)^2\right]}, \tag{4.4}$$

where $c > 0$, which gives

$$\dot V = -\frac{W_d}{2}w(1,t)^2 - c\int_0^1 |w_x(x,t)|^3\,\mathrm{d}x - cW_d w_x(0,t)^2\,. \tag{4.5}$$

From the Poincaré inequality [a tighter version than (A.1) which uses Eq. (4.3) and the Cauchy–Schwartz inequality], with the help of Holder's inequality, it follows that

$$\begin{aligned}\int_0^1 w(x)^2\,\mathrm{d}x &\leqslant 2\int_0^1 w_x(x)^2\,\mathrm{d}x\\ &\leqslant 2\left(\int_0^1 |w_x|^3\,\mathrm{d}x\right)^{2/3}\left(\int_0^1 1^3\,\mathrm{d}x\right)^{2/3}\\ &\leqslant 2\left(\int_0^1 |w_x|^3\,\mathrm{d}x\right)^{2/3}.\end{aligned} \tag{4.6}$$

By substituting Eq. (4.6) into Eq. (4.5) we get

$$\dot V \leqslant -cV^{3/2}\,. \tag{4.7}$$

It then follows that

$$\int_0^1 w(x,t)^2\,\mathrm{d}x \leqslant \frac{\int_0^1 w(x,0)^2\,\mathrm{d}x}{\left(1 + t\frac{c}{2^{3/2}}\sqrt{\int_0^1 w(x,0)^2\,\mathrm{d}x}\right)^2}, \tag{4.8}$$

i.e., the equilibrium $w(x,t) \equiv 0$ is globally asymptotically stable in $L_2[0,1]$.

4.2. Boundedness

The stability analysis does not guarantee that $w(x,t)$ is bounded and converges to zero for all $x \in [0,1]$. We now prove that $\int_0^1 w_x(x,t)^2\,\mathrm{d}x$ is bounded and then use Agmon's inequality (Lemma A.2),

$$\max_{x\in[0,1]} |w(x,t)| \leqslant \sqrt{2}\sqrt[4]{\int_0^1 w(x,t)^2\,\mathrm{d}x}\sqrt[4]{\int_0^1 w_x(x,t)^2\,\mathrm{d}x}\,, \tag{4.9}$$

to draw conclusions about boundedness and convergence in $L_\infty[0,1]$. Let us start by taking the L_2-inner product of (4.1) with w_{xx}, which, after some calculations, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 w_x(x,t)^2\,\mathrm{d}x\right) = -(W_d + w(1,t))w_x(1,t)^2 + (W_d + w(0,t))w_x(0,t)^2 - \int_0^1 w_x(x,t)^3\,\mathrm{d}x\,. \tag{4.10}$$

From Eq. (4.4) it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 w_x(x,t)^2\,\mathrm{d}x\right) &= -\left[W_d + \sqrt[3]{3c\left(\int_0^1 |w_x(x,t)|^3\,\mathrm{d}x + W_d w_x(0,t)^2\right)}\right] w_x(1,t)^2 \\ &\quad + W_d w_x(0,t)^2 - \int_0^1 w_x(x,t)^3\,\mathrm{d}x \\ &\leqslant W_d w_x(0,t)^2 + \int_0^1 |w_x(x,t)|^3\,\mathrm{d}x\,. \end{aligned} \tag{4.11}$$

By integrating (4.5) and (4.11) over $[0,t]$ and combining, we get

$$\begin{aligned}\int_0^1 w_x(x,t)^2\,\mathrm{d}x &\leqslant \int_0^1 w_x(x,0)^2\,\mathrm{d}x + \frac{1}{2c}\int_0^1 w(x,0)^2\,\mathrm{d}x \\ &\leqslant \left(1+\frac{2}{c}\right)\int_0^1 w_x(x,0)^2\,\mathrm{d}x\,, \end{aligned} \tag{4.12}$$

where the second inequality follows from the Poincaré inequality [first line of (4.6)]. Inequalities (4.8), (4.9), and (4.12) imply the following global result.

Theorem 4.1. *Consider the system* (4.1), (4.3), (4.4). *If* $\int_0^1 w_x(x,0)^2\,\mathrm{d}x$ *is finite, then*
1. $\sup_{(x,t)\in[0,1]\times[0,\infty)} |w(x,t)| < \infty$,
2. $\lim_{t\to\infty}\max_{x\in[0,1]} |w(x,t)| = 0$.

