Adaptive rejection of harmonic disturbance anticollocated with control in 1D wave equation

Wei Guo a, Zhi-Chao Shao a, Miroslav Krstic b

a School of Statistics, University of International Business and Economics, Beijing 100029, PR China
b Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA

Article history:
Received 7 April 2016
Received in revised form 12 September 2016
Accepted 9 January 2017
Available online 2 March 2017

Keywords:
Distributed parameter system
Disturbance rejection
Output regulation

1. Introduction

The output regulation problem, or alternatively the servomechanism, is one of the central problems in control theory, which addresses designing of a feedback controller to achieve asymptotic tracking of prescribed reference signals and asymptotic rejection of undesired disturbances in an uncertain system while maintaining closed-loop stability. In the finite-dimensional system (linear or nonlinear) setting, there are many classical results which include internal model principle to address this problem such as Callier and Desoer (1980), Davison (1976), Desoer and Lin (1985), Francis (1977), Francis and Wonham (1976) and Isidori and Byrnes (1990), among many others. Much work has been done to extend these classical results to infinite-dimensional systems, see, for instance Byrnes, Laukó, Gilliam, and Shubov (2000), Deutscher (2011), Hämäläinen and Pohjolainen (2010), Paunonen and Pohjolainen (2010) and Rebarber and Weiss (2003). Most of the above works about output regulation problem focus on the extension of internal model principle theory to infinite-dimensional systems where reference signal and disturbance are generated by finite-dimensional or infinite-dimensional exosystem.

Also some attempts concerned with adaptive servomechanism design for infinite-dimensional systems have been made. Earlier work on applying adaptive servomechanism design for infinite-dimensional systems is reported in Logemann and Ilchmann (1994). In paper Kobayashi and Oya (2002), an adaptive servomechanism control is designed based on passivity principle for a class of distributed parameter system where the input and output operators are collocated and the disturbance is colocated with control. Those works build a theoretical framework which covers a large class of real systems. However, many real control systems are not included in those abstract frameworks, such as the boundary control PDE (Partial Differential Equation) system which is anticollocated, or is unstable or even antistable itself. In this situation, the input–output operator is typically no longer passive or uncontrolled system has real positive eigenvalues, which results in that the passivity principle cannot be directly applied anymore. In the last few years, there have been a few works contributed to the boundary feedback stabilization and observer design of these systems. An observer-based compensator which exponentially stabilizes the string system with an anticollocated actuator/ sensor configuration is proposed in Guo and Xu (2007). A lot of papers show that the backstepping method introduced in PDEs Krstic and Smyshlyaev (2008a) is very powerful in dealing with unstable or even antistable PDE system. In Krstic, Guo, Balogh, and Smyshlyaev (2008b), an observer-based controller is designed using both the displacement and velocity...
measurement via the backstepping method to exponentially stabilize an unstable one-dimensional wave equation. A breakthrough was made recently in Smyshlyaev and Krstic (2009), where the anti-stable wave equation is exponentially stabilized through a novel backstepping transformation method. The stabilization of unstable shear beam equation is addressed in Krstic, Guo, Balogh, and Smyshlyaev (2008a), where the non-collocated boundary stabilization is discussed by using the backstepping method and observer-based feedback.

Almost at the same time, much more attention has also been paid to the adaptive boundary control by using backstepping method for these anticollocated, unstable or even antistable PDE systems. The early efforts on applying the backstepping method to the design of adaptive boundary state feedback controllers or output feedback controllers for partial differential equation control systems have been made in Krstic and Smyshlyaev (2008b) and Smyshlyaev and Krstic (2007a,b), particularly for parabolic PDEs with boundary control and unknown parameters that may cause instability of the system and affect the interior of the domain. Adaptive boundary state feedback stabilization and recent output feedback stabilization for the most challenging anti-stable wave equation with unknown anti-damping in the uncontrolled boundary can be found in Krstic (2010) and Bresch-Pietri and Krstic (2014) respectively. A recent result on adaptive anti-stable wave equation with unknown anti-damping in the uncontrolled output feedback stabilization for the most challenging anti-stable wave equation is presented in Smyshlyaev and Krstic (2009), where the stabilization of the disturbances and then construct an auxiliary system in which the control and the anti-collocated disturbance become collocated. Our adaptive observer and controller do not seem to employ the internal model principle and the regulator equations associated with the output regulation approach.

The key characteristic of our approach is the use of the measurement $y_t(0, t), y(0, t)$ to first give a backstepping-based adaptive observer of the state and the estimator of the parameters of the disturbances and then construct an auxiliary system in which the control and the anti-collocated disturbance become collocated. The applications of adaptive cancellation of anti-collocated disturbances arise in many applications, from oil drilling to atomic force microscopy, where disturbances appear on the opposite boundary relative to the actuator.

The problem is organized as follows. In next section, Section 2, we give the adaptive disturbance cancellation controller design. The asymptotic stability of the error system is given in Section 3. In Section 4, we give the convergence analysis of the auxiliary system. We present some illustrative simulation results in Section 5. Conclusions are made in Section 6.

2. Adaptive disturbance cancellation controller design

We now design an adaptive observer for system (1) using the measurement of $y_{out}(t) = (y(0, t), y_t(0, t))$. Since the plant is second-order in time, $y_t(0, t)$ is not to be thought as a derivative of the displacement $y(0, t)$ but as a measured part of the overall state $(y, y_t)$. Indeed it is common to virtually all results in boundary control of flexible structures to employ measurements of the velocity at the boundary.

Our adaptive observer is given as:

\[
\begin{align*}
\hat{y}_t(x, t) &= \hat{y}_t(x, t), \\
\hat{y}_t(0, t) &= \hat{a}(t) \cos \alpha t + \hat{b}(t) \sin \alpha t \\
+ k_1(\hat{y}_t(0, t) - y_t(0, t)) + k_2(\hat{y}_t(0, t) - y_t(0, t)), \\
\hat{y}_t(1, t) &= U(t), \\
\hat{a}(t) &= -r(y_t(0, t) - \hat{y}_t(0, t)) \cos \alpha t, \\
\hat{b}(t) &= -r(y_t(0, t) - \hat{y}_t(0, t)) \sin \alpha t, \\
\hat{a}(0) &= \hat{a}_0, \\
\hat{b}(0) &= \hat{b}_0, \\
\hat{y}_t(x, 0) &= \hat{y}_t(x, 0) = \hat{y}_t(x, 0).
\end{align*}
\]

where $k_1, k_2, r > 0$ are design parameters. In the rest of the paper, we omit the (obvious) domains for $t$ and $x$.

