Output-Feedback Stochastic Nonlinear Stabilization

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***Abstract*—The authors present the first result on *global output-feedback* stabilization (in probability) for stochastic nonlinear continuous-time systems. The class of systems that they consider is a stochastic counterpart of the broadest class of deterministic systems for which globally stabilizing controllers are currently available. Their controllers are "inverse optimal" and possess an infinite gain margin. A reader of the paper needs no prior familiarity with techniques of stochastic control.**

***Index Terms*—Backstepping, control Lyapunov functions, inverse optimality, stochastic nonlinear output-feedback systems, stochastic stabilization.**

I. Introduction

Despite huge popularity of the linear-quadratic-Gaussian control problem, the stabilization problem for *nonlinear stochastic* systems has been receiving relatively little attention until recently. Efforts toward (global) *stabilization* of stochastic nonlinear systems have been initiated in the work of Florchinger [4]–[6] who, among other things, extended the concept of control Lyapunov functions and Sontag's stabilization formula [25] to the stochastic setting. A breakthrough toward arriving at *constructive* methods for stabilization of broader classes of stochastic nonlinear systems came with the result of Pan and Basar [22] who derived a backstepping design for strict-feedback systems motivated by a risk-sensitive cost criterion [1], [11], [20], [24] (for other types of optimal control problems, see, e.g., [9] and [10]). In [2] and [3], we designed simpler inverse optimal control laws for strict-feedback systems which guarantee global asymptotic stability in probability and whose algorithms can be directly coded in symbolic software.

In this paper, we address the *output-feedback* global stabilization problem for stochastic nonlinear systems. The output-feedback problem has received considerable attention in the recent robust and adaptive nonlinear control literature [12], [14], [16], [19], [23], [26]. The present paper is the first to address the output-feedback problem in the stochastic setting.

We present two results. First, in Section II, we design an output-feedback (observer-based) backstepping control law which guarantees global asymptotic stability in probability. Second, in Section III, based on a theorem derived in [3], we design stabilizing control laws which are also optimal with respect to meaningful cost functionals. The class of systems that we consider is the stochastic version of the *output-feedback form*, which is the broadest class for which *global* output-feedback controllers currently exist in the deterministic setting. Finally, in Section IV, we give a second-order simulation example.

A. Preliminaries on Stability in Probability

Consider the nonlinear stochastic system

$$d\chi = f(\chi)\,dt + g(\chi)\,dw \tag{1}$$

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where $\chi \in \mathbb{R}^n$ is the state, w is an r-dimensional independent standard Wiener process, and f: $\mathbb{R}^n \to \mathbb{R}^n$ and g: $\mathbb{R}^n \to \mathbb{R}^{n\times r}$ are locally Lipschitz and satisfy $f(0) = 0$, $g(0) = 0$.

Definition 1.1: The equilibrium $\chi = 0$ of (1) is said to be globally asymptotically stable in probability if for any $t_0 \geq 0$ and $\epsilon > 0$, $\lim_{\chi(t_0)\to 0} P\{\sup_{t\geq t_0} |\chi(t)| > \epsilon\} = 0$, and for any initial condition $\chi(t_0)$, $P\{\lim_{t\to\infty} \chi(t) = 0\} = 1$.

Theorem 1.1—Khas'minskii [15], Kushner [17], and Mao [18]: Consider system (1) and suppose there exists a positive definite, radially unbounded, twice continuously differentiable function $V(\chi)$ such that the infinitesimal generator

$$\mathcal{L}V(\chi) = \frac{\partial V}{\partial \chi} f + \frac{1}{2}\operatorname{Tr}\left\{g^T \frac{\partial^2 V}{\partial \chi^2} g\right\} \tag{2}$$

is negative definite. Then the equilibrium $\chi = 0$ of (1) is globally asymptotically stable in probability.

II. Output-Feedback Stabilization in Probability

In this section we deal with nonlinear *output-feedback* systems driven by white noise. This class of systems is given by the following nonlinear stochastic differential equations:

$$\begin{aligned} dx_i &= x_{i+1}\,dt + \varphi_i(y)^T\,dw, \qquad i = 1, \cdots, n-1\\ dx_n &= u\,dt + \varphi_n(y)^T\,dw\\ y &= x_1 \end{aligned} \tag{3}$$

where $\varphi_i(y)$ are r-vector-valued smooth functions with $\varphi_i(0) = 0$, and w is an independent r-dimensional standard Wiener process.