This stabilization result is robust because it holds for any controller gain $c > 0$. However, the optimality properties of the control law (4.3), (4.4) are not clear.

4.3. Control of stochastic inviscid Burgers' equation

The stochastic inviscid Burgers equation in the Ito form is

$$\mathrm{d}w = (-W_d w_x - w w_x)\,\mathrm{d}t + \mathrm{d}n(x,t)\,, \tag{4.13}$$

where $n(x,t)$ is an independent Wiener process with $E\left\{\int_{x=0}^{x=1}\mathrm{d}n(x,t)^2\right\} = \sigma(t)^2\,\mathrm{d}t$. Taking the control law (4.3), (4.4) and the Lyapunov function (3.1), using the Ito differentiation rule, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E\{V\} \leqslant -cE\{V^{3/2}\} + \frac{1}{2}\sigma(t)^2\,. \tag{4.14}$$

According to the results of [4], this implies that for all $\delta > 0$ (arbitrarily small) there exist functions $\gamma(\cdot)\in\mathscr{K}$ (a function in class $\mathscr{K}$ is continuous, vanishing at zero, and monotonically increasing) and $\beta(\cdot,\cdot)\in\mathscr{KL}$ (a function in class $\mathscr{KL}$ is in class $\mathscr{K}$ in its first argument and is continuous and monotonically decreasing to zero in its second argument) such that

$$P\left\{\int_0^1 w(x,t)^2\,\mathrm{d}x < \beta\left(\int_0^1 w(x,0)^2\,\mathrm{d}x, t\right) + \gamma\left(\sup_{0\le\tau\leqslant t}\sigma(\tau)^2\right)\right\} \geqslant 1-\delta \tag{4.15}$$

for all $t \geqslant 0$. This means, in particular, that, with probability arbitrarily close to one, the solutions will converge to an L_2-ball proportional to the variance of the noise.

5. Dirichlet boundary control for viscous Burgers' equation

5.1. Global stabilization

We start from (3.2)

$$\begin{aligned}\dot{V} = & -\varepsilon \int_0^1 (w_x)^2 \mathrm{d}x \\ & -w(0,t)\left[\varepsilon w_x(0,t) - \frac{W_d}{2} w(0,t) - \frac{1}{3} w(0,t)^2\right] \\ & +w(1,t)\left[\varepsilon w_x(1,t) - \frac{W_d}{2} w(1,t) - \frac{1}{3} w(1,t)^2\right]. \end{aligned} \tag{5.1}$$

On the 0-boundary we select

$$w(0,t) = 0 \tag{5.2}$$

and seek control $w(1,t)$ on the 1-boundary. By applying Young's inequality

$$\varepsilon w(1,t) w_x(1,t) \leqslant \frac{1}{6}|w(1,t)|^3 + \frac{1}{3}|2\varepsilon w_x(1,t)|^{3/2}, \tag{5.3}$$

we get

$$\dot{V} \leqslant -\varepsilon \int_0^1 (w_x)^2\,\mathrm{d}x - \frac{W_d}{2} w(1,t)^2 + \frac{1}{6}|w(1,t)|^3 + \frac{1}{3}|2\varepsilon w_x(1,t)|^{3/2} - \frac{1}{3} w(1,t)^3. \tag{5.4}$$

Picking

$$w(1,t) = 2^{5/6}\sqrt{\varepsilon |w_x(1,t)|}, \tag{5.5}$$

we obtain

$$\dot{V} \leqslant -\varepsilon \int_0^1 (w_x)^2\,\mathrm{d}x - \frac{W_d}{2} w(1,t)^2. \tag{5.6}$$

This is sufficient to conclude global asymptotic stability in L_2 but not to conclude boundedness of $w(x,t)$ for all $(x,t) \in [0,1] \times [0,\infty)$. For this reason, instead of the control law (5.5), (5.2), we employ

$$w(0,t) = 0, \tag{5.7}$$

$$w(1,t) = \sqrt[3]{3|2\varepsilon w_x(1,t)|^{3/2} + \mu |w_{xx}(1,t)|^3}, \tag{5.8}$$

where $\mu > 0$. This results in

$$\dot{V} \leqslant -\varepsilon \int_0^1 (w_x)^2\,\mathrm{d}x - \frac{W_d}{2} w(1,t)^2 - \frac{1}{6}|2\varepsilon w_x(1,t)|^{3/2} - \frac{\mu}{6}|w_{xx}(1,t)|^3. \tag{5.9}$$

By Lemma A.1, we get $\dot{V} \leqslant -cV$, which guarantees that the equilibrium $w(x,t) \equiv 0$ is globally exponentially stable in $L_2[0,1]$.