Remark 2.1. The observer design for the case $d(t) = \sum_{j=1}^m [a_j \cos \omega_0 t + b_j \sin \omega_0 t]$ is still valid because we only need to write the term $\hat{a}(t) \cos \alpha t + \hat{b}(t) \sin \alpha t$ in (2) to be $\sum_{j=1}^m \hat{a}_j(t) \cos \omega_0 t + \hat{b}_j(t) \sin \alpha t$ and the update law to be $\hat{a}_j(t) = -r_j(y_t(0, t) - \hat{y}_t(0, t)) \cos \omega_0 t, \hat{b}_j(t) = -r_j(y_t(0, t) - \hat{y}_t(0, t)) \sin \omega_0 t, j = 1, 2, \ldots, m.$
Let \( \varepsilon = y - \hat{y} \) and \( \tilde{a} = a - \hat{a}(t) \), \( \tilde{b} = b - \hat{b}(t) \) be parameter estimation error, then from (1) and (2), \( \varepsilon \) is governed by
\[
\begin{align*}
\varepsilon_x(t) &= \varepsilon_x(x, t), \\
\varepsilon_0(t) &= k_1\varepsilon_0(0, t) + k_2\varepsilon(0, t) + \tilde{a}(t)\cos \omega t + \tilde{b}(t)\sin \omega t, \\
\varepsilon_1(1, t) &= 0, \\
\tilde{a}(t) &= r\varepsilon_1(0, t)\cos \omega t, \\
\tilde{b}(t) &= r\varepsilon_1(0, t)\sin \omega t, \\
\varepsilon(x, 0) &= \varepsilon_0(x), \\
\varepsilon_t(x, 0) &= \varepsilon_1(x),
\end{align*}
\]
where
\[
\varepsilon(x, 0) = y_0(x) - \hat{y}_0(x), \quad \varepsilon_t(x, 0) = y_1(x) - \hat{y}_1(x).
\]
Define the energy function for system (3) as follows:
\[
E_\varepsilon(t) = \frac{1}{2} \int_0^1 [\varepsilon_x^2(t, x) + \varepsilon_t^2(t, x)] dx + \frac{k_2}{2} [\varepsilon_0(t, 0)^2 + \frac{1}{2r} \tilde{a}^2(t) + \tilde{b}^2(t)].
\]
A simple computation of the derivative of \( E_\varepsilon(t) \) with respect to \( t \) along the solution to (3) shows that
\[
\dot{E}_\varepsilon(t) = -k_1\varepsilon_0(1, 0)^2 \leq 0,
\]
from which we obtain the update law of \( \hat{a}(t) \) and \( \hat{b}(t) \) in system (2). By the update law of \( \hat{a}(t) \) and \( \hat{b}(t) \) in system (2), a formal computation gives
\[
\begin{align*}
\frac{d}{dt} [\hat{a}(t)\cos \omega t + \hat{b}(t)\sin \omega t + r\varepsilon(0, t)] &= \omega \hat{b}(t)\cos \omega t - \hat{a}(t)\sin \omega t, \\
\frac{d^2}{dt^2} [\hat{a}(t)\cos \omega t + \hat{b}(t)\sin \omega t + r\varepsilon(0, t)] &= -\omega^2 [\hat{a}(t)\cos \omega t + \hat{b}(t)\sin \omega t].
\end{align*}
\]
Let
\[
\begin{align*}
\hat{z}(x, t) &= \hat{y}(x, t) - \frac{1}{\omega} \sin \omega x [\hat{a}(t)\cos \omega t + \hat{b}(t)\sin \omega t] + \hat{b}(t)\sin \omega t + r\varepsilon(0, t), \\
(x, t) &\in [0, 1] \times [0, \infty).
\end{align*}
\]
Then from (2) and (7), we can get the following auxiliary system:
\[
\begin{align*}
\begin{aligned}
\hat{z}_x(t, x) &= \hat{z}(x, t) - \omega\sin \omega x [\hat{a}(t)\cos \omega t + \hat{b}(t)\sin \omega t]
\end{aligned}, \\
\hat{z}_x(0, t) &= -k_1\hat{e}_1(0, t) - (k_2 + r)\varepsilon_0(0, t), \\
\hat{z}_x(1, t) &= U(t) - \cos \omega [\hat{a}(t)\cos \omega t + \hat{b}(t)\sin \omega t] - r\cos \omega \varepsilon(0, t), \\
\hat{z}(x, 0) &= \hat{z}_0(x), \quad \hat{z}_x(0, 0) = \hat{z}_1(x),
\end{align*}
\]
where
\[
\begin{align*}
\hat{z}_0(x) &= \hat{y}_0(x) - \frac{\hat{a}_0 + r\varepsilon_1(0)}{\omega}\sin \omega x, \\
\hat{z}_1(x) &= \hat{y}_1(x) - \hat{b}_0 \sin \omega x.
\end{align*}
\]
Moreover,
\[
\hat{z}(0, t) = \hat{y}(0, t).
\]

**Remark 2.2.** (a) The motivation of constructing auxiliary system (9) is to make the control and the anticollocated disturbance to be collocated and obtain \( \hat{z}(0, t) = \hat{y}(0, t) \).

(b) Here we consider the auxiliary system (9) without the dynamic equations for \( \hat{a}(t) \) and \( \hat{b}(t) \) since they have been determined by the error system (3) already.

(c) This construction is still valid for the case \( d(t) = \sum_{j=1}^{m} [\alpha_j \cos \omega_j t + b_j \sin \omega_j t] \) if we let \( \hat{z}(x, t) = \hat{y}(x, t) - \sum_{j=1}^{m} \frac{1}{\omega_j} \sin \omega_j x [\hat{a}_j(1) \cos \omega_j t + \hat{b}_j(1) \sin \omega_j t] \).