Since the states $x_2, \cdots, x_n$ are not measured, we first design an observer which would provide exponentially convergent estimates of the unmeasured states in the absence of noise. The observer is designed as

$$\dot{\hat{x}}_i = \hat{x}_{i+1} + k_i(y - \hat{x}_1), \qquad i = 1, \cdots, n \tag{4}$$

where $\hat{x}_{n+1} = u$. The observation errors $\tilde{x} = x - \hat{x}$ satisfy

$$\begin{aligned} d\tilde{x} &= \begin{bmatrix} -k_1 & & \\ \vdots & I & \\ -k_n & 0 \quad \cdots & 0 \end{bmatrix} \tilde{x}\,dt + \varphi(y)^T\,dw\\ &= A_0\tilde{x}\,dt + \varphi(y)^T\,dw \end{aligned} \tag{5}$$

where A_0 is designed to be asymptotically stable. Now, the entire system can be expressed as

$$\begin{aligned} d\tilde{x} &= A_0\tilde{x}\,dt + \varphi(y)^T\,dw\\ dy &= (\hat{x}_2 + \tilde{x}_2)\,dt + \varphi_1(y)^T\,dw\\ d\hat{x}_2 &= [\hat{x}_3 + k_2(y - \hat{x}_1)]\,dt\\ &\;\;\vdots\\ d\hat{x}_n &= [u + k_n(y - \hat{x}_1)]\,dt. \end{aligned} \tag{6}$$

Our output-feedback design will consist of applying a backstepping procedure to the system $(y, \hat{x}_2, \cdots, \hat{x}_n)$, which also takes care of the feedback connection through the $\tilde{x}$ system.

In the standard backstepping method for deterministic systems [7] (where dw/dt would be a bounded deterministic disturbance), a sequence of stabilizing functions $\alpha_i(\overline{\hat{x}}_i, y)$, where $\overline{\hat{x}}_i = [\hat{x}_2, \cdots, \hat{x}_i]^T$, is constructed recursively to build a Lyapunov function of the form

$$V = \sum_{i=1}^{n} \tfrac{1}{2} z_i^2 + \tilde{x}^T P \tilde{x} \tag{7}$$

where P is a positive definite matrix which satisfies $A_0^T P + PA_0 = -I$, and the error variables z_i are given by

$$z_1 = y \tag{8}$$

$$z_i = \hat{x}_i - \alpha_{i-1}(\overline{\hat{x}}_{i-1}, y), \qquad i = 2, \cdots, n. \tag{9}$$

The Lyapunov design for stochastic systems cannot be performed using the quadratic Lyapunov function (7) because of the term $\frac{1}{2}\mathrm{Tr}\{g^T(\partial^2 V/\partial \chi^2)g\}$ in (2). We instead employ *quartic* (fourth-order) Lyapunov functions

$$V = \sum_{i=1}^{n} \tfrac{1}{4} z_i^4 + (\tilde{x}^T P \tilde{x})^2. \tag{10}$$

Our presentation of the backstepping procedure here is very concise: instead of introducing the stabilizing functions α_i in a step-by-step fashion, we derive them simultaneously. A reader who is a novice to the technique of backstepping is referred to [16].

We start by an important preparatory comment. Since $\varphi_i(0) = 0$, the α_i's will vanish at $\overline{\hat{x}}_i = 0$, $y = 0$, as well as at $\overline{z}_i = 0$, where $\overline{z}_i = [z_1, \cdots, z_i]^T$. Thus, by the mean value theorem, $\alpha_i(\overline{\hat{x}}_i, y)$ and $\varphi(y)$ can be expressed, respectively, as

$$\alpha_i(\overline{\hat{x}}_i, y) = \sum_{l=1}^{i} z_l \alpha_{il}(\overline{\hat{x}}_i, y) \tag{11}$$

$$\varphi(y) = y\psi(y) \tag{12}$$

where $\alpha_{il}(\overline{\hat{x}}_i, y)$ and $\psi(y)$ are smooth functions.

Now, we are ready to start the backstepping design procedure. According to Itô's differentiation rule [21], we have

$$dz_1 = (\hat{x}_2 + \tilde{x}_2)\, dt + \varphi_1(y)^T\, dw \tag{13}$$

$$\begin{aligned} dz_i = \Bigg[& \hat{x}_{i+1} + k_i \tilde{x}_1 - \sum_{l=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_l} (\hat{x}_{l+1} + k_l \tilde{x}_1) \\ & - \frac{\partial \alpha_{i-1}}{\partial y} (\hat{x}_2 + \tilde{x}_2) - \frac{1}{2}\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\right) \varphi_1(y)^T \varphi_1(y) \Bigg] dt \\ & - \frac{\partial \alpha_{i-1}}{\partial y} \varphi_1(y)^T\, dw, \qquad i = 2, \cdots, n. \end{aligned} \tag{14}$$

As we announced previously, we employ a Lyapunov function of a quartic form

$$V(z, \tilde{x}) = \frac{1}{4} y^4 + \frac{1}{4} \sum_{i=2}^{n} z_i^4 + \frac{b}{2} (\tilde{x}^T P \tilde{x})^2 \tag{15}$$

where b is a positive constant. This form of the Lyapunov function clearly indicates that we view the system as a feedback connection in Fig. 1. The first two terms in (15) constitute a Lyapunov function for the $(y, \hat{x}_2, \cdots, \hat{x}_n)$-system, while the third term in (15) is a Lyapunov function for the $\tilde{x}$-system. Even though not obvious from the calculations that follow, we achieve a nonlinear small-gain global stabilization (in probability) in the style of [13].

