# Inverse Optimal Stabilization of a Rigid Spacecraft

Miroslav Krstić and Panagiotis Tsiotras

***Abstract*—The authors present an approach for constructing optimal feedback control laws for regulation of a rotating rigid spacecraft. They employ the inverse optimal control approach which circumvents the task of solving a Hamilton–Jacobi equation and results in a controller optimal with respect to a meaningful cost functional. The inverse optimality approach requires the knowledge of a control Lyapunov function and a stabilizing control law of a particular form. For the spacecraft problem, they are both constructed using the method of integrator backstepping. The authors give a characterization of (nonlinear) stability margins achieved with the inverse optimal control law.**

***Index Terms*—Attitude control, backstepping, inverse optimality, stability margins, stabilization.**

## I. INTRODUCTION

Optimal control of rigid bodies has a long history stemming from interest in the control of rigid spacecraft and aircraft [1]–[5]. The main thrust of this research has been directed, however, toward the time-optimal and fuel-optimal control problems [6]–[11]. The optimal regulation problem over a finite or infinite horizon has been treated in the past mainly for the angular velocity subsystem and for special quadratic costs [10], [12]–[16]. The case of general quadratic costs has also been addressed in [17]. Optimal control for the complete attitude problem, i.e., including the orientation equations, is more difficult and has been addressed in terms of trajectory planning [18], [19] or in semifeedback form [20]. The main obstruction in constructing feedback control laws in this case stems from the difficulty in solving the Hamilton–Jacobi equation, especially when the cost includes a penalty term on the control effort. In [21] the authors obtain closed-form optimal solutions for special cases of quadratic costs without penalty on the control effort. These control laws asymptotically recover the optimal cost for the kinematics but may lead to high-gain controllers. When a control penalty is included in the performance index, linear control laws have been constructed which provide an upper bound for a quadratic cost in some specified compact set of initial conditions. Suboptimal results can be obtained by minimizing this upper bound [21]. Alternatively, one can penalize only the high-gain portion of the control input. This approach is based on the optimality results of [22] and it has been used both for axisymmetric [23] and nonsymmetric bodies [24]. The most advanced efforts toward designing optimal feedback controllers have been made in [26] and [27] in the framework on nonlinear $\mathcal{H}_\infty$ design. However, the authors in [27] solve the Hamilton–Jacobi–Isaacs *inequality* which, in general, only guarantees an upper bound of the cost for the zero-disturbance case.

In this paper we follow an alternative approach in order to derive optimal feedback control laws for the complete rigid body system. We employ the *inverse optimal* control approach which circumvents the task of solving a Hamilton–Jacobi equation and results in a controller optimal with respect to a meaningful cost functional. This approach, originated by Kalman to establish certain gain and phase margins of linear quadratic regulators [28], was introduced into nonlinear control in [29], and has been long dormant until it was recently revived in [30] to develop a methodology for design of *robust* nonlinear controllers. While [29] establishes a certain nonlinear "return difference" inequality which implies robustness to some input nonlinearities, the full analogy with the linear stability margins was only recently established in [31].

The inverse optimality approach used in this paper requires the knowledge of a control Lyapunov function and a stabilizing control law of a particular form. For the spacecraft problem, we construct them both using the method of integrator backstepping [32]. The resulting design includes a penalty on the angular velocity, orientation, *and* the control torque. The weight in the penalty on the control depends on the current state and decreases for states away from the origin. We also present a result which puts a constant (identity) weight on control and possesses stability margins analogous to the infinite gain margin and the 60° phase margins for the linear quadratic regulators. It should be pointed out that global stabilizing controllers using the inverse optimality approach of [30] have also been presented in [33].

The paper is organized as follows. Section II reviews the basics of the inverse optimality approach and presents it in a format convenient for *design* of controllers. Section IV contains the main result—the construction of the inverse optimal feedback law for a rigid spacecraft, which is specialized in Section IV-B to the case of a symmetric spacecraft. A numerical example in Section V illustrates the theoretical result of the paper.

## II. INVERSE OPTIMAL CONTROL APPROACH

We consider nonlinear systems affine in the control variable

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $f$: $\mathbb{R}^n \to \mathbb{R}^n$ and $g$: $\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth, vector- and matrix-valued functions, respectively, with $f(0) = 0$. Moreover, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote the state and control vectors, respectively.

*Proposition 1 [29], [31]:* Assume that the static state feedback control law

$$u = \kappa(x) := -R^{-1}(x)\left(\frac{\partial V}{\partial x}\, g(x)\right)^T \tag{2}$$

where $R$: $\mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a positive definite matrix-valued function (i.e., $R(x) = R^T(x) > 0$ for all $x \in \mathbb{R}^n$), stabilizes the system in (1) with respect to a positive definite radially unbounded Lyapunov function $V(x)$. Then the control law

$$u = \kappa^*(x) := \beta\,\kappa(x), \qquad \beta \geq 2 \tag{3}$$

is optimal with respect to the cost

$$\mathcal{J} = \int_0^\infty \{l(x) + u^T R(x) u\}\, dt \tag{4}$$

where

$$\begin{aligned} l(x) = &-2\beta\,\frac{\partial V}{\partial x}\,(f(x) + g(x)\kappa(x)) \\ &+ \beta(\beta-2)\,\frac{\partial V}{\partial x}\, g(x) R^{-1}(x)\left(\frac{\partial V}{\partial x}\, g(x)\right)^T. \end{aligned} \tag{5}$$

Manuscript received October 2, 1996. Recommended by Associate Editor, C. Canudas de Wit. The work of M. Krstić was supported in part by the National Science Foundation under Grant ECS-9624386 and in part by the Air Force Office of Scientific Research under Grant F496209610223. The work of P. Tsiotras was supported by the National Science Foundation under Grant CMS-9624188.