We present the controller for (9) as follows:
\[
U(t) = \cos \omega [\hat{a}(t)\cos \omega t + \hat{b}(t)\sin \omega t] - c_0\hat{z}(1, t)
\]
\[
- c_1\hat{z}_1(1, t) - c_0c_1 \int_0^1 \hat{z}_1(\xi, t) d\xi + r \cos \omega \varepsilon(0, t),
\]
where \( c_0, c_1 \) are positive design parameters. The closed-loop system of (8) corresponding to controller (11) is
\[
\begin{align*}
\hat{z}_x(t, x) &= \hat{z}(x, t) - \omega \sin \omega x \varepsilon(0, t), \\
\hat{z}_x(0, t) &= -k_1\hat{e}_1(0, t) - (k_2 + r)\varepsilon_0(0, t), \\
\hat{z}_x(1, t) &= -c_0\hat{z}(1, t) - c_1\hat{z}_1(1, t) - c_0c_1 \int_0^1 \hat{z}_1(\xi, t) d\xi, \\
\hat{z}(x, 0) &= \hat{z}_0(x), \quad \hat{z}_x(0, 0) = \hat{z}_1(x),
\end{align*}
\]
Introduce the following transformation (see Krstic et al., 2008a or page 83 in Krstic & Smyshlyaev, 2008a)
\[
\tilde{z}(x, t) = \hat{z}(x, t) - c_0 \int_0^x \hat{z}(\xi, t) d\xi,
\]
which is invertible. The inverse is given by
\[
\hat{z}(x, t) = \tilde{z}(x, t) - c_0 \int_0^x e^{-c_0(\xi-x)\tilde{z}(\xi, t)} d\xi.
\]
It is seen that transformation (13) converts system (12) into
\[
\begin{align*}
\hat{z}_x(t, x) &= \tilde{z}_x(t, x) - [\omega \sin \omega x \varepsilon(0, t), \\
\hat{z}_x(0, t) &= -k_1\hat{e}_1(0, t) - (k_2 + r)\varepsilon_0(0, t), \\
\hat{z}_x(1, t) &= -c_0\hat{z}(1, t) - c_1\hat{z}_1(1, t) - c_0c_1 \int_0^1 \hat{z}_1(\xi, t) d\xi, \\
\hat{z}(x, 0) &= \hat{z}_0(x), \quad \hat{z}_x(0, 0) = \tilde{z}_1(0, 0),
\end{align*}
\]
where
\[
\begin{align*}
\hat{z}_0(x) &= \hat{z}_0(x) + c_0 \int_0^x \hat{z}_0(\xi) d\xi, \\
\hat{z}_1(x) &= \hat{z}_1(x) + c_0 \int_0^x \hat{z}_1(\xi) d\xi.
\end{align*}
\]
Then controller (11) is obtained in the process of transforming (12) into (14) under the Backstepping transformation (13). Notice that controller (11) is expressed by variable \( \tilde{z} \). In order to get the closed-loop of system (1), it is necessary to write controller (11) to be expressed by variable \( \hat{y} \). Then by (8), we rewrite the controller (11) to be
\[
U(t) = -c_0 \sqrt{y}(1, t) - c_1 \sqrt{y}_1(1, t)
\]
\[
- c_0c_1 \int_0^1 \sqrt{y}(\xi, t) d\xi + \left[ c_0 \frac{\sin \omega \omega \cos + \cos}{\omega} \right]
\]
\[
\times [\hat{a}(t)\cos \omega t + \hat{b}(t)\sin \omega t]
\]
\[
+ r \left[ c_0 \frac{\sin \omega \omega \cos + \cos}{\omega} \right] y(0, t) - \hat{y}(0, t)
\]
\[
+ c_1 \left[ \frac{\sin \omega + c_0(1 - \cos \omega)}{\omega} \right] \hat{b}(t) \cos \omega t - \hat{a}(t) \sin \omega t.
\]
Combined by (1),(2) and (16), the resulting closed-loop is governed by
\[
\begin{align*}
y_{I}(t) &= y_{\omega}(x, t), \\
y_{2}(0, t) &= a \cos \omega t + b \sin \omega t, \\
y_{1}(t) &= \tilde{y}_{2}(t), \\
\tilde{y}_{1}(t) &= \tilde{y}_{2}(x, t), \\
\tilde{y}_{2}(t) &= \tilde{a}(t) \cos \omega t + \tilde{b}(t) \sin \omega t + k_{1}(\tilde{y}_{2}(t) - y(0, t)) + k_{2}(\tilde{y}_{2}(t) - y(0, t)), \\
y_{2}(1, t) &= -c_{0}\tilde{y}_{2}(t) - c_{1}y_{2}(1, t), \\
y_{1}(1, t) &= \frac{c_{0} \sin \omega t + \cos \omega t}{\omega} + y_{I}(t), \\
\tilde{y}_{2}(t) &= \tilde{a}(t) \cos \omega t + \tilde{b}(t) \sin \omega t + y_{1}(t), \\
\tilde{y}_{1}(0, t) &= \tilde{a}(t) \cos \omega t + \tilde{b}(t) \sin \omega t + \tilde{y}_{2}(0, t) - y(0, t), \\
\tilde{y}_{1}(t) &= \frac{c_{0} \sin \omega t + \cos \omega t}{\omega} + \tilde{y}_{2}(t).
\end{align*}
\]

(17)

Eq. (19) is a nonlinear autonomous evolution system. However, same as Guo and Guo (2013a), it seems hard to use nonlinear semigroup to prove its well-posedness due to the lack of dissipativity of $\omega$ or any other kind of $\omega + \mu t$ for constant $\mu \in \mathbb{R}$. Hence we invoke the operator method to establish the existence and uniqueness for the solution of Eq. (3). To do this, we need a basis to construct a Galerkin approximation, which can be realized by the operator $A$ defined in $L^{2}(0, 1)$ as follows:
\[
\begin{align*}
A \phi &= -\phi''', \\
D(A) &= \{ \phi \in L^{2}(0, 1) | \hat{\phi}(0) = 0, \phi(1) = 0 \}.
\end{align*}
\]

It is seen that $A$ is unbounded self-adjoint positive definite in $L^{2}(0, 1)$ with compact resolvent. A simple computation shows that the eigenpairs $(\lambda_{n}, \phi_{n})_{n=1}^{\infty}$ of $A$ are
\[
\begin{align*}
\lambda_{n} &= -\omega_{n}^{2}, \\
\omega_{n} &= \left( n + \frac{1}{2} \right) \pi, \\
\phi_{n}(x) &= 2 \cos \omega_{n}x = 2 \cos \left( n + \frac{1}{2} \right) \pi x.
\end{align*}
\]

Since $(\phi_{n}(x))_{n=1}^{\infty}$ defined by (20) forms an orthogonal basis for $L^{2}(0, 1)$, we can then follow the steps as those in Guo and Guo (2013a) to construct a Galerkin scheme to prove the existence and uniqueness for the classical solution to error system (3).

Let $V = H^{1}(0, 1) \cap D(A)$.

**Theorem 3.1.** Suppose that $(\varepsilon_{0}, \varepsilon_{1}, \tilde{\varepsilon}_{0}, \tilde{\varepsilon}_{1}, \tilde{\varepsilon}_{2}, \tilde{\varepsilon}_{3}, \tilde{\varepsilon}_{4}) \in V \times V \times \mathbb{R}^{2}$, and they satisfy the following compatible condition:
\[
k_{1}\varepsilon_{1}(0) + k_{2}\tilde{\varepsilon}_{3}(0) + \tilde{\varepsilon}_{0} = 0
\]
and
\[
k_{1}\varepsilon_{1}'(0) - \omega \tilde{\varepsilon}_{0} = 0.
\]

Then system (3) admits a unique classical solution $\varepsilon$. That is to say, for any time $T > 0$,
\[
\begin{align*}
\varepsilon &\in L^{1}(0, T; H^{1}(0, 1)), \\
\varepsilon_{1} &\in L^{1}(0, T; H^{1}(0, 1)), \\
\tilde{\varepsilon}_{0} &\in L^{1}(0, T; H^{2}(0, 1)), \\
\tilde{\varepsilon}_{1} &\in L^{1}(0, T; V), \\
\tilde{\varepsilon}_{2} &\in L^{1}(0, T; V^{\ast}), \\
\tilde{\varepsilon}_{3} &\in L^{1}(0, T; V^{\ast}), \\
\tilde{\varepsilon}_{4} &\in L^{1}(0, T; V^{\ast}),
\end{align*}
\]
\[
\begin{align*}
\varepsilon_{1}(x, t) &= \varepsilon_{2}(x, t) = \varepsilon_{3}(x, t) = \varepsilon_{4}(x, t) = 0, \\
\tilde{\varepsilon}_{0}(x, t) &= \tilde{\varepsilon}_{1}(x, t) = \tilde{\varepsilon}_{2}(x, t) = \tilde{\varepsilon}_{3}(x, t) = \tilde{\varepsilon}_{4}(x, t) = 0.
\end{align*}
\]

By the Sobolev embedding theorem, it follows that $\varepsilon \in C([0, 1] \times [0, T])$.