The control law (5.7), (5.8) requires only boundary measurements. If, like in Section 4, we assume that measurements in the interior of the interval $[0,1]$ are available, then we could add, for example, a term of the form $\int_0^1 (w_x)^2\,\mathrm{d}x$ under the root in Eq. (5.8) to enhance stability. We postpone this for Section 6 where we employ an observer rather than require interior measurements.

5.2. Boundedness

The stability analysis does not guarantee that $w(x,t)$ is bounded and converges to zero for all $x \in [0,1]$. We now prove that $\int_0^1 w_x(x,t)^2\,\mathrm{d}x$ is bounded and then use Agmon's inequality (Lemma A.2),

$$\max_{x\in[0,1]} |w(x,t)| \leqslant \sqrt{2}\sqrt[4]{\int_0^1 w(x,t)^2\,\mathrm{d}x}\sqrt[4]{\int_0^1 w_x(x,t)^2\,\mathrm{d}x}\,, \tag{5.10}$$

to draw conclusions about boundedness and convergence in $L_\infty[0,1]$. By taking the L_2-inner product of Eq. (2.3) with w_{xx}, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 w_x(x,t)^2\,\mathrm{d}x\right) &= 2w_x(1,t)w_t(1,t) - 2w_x(0,t)w_t(0,t)\\ &\quad -2\varepsilon\int_0^1 w_{xx}^2\,\mathrm{d}x + \int_0^1 2(W_d+w)w_x w_{xx}\,\mathrm{d}x\,.\end{aligned} \tag{5.11}$$

By substituting Eqs. (5.7) and (5.8), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 w_x(x,t)^2\,\mathrm{d}x\right) &= 2\varepsilon w_x(1,t)w_{xx}(1,t) - \left[W_d + \sqrt[3]{3|2\varepsilon w_x(1,t)|^{3/2} + \mu|w_{xx}(1)|^3}\right] w_x(1,t)^2\\ &\quad -2\varepsilon\int_0^1 w_{xx}^2\,\mathrm{d}x + \int_0^1 2(W_d+w)w_x w_{xx}\,\mathrm{d}x\\ &\leqslant 2\varepsilon w_x(1,t)w_{xx}(1,t)\\ &\quad -2\varepsilon\int_0^1 w_{xx}^2\,\mathrm{d}x + \int_0^1\left[2\varepsilon w_{xx}^2 + \frac{1}{2\varepsilon}(W_d+w)^2 w_x^2\right]\mathrm{d}x\,.\end{aligned} \tag{5.12}$$

From Young's inequality it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 w_x(x,t)^2\,\mathrm{d}x\right) &\leqslant \frac{1}{3}|w_{xx}(1,t)|^3 + \frac{2}{3}|2\varepsilon w_x(1,t)|^{3/2}\\ &\quad +\frac{1}{2\varepsilon}\int_0^1 (W_d+w)^2 w_x^2\,\mathrm{d}x\\ &\leqslant \frac{1}{3}|w_{xx}(1,t)|^3 + \frac{2}{3}|2\varepsilon w_x(1,t)|^{3/2}\\ &\quad +\frac{1}{\varepsilon}\left(W_d^2 + \max_{x\in[0,1]} w(x,t)^2\right)\int_0^1 w_x^2\,\mathrm{d}x\,.\end{aligned} \tag{5.13}$$

By combining Lemmas A.2 and A.1, it is easy to see that there exists a positive constant q such that

$$\max_{x\in[0,1]} w(x,t)^2 \leqslant q\int_0^1 w_x(x,t)^2\,\mathrm{d}x\,. \tag{5.14}$$

Thus we can write (5.13) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 w_x(x,t)^2\,\mathrm{d}x\right) \leqslant q\left[\int_0^1 w_x(x,t)^2\,\mathrm{d}x\right]^2 + \frac{1}{3}|w_{xx}(1,t)|^3 + \frac{2}{3}|2\varepsilon w_x(1,t)|^{3/2} + \frac{W_d^2}{\varepsilon}\int_0^1 w_x(x,t)^2\,\mathrm{d}x. \tag{5.15}$$