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Publisher Item Identifier S 0018-9286(99)02117-0.

Because $\partial V/\partial x(f(x) + g(x)\kappa(x)) < 0, \forall x \neq 0$, we have $l(x) > 0$ for all $x \neq 0$ and the performance index in (5) represents a meaningful cost, in the sense that it includes a positive penalty on the state and a positive penalty on the control for each $x$.

The cost (5) depends on the particular system dynamics. This is understandable, since by requiring *closed-form solutions* to a nonlinear optimal feedback problem it is sensible to choose costs which are compliant with the system dynamics. In other words, the cost should reflect somehow, and take into account, the form of the nonlinearity of the system. This restricts of course the choice of performance indexes. On the other hand, one avoids solving the often formidable Hamilton–Jacobi equation.

The result of Proposition 1 was given in [31] for $\beta = 2$. The extension that we give here for $\beta \geq 2$ is straightforward and given without proof. However, this extension already establishes an infinite gain margin of the inverse optimal controller, a well-known property of linear quadratic regulators [28]. An equivalent of the phase margin was also given in [31] and it requires that the function $R^{-1}(x)$ be locally bounded. Under this condition, there exists a continuous positive function $\eta(\cdot)$ such that

$$R^{-1}(x) \leq \eta(V(x))I, \qquad \forall x \in \mathbb{R}^n \tag{6}$$

which follows from the radial unboundedness of $V(x)$. With this definition, we state the main result on robustness margins achievable using the inverse optimality approach. In the linear case, this result gives precisely the infinite gain margin[1] and the 60° phase margin.

*Proposition 2 [31]:* Under the conditions of Proposition 1 and assuming that $R^{-1}(x)$ is locally bounded, the control law

$$v = \kappa^*(x) := -\beta\,\eta(V(x))\left(\frac{\partial V}{\partial x}\,g(x)\right)^T, \qquad \beta \geq 2 \tag{7}$$

is globally asymptotically stabilizing for (1) with the input dynamics $u = a(I + \mathcal{P})v$, where $a \geq 1/\beta$ is a constant and $\mathcal{P}$ is a strictly passive[2] (possibly nonlinear) system.

Note that the form of the control law (7) is

$$\kappa^*(x) := -\beta\left(\frac{\partial \hat{V}}{\partial x}\,g(x)\right)^T \tag{8}$$

where

$$\hat{V}(x) = \int_0^{V(x)} \eta(r)\,dr \tag{9}$$

is a positive definite and radially unbounded Lyapunov function. The control law (7) minimizes the cost functional

$$\mathcal{J} = \int_0^{\infty} \{\hat{l}(x) + u^T u\}\,dt \tag{10}$$

where $\hat{l}(x) \geq \eta(V)l(x)$ is positive definite.

## III. The Rigid Body Model

In this section we use the inverse optimal results of Proposition 1 in order to derive control laws which are optimal with respect to a cost which includes a penalty on the control input as well as the angular position and velocity of a rigid spinning spacecraft. The complete attitude motion of a rigid spacecraft can be described by the state equations [24], [25]

$$\dot{\omega} = J^{-1}S(\omega)J\omega + J^{-1}u \tag{11a}$$

$$\dot{\rho} = H(\rho)\omega \tag{11b}$$

[1] See also [35] for a discussion on gain margins for nonlinear optimal regulators.

[2] In the sense of the definition in [34].

where $\omega \in \mathbb{R}^3$ is the angular velocity vector in a body-fixed frame, $\rho \in \mathbb{R}^3$ is the Cayley–Rodrigues parameters vector [25] describing the body orientation, $u \in \mathbb{R}^3$ is the acting control torque, and $J$ is the (positive definite) inertia matrix. The symbol $S(\cdot)$ denotes a $3 \times 3$ skew-symmetric matrix, that is

$$S(\omega) := \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{12}$$

and the matrix-valued function $H$: $\mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ denotes the kinematics Jacobian matrix for the Cayley–Rodrigues parameters, given by

$$H(\rho) := \tfrac{1}{2}\left(I - S(\rho) + \rho\rho^T\right) \tag{13}$$

where $I$ denotes the $3 \times 3$ identity matrix. The matrix $H(\rho)$ satisfies the following identity [24]:

$$\rho^T H(\rho)\omega = \left(\frac{1 + \|\rho\|^2}{2}\right)\rho^T\omega \tag{14}$$

for all $\omega, \rho \in \mathbb{R}^3$, where $\|\cdot\|$ denotes the Euclidean norm, i.e., $\|x\|^2 = x^T x$, for $x \in \mathbb{R}^n$.

Observe that the system in (11) is in cascade interconnection, that is, the kinematics subsystem (11b) is controlled only indirectly through the angular velocity vector $\omega$. Stabilizing control laws for systems in this hierarchical form can be efficiently designed using the method of *backstepping* [32]. According to this approach, one thinks of $\omega$ as the *virtual control* in (11b) and designs a control law, say $\omega_d(\rho)$, which stabilizes this system. Subsequently, one designs the actual control input $u$ so as to stabilize the system in (11a) without destabilizing the system in (11b) by forcing, for example, $\omega \to \omega_d$. The main benefits of this methodology is that it is flexible and lends itself to a systematic construction of stabilizing control laws along with the corresponding Lyapunov functions.