**Remark 3.1.** In Theorem 3.1, condition (21) is the natural compatible condition for the classical solution of (3), and condition (22) is for the existence of the more smoother solution that we shall need in the proof of Theorem 3.2.

Next, we establish the convergence of error system (3). To do this, we need the weak solution of (3).

**Definition 3.1.** For any initial data $(\varepsilon_{0}, \varepsilon_{1}, \tilde{\varepsilon}_{0}, \tilde{\varepsilon}_{1}, \tilde{\varepsilon}_{2}, \tilde{\varepsilon}_{3}, \tilde{\varepsilon}_{4}) \in V \times V \times \mathbb{R}^{2}$, the weak solution $(\varepsilon, \varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}, \varepsilon_{4})$ of (3) is defined as the limit of any convergent subsequence of $(\varepsilon_{n}, \varepsilon_{1}^{n}, \varepsilon_{2}^{n}, \varepsilon_{3}^{n}, \varepsilon_{4}^{n})$ in the space $L^{p}(0, T; V)$, where $(\varepsilon_{n}, \varepsilon_{1}^{n}, \varepsilon_{2}^{n}, \varepsilon_{3}^{n}, \varepsilon_{4}^{n})$ is the classical solution (ensured by Theorem 3.1) with the initial condition (for all $x \in (0, 1)$)
\[
(\varepsilon_{n}^{0}(x), \varepsilon_{1}^{n}(x), \varepsilon_{2}^{n}(x), \varepsilon_{3}^{n}(x), \varepsilon_{4}^{n}(x), \tilde{\varepsilon}_{0}^{n}(x), \tilde{\varepsilon}_{1}^{n}(x), \tilde{\varepsilon}_{2}^{n}(x), \tilde{\varepsilon}_{3}^{n}(x), \tilde{\varepsilon}_{4}^{n}(x)) \in V \times V \times \mathbb{R}^{2},
\]

where
which satisfies
\[ \lim_{n \to \infty} \left\| \left( \epsilon_0^n(x), \epsilon_1^n(x), \vec{a}_0^n, \vec{b}_0^n \right) - (\epsilon_0, \epsilon_1, \vec{a}_0, \vec{b}_0) \right\|_\gamma = 0. \]

By (5) and (6), the above weak solution is well defined, since it does not depend on the choice of initial sequence \((\epsilon^n(x, 0), \epsilon^n_1(x, 0), \vec{a}^n(0), \vec{b}^n(0))\).

**Theorem 3.2. Suppose that**
\[ \omega \neq 0, \pi \pm \frac{\pi}{2}, \quad n \in \mathbb{Z}. \] (23)

Then for any initial value \((\epsilon_0, \epsilon_1, \vec{a}_0, \vec{b}_0) \in \gamma\), the solution of system (3) is asymptotically stable in the sense that
\[ \lim_{t \to \infty} \left[ \frac{1}{2} \int_0^t \left[ \epsilon_1^2(x, t) + \epsilon_2^2(x, t) \right] dx + k_2 \epsilon^2(0, t) \right] = 0 \] and
\[ \lim_{t \to \infty} \vec{a}(t) = a, \quad \lim_{t \to \infty} \vec{b}(t) = b. \]

**Proof.** By density argument, we may assume without loss of generality that the initial value \((\epsilon_0, \epsilon_1, \vec{a}_0, \vec{b}_0)\) belongs to \(V \times V \times \mathbb{R}^2\) and satisfies compatible conditions (5) and (6). Construct the Lyapunov functional \(V_r(t)\) for system (19) as follows:
\[ V_r(t) = \frac{1}{2} \int_0^t \left[ \epsilon_1^2(x, t) + \epsilon_2^2(x, t) \right] dx + k_2 \epsilon^2(0, t) \] + \frac{1}{2r} \left[ \vec{a}^2(t) + \vec{b}^2(t) \right] + [\epsilon_2^2(t) + \eta_1(t)]. \] (24)

A simple computation gives the time derivative of \(V_r(t)\) along the solution of system (19),
\[ \dot{V}_r(t) = -k_1 \epsilon_1(0, t)^2. \]

This concludes that \(V_r(t) \leq V_r(0)\), hence
\[ \sup_{t \geq 0} \left[ \frac{1}{2} \int_0^t \left[ \epsilon_1^2(x, t) + \epsilon_2^2(x, t) \right] dx + k_2 \epsilon^2(0, t) \right] + \left[ |\vec{a}(t)| + |\vec{b}(t)| \right] < \infty. \] (25)

In particular, one has
\[ \epsilon_1(0, t) \in L^2(0, \infty). \] (26)

Similarly, define
\[ U(t) = \frac{1}{2} \int_0^t \left[ \epsilon_1^2(x, t) + \epsilon_2^2(x, t) \right] dx + \frac{k_2 + r}{2} \epsilon^2(0, t). \]

The time derivative of \(U(t)\) along the solution of (3) can be found as
\[ \dot{U}(t) = -k_1 \epsilon_1(0, t)^2 + \omega \epsilon_2(0, t) |\vec{a}(t)| \sin \omega t - \vec{b}(t) \cos \omega t. \] (27)

Integrating over \([0, t]\) on both sides of (27) gives
\[ U(t) = -k_1 \int_0^t [\vec{a}(0, s)]^2 ds + \omega \int_0^t [\vec{a}(0, s)] \sin \omega s ds + U(0) \] \[ - k_1 \int_0^t [\vec{a}(0, s)]^2 ds + \omega \int_0^t [\vec{a}(0, s)] \sin \omega s ds - \vec{b}(t) \cos \omega t - \omega \epsilon_1(0) \vec{b}_0 + \omega \int_0^t [\vec{a}(0, s)] \sin \omega s ds \] \[ + \vec{b}(s) \sin \omega s ds = -k_1 \int_0^t [\vec{a}(0, s)]^2 ds + \omega \int_0^t [\vec{a}(0, s)] \sin \omega s ds + \omega \epsilon_1(0) \vec{b}_0 + \frac{\omega}{2r} [\vec{a}^2(t) + \vec{b}^2(t)] + U(0). \] (28)

Use Young Inequality in (28) to obtain
\[ U(t) \leq \frac{1}{\delta_2 k_2} [\vec{a}(t) \sin \omega t - \vec{b}(t) \cos \omega t]^2 + \delta_2 k_2 \epsilon_1^2(0, t) \] \[ + |\omega \epsilon_1(0) \vec{b}_0| + \frac{\omega}{2r} [\vec{a}^2(t) + \vec{b}^2(t)] + U(0). \] (29)

Taking \(0 < \delta_2 < \frac{1}{4}\) in (29) gives
\[ U(t) - \delta_2 k_2 \epsilon_1^2(0, t) \geq \frac{1}{2} U(t), \]
which implies
\[ U(t) \leq \frac{2}{\delta_2 k_2} [\vec{a}(t) \sin \omega t - \vec{b}(t) \cos \omega t]^2 + 2|\omega \epsilon_1(0) \vec{b}_0| + \frac{\omega}{r} [\vec{a}^2(t) + \vec{b}^2(t)] + 2U(0). \] (30)