Now we start the process of selecting the functions $\alpha_i(\overline{\hat{x}}_i, y)$ to make $\mathcal{L}V$ negative definite. Along the solutions of (5), (13), and (14), we have

$$\begin{aligned} \mathcal{L}V =\, & y^3(\hat{x}_2 + \tilde{x}_2) + \tfrac{3}{2} y^2 \varphi_1(y)^T \varphi_1(y) \\ & + \sum_{i=2}^{n} z_i^3 \Bigg[\hat{x}_{i+1} + k_i \tilde{x}_1 - \sum_{l=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_l} (\hat{x}_{l+1} + k_l \tilde{x}_1) \\ & \qquad - \frac{\partial \alpha_{i-1}}{\partial y} (\hat{x}_2 + \tilde{x}_2) - \frac{1}{2}\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\right) \varphi_1(y)^T \varphi_1(y) \Bigg] \\ & + \frac{3}{2} \sum_{i=2}^{n} z_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \varphi_1(y)^T \varphi_1(y) - b\tilde{x}^T P \tilde{x} |\tilde{x}|^2 \\ & + 2b\, \mathrm{Tr}\{\varphi(y)(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P \tilde{x} P)\varphi(y)^T\} \end{aligned}$$

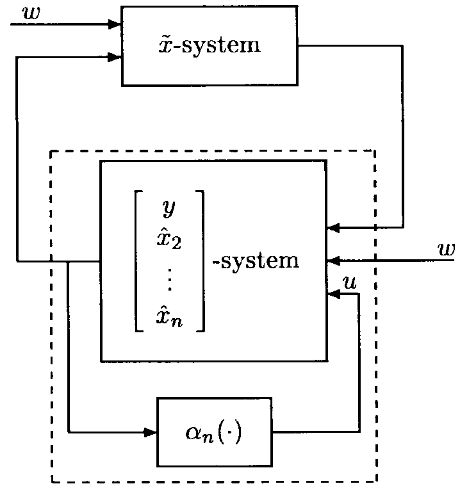

Fig. 1. Feedback structure of the system (6).

$$\begin{aligned} =\, & - b\tilde{x}^T P \tilde{x} |\tilde{x}|^2 + 2b\, \mathrm{Tr}\{\varphi(y)(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P \tilde{x} P)\varphi(y)^T\} \\ & + y^3(\alpha_1 + z_2 + \tilde{x}_2) + \tfrac{3}{2} y^2 \varphi_1(y)^T \varphi_1(y) \\ & + \sum_{i=2}^{n} z_i^3 \Bigg[\alpha_i + z_{i+1} + k_i \tilde{x}_1 - \sum_{l=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_l} (\hat{x}_{l+1} + k_l \tilde{x}_1) \\ & \qquad - \frac{\partial \alpha_{i-1}}{\partial y} (\hat{x}_2 + \tilde{x}_2) - \frac{1}{2}\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\right) \varphi_1(y)^T \varphi_1(y) \Bigg] \\ & + \frac{3}{2} \sum_{i=2}^{n} z_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \varphi_1(y)^T \varphi_1(y) \\ \le\, & - \left[b\lambda - 3bn\sqrt{n}\epsilon_2^2 |P|^4 - \frac{1}{4} \sum_{i=2}^{n} \frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} \right] |\tilde{x}|^4 \\ & + y^3 \Bigg[\alpha_1 + \tfrac{3}{2} \psi_1(y)^T \psi_1(y) y + \tfrac{3}{4} \delta_1^{4/3} y + \tfrac{3}{4} \epsilon_1^{4/3} y \\ & \qquad + \frac{3}{4} \sum_{i=2}^{n} \xi_i^2 (\psi_1(y)^T \psi_1(y))^2 y + \frac{3bn\sqrt{n}}{\epsilon_2^2} |\psi(y)|^4 y \Bigg] \\ & + \sum_{i=2}^{n-1} z_i^3 \Bigg[\alpha_i + k_i \tilde{x}_1 - \sum_{l=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_l} (\hat{x}_{l+1} + k_l \tilde{x}_1) \\ & \qquad - \frac{\partial \alpha_{i-1}}{\partial y} \hat{x}_2 - \frac{1}{2} \frac{\partial^2 \alpha_{i-1}}{\partial y^2} \varphi_1(y)^T \varphi_1(y) + \frac{3}{4} \delta_i^{4/3} z_i \\ & \qquad + \frac{1}{4\delta_{i-1}^4} z_i + \frac{3}{4} \eta_i^{4/3} \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^{4/3} z_i \\ & \qquad + \frac{3}{4\xi_i^2} \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^4 z_i \Bigg] \\ & + z_n^3 \Bigg[u + k_n \tilde{x}_1 - \sum_{l=2}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} (\hat{x}_{l+1} + k_l \tilde{x}_1) \\ & \qquad - \frac{\partial \alpha_{n-1}}{\partial y} \hat{x}_2 - \frac{1}{2} \frac{\partial^2 \alpha_{n-1}}{\partial y^2} \varphi_1(y)^T \varphi_1(y) + \frac{1}{4\delta_{n-1}^4} z_n \\ & \qquad + \frac{3}{4} \eta_n^{4/3} \left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} z_n + \frac{3}{4\xi_n^2} \left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^4 z_n \Bigg] \end{aligned} \tag{16}$$

where $\lambda > 0$ is the smallest eigenvalue of P. The second equality comes from substituting $\hat{x}_i = z_i + \alpha_{i-1}$, and the inequality comes from Young's inequalities in Appendix A. At this point, we can see that all the terms can be cancelled by u and α_i. If we choose ϵ_1,