## IV. Control Design

### A. Backstepping

The first step for applying the results of Proposition 1 is to construct a control-Lyapunov function for the system in (11). For systems with cascade interconnection structure, such as the rigid body equations, one can use the method of integrator backstepping to achieve this objective. Sontag and Sussmann were the first to notice this property for the rigid body in [36], where they used backstepping to design smooth feedback control laws for an underactuated rigid body. The same technique was also used in [37] for stabilization of an axisymmetric spacecraft using two control torques. Here we use backstepping in order to derive a control-Lyapunov function, along with a stabilizing controller of a particular form for the system in (11).

*Control of the Kinematic Subsystem:* Consider the kinematics subsystem in (11b) with $\omega$ promoted to a control input and let the control law

$$\omega_d = -k_1\rho, \qquad k_1 > 0. \tag{15}$$

With this control law the closed-loop system becomes

$$\dot{\rho} = -k_1 H(\rho)\rho. \tag{16}$$

The system in (16) is globally exponentially stable. To see this, consider the following Lyapunov function:

$$V_1(\rho) = \tfrac{1}{2}\|\rho\|^2. \tag{17}$$

Using (14) the derivative of $V_1$ along the trajectories of (16) is given by

$$\dot{V}_1 = -\frac{k_1}{2}(1+\|\rho\|^2)\|\rho\|^2 \le -k_1 V_1 < 0, \qquad \forall \rho \ne 0. \tag{18}$$

Global exponential stability with rate of decay $k_1/2$ follows.

*Control of the Full Rigid Body Model:* Consider now the error variable

$$z = \omega - \omega_d = \omega + k_1\rho. \tag{19}$$

The differential equation for the kinematics is written as

$$\dot{\rho} = -k_1 H(\rho)\rho + H(\rho)z \tag{20}$$

and, as shown above, it is globally exponentially stable for $z = 0$. The differential equation for $z$ is

$$\begin{aligned}\dot{z} &= \big(J^{-1}S(\omega)J + k_1 H(\rho)\big)z \\ &\quad - k_1\big(J^{-1}S(\omega)J + k_1 H(\rho)\big)\rho + J^{-1}u.\end{aligned} \tag{21}$$

We want to find $u = u(\rho, z)$ such that the system of (20) and (21) is globally asymptotically stable. To this end, consider the following candidate Lyapunov function:

$$V(\rho, z) = k_1^2 V_1(\rho) + \frac{1}{2}\|z\|^2 = \frac{k_1^2}{2}\|\rho\|^2 + \frac{1}{2}\|z\|^2. \tag{22}$$

In order to use the results of Proposition 1 we need a stabilizing control law of the form in (2). Noticing that with $V$ as in (22) one has

$$\frac{\partial V}{\partial z} J^{-1} = z^T J^{-1} \tag{23}$$

we are looking for a control law of the form

$$u = -R^{-1}(\rho, \omega)J^{-1}z \tag{24}$$

where $R(\rho, \omega) > 0$, $\forall \rho, \omega \in \mathbb{R}^3$. Taking the derivative of $V$ along the trajectories of (20) and (21) one obtains

$$\begin{aligned}\dot{V} = &-\frac{k_1^3}{2}(1+\|\rho\|^2)\|\rho\|^2 - k_1 z^T J^{-1}S(\omega)J\rho + z^T J^{-1}S(\omega)Jz \\ &+ z^T\left(\frac{k_1}{2}(I+\rho\rho^T)z + J^{-1}u\right)\end{aligned} \tag{25}$$

and upon completion of squares

$$\begin{aligned}\dot{V} = &-\frac{k_1^3}{4}(1+2\|\rho\|^2)\|\rho\|^2 - \frac{k_1^3}{4}\left\|\rho - \frac{2}{k_1^2}JS(\omega)J^{-1}z\right\|^2 \\ &- \frac{k_1}{4}\left\|\left(I + \frac{2}{k_1}JS(\omega)J^{-1}\right)z\right\|^2 \\ &+ z^T\left\{\left[\frac{k_1}{2}\left(\frac{3}{2}I + \rho\rho^T\right)\frac{2}{k_1}J^{-1}S(\omega)^T J^2 S(\omega)J^{-1}\right]z \right.\\ &\left.\quad + J^{-1}u\right\}.\end{aligned} \tag{26}$$

Denote

$$\begin{aligned}R(\rho, \omega) = J^{-1}&\left[\left(k_2 + \frac{3}{4}k_1\right)I + \frac{k_1}{2}\rho\rho^T \right.\\ &\left. + \frac{2}{k_1}\big(S(\omega)J^{-1}\big)^T J^2 S(\omega)J^{-1}\right]^{-1} J^{-1}\end{aligned} \tag{27}$$

where $k_2 > 0$. Then (26) becomes

$$\begin{aligned}\dot{V} = &-\frac{k_1^3}{4}(1+2\|\rho\|^2)\|\rho\|^2 - \frac{k_1^3}{4}\left\|\rho - \frac{2}{k_1^2}JS(\omega)J^{-1}z\right\|^2 \\ &- \frac{k_1}{4}\left\|\left(I + \frac{2}{k_1}JS(\omega)J^{-1}\right)z\right\|^2 - k_2\|z\|^2 \\ &+ z^T J^{-1}\big\{R^{-1}(\rho, \omega)J^{-1}z + u\big\}.\end{aligned} \tag{28}$$