It follows from (25), (26) and (30) that
\[ \sup_{t \geq 0} U(t) < \infty, \]
which implies that the trajectory of system (3)
\[ \gamma'(\epsilon_0) = \{(\epsilon, \epsilon_1, \vec{a}(t), \vec{b}(t), \eta(t), \xi(t)) | t \geq 0\} \]
is precompact in \(\gamma\). In the light of Lasalle’s invariance principle (Walker, 1980), any solution of system (3) tends to the maximal invariant set of the following:
\[ S = \{(\epsilon, \epsilon_1, \vec{a}(t), \vec{b}(t), \eta(t), \xi(t)) \in \gamma | \dot{V}_r(t) = 0\}. \]

Now, since \(\dot{V}_r(t) = 0\), it follows that \(\epsilon_1(0, t) = 0\), which implies \(\vec{a}(t) \equiv 0\) and \(\vec{b}(t) \equiv 0\). So \(\vec{a} \equiv \vec{a}_0\) and \(\vec{b} \equiv \vec{b}_0\). Thus the solution reduces to
\[ \begin{align*}
\epsilon_1(t) & = \epsilon_1(0), \\
\epsilon_1(1) & = 0, \\
\epsilon_1(2) & = 0.
\end{align*} \] (31)

The proof will be completed if we can prove that (31) admits zero solution only. To this end, we first consider the equation
\[ \begin{align*}
\epsilon_1(t) & = \epsilon_1(0), \\
\epsilon_1(1) & = 0, \\
\epsilon_1(2) & = 0.
\end{align*} \] (32)

Introduce a Hilbert space \(H = H^1([0, 1]) \times L^2(0, 1)\) with the inner product
\[ \langle (y_1, z_1), (y_2, z_2) \rangle_\gamma = \int_0^1 [y_1'(x)z_2'(x) + z_1(x)z_2(x)] dx, \]
where \(H^1([0, 1]) = \{ u \in H^1(0, 1) | u(0) = 0 \}\). Define a linear operator \(A\) in \(H\) associated to system (32)
\[ \begin{align*}
A(y, z) & = (2, y''), \\
D(A) & = \{(y, z) \in H^2(0, 1) \times H^1(0, 1) | y'(0) = 0, z(0) = 0\}.
\end{align*} \] (33)

It is a simple exercise to show that \((\mu I - A)^{-1}\) is compact on \(H\) for some \(\mu > 0\). Hence \(A\) is a skew-adjoint operator with compact resolvent on \(H\). Consequently, the spectrum of \(A\) consists of isolated eigenvalues on the imaginary axis only, and from a general result of functional analysis, the algebraic multiplicity of each eigenvalue of
\[ \hat{A} \text{ is equal to its geometric multiplicity. Solve the eigenvalue problem} \]
\[ \lambda, \phi, \psi = \lambda, \phi, \psi \]
for any \( \lambda \in \sigma_p(\hat{A}) \). The solution is \( \psi = \lambda \phi \) with \( \phi \neq 0 \) satisfying
\[ \begin{cases} \lambda^2 \phi(x) - \phi''(x) = 0, \\ \phi(1) = 0, \\ \lambda \phi(0) = 0. \end{cases} \] (34)

Solve (34) in the case where \( \lambda = 0 \) to give
\[ \phi(x) = c, \]
where \( c \) is a constant. When \( \lambda \neq 0 \),
\[ \phi(x) = e^{\lambda x} - e^{-\lambda x} \] (36)
with
\[ e^{2\lambda} = -1. \] (37)

So \( \lambda \) is geometrically simple. So each \( \lambda \) is algebraically simple.

Furthermore, from (36), we can obtain eigen-pairs of \( \hat{A} \):
\[ \begin{cases} \lambda_n = \left(n \pi + \frac{\pi}{2}\right)i, \\ \phi_n = (\lambda_n^{-1} \phi_n, \phi_n), \end{cases} \] (38)
where
\[ \phi_n(x) = \sin \left(n + \frac{1}{2}\right) \pi x, \quad n \in \mathbb{Z}. \]

By general theory of functional analysis, \( \{\phi_n\}_{n \in \mathbb{Z}} \) forms an orthogonal basis for \( \mathbb{H} \). Therefore, the solution of (32) can be represented as
\[ (\varepsilon(\cdot, t), \tilde{\varepsilon}(\cdot, t)) = k_2 d_0 (c, 0) + \sum_{n=1}^{\infty} d_n e^{\lambda_n t} \phi_n + \sum_{n=1}^{\infty} d_n e^{-\lambda_n t} \phi_n, \]
where the constants \( \{d_n\}_{n \in \mathbb{Z}} \) are determined by the initial condition. That is,
\[ \varepsilon_0 = d_0 c + \sum_{n=1}^{\infty} d_n \phi_n, \quad \varepsilon_1 = \sum_{n=1}^{\infty} d_n \phi_n. \]

Hence
\[ \varepsilon(x, t) = \sum_{n=1}^{\infty} d_n \frac{\phi_n(0)}{\lambda_n} e^{\lambda_n t} + \sum_{n=1}^{\infty} d_n \frac{\phi_n'(0)}{\lambda_n} e^{-\lambda_n t} = k_2 d_0 c + \tilde{\varepsilon}_0 \cos \omega t + \tilde{\varepsilon}_0 \sin \omega t. \]

Therefore,
\[ \begin{align*}
- k_2 d_0 c &+ \sum_{n=1}^{\infty} d_n \frac{\phi_n'(0)}{\lambda_n} e^{\lambda_n t} \\
&+ \sum_{n=1}^{\infty} d_n \frac{\phi_n'(0)}{\lambda_n} e^{-\lambda_n t} - \frac{1}{2} (\tilde{\varepsilon}_0 - \tilde{\varepsilon}_0) e^{\omega t} \\
&- \frac{1}{2} (\tilde{\varepsilon}_0 + \tilde{\varepsilon}_0) e^{-\omega t} = 0.
\end{align*} \] (39)

We now show that \( d_{\pm n} = 0 \), for all \( n \geq 1 \). Since otherwise, if there exists \( n_0 \geq 1 \) such that \( d_{n_0} = 0 \), then \( d_{n_0} = 0 \) due to the fact \( \phi_n'(0) \neq 0 \) for all \( n \). Furthermore, the smoothness of the initial value guarantees that \( \sum_{n \in \mathbb{Z}, n \neq 0} \| d_{n} \phi_n'(0) \|_{\lambda_n} < \infty \), which implies that there exists an integer \( N > n_0 \) such that
\[ \sum_{n=N}^{\infty} \| d_{n} \phi_n'(0) \|_{\lambda_n} < \frac{1}{4} \| d_{n_0} \phi_{n_0}'(0) \|_{\lambda_{n_0}}, \]
\[ + \sum_{n=N+1}^{\infty} \| d_{n} \phi_n'(0) \|_{\lambda_n} < \frac{1}{4} \| d_{n_0} \phi_{n_0}'(0) \|_{\lambda_{n_0}}. \] (40)