By integrating Eq. (5.9) over $[0,t]$ and employing Lemma B.6 in [8], it readily follows that

$$\int_0^1 w_x(x,t)^2\,\mathrm{d}x < \infty, \qquad \forall t \geqslant 0 \tag{5.16}$$

whenever $\int_0^1 w_x(x,0)^2\,\mathrm{d}x$ is finite. From inequalities (5.10) and (5.16), combined with exponential stability in $L_2[0,1]$, it follows that we have proved the following global result.

Theorem 5.1. *Consider the system* (2.3), (5.7), (5.8). *If* $\int_0^1 w_x(x,0)^2\,\mathrm{d}x$ *is finite, then*
1. $\sup_{(x,t)\in[0,1]\times[0,\infty)} |w(x,t)| < \infty$,
2. $\lim_{t\to\infty}\max_{x\in[0,1]} |w(x,t)| = 0$,
3. $\int_0^1 w(x,t)^2\,\mathrm{d}x$ *decays to zero exponentially.*

5.3. Robust, adaptive, and stochastic control

Suppose now that the viscosity ε is not known but an upper bound $\bar{\varepsilon}$ is known. Choosing the control as

$$w(0,t) = 0, \tag{5.17}$$

$$w(1,t) = \sqrt[3]{3|2\bar{\varepsilon} w_x(1,t)|^{3/2} + \mu|w_{xx}(1,t)|^3}, \tag{5.18}$$

we get

$$\dot{V} \leqslant -\varepsilon\int_0^1 (w_x)^2\,\mathrm{d}x - \frac{W_d}{2}w(1,t)^2 - \frac{1}{6}|2\varepsilon w_x(1,t)|^{3/2} - \frac{\mu}{6}|w_{xx}(1,t)|^3. \tag{5.19}$$

Since a control using $\bar{\varepsilon}$ involves high gain, we are motivated to look for an adaptive controller which employs an on-line estimate $\widehat{\varepsilon^{3/2}}$ of $\varepsilon^{3/2}$:

$$w(0,t) = 0, \tag{5.20}$$

$$w(1,t) = \sqrt[3]{3\widehat{\varepsilon^{3/2}}|2w_x(1,t)|^{3/2} + \mu|w_{xx}(1,t)|^3}. \tag{5.21}$$

Consider the Lyapunov function

$$U = \frac{1}{2}\int_0^1 w(x,t)^2\,\mathrm{d}x + \frac{\sqrt{2}}{3\gamma}\left(\varepsilon^{3/2} - \widehat{\varepsilon^{3/2}}\right)^2$$

with $\gamma > 0$. Its derivative along the solutions of the Burgers' equation with controls (5.20) and (5.21) is given by

$$\dot{U} \leqslant -\varepsilon\int_0^1 (w_x)^2\,\mathrm{d}x - \frac{W_d}{2}w(1,t)^2 - \frac{1}{6}|2\varepsilon w_x(1,t)|^{3/2} - \frac{\mu}{6}|w_{xx}(1,t)|^3 + \left(\varepsilon^{3/2} - \widehat{\varepsilon^{3/2}}\right)\frac{2^{3/2}}{3}\left[|w_x(1,t)|^{3/2} - \frac{1}{\gamma}\dot{\widehat{\varepsilon^{3/2}}}\right]. \tag{5.22}$$

Taking the update law

$$\dot{\widehat{\varepsilon^{3/2}}} = \gamma|w_x(1,t)|^{3/2} \tag{5.23}$$

yields

$$\dot{U} \leqslant -\varepsilon\int_0^1 (w_x)^2\,\mathrm{d}x - \frac{W_d}{2}w(1,t)^2 - \frac{1}{6}|2\varepsilon w_x(1,t)|^{3/2} - \frac{\mu}{6}|w_{xx}(1,t)|^3\,. \tag{5.24}$$

By the Invariance Principle [10, Theorem IV.4.2], the equilibrium solution $w(x,t) \equiv 0, \widehat{\varepsilon^{3/2}}(t) \equiv \varepsilon^{3/2}$ is stable and

$$\lim_{t\to\infty}\int_0^1 w(x,t)^2\,\mathrm{d}x = 0\,.$$

The controller (5.7), (5.8) applied to the stochastic viscous Burgers equation achieves the same result as in Section 3.5.