ϵ_2, and η_i to satisfy

$$b\lambda - 3bn\sqrt{n}\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} = p > 0 \tag{17}$$

and α_i and u as

$$\begin{aligned}\alpha_1 = &-c_1 y - \tfrac{3}{2}\,\psi_1(y)^T\psi_1(y)y - \tfrac{3}{4}\,\delta_1^{4/3}y - \tfrac{3}{4}\,\epsilon_1^{4/3}y - \tfrac{3}{4}\\ &\cdot\sum_{i=2}^{n}\xi_i^2(\psi_1(y)^T\psi_1(y))^2 y - \frac{3bn\sqrt{n}}{\epsilon_2^2}\,|\psi(y)|^4 y\end{aligned} \tag{18}$$

$$\begin{aligned}\alpha_i = &-c_i z_i - k_i\tilde{x}_1 + \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}\,(\hat{x}_{l+1} + k_l\tilde{x}_1)\\ &+ \frac{\partial\alpha_{i-1}}{\partial y}\,\hat{x}_2 + \frac{1}{2}\,\frac{\partial^2\alpha_{i-1}}{\partial y^2}\,\varphi_1(y)^T\varphi_1(y) - \frac{3}{4}\,\delta_i^{4/3}z_i\\ &- \frac{1}{4\delta_{i-1}^4}\,z_i - \frac{3}{4}\,\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3}z_i - \frac{3}{4\xi_i^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i\end{aligned} \tag{19}$$

$$\begin{aligned}u = &-c_n z_n - k_n\tilde{x}_1 + \sum_{l=2}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\,(\hat{x}_{l+1} + k_l\tilde{x}_1)\\ &+ \frac{\partial\alpha_{n-1}}{\partial y}\hat{x}_2 + \frac{1}{2}\,\frac{\partial^2\alpha_{n-1}}{\partial y^2}\,\varphi_1(y)^T\varphi_1(y) - \frac{1}{4\delta_{n-1}^4}\,z_n\\ &- \frac{3}{4}\,\eta_n^{4/3}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3}z_n - \frac{3}{4\xi_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^4 z_n\end{aligned} \tag{20}$$

where $c_i > 0$, then the infinitesimal generator of the closed-loop system (5), (13), (14), and (20) is negative definite

$$\mathcal{L}V \leq -\sum_{i=1}^{n}c_i z_i^4 - p|\tilde{x}|^4. \tag{21}$$

With (21), we have the following stability result.

Theorem 2.1: The equilibrium at the origin of the closed-loop stochastic system (6), (20) is globally asymptotically stable in probability.

III. Inverse Optimal Output-Feedback Stabilization

This section first reviews some definitions and theorems established in [3], which are then used in the design of an inverse optimal stabilizing control law. Consider the system

$$d\chi = f(\chi)\,dt + g_1(\chi)\,dw + g_2(\chi)u\,dt \tag{22}$$

where $f(0) = 0$, $g_1(0) = 0$, and $u \in \mathbb{R}^m$.

Definition 3.1 [3]: The problem of *inverse optimal stabilization in probability* for system (22) is solvable if there exist a class $\mathcal{K}_\infty$ function[1] γ_2 whose derivative γ_2' is also a class $\mathcal{K}_\infty$ function, a matrix-valued function $R_2(\chi)$ such that $R_2(\chi) = R_2(\chi)^T > 0$ for all χ, a positive definite radially unbounded function $l(\chi)$, and a feedback control law $u = \alpha(\chi)$ continuous away from the origin with $\alpha(0) = 0$, which guarantees global asymptotic stability in probability of the equilibrium $\chi = 0$ and minimizes the cost functional

$$J(u) = E\left\{\int_0^\infty [l(\chi) + \gamma_2(|R_2(\chi)^{1/2}u|)]\,d\tau\right\}. \tag{23}$$

Theorem 3.1 [3]: Consider the control law

$$u = \alpha(\chi) = -R_2^{-1}(L_{g_2}V)^T\,\frac{\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|)}{|L_{g_2}VR_2^{-1/2}|^2} \tag{24}$$

where $V(\chi)$ is a Lyapunov function candidate, γ_2 is a class $\mathcal{K}_\infty$ function whose derivative is also a class $\mathcal{K}_\infty$ function, $R_2(\chi)$ is a matrix-valued function such that $R_2(\chi) = R_2(\chi)^T > 0$, and $\ell\gamma_2$ is the Legandre–Fenchel transform defined as

$$\ell\gamma_2 = \int(\gamma_2')^{-1}. \tag{25}$$

If the control law (24) achieves global asymptotic stability in probability for the system (22) with respect to $V(\chi)$, then the control law