With the choice of the feedback control law in (24) and (27), (26) yields

$$\begin{aligned}\dot{V} = &-\frac{k_1^3}{4}(1+2\|\rho\|^2)\|\rho\|^2 - \frac{k_1^3}{4}\left\|\rho - \frac{2}{k_1^2}JS(\omega)J^{-1}z\right\|^2 \\ &- \frac{k_1}{4}\left\|\left(I + \frac{2}{k_1}JS(\omega)J^{-1}\right)z\right\|^2 - k_2\|z\|^2\end{aligned} \tag{29}$$

and the equilibrium $\rho = \omega = 0$ is rendered globally asymptotically stable.

From Proposition 1, for $\beta = 2$, we get the following result.

*Theorem 1:* The control law

$$\begin{aligned}u^* = -J&\left[\left(2k_2 + \tfrac{3}{2}k_1\right)I + k_1\rho\rho^T \right.\\ &\left. + \frac{4}{k_1}J^{-1}S(\omega)^T J^2 S(\omega)J^{-1}\right]z\end{aligned} \tag{30}$$

minimizes the cost functional

$$\mathcal{J} = \int_0^\infty \{l(\rho, \omega) + u^T R(\rho, \omega)u\}\,dt \tag{31}$$

where

$$\begin{aligned}l(\rho, \omega) = &k_1^3(1+2\|\rho\|^2)\|\rho\|^2 + 4k_2\|\omega + k_1\rho\|^2 \\ &+ k_1^3\left\|\rho - \frac{2}{k_1^2}JS(\omega)J^{-1}(\omega + k_1\rho)\right\|^2 \\ &+ k_1\left\|\left(I + \frac{2}{k_1}JS(\omega)J^{-1}\right)(\omega + k_1\rho)\right\|^2\end{aligned} \tag{32}$$

and $R(\rho, \omega)$ as in (27).

The performance index in (31) represents a meaningful cost since $l(\rho, \omega) > 0$ and $R(\rho, \omega) > 0$ for all $(\rho, \omega) \ne (0, 0)$; therefore, it penalizes both the states $\rho$ and $\omega$, as well as the control effort $u$. As $\rho$ and $\omega$ increase, the penalty on the control decreases. This is a desirable feature of the optimal control law, since it implies more aggressive control action far away from the equilibrium. Indeed, as the system state starts deviating from the intended operating point the controller allows for increasingly corrective action. For $\rho$ and $\omega$ large we have

$$l(\rho, \omega) \sim 2k_1^3\|\rho\|^4 + \frac{8}{k_1}\left\|JS(\omega)J^{-1}(\omega + k_1\rho)\right\|^2 \tag{33a}$$

$$R(\rho, \omega) \sim \left[\frac{k_1}{2}J\rho\rho^T J + \frac{2}{k_1}S(\omega)^T J^2 S(\omega)\right]^{-1}. \tag{33b}$$

One can see that $k_2$ has no effect on the large-signal performance. In addition, larger values of $k_1$ tend to put more penalty on $\rho$ while smaller values of $k_1$ tend to put more penalty on $\omega$. At the same time, for $\rho$ and $\omega$ small we have that

$$l(\rho, \omega) \sim 2k_1^3\|\rho\|^2 + (4k_2 + k_1)\|\omega + k_1\rho\|^2 \tag{34a}$$

$$R(\rho, \omega) \sim \left(k_2 + \tfrac{3}{4}k_1\right)^{-1} J^{-2} \tag{34b}$$

so, close to the origin, the control law reduces to a linear quadratic regulator (LQR)-type linear control law. The control law in this case minimizes the LQR cost

$$\mathcal{J} = \int_0^\infty \left\{[\omega^T \rho^T]Q\begin{bmatrix}\omega \\ \rho\end{bmatrix} + u^T R u\right\}dt \tag{35}$$

where

$$\begin{aligned}Q &= \begin{bmatrix} 4k_2 + k_1 & k_1(4k_2 + k_1) \\ k_1(4k_2 + k_1) & k_1^2(3k_1 + 4k_2)\end{bmatrix} \\ R &= \left(\frac{4}{4k_2 + 3k_1}\right)J^{-2}.\end{aligned} \tag{36}$$

It is important to realize that the optimal control law in (30) avoids the cancellation of the nonlinearities. Notice, for example, that from (25)