Since \( \lambda_n \neq \lambda_m \) for any \( n, m \in \mathbb{Z}, n \neq m \) and \( |\lambda_{n+1} - \lambda_n| = \pi, n \in \mathbb{Z}, \) one has, for \( t > 0 \),
\[ d_{n_0} \frac{\phi_{n_0}'(0)}{\lambda_{n_0}} + \sum_{n=N+1}^{\infty} d_n \frac{\phi_n'(0)}{\lambda_n} e^{\lambda_n t} + \sum_{n=N+1}^{\infty} d_n \frac{\phi_n'(0)}{\lambda_n} e^{-\lambda_n t} \\
= - k_2 d_0 e^{-\lambda_{n_0} t} - \frac{1}{2} (\tilde{\varepsilon}_0 - \tilde{\varepsilon}_0) e^{\omega t} - \frac{1}{2} (\tilde{\varepsilon}_0 + \tilde{\varepsilon}_0) e^{-\omega t} = 0. \] (41)

Integrating over \([0, t]\) on both sides of (41) and using (40), and the fact \( \text{Re} \lambda_n = 0 \), we have
\[ \left| d_{n_0} \frac{\phi_{n_0}'(0)}{\lambda_{n_0}} \right| t \leq \left( \int_0^t \sum_{n=1}^{N} d_n \frac{\phi_n'(0)}{\lambda_n} e^{\lambda_n t} ds \right) + \left( \int_0^t \sum_{n=1}^{N} d_n \frac{\phi_n'(0)}{\lambda_n} e^{-\lambda_n t} ds \right) \]
\[ + \left( \int_0^t k_2 d_0 e^{-\lambda_{n_0} t} ds \right) + \left( \int_0^t (\tilde{\varepsilon}_0 - \tilde{\varepsilon}_0) e^{\omega t} ds \right) \]
\[ + \left( \int_0^t (\tilde{\varepsilon}_0 + \tilde{\varepsilon}_0) e^{-\omega t} ds \right). \]

Since the right side of the above equation has an upper bound for all \( t \geq 0 \), we conclude that \( d_{n_0} = 0 \), which is a contradiction. Hence \( d_{n_0} = 0, n = 1, 2, \ldots \) and by (39), \( d_0 = \tilde{\varepsilon}_0 = 0 \). We have thus proved that \( S = \{0, 0, 0, 0, 1, 0, 0\} \), that is
\[ \lim_{t \to \infty} \left[ \frac{1}{2} \int_0^t \left( \frac{1}{2} \right) + \frac{1}{2} \right] = 0. \]

The proof is complete. \( \Box \)

4. Convergence of the auxiliary system

Define \( \mathbb{H} = H^1(0, 1) \times L^2(0, 1) \), which is a Hilbert Space with the two following equivalent norms induced by the inner product:
\[ \| p, q \|_{\mathbb{H}_1}^2 = \int_0^1 \left[ |p'(x)|^2 dx + |q(x)|^2 dx + c_0 |p(1)|^2 \right], \quad \forall (p, q) \in \mathbb{H} \]
and
\[ \| p, q \|_{\mathbb{H}_2}^2 = \int_0^1 \left[ |p'(x)|^2 dx + |q(x)|^2 dx + c_0 |p(0)|^2 \right], \quad \forall (p, q) \in \mathbb{H}. \]
In the rest of the paper, we write norm $\| \cdot \|_H$ without discrimination.

**Theorem 4.1.** For any initial value $(\tilde{z}_0, \tilde{z}_1) \in H$, there exists a unique (weak) solution to (12) such that $(\tilde{z}, \tilde{z}_t) \in C(0, \infty; H)$. Moreover, the solution of (12) is asymptotically stable in the sense that

$$
\lim_{t \to \infty} E_2(t) = \lim_{t \to \infty} \left[ \frac{1}{2} \int_{H} \left( \tilde{z}_t^2(x, t) + \tilde{z}_t^2(x, t) \right) dx + \frac{1}{2} c_0(\tilde{z}(1, t))^2 \right] = 0.
$$

**Proof.** For any initial value $(\tilde{z}_0, \tilde{z}_1) \in H$, it follows from (15) that $(\tilde{z}_0, \tilde{z}_1) \in H$. By the transformation (13), it is sufficient to prove system (14) has a unique (weak) solution $(\tilde{z}, \tilde{z}_t) \in C(0, \infty; H)$ and asymptotically stabilization of system (14) in the sense that

$$
\lim_{t \to \infty} E_2(t) = \lim_{t \to \infty} \left[ \frac{1}{2} \int_{H} \left( \tilde{z}_t^2(x, t) + \tilde{z}_t^2(x, t) \right) dx + \frac{1}{2} c_0(\tilde{z}(0, t))^2 \right] = 0.
$$

(42)

Define an operator $A : D(A) \to H$ by

$$
\begin{align*}
A(u, v) &= (v, u')^T, \\
D(A) &= \{(u, v) \in H | A(u, v) \in H \}
\end{align*}
$$

(43)

Then system (14) can be written as

$$
\begin{align*}
\frac{d}{dt} (\tilde{z}(t), \tilde{z}_t(t)) &= A (\tilde{z}(t), \tilde{z}_t(t)) + f(t), \\
&= B [e^{-k_1(0, t)} - (k_2 + r) e(0, t)],
\end{align*}
$$

where $f(x, t) = -e(0, t) [\omega \sin \omega x - \mu \cos \omega x + 2c_0 \mu + c_0 k_2] + c_0 k_1 e(0, t)$. $B = \{0, -e(0, t)^T\}$.

It is well-known that $A$ generates an exponential stable $C_0$-semigroup. Then there exist $K, \mu > 0$ such that

$$
\|e^{At}\| \leq Ke^{-\mu t}.
$$

(44)

It is a routine exercise that $B$ are admissible for $A$. By Weiss (1989), it concludes that for any initial value $(\tilde{z}_0, \tilde{z}_1) \in H$, there exists a unique solution $(\tilde{z}, \tilde{z}_t)(t) \in C(0, \infty; H)$ to system (14) in $H$, which takes the form

$$
\begin{align*}
(\tilde{z}(t), \tilde{z}_t(t)) &= e^{At} (\tilde{z}_0(t), \tilde{z}_1(t)) \\
&+ \int_0^t e^{A(t-s)} B e(0, s) ds - k_1 \int_0^t e^{A(t-s)} B e(0, s) ds \\
&- (k_2 + r) \int_0^t e^{A(t-s)} B e(0, s) ds.
\end{align*}
$$

(45)

The first part is proved.