$$\begin{aligned}u^* &= \alpha^*(\chi)\\ &= -\frac{\beta}{2}\,R_2^{-1}(L_{g_2}V)^T\,\frac{(\gamma_2')^{-1}(|L_{g_2}VR_2^{-1/2}|)}{|L_{g_2}VR_2^{-1/2}|}, \qquad \beta \geq 2\end{aligned} \tag{26}$$

solves the problem of inverse optimal stabilization in probability for the system (22) by minimizing the cost functional

$$J(u) = E\left\{\int_0^\infty \left[l(\chi) + \beta^2\gamma_2\left(\frac{2}{\beta}\,|R_2^{1/2}u|\right)\right]d\tau\right\} \tag{27}$$

where[2]

$$\begin{aligned}l(\chi) = 2\beta&\left[\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|) - L_fV - \frac{1}{2}\mathrm{Tr}\left\{g_1^T\,\frac{\partial^2 V}{\partial\chi^2}\,g_1\right\}\right]\\ &+ \beta(\beta-2)\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|).\end{aligned} \tag{28}$$

Now we return to the output-feedback system (3) and redesign the control law (20) to make it inverse optimal. The following result is instrumental.

Corollary 3.1 [3]: If there exists a continuous positive function $M(y,\,\hat{x})$ such that the control law

$$u = \alpha(y,\,\hat{x}) = -M(y,\,\hat{x})z_n \tag{29}$$

globally asymptotically stabilizes the system (6) in probability with respect to the Lyapunov function (15), then the control law

$$u^* = \alpha^*(y,\,\hat{x}) = \beta\alpha(y,\,\hat{x}), \qquad \beta \geq \tfrac{4}{3} \tag{30}$$

solves the problem of inverse optimal stabilization in probability.

From Corollary 3.1, we know that if we can design a stabilizing control law that has z_n as a factor, we can easily find another control law which solves the problem of inverse optimal stabilization in probability. If we consider carefully the last bracket of (16), every term except the second, the third, the fourth, and the fifth has z_n as a factor. With the help of Young's inequalities in Appendix B, we have

$$\begin{aligned}\mathcal{L}V \leq -&\left[b\lambda - 3bn\sqrt{n}\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4}\right.\\ &\left.- \frac{1}{4\epsilon_1^4} - \frac{1}{4\epsilon_4^4} - \frac{1}{4\epsilon_3^4}\,k_n^4\right]|\tilde{x}|^4\\ &+ y^3\left[\alpha_1 + \tfrac{3}{2}\,\psi_1(y)^T\psi_1(y)y + \tfrac{3}{4}\,\delta_1^{4/3}y + \tfrac{3}{4}\,\epsilon_1^{4/3}y\right.\\ &+ \frac{3}{4}\sum_{i=2}^{n}\xi_i^2(\psi_1(y)^T\psi_1(y))^2y + \frac{3bn\sqrt{n}}{\epsilon_2^2}|\psi(y)|^4y\\ &\left.+ \frac{1}{4\epsilon_7^4}\,y + \frac{1}{8\epsilon_5^4}\,y\right]\end{aligned}$$

[1] A function α: $\mathbb{R}_+ \to \mathbb{R}_+$ is said to be in class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, and $\lim_{r\to\infty}\alpha(r) = \infty$.

[2] The function $l(\chi)$ is positive definite because, by assumption of the theorem, the bracketed term is positive definite, $\ell\gamma_2(\cdot)$ is in class $\mathcal{K}_\infty$, and $\beta \geq 2$.

$$
\begin{aligned}
&+\sum_{i=2}^{n-1} z_i^3 \Bigg[\alpha_i + k_i\tilde{x}_1 - \sum_{l=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_l}\,(\hat{x}_{l+1} + k_l\tilde{x}_1)\\
&\qquad - \frac{\partial \alpha_{i-1}}{\partial y}\,\hat{x}_2 - \frac{1}{2}\,\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\varphi_1(y)^T\varphi_1(y) + \frac{3}{4}\,\delta_i^{4/3} z_i\\
&\qquad + \frac{1}{4\delta_{i-1}^4}\,z_i + \frac{3}{4}\,\eta_i^{4/3}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^{4/3} z_i\\
&\qquad + \frac{3}{4\xi_i^2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^4 z_i + \frac{1}{4\epsilon_6^4}\,z_i + \frac{1}{4\epsilon_7^4}\,z_i\Bigg]\\
&+ z_n^3 \Bigg[u + \frac{3}{4}\sum_{l=2}^{n-1}\left(\epsilon_6\,\frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\right)^{4/3} z_n\\
&\qquad + \frac{3}{4}\left(\epsilon_6\,\frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} z_n + \frac{1}{4\epsilon_6^4}\,z_n\\
&\qquad + \frac{3}{4}\sum_{k=1}^{n-1}\left(\epsilon_7\sum_{l=k}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\,\alpha_{lk}\right)^{4/3} z_n\\
&\qquad + \frac{3}{8}\left(\epsilon_5\,\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\,\psi_1(y)^T\psi_1(y)\right)^{4/3} z_n\\
&\qquad + \frac{3}{4}\left(\epsilon_4\sum_{l=2}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\,k_l\right)^{4/3} z_n + \frac{3}{4}\,\epsilon_3^{4/3} z_n + \frac{1}{4\delta_{n-1}^4}\,z_n\\
&\qquad + \frac{3}{4}\,\eta_n^{4/3}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} z_n + \frac{3}{4\xi_n^2}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^4 z_n\Bigg].
\end{aligned}
\tag{31}
$$