6. Observer-based Dirichlet boundary control for viscous Burgers' equation

6.1. Global stabilization

In Section 4 we designed a feedback controller for the inviscid Burgers equation which effectively adds viscosity and strengthens the stability of the system. Similar feedback which strengthens stability would be of interest for the viscous Burgers equation. For example, one would replace (5.7), (5.8) by

$$w(0,t) = 0, \tag{6.1}$$

$$w(1,t) = \sqrt[3]{3|2\varepsilon w_x(1,t)|^{3/2} + \mu|w_{xx}(1,t)|^3 + \nu\int_0^1 w_x(x,t)^2\,\mathrm{d}x}\,, \tag{6.2}$$

to get

$$\dot{V} \leqslant -\left(\varepsilon + \frac{\nu}{6}\right)\int_0^1 (w_x)^2\,\mathrm{d}x - \frac{W_d}{2}w(1,t)^2 - \frac{1}{6}|2\varepsilon w_x(1,t)|^{3/2} - \frac{\mu}{6}|w_{xx}(1,t)|^3\,. \tag{6.3}$$

Unfortunately, Eq. (6.2) requires measurements from the interior of the domain. Since these measurements may be impossible to obtain, we are motivated to look for an observer which estimates $w(x,t)$ or $w_x(x,t)$ inside the domain.

For example, consider the following nonlinear observer

$$\hat{w}_t - \varepsilon\hat{w}_{xx} + W_d\hat{w}_x + \hat{w}\hat{w}_x = 0 \tag{6.4}$$

with boundary conditions

$$\hat{w}(0,t) = w(0,t), \tag{6.5}$$

$$\hat{w}(1,t) = w(1,t)\,. \tag{6.6}$$

We select the control law as

$$w(0,t) = 0, \tag{6.7}$$

$$w(1,t) = \sqrt[3]{3|2\varepsilon\hat{w}_x(1,t)|^{3/2} + \mu|\hat{w}_{xx}(1,t)|^3 + \nu\int_0^1 \hat{w}_x(x,t)^2\,\mathrm{d}x}\,. \tag{6.8}$$

This controller guarantees

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \hat{w}(x,t)^2\,\mathrm{d}x\right) \leqslant -\left(\varepsilon+\frac{\nu}{6}\right)\int_0^1 (\hat{w}_x)^2\,\mathrm{d}x - \frac{W_d}{2}\hat{w}(1,t)^2$$

$$-\frac{1}{6}|2\varepsilon\hat{w}_x(1,t)|^{3/2} - \frac{\mu}{6}|\hat{w}_{xx}(1,t)|^3\,. \tag{6.9}$$

We will return to this expression later. Thus, employing Poincaré's inequality, we conclude that the observer has a globally $L_2[0,1]$-exponentially stable equilibrium $\hat{w}(x,t)\equiv 0$. The proof of the boundedness of $\hat{w}(x,t)$ follows the same lines as the proof of the boundedness of $w(x,t)$ in Section 5.2. In addition, we have that

$$\sup_{t\in[0,\infty)}\int_0^1 \hat{w}_x(x,t)^2\,\mathrm{d}x<\infty, \tag{6.10}$$

$$\int_0^\infty\int_0^1 \hat{w}_x(x,t)^2\,\mathrm{d}x\,\mathrm{d}t<\infty\,. \tag{6.11}$$

The observer error $\tilde{w}=w-\hat{w}$ is governed by the equation

$$\tilde{w}_t-\varepsilon\tilde{w}_{xx}+W_d\tilde{w}_x+\tilde{w}\tilde{w}_x+\hat{w}\tilde{w}_x+\hat{w}_x\tilde{w}=0 \tag{6.12}$$