Now we are in a position to prove system (14) is asymptotically stable. By (44), the norm of the first term on the right side of (45) can be estimated as

$$
\begin{align*}
\|e^{At} (\tilde{z}_0(t), \tilde{z}_1(t))\|_H &\leq \|e^{At}\| \| (\tilde{z}_0(t), \tilde{z}_1(t))\|_H \\
&\leq Ke^{-\mu t} \| (\tilde{z}_0(t), \tilde{z}_1(t))\|_H.
\end{align*}
$$

(46)

We rewrite the second term on the right side of (45) to be

$$
\begin{align*}
&\int_0^t e^{A(t-s)} f(s) ds \\
&+ \int_0^t e^{A(t-s)} \left( \begin{array}{c} 0 \\ c_0 k_1 \end{array} \right) \tilde{e}(0, s) ds,
\end{align*}
$$

(47)

where $m(x) = \omega \sin \omega x - \mu \cos \omega x + 2c_0 \mu + c_0 k_2$. For the first term on the right side of (47), one has

$$
\begin{align*}
\int_0^t e^{A(t-s)} ((0, m(s)) \tilde{e}(0, s) ds &\leq \int_0^t \|e^{A(t-s)}\| \|m(s)\|_2 \|e(0, s)\| ds \\
&\leq K \|m(s)\|_2 \int_0^t e^{-\mu(t-s)} \|e(0, s)\| ds.
\end{align*}
$$

(48)

By the fact

$$
\tilde{e}(0, t) \leq \frac{2}{k_2} \tilde{E}(t) \to 0, \quad \text{as } t \to \infty.
$$

For any given $\eta > 0$, there exists $\tau_0 > 0$, such that $|\tilde{e}(0, t)| < \frac{\mu \eta}{2}$, for all $t > \tau_0$. Then

$$
\int_0^t e^{-\mu(t-s)} \|e(0, s)\| ds \leq \int_0^{\tau_0} e^{-\mu(t-s)} \|e(0, s)\| ds \\
+ \int_{\tau_0}^t e^{-\mu(t-s)} \|e(0, s)\| ds \\
\leq \frac{\sqrt{2}}{k_2} \tilde{E}(0) \mu^{-1} e^{\mu(\tau_0)} + \frac{\eta}{2}.
$$

Choosing $t > \tau_0$ large enough, the first term on the right-hand side above will be less than $\frac{\eta}{2}$, and thus for $t$ large enough

$$
\int_0^t e^{-\mu(t-s)} \|e(0, s)\| ds < \eta,
$$

which implies that

$$
\lim_{t \to \infty} \int_0^t e^{-\mu(t-s)} \|e(0, s)\| ds = 0.
$$

(49)

This together with (48) concludes that

$$
\lim_{t \to \infty} \left\| \int_0^t e^{A(t-s)} \left( \begin{array}{c} 0 \\ c_0 k_1 \end{array} \right) \tilde{e}(0, s) ds \right\|_H = 0.
$$

(49)

On one hand, for the second term on righthand of (47), we have

$$
\begin{align*}
&\int_0^t e^{A(t-s)} \left( \begin{array}{c} 0 \\ c_0 k_1 \end{array} \right) \tilde{e}(0, s) ds \\
&\leq c_0 k_1 \int_0^t e^{-\mu(t-s)} \|\tilde{e}(0, s)\| ds.
\end{align*}
$$

On the other hand,

$$
\begin{align*}
&\int_0^t e^{-\mu(t-s)} \|\tilde{e}(0, s)\| ds = \int_0^t e^{-\mu(t-s)} \|\tilde{e}(0, s)\| ds \\
&+ \int_{\tau_0}^t e^{-\mu(t-s)} \|\tilde{e}(0, s)\| ds \\
&\leq \sqrt{\frac{2}{\mu \eta}} \left[ \int_{\tau_0}^t \|\tilde{e}(0, s)\|^2 ds \right]^{1/2} \\
&+ \left[ \int_{\tau_0}^t \|\tilde{e}(0, s)\|^2 ds \right]^{1/2}.
\end{align*}
$$

(49)
This together with the fact \( \varepsilon_t(0, t) \in L^2(0, \infty) \) yields

\[
\left\| \int_0^t e^{A(t-s)} \left( \begin{array}{c} 0 \\ C_0 k_t \end{array} \right) \varepsilon_t(0, s) ds \right\|_H \\
\leq C_0 k_t \int_0^t e^{-\mu(t-s)} |\varepsilon_t(0, s)| ds \xrightarrow{t \to \infty} 0,
\]

as \( t \to \infty \). \( \tag{50} \)

The admissibility of \( B \) implies that

\[
\left\| \int_0^t e^{A(t-s)} B \varepsilon_t(0, s) ds \right\|_H^2 \leq C_0 \| \varepsilon(s) \|_H^2,
\]

\[
\leq \int_0^t C_0 \| \varepsilon(s) \|_H^2 ds,
\]

where \( C_0 \) is a positive constant that is independent of \( \varepsilon(0, s) \). From the fact that \( e^A \) is exponentially stable, it follows from Proposition 2.5 of Weiss (1989) that

\[
\left\| \int_0^t e^{A(t-s)} B \varepsilon_t(0, s) ds \right\|_H \leq K_0 \| \varepsilon(t) \|_H \xrightarrow{t \to \infty} \frac{\eta}{2},
\]

where \( K_0 \) is a constant that is independent of \( \varepsilon(0, s) \), and

\[
(u, v)(t) = \begin{cases} u(t), & 0 \leq t \leq \tau \\ v(t-\tau), & t > \tau. \end{cases}
\]

Thus one has

\[
\left\| \int_0^t e^{A(t-s)} B \varepsilon_t(0, s) ds \right\|_H \leq K_0 \| \varepsilon(t) \|_H \xrightarrow{t \to \infty} \frac{\eta}{2},
\]

\( \tag{51} \)

Choose \( t \) large enough to let the first term on the right-hand side above be less than \( \frac{\eta}{2} \), and thus for \( t \) large enough

\[
\left\| \int_0^t e^{A(t-s)} B \varepsilon_t(0, s) ds \right\|_H < \eta,
\]

which implies that

\[
\lim_{t \to \infty} \left\| \int_0^t e^{A(t-s)} B \varepsilon_t(0, s) ds \right\|_H = 0.
\]

\( \tag{53} \)

By the fact that \( \varepsilon_t(1, t) \in L^2(0, \infty) \), it follows that for any given \( \sigma > 0 \),

\[
\int_{t_1}^\infty |\varepsilon_t(1, s)|^2 ds \leq \sigma.
\]

\( \tag{54} \)

The admissibility of \( B \) implies that

\[
\left\| \int_0^t e^{A(t-s)} B \varepsilon_t(0, s) ds \right\|_H \leq K e^{-\omega(t-\tau)} \int_0^t e^{A(t-s)} B \varepsilon_t(0, s) ds + K \| \varepsilon_t(1, s) \|_{L^2(0, t)} \| \phi(t) \|_{L^2(0, t)}
\]

\[
\leq K e^{-\omega(t-\tau)} \sqrt{K_0} |\varepsilon_t(1, s)|_{L^2(0, t)} + K \| \varepsilon_t(1, s) \|_{L^2(0, t)} + K \sqrt{\sigma}.
\]

\( \tag{55} \)

Same to \( \text{(52)} \), we have

\[
\lim_{t \to \infty} \left\| \int_0^t e^{A(t-s)} B \varepsilon_t(0, s) ds \right\|_H = 0.
\]

\( \tag{56} \)

It follows from \( \text{(45), (46), (49), (50), (53) and (56)} \) that

\[
(42) \text{ is then proved. } \blacksquare
\]

5. Main result

We now go back to the closed-loop \( \text{(17)} \). Define \( \mathcal{X} = H \times \mathcal{Y} \). Let us consider system \( \text{(17)} \) in space \( \mathcal{X} \).