If $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7$, and η_i are chosen to satisfy

$$
b\lambda - 3bn\sqrt{n}\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4\epsilon_4^4} - \frac{1}{4\epsilon_3^4}\,k_n^4 = p > 0 \tag{32}
$$

$$
\frac{1}{4\epsilon_7^4} + \frac{1}{8\epsilon_5^4} = \frac{c_1}{2} \tag{33}
$$

$$
\frac{1}{4\epsilon_6^4} + \frac{1}{4\epsilon_7^4} = \frac{c_i}{2} \tag{34}
$$

where c_1 and c_i are those in (18) and (19), and

$$
u = -M(y, \hat{x})z_n \tag{35}
$$

$$
\begin{aligned}
M(y, \hat{x}) = {} & c_n + \frac{3}{4}\sum_{l=2}^{n-1}\left(\epsilon_6\,\frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\right)^{4/3} + \frac{3}{4}\left(\epsilon_6\,\frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3}\\
& + \frac{1}{4\epsilon_6^4} + \frac{3}{4}\sum_{k=1}^{n-1}\left(\epsilon_7\sum_{l=k}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\,\alpha_{lk}\right)^{4/3}\\
& + \frac{3}{8}\left(\epsilon_5\,\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\right)^{4/3} + \frac{3}{4}\left(\epsilon_4\sum_{l=2}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\,k_l\right)^{4/3}\\
& + \frac{3}{4}\,\epsilon_3^{4/3} + \frac{1}{4\delta_{n-1}^4} + \frac{3}{4}\,\eta_n^{4/3}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3}\\
& + \frac{3}{4\xi_n^2}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^4
\end{aligned}
\tag{36}
$$

with (18), (19), and (35), we get

$$
\mathcal{L}V \le -\tfrac{1}{2}\sum_{i=1}^{n} c_i z_i^4 - p|\tilde{x}|^4. \tag{37}
$$

Thus, according to Corollary 3.1, we achieve not only global asymptotic stability in probability, but also inverse optimality.

Theorem 3.2: The control law

$$
u^* = -\beta M(y, \hat{x})z_n, \qquad \beta \ge \tfrac{4}{3} \tag{38}
$$

guarantees that the equilibrium at the origin of the system (3), (5) is globally asymptotically stable in probability and also minimizes the cost functional

$$
J(u) = E\left\{\int_0^\infty \left[l(x, \tilde{x}) + \frac{27}{16\beta^2}\,M(y, \hat{x})^{-3}u^4\right]d\tau\right\} \tag{39}
$$

for some positive definite radially unbounded function $l(x, \tilde{x})$ parameterized by β.

Proof: Let $\gamma_2(r) = \frac{1}{4}r^4$, $R_2 = (\frac{4}{3}M)^{-(3/2)}$. Applying Theorem 3.1, the result follows readily. □

Remark 3.1: The inverse optimal control law has infinite upper gain margin and lower gain margin of 75% because $u = ku^*$ is globally asymptotically stabilizing for $k \in [\frac{3}{4}, \infty)$. The function $l(x, \tilde{x})$ is lower bounded by $2\sum_{i=1}^{n} c_i z_i^4 + 4p|\tilde{x}|^4$ (which means that it is a positive definite and radially unbounded function of x and $\tilde{x}$). □

IV. EXAMPLE

We give a second-order example to illustrate Theorem 2.1. Consider the system

$$
\begin{aligned}
dx_1 &= x_2\,dt + \tfrac{1}{2}x_1^2\,dw\\
dx_2 &= u\,dt\\
y &= x_1.
\end{aligned}
\tag{40}
$$

For this system, the estimator is

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + k_1(y - \hat{x}_1)\\
\dot{\hat{x}}_2 &= u + k_2(y - \hat{x}_1).
\end{aligned}
\tag{41}
$$

The virtual control α_1 and control u are

$$
\alpha_1 = -c_1 y - \frac{3}{8}\,y^3 - \frac{3}{4}\,\delta_1^{4/3}y - \frac{3}{4}\,\epsilon_1^{4/3}y - \frac{3}{64}\,\xi_2^2 y^5 - \frac{6\sqrt{2}b}{16\epsilon_2^2}\,y^5 \tag{42}
$$

$$
\begin{aligned}
u = {} & -c_2 z_2 - k_2\tilde{x}_1 + \frac{\partial \alpha_1}{\partial y}\,\hat{x}_2 + \frac{1}{8}\,\frac{\partial^2 \alpha_1}{\partial y^2}\,y^4 - \frac{1}{4\delta_1^4}\,z_2\\
& - \frac{3}{4}\,\eta_2^{4/3}\left(\frac{\partial \alpha_1}{\partial y}\right)^{4/3} z_2 - \frac{3}{4\xi_2^2}\left(\frac{\partial \alpha_1}{\partial y}\right)^4 z_2.
\end{aligned}
\tag{43}
$$