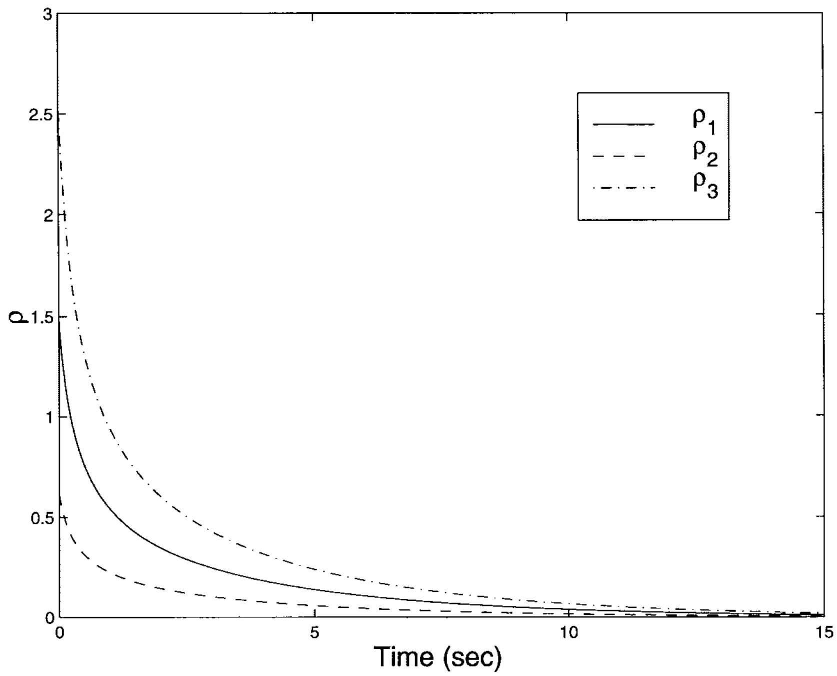


Fig. 1. Orientation parameters for the kinematics.

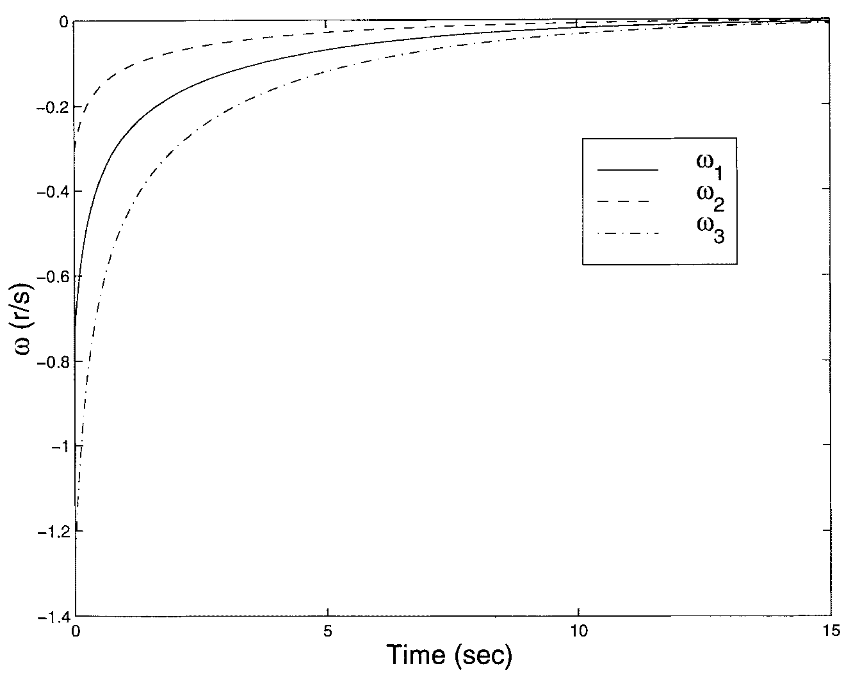


Fig. 2. Angular velocity for the kinematics.

one can globally asymptotically stabilize the system by choosing the control law

$$u = -k_2 J z - \frac{k_1}{2} J(I + \rho\rho^T)z - S(\omega)J\omega \tag{37}$$

which renders

$$\dot{V} = -\frac{k_1^3}{2}(1+\|\rho\|^2)\|\rho\|^2 - k_2\|z\|^2 < 0, \quad \forall (\rho,\, z) \neq (0,\, 0). \tag{38}$$

There are no obvious optimality characteristics associated with this control law. In fact, as was pointed out in [31] and [38], controllers which cancel nonlinearities are, in general, *non*optimal since the nonlinearity may be actually beneficial in meeting the stabilization and/or performance objectives.

An undesirable feature of the optimal control law in (30) is that it depends on the moment of inertia matrix $J$, which may not be always accurately known. The robustness properties of the optimal control law will be addressed in the future.

### *B. The Symmetric Case*

When the rigid body is symmetric, its inertia matrix is a multiple of the identity matrix and

$$S(\omega)J\omega \equiv 0, \qquad \forall\, \omega \in \mathbb{R}^3. \tag{39}$$

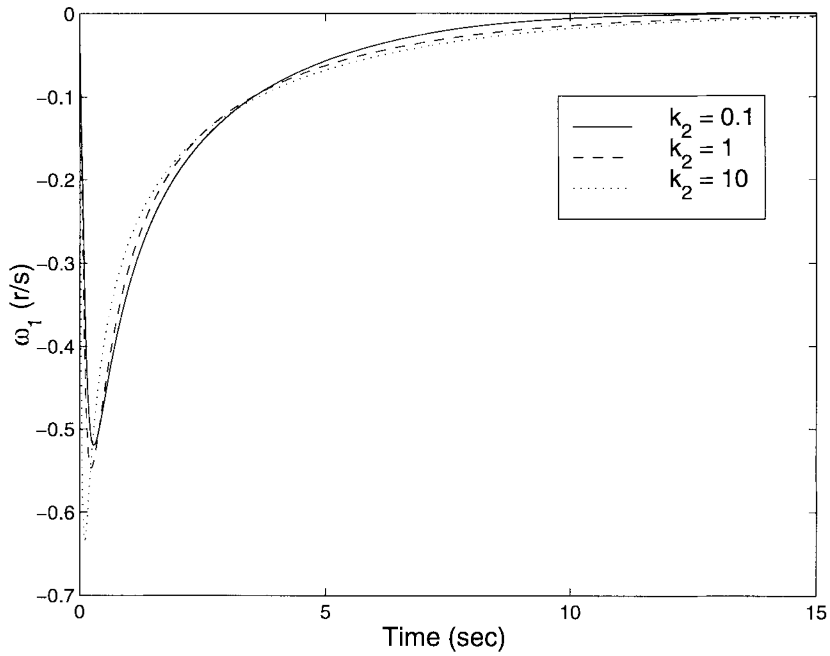


Fig. 3. Angular velocity $\omega_1$.