with boundary conditions

$$\tilde{w}(0,t)=\tilde{w}(1,t)\equiv 0\,. \tag{6.13}$$

Along the solutions of Eqs. (6.12) and (6.13) we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x\right) &= -\varepsilon\int_0^1 \tilde{w}_x(x,t)^2\,\mathrm{d}x\\ &\quad+\int_0^1 \hat{w}(x,t)\tilde{w}_x(x,t)\tilde{w}(x,t)\,\mathrm{d}x+\int_0^1 \hat{w}_x(x,t)\tilde{w}(x,t)^2\,\mathrm{d}x\\ &= -\varepsilon\int_0^1 \tilde{w}_x(x,t)^2\,\mathrm{d}x\\ &\quad+\int_0^1 \hat{w}(x,t)\tilde{w}_x(x,t)\tilde{w}(x,t)\,\mathrm{d}x+\hat{w}\;\tilde{w}^2\big|_0^1-2\int_0^1 \hat{w}(x,t)\tilde{w}_x(x,t)\tilde{w}(x,t)\,\mathrm{d}x\\ &= -\varepsilon\int_0^1 \tilde{w}_x(x,t)^2\,\mathrm{d}x-\int_0^1 \hat{w}(x,t)\tilde{w}_x(x,t)\tilde{w}(x,t)\,\mathrm{d}x\,,\end{aligned} \tag{6.14}$$

where we have used integration by parts and the boundary conditions (6.13). Then with Young's inequality we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x\right) &\leqslant -\varepsilon\int_0^1 \tilde{w}_x(x,t)^2\,\mathrm{d}x\\ &\quad+\frac{\varepsilon}{2}\int_0^1 \tilde{w}_x(x,t)^2\,\mathrm{d}x+\frac{1}{2\varepsilon}\int_0^1 \hat{w}(x,t)^2\tilde{w}(x,t)^2\,\mathrm{d}x\\ &\leqslant -\frac{\varepsilon}{2}\int_0^1 \tilde{w}_x(x,t)^2\,\mathrm{d}x+\frac{1}{2\varepsilon}\max_{x\in[0,1]}\hat{w}(x,t)^2\int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x\,.\end{aligned} \tag{6.15}$$

From Agmon's and Poincaré's inequalities we have

$$\max_{x\in[0,1]} \hat{w}(x,t)^2 \leqslant q \int_0^1 \hat{w}_x(x,t)^2\,\mathrm{d}x\,, \tag{6.16}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x\right) \leqslant -\varepsilon \int_0^1 \tilde{w}_x(x,t)^2\,\mathrm{d}x + \frac{q}{2\varepsilon}\int_0^1 \hat{w}_x(x,t)^2\,\mathrm{d}x \int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x\,. \tag{6.17}$$

By applying Poincaré's inequality to the first term on the right hand side, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x\right) \leqslant -\left(\frac{\varepsilon}{2} - \frac{q}{\varepsilon}\int_0^1 \hat{w}_x(x,t)^2\,\mathrm{d}x\right)\int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x\,. \tag{6.18}$$

Inequality (6.9) establishes that $\int_0^1 \hat{w}_x(x,t)^2\,\mathrm{d}x$ is integrable in t over $[0,\infty)$. Then by Lemma B.6 in [8] it follows that

$$\sup_{t\in[0,\infty)} \int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x < \infty \tag{6.19}$$

$$\int_0^\infty \int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x\mathrm{d}t < \infty\,. \tag{6.20}$$

In addition, from (6.18), (6.10), and (6.19), it follows that $\int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x$ is uniformly continuous in t. Thus, by Barbalat's lemma [7],

$$\lim_{t\to\infty} \int_0^1 \tilde{w}(x,t)^2\,\mathrm{d}x = 0\,. \tag{6.21}$$

By integrating (6.17) it also follows that

$$\int_0^\infty \int_0^1 \tilde{w}_x(x,t)^2\,\mathrm{d}x\,\mathrm{d}t < \infty\,. \tag{6.22}$$

Since $w = \tilde{w} + \hat{w}$, the above properties guarantee that $w(x,t)$ is bounded and

$$\sup_{t\in[0,\infty)} \int_0^1 w(x,t)^2\,\mathrm{d}x < \infty \tag{6.23}$$

$$\int_0^\infty \int_0^1 w(x,t)^2\,\mathrm{d}x\,\mathrm{d}t < \infty \tag{6.24}$$

$$\lim_{t\to\infty} \int_0^1 w(x,t)^2\,\mathrm{d}x = 0 \tag{6.25}$$

$$\int_0^\infty \int_0^1 w_x(x,t)^2\,\mathrm{d}x\,\mathrm{d}t < \infty\,. \tag{6.26}$$

6.2. Boundedness

To conclude the boundedness of $w(x,t)$, it remains to prove that $\tilde{w}(x,t)$ is bounded.