We choose $k_1 = 3$, $k_2 = 4.5$, $c_1 = 0.01$, $c_2 = 0.1$, $\delta_1 = 0.1$, $\xi_2 = 0.8$, $\epsilon_1 = 0.01$, $\eta_2 = 0.1$, $b = 0.1$, $\epsilon_2 = 50$, and set the initial condition at $x_1(0) = 1.3$, $x_2(0) = 0$, $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = \alpha_1(0)$, the states and control of the system are shown in Fig. 2. From Fig. 2, we can see that the output converges to zero. It is also interesting to note how the solutions become less noisy as they approach zero—a consequence of the fact that the noise vector field vanishes at zero.

APPENDIX A

In this and the following Appendix, we use Young's inequality [8, Th. 156]

$$
xy \le \frac{\epsilon^p}{p}\,|x|^p + \frac{1}{q\epsilon^q}\,|y|^q \tag{A.1}
$$

where $\epsilon > 0$, the constants $p > 1$ and $q > 1$ satisfy $(p-1)(q-1) = 1$, and $(x, y) \in \mathbb{R}^2$. Applying these inequalities leads to

$$
y^3 z_2 \le \frac{3}{4}\,\delta_1^{4/3}y^4 + \frac{1}{4\delta_1^4}\,z_2^4 \tag{A.2}
$$

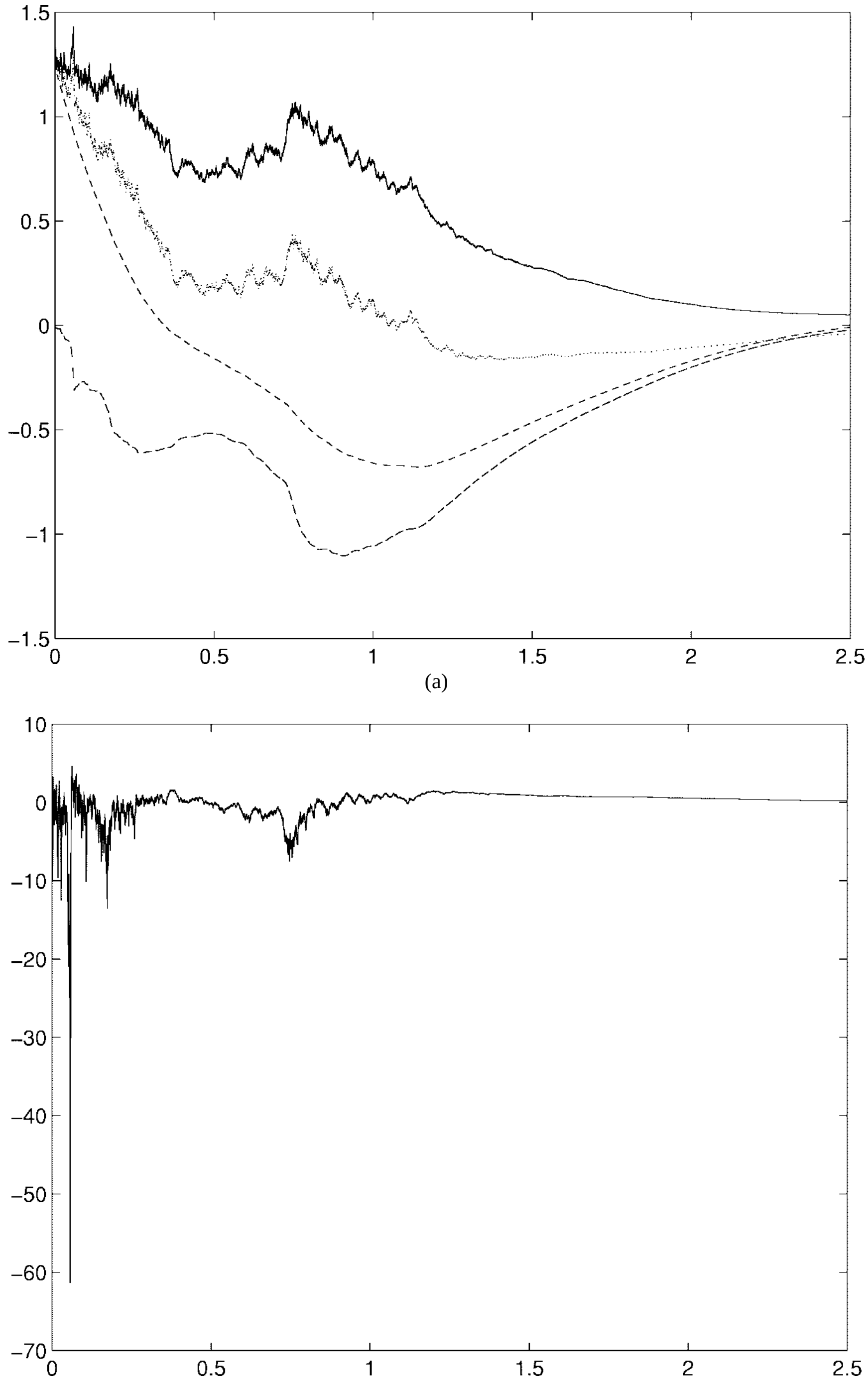

Fig. 2. The states and control effort of the output-feedback system.