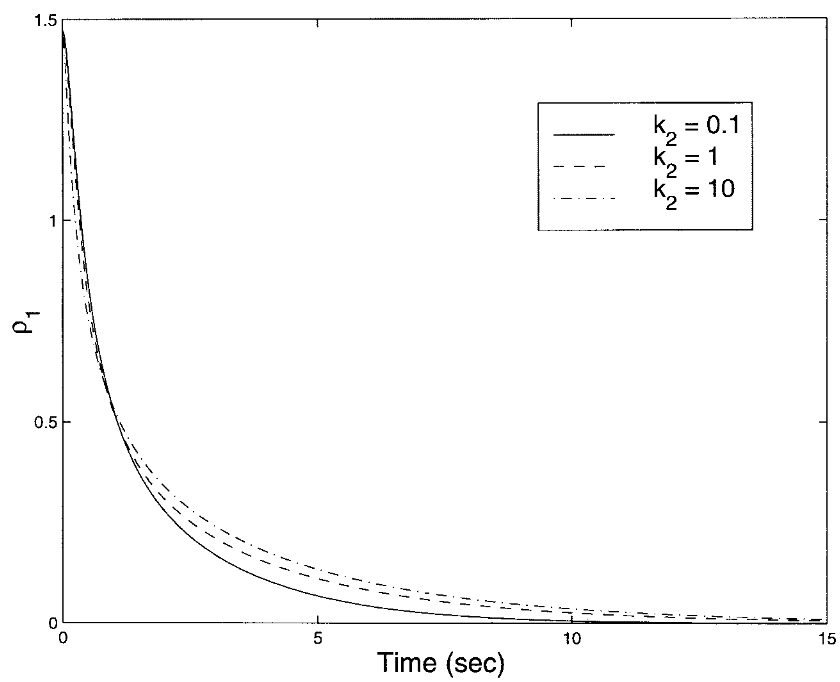


Fig. 4. Orientation parameter $\rho_1$.

In this case the optimal control law simplifies to

$$u^* = -J\left[(2k_2 + k_1)I + k_1\,\rho\rho^T\right]z \tag{40}$$

which minimizes the cost in (4) where

$$l(\omega,\,\rho) = 2k_1^3(1 + \|\rho\|^2)\|\rho\|^2 + 4\,k_2\|\omega + k_1\rho\|^2 \tag{41a}$$

$$R(\omega,\,\rho) = J^{-1}\left[\left(k_2 + \frac{k_1}{2}\right)I + \frac{k_1}{2}\,\rho\rho^T\right]^{-1} J^{-1}. \tag{41b}$$

This control law reduces to an LQR-type feedback control law close to the origin with

$$Q = \begin{bmatrix} 4k_2 & 4k_1k_2 \\ 4k_1k_2 & 2k_1^2(k_1 + 2k_2) \end{bmatrix}$$

and

$$R = \left(\frac{2}{2k_2 + k_1}\right)J^{-2}. \tag{42}$$

We note that the symmetric case has been previously addressed by Wie *et al.* [39], where a Euler parameter description for the kinematics was used.

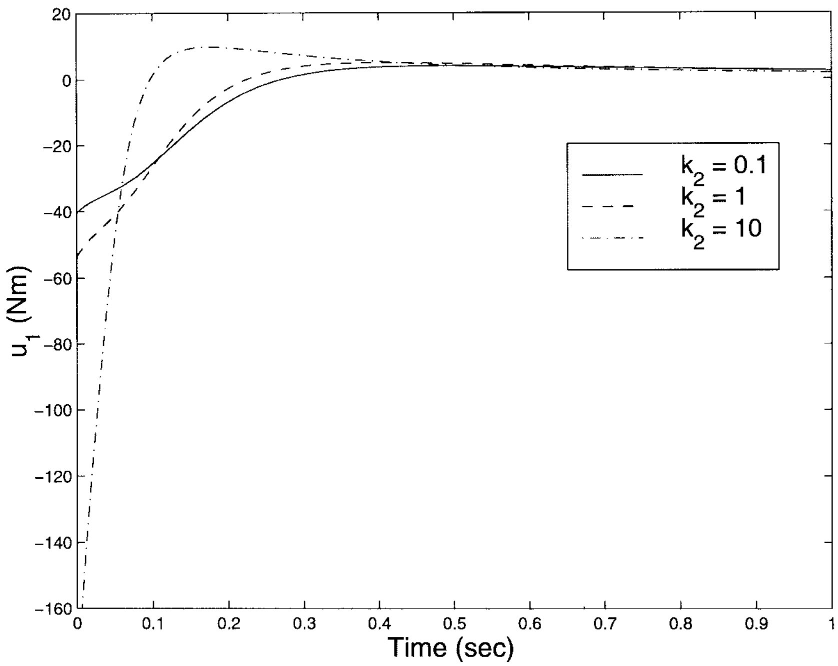


Fig. 5. Control input $u_1$.