By taking the L_2-inner product of Eq. (6.4) with w_{xx}, using Eq. (6.13), we get

$$\begin{aligned}\frac{d}{dt}\left(\int_0^1 \tilde{w}_x(x,t)^2\,dx\right) &= -2\varepsilon\int_0^1 \tilde{w}_{xx}^2\,dx + \int_0^1 2(W_d+w)\tilde{w}_x\tilde{w}_{xx}\,dx + \int_0^1 2\hat{w}_x\tilde{w}\tilde{w}_{xx}\,dx\\
&\leqslant -2\varepsilon\int_0^1 \tilde{w}_{xx}^2\,dx + \int_0^1\left[\varepsilon\tilde{w}_{xx}^2 + \frac{1}{\varepsilon}(W_d+w)^2\tilde{w}_x^2\right]dx\\
&\quad + \int_0^1\left[\varepsilon\tilde{w}_{xx}^2 + \frac{1}{\varepsilon}\hat{w}_x^2\tilde{w}^2\right]dx\\
&\leqslant \frac{1}{\varepsilon}\int_0^1 (W_d+w)^2\tilde{w}_x^2\,dx + \frac{1}{\varepsilon}\int_0^1 \hat{w}_x^2\,\tilde{w}^2\,dx\\
&\leqslant \frac{2}{\varepsilon}\left(W_d^2 + \max_{x\in[0,1]} w(x,t)^2\right)\int_0^1 \tilde{w}_x^2\,dx\\
&\quad + \frac{1}{\varepsilon}\max_{x\in[0,1]}\tilde{w}(x,t)^2\int_0^1 \hat{w}_x^2\,dx\,.\end{aligned}\tag{6.27}$$

Using the inequality (6.16) applied to w and $\tilde{w}$, we get

$$\begin{aligned}\frac{d}{dt}\left(\int_0^1 \tilde{w}_x(x,t)^2\,dx\right) &\leqslant \frac{2}{\varepsilon}\left(W_d^2 + q\int_0^1 w_x(x,t)^2\right)\int_0^1 \tilde{w}_x^2\,dx\\
&\quad + \frac{q}{\varepsilon}\int_0^1 \tilde{w}_x(x,t)^2\int_0^1 \hat{w}_x^2\,dx\,.\end{aligned}\tag{6.28}$$

Rearranging the terms, we arrive at

$$\begin{aligned}\frac{d}{dt}\left(\int_0^1 \tilde{w}_x(x,t)^2\,dx\right) &\leqslant \frac{q}{\varepsilon}\left(\int_0^1 \hat{w}_x(x,t)^2\,dx + 2\int_0^1 w_x(x,t)^2\right)\int_0^1 \tilde{w}_x(x,t)^2\,dx\\
&\quad + \frac{2}{\varepsilon}W_d^2\int_0^1 \tilde{w}_x(x,t)^2\,.\end{aligned}\tag{6.29}$$

By (6.11), (6.24), and (6.26), employing Lemma B.6 in [8], it readily follows that

$$\sup_{t\in[0,\infty)}\int_0^1 \tilde{w}_x(x,t)^2\,dx < \infty\,,\tag{6.30}$$

and hence, that

$$\sup_{t\in[0,\infty)}\int_0^1 w_x(x,t)^2 dx < \infty\,,\tag{6.31}$$

With Agmon's inequality, we establish the following global result.

Theorem 6.1. *Consider the system* (2.3), (6.4), (6.7), (6.8). *If* $\int_0^1 w_x(x,0)^2\,dx$ *and* $\int_0^1 \hat{w}_x(x,0)^2\,dx$ *are finite, then*

1. $\sup_{(x,t)\in[0,1]\times[0,\infty)} (|w(x,t)| + |\hat{w}(x,t)|) < \infty,$
2. $\lim_{t\to\infty}\max_{x\in[0,1]} (|w(x,t)| + |\hat{w}(x,t)|) = 0.$

Appendix A

Lemma A.1 (Poincaré's inequality)**.** *For any* $w \in C^1[0,1]$*, the following inequalities hold*:

$$\int_0^1 w(x)^2\,\mathrm{d}x \leqslant 2w(0)^2 + 4\int_0^1 w_x(x)^2\,\mathrm{d}x \tag{A.1}$$

$$\int_0^1 w(x)^2\,\mathrm{d}x \leqslant 2w(1)^2 + 4\int_0^1 w_x(x)^2\,\mathrm{d}x\,. \tag{A.2}$$