**Theorem 5.1.** Suppose that \( \omega \neq 0, n \pi + \frac{\pi}{2}, n \in \mathbb{Z} \). For any initial value \( (y_0, y_1, \tilde{y}_0, \tilde{y}_1, \tilde{a}_0, \tilde{b}_0) \in \mathcal{X} \), there exists a unique (weak) solution to \( \text{(17)} \) such that \( (y(t), y_1(t), \tilde{y}_1(t), \tilde{a}(t), \tilde{b}(t)) \in C([0, \infty); \mathcal{X}) \). Moreover, this closed-loop solution has the following properties: (i) \( \sup_{t \geq 0} \int_0^t |y(t)|^2 + |y_1(t)|^2 + |\tilde{y}_1(t)|^2 + |\tilde{a}(t)|^2 + |\tilde{b}(t)|^2 < \infty \), (ii) \( \lim_{t \to \infty} \int_0^t (|y(t)|^2 + |y_1(t)|^2 + |\tilde{y}_1(t)|^2 + |\tilde{a}(t)|^2 + |\tilde{b}(t)|^2) dt = 0 \), (iii) \( \lim_{t \to \infty} \tilde{a}(t) = a, \lim_{t \to \infty} \tilde{b}(t) = b \), (iv) \( \lim_{t \to \infty} y(t, 0) = 0 \), (v) When \( a = b = 0 \), \( \int_0^t (|y(t)|^2 + |y_1(t)|^2 + |\tilde{y}_1(t)|^2 + |\tilde{a}(t)|^2 + |\tilde{b}(t)|^2) dt \xrightarrow{t \to \infty} 0 \).

**Proof.** For any initial value \( (y_0, y_1, \tilde{y}_0, \tilde{y}_1, \tilde{a}_0, \tilde{b}_0) \in \mathcal{X} \), it is easy to verify from \( \text{(4) and (10)} \) that \( (\phi_0, \psi_1, \tilde{a}_0, \tilde{b}_0) \in \mathcal{Y} \) and \( (\tilde{z}, \tilde{y}) \in \mathcal{H} \), which implies that there exist a unique solution \( (\phi, \psi, \tilde{a}(t), \tilde{b}(t)) \in C([0, \infty); \mathcal{Y}) \) to \( \text{(3)} \) and a unique solution \( (\tilde{y}, \tilde{z}) \in C([0, \infty); \mathcal{H}) \) to \( \text{(9)} \), respectively. It follows from \( \text{(8)} \) and \( y = \tilde{y} + \varepsilon \) that system \( \text{(17)} \) has a unique solution (weak) \( (y(t), y_1(t), \tilde{y}(t), \tilde{y}_1(t), \tilde{a}(t), \tilde{b}(t)) \in C([0, \infty); \mathcal{X}) \). The first part is proved.

From \( \text{(8)} \), we have

\[
\left\| \begin{array}{c} \tilde{y}(t, 0) \\ \tilde{y}_1(t, 0) \\ \tilde{a}(t) \\ \tilde{b}(t) \end{array} \right\|_\mathcal{X} \leq \left\| \begin{array}{c} \tilde{z}(t, 0) \\ \tilde{z}_1(t, 0) \end{array} \right\|_\mathcal{H} + \| \cos \omega x \|_{L^2(0, t)} |\tilde{a}(t)| \cos o t + |\tilde{b}(t)| \sin o t + r \varepsilon(t) + c \omega x \|_{L^2(0, t)} |\tilde{a}(t)| \cos o t - |\tilde{b}(t)| \sin o t.
\]

This together with Theorems 3.2, 4.1 and the fact \( \tilde{y}(0, t) = \tilde{z}(0, t) \) gives property (i)-(iv), (v) directly follows from (ii) and the stability of \( \varepsilon \) and \( \tilde{y} \).

**Remark 5.1.** The technique in this paper can be used to track general sinusoidal and cosinusoid reference signals. For tracking the reference \( y_{ref}(t, a) = A \cos(\omega t + \phi) + B \sin(\beta t + \psi) + C \) with the output \( y(t, 0) \), we only need \( \tilde{z} = \tilde{y} - \frac{1}{\omega} \sin \omega x |\tilde{a}(t)| \cos o t + |\tilde{b}(t)| \sin o t + r \varepsilon(t) + c \omega x \|_{L^2(0, t)} |\tilde{a}(t)| \cos o t - |\tilde{b}(t)| \sin o t \).
6. Numerical simulation

In this section, we present some numerical simulations to illustrate the theory results. In the simulation, the second order equations in time are firstly converted into a system of two first order equations, and then the backward Euler method in time and the Chebyshev spectral method in space are used. The grid size is taken as \( N = 20 \) and time step \( dt = 10^{-3} \). We choose \( k_1 = 0.9, k_2 = 4, \) \( c_0 = 2, c_1 = 0.9 \) and \( r = 1 \). The other parameters are taken as \( a = 0.1, b = -0.2, \) and \( \omega = 3 \). We can choose the proper initial conditions which are not listed here for simplicity to give the following simulation results. The numerical results for \( \varepsilon(x, t) \) and \( z(x, t) \) are presented in Fig. 1. It is can be seen that system (3) and (14) are indeed asymptotically stable, which implies that the convergence of \( \varepsilon(0, t) \) and \( z(0, t) \). Fig. 2(left) shows approximation of the parameters. It is seen that the estimates \( \hat{a}(t) \) and \( \hat{b}(t) \) with initial values \( \hat{a}_0 = 0.2 \) and \( \hat{b}_0 = 0.1 \) approximate, respectively, the system parameters \( a = 0.1 \) and \( b = -0.2 \). Fig. 2(right) shows that the convergence of \( \hat{z}(0, t) \). The convergence of \( \hat{z}(0, t) \) and \( \hat{z}(0, t) \) together with the fact \( \varepsilon(0, t) = y(0, t) - \hat{y}(0, t), \hat{y} = \hat{z}(0, t) = \hat{z}(0, t) \) shows that \( y(0, t) \) is convergent to zero as time goes to infinity. 

7. Concluding remarks

This paper is concerned with output regulation and disturbance rejection for a wave equation with external harmonic disturbance anticollocated with control. An adaptive observer by the measured output is designed to estimate unknown parameters of the disturbance and recover the system state. An auxiliary system is constructed by using the observer and parameter estimators to make the control and the anticollocated disturbance to be collocated. By applying the Backstepping method for finite dimensional system, we design an adaptive output feedback controller which regulates the output to zero and keep all the states bounded. This design method used in this paper also can be applied to an unstable wave equation presented in Krstic et al. (2008a). In future works, applying our approach to beam equation seems interesting, and relaxing the harmonic to general bounded disturbance is also interesting problem.

References