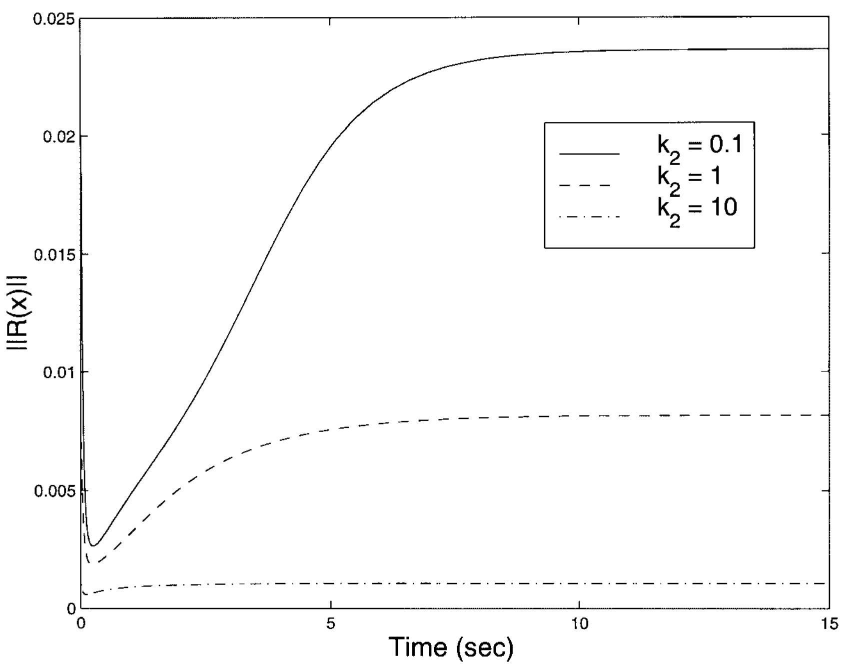


Fig. 6. Norm of $R(\omega, \rho)$.

*C. A Controller with Stability Margins*

We now set out to derive a control law that has stability margins described in Proposition 2. Lengthy calculations show that

$$R^{-1}(\rho, \omega) \le \lambda_{\max}^2(J)\left[k_2 + \frac{3}{4}k_1 + \frac{9}{k_1}V(\rho, z)\right]I, \quad \forall \rho, \omega \in \mathbb{R}^3. \tag{43}$$

By Proposition 2, the control law

$$u^* = -\lambda_{\max}^2(J)\left[k_2 + \frac{3}{4}k_1 + \frac{9}{2k_1}\left(k_1^2\|\rho\|^2 + \|\omega + k_1\rho\|^2\right)\right] \times J^{-1}(\omega + k_1\rho) \tag{44}$$

where $\lambda_{\max}(J)$ is the maximum eigenvalue of the matrix $J$, is robust to the input dynamics $a(I + \mathcal{P})$, where $a \ge 1/2$ is a constant and $\mathcal{P}$ is a strictly passive (possibly nonlinear) system. For example, the controller (44) will be stabilizing when passed through linear input dynamics $a(s+z)/(s+p)$ for any $z \ge p > 0$ and any $a \ge 1/2$ because the transfer function $(z-p)/(s+p)$ is strictly positive real.

## V. Numerical Example

Numerical simulations were performed to establish the validity of the theory. We assume a rigid spacecraft with inertia matrix $J = \text{diag}(10, 15, 20)$ kg m. A rest-to-rest maneuver is considered, thus $\omega(0) = 0$. First, we consider the kinematics subsystem in (11b) with $\omega$ regarded as the control input. Let the initial conditions $\rho(0) = [1.4735, 0.6115, 2.5521]^T$ in terms of the Cayley–Rodrigues parameters. These initial conditions correspond to a principal axis/angle pair $\hat{e} = [0.4896, 0.2032, 0.8480]^T$ and $\Phi = 2.5$ rad and describe an almost "upside-down" initial orientation. The trajectories of the system with the control law in (15) with $k_1 = 0.5$ are shown in Figs. 1 and 2. The exponential stability of the closed-loop system is evident from these figures. At this step the choice of $k_1$ is basically dictated by the required speed for the completion of the rest-to-rest maneuver.

For the stabilization of the complete system we use the control law in (30). The state trajectories for different values of the gain $k_2$ are depicted in Figs. 3 and 4. The optimal trajectories have a very uniform behavior which is essentially independent of the value of $k_2$ and they follow very closely the corresponding trajectories for the kinematics subsystem. From Fig. 5 it is seen that the control action varies a great deal, however, with $k_2$. The initial control action consists, essentially, in making $\omega \rightarrow -k_1\rho$. This is clearly shown in Fig. 3.

Finally, Fig. 6 shows the time history of the Frobenious norm of the control penalty matrix $R(\omega, \rho)$. The control penalty is decreased rapidly at the initial portion of the trajectory when increased control action is necessary in order to "match" $\omega$ with $\omega_d$ within a short period of time.