Proof. We prove (A.2). The proof of (A.1) is identical by introducing a change of coordinate $\xi = 1 - x$. By integration by parts we have

$$\begin{aligned}\int_0^1 w(x)^2\,\mathrm{d}x &= w(x)^2 x\Big|_0^1 - 2\int_0^1 x w(x) w_x(x)\,\mathrm{d}x \\ &\leqslant w(1)^2 + \frac{1}{2}\int_0^1 w(x)^2\,\mathrm{d}x + 2\int_0^1 x^2 w_x(x)^2\,\mathrm{d}x \\ &\leqslant w(1)^2 + \frac{1}{2}\int_0^1 w(x)^2\,\mathrm{d}x + 2\int_0^1 w_x(x)^2\,\mathrm{d}x\,,\end{aligned} \tag{A.3}$$

which yields the result. □

Lemma A.2 (Agmon's inequality)**.** *For any* $w \in C^1[0,\ 1]$, *the following inequalities hold*:

$$\max_{x\in[0,1]} w(x)^2 \leqslant w(0)^2 + 2\sqrt{\int_0^1 w(x)^2\,\mathrm{d}x}\sqrt{\int_0^1 w_x(x)^2\,\mathrm{d}x}, \tag{A.4}$$

$$\max_{x\in[0,1]} w(x)^2 \leqslant w(1)^2 + 2\sqrt{\int_0^1 w(x)^2\,\mathrm{d}x}\sqrt{\int_0^1 w_x(x)^2\,\mathrm{d}x}\,. \tag{A.5}$$

Proof. We prove (A.4). The proof of (A.5) is identical by introducing a change of coordinate $\xi = 1 - x$. From the fundamental theorem of calculus, we get

$$\begin{aligned}w(x)^2 &= w(0)^2 + 2\int_0^x w(\zeta) w_\zeta(\zeta)\,\mathrm{d}\zeta \\ &\leqslant w(0)^2 + 2\sqrt{\int_0^x w(\zeta)^2\,\mathrm{d}\zeta}\sqrt{\int_0^x w_\zeta(\zeta)^2\,\mathrm{d}\zeta}\,,\end{aligned} \tag{A.6}$$

where the inequality follows from the Cauchy–Schwartz inequality. □

Lemma A.3 (Young's inequality)**.** *For* $a,b \geqslant 0$, $\lambda > 0$, *and* $(1/p) + (1/q) = 1$ *the following holds*

$$ab \leqslant \frac{\lambda^p}{p} a^p + \frac{1}{q\lambda^q} b^q\,. \tag{A.7}$$

References

[1] J.A. Burns, S. Kang, A control problem for Burgers' equation with bounded input/output, Nonlinear Dynamics 2 (1991) 235–262.
[2] C.I. Byrnes, D.S. Gilliam, V.I. Shubov, On the global dynamics of a controlled viscous Burgers' equation, J. Dynam. Control Systems 4 (4) (1998) 457–519.

[3] H. Choi, R. Temam, P. Moin, J. Kim, Feedback control for unsteady flow and its application to the stochastic Burgers' equation, J. Fluid Mech. 253 (1993) 509–543.
[4] H. Deng, M. Krstić, Stabilization of stochastic nonlinear systems driven by noise of unknown covariance, IEEE Trans. Automat. Control, submitted for publication.
[5] K. Ito, S. Kang, A dissipative feedback control for systems arising in fluid dynamics, SIAM J. Control Optim. 32 (1994) 831–854.
[6] K. Ito, Y. Yan, Viscous scalar conservation laws with nonlinear flux feedback and global attractors, J. Math. Anal. Appl. 227 (1998) 271–299.
[7] H. Khalil, Nonlinear Systems, Prentice-Hall, Englewood Cliffs, NJ, 1995.
[8] M. Krstić, I. Kanellakopoulos, P.V. Kokotović, Nonlinear and Adaptive Control Design, Wiley, New York, 1995.
[9] H. Van Ly, K.D. Mease, E.S. Titi, Distributed and boundary control of the viscous Burgers' equation, Numer. Funct. Anal. Optim. 18 (1997) 143–188.
[10] J.A. Walker, Dynamical Systems and Evolution Equations, Plenum Press, New York, 1980.