## VI. Conclusions

Due to the difficulty in obtaining closed-form solutions to the Hamilton–Jacobi–Bellman equation, the *direct* optimal control problem for nonlinear systems remains open. However, the knowledge of a control Lyapunov function allows us to solve the *inverse* optimal control problem, i.e., find a controller which is optimal with respect to a meaningful cost. The inverse optimal stabilization design for a rigid spacecraft in this paper is, to the authors' knowledge, the first feedback control law that minimizes a cost that incorporates a penalty on both the state (angular velocity and orientation) and the control effort (torque).

## Acknowledgment

The authors would like to thank P. Kokotović for critical reading and helpful comments.

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# Systems with Finite Communication Bandwidth Constraints—II: Stabilization with Limited Information Feedback

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***Abstract*—In this paper a new class of feedback control problems is introduced. Unlike classical models, the systems considered here have communication channel constraints. As a result, the issue of coding and communication protocol becomes an integral part of the analysis. Since these systems cannot be asymptotically stabilized if the underlying dynamics are unstable, a weaker stability concept called containability is introduced. A key result connects containability with an inequality equation involving the communication data rate and the rate of change of the state.**

***Index Terms*—Asymptotic stability, containability, feedback control, Kraft inequality.**

## I. Introduction

In the early part of this decade several papers appeared which investigated various information related aspects of decision and control. These included work by Delchamps [4], Kabamba and Hara [6], and Williamson [10]. For example, Delchamps studied the problem of stabilizing a discrete-time linear system with quantized state feedback. Quantization is, of course, a crucial consideration, but informationally related issues involve a much wider range of questions. In a previous paper by the authors [11], a class of estimation problems with communication constraints was introduced and analyzed. It was shown, in particular, that the performance of estimation algorithms is closely related to the data rate and the time scale of the underlying dynamical system. The motivation for investigating these systems came from a variety of sources including neurobiological systems, social-economical systems, and remotely controlled systems, (see, for example, the problem of power control in wireless communication studied in [9]). This class of systems is substantially different from those studied in [4], [6], and [10] because the issues of coding, communication protocol, and delays are not only explicitly considered but actually form the focal point of the investigation. Recent papers by Borkar and Mitter [2] and Li and Wong [8] also adopt a similar perspective.

In this paper, we continue the analysis of communication constrained systems, studying the effect of the communication rate on a class of linear feedback control problems. The model studied here can be viewed as a variant of the classical linear feedback control problem. However, unlike the classical problem where the assumption is that the plant and the feedback controller are either colocated or they can communicate with each other over a channel with infinite capacity, the crux of the problem studied here is that the plant and the feedback controller communicate over a digital channel with finite capacity. This simple change in the basic assumption has a substantial effect on the complexity of the problem. First of all, the issue of coding and communication protocol becomes an integral part of the analysis and cannot be decoupled from the control law design. Second, there is an inherent delay in the feedback control that further complicates matters. In particular, one simple consequence is that such communication constrained systems can never be asymptotically stabilized if the uncontrolled dynamics are unstable. Instead, a weaker stability concept called *containability* is introduced. The concept of containability is closely related to what has been called *practical stability* [7]. A key result in this paper connects containability with the Kraft inequality [3] and a newly derived inequality that involves the communication data rate and the rate of change of the state.

Manuscript received December 19, 1996. Recommended by Associate Editor, W.-B. Gong. This work was supported by a grant from the Hong Kong Research Grants Council.

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Publisher Item Identifier S 0018-9286(99)02118-2.

## II. The Finite Communication Control Problem

Consider a system with linear dynamics

$$\begin{cases} \dfrac{d}{dt}\, x(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where the state $x(t)$ is an element in $\Re^n$, $u(t)$ is a $m$-dimensional vector of control, $y(t)$ is a $p$-dimensional observation, and $A$, $B$, and $C$ are $n$ by $n$, $n$ by $m$, and $p$ by $n$ constant matrices, respectively.

The observation of $x(t)$, $y(t)$ is transmitted to a remote decision-maker for computing the appropriate level of feedback control. The communication channel is assumed to have a data rate of $R$ bits per second. For simplicity, we ignore the detailed implementation issues in the communication protocol and simply assume that it takes $\delta = 1/R$ s to send one bit from the plant to the controller and vice versa from the controller to the plant. Hence, if a bit is sent at time zero, it will be received at time $\delta$ at the receiver. Unlike classical models, the observed information is not transmitted continuously. Hence, we assume that $x(t)$ is sampled at time instances $\{r_i\}_{i=0}^{\infty}$ with $r_0 = 0$; the other sample instances will be defined later. Before an observation can be transmitted, it must be quantized and coded for the transmission. We assume that *prefix codes* are used so that the termination of a codeword is immediately recognizable [3]. The quantization and coding function can be symbolically represented by a function $h$ from the state space $\mathcal{R}^p$ to $\mathcal{B}$ where $\mathcal{B}$ stands for the set of finite length strings of symbols from a $D$-ary symbol set. $c_i$, the $i$th transmitted codeword from the plant to the controller, can be represented as

$$c_i = h(y(r_i)). \tag{2}$$

It is assumed in this paper that $h$ is a measurable function so that $h^{-1}(c)$ for any codeword $c$ is measurable.

We use variable length codewords. The codeword length function is denoted by $l$. Denote the time the $i$th codeword $c_i$ is received at the feedback decision-maker by $s_i$. Once the coded observation is received, it is decoded and the feedback control is computed and then coded for transmission back to the plant. We assume there is no computation delay. However, there is a transmission delay due