$$y^3\tilde{x}_2 \le \frac{3}{4}\,\epsilon_1^{4/3}y^4 + \frac{1}{4\epsilon_1^4}\,\tilde{x}_2^4 \le \frac{3}{4}\,\epsilon_1^{4/3}y^4 + \frac{1}{4\epsilon_1^4}\,|\tilde{x}|^4 \tag{A.3}$$

$$\sum_{i=2}^{n} z_i^3 z_{i+1} \le \frac{3}{4}\sum_{i=2}^{n-1}\delta_i^{4/3} z_i^4 + \frac{1}{4}\sum_{i=3}^{n}\frac{1}{\delta_{i-1}^4}\,z_i^4 \tag{A.4}$$

$$\begin{aligned} -\sum_{i=2}^{n} z_i^3\,\frac{\partial\alpha_{i-1}}{\partial y}\tilde{x}_2 &\le \frac{3}{4}\sum_{i=2}^{n}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i^4 + \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4}\,\tilde{x}_2^4 \\ &\le \frac{3}{4}\sum_{i=2}^{n}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i^4 + \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4}\,|\tilde{x}|^4 \end{aligned} \tag{A.5}$$

$$\begin{aligned} &\frac{3}{2}\sum_{i=2}^{n} z_i^2\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 \varphi_1(y)^T\varphi_1(y) \\ &\quad\le \frac{3}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i^4 + \frac{3}{4}\sum_{i=2}^{n}\xi_i^2(\varphi_1(y)^T\varphi_1(y))^2 \end{aligned} \tag{A.6}$$

$$\begin{aligned} &2b\,\mathrm{Tr}\{\varphi(y)(2P\tilde{x}\tilde{x}^TP + \tilde{x}^TP\tilde{x}P)\varphi(y)^T\} \\ &\quad\le 2bn|\varphi(y)(2P\tilde{x}\tilde{x}^TP + \tilde{x}^TP\tilde{x}P)\varphi(y)^T|_\infty \\ &\quad\le 2bn\sqrt{n}|\varphi(y)(2P\tilde{x}\tilde{x}^TP + \tilde{x}^TP\tilde{x}P)\varphi(y)^T| \\ &\quad\le 6bn\sqrt{n}y^2|\psi(y)|^2|P|^2|\tilde{x}|^2 \qquad \text{(cf. (12))} \\ &\quad\le \frac{3bn\sqrt{n}}{\epsilon_2^2}\,y^4|\psi(y)|^4 + 3bn\sqrt{n}\epsilon_2^2|P|^4|\tilde{x}|^4 \end{aligned} \tag{A.7}$$

where the ϵ's, δ's, η's, and ξ's are positive constants to be chosen.

Appendix B

Similar to Appendix A, in the following inequalities, ϵ's are constants to be chosen:

$$z_n^3 k_n \tilde{x}_1 \le \frac{3}{4}\epsilon_3^{4/3} z_n^4 + \frac{1}{4\epsilon_3^4} k_n^4 \tilde{x}_1^4 \le \frac{3}{4}\epsilon_3^{4/3} z_n^4 + \frac{1}{4\epsilon_3^4} k_n^4 |\tilde{x}|^4 \quad \text{(B.1)}$$

$$\begin{aligned} -z_n^3 \sum_{l=2}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} k_l \tilde{x}_1 &\le \frac{3}{4}\left(\epsilon_4 \sum_{l=2}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} k_l\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_4^4}\tilde{x}_1^4 \\ &\le \frac{3}{4}\left(\epsilon_4 \sum_{l=2}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} k_l\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_4^4}|\tilde{x}|^4 \qquad \text{(B.2)} \end{aligned}$$

$$\begin{aligned} -\frac{1}{2} z_n^3 \frac{\partial^2 \alpha_{n-1}}{\partial y^2} \varphi_1(y)^T \varphi_1(y) &= -\frac{1}{2} z_n^3 \frac{\partial^2 \alpha_{n-1}}{\partial y^2} \psi_1(y)^T \psi_1(y) y^2 \\ &\le \frac{3}{8}\left(\epsilon_5 \frac{\partial^2 \alpha_{n-1}}{\partial y^2} \psi_1(y)^T \psi_1(y)\right)^{4/3} z_n^4 + \frac{1}{8\epsilon_5^4} y^4 \qquad \text{(B.3)} \end{aligned}$$

$$\begin{aligned} -z_n^3 \frac{\partial \alpha_{n-1}}{\partial y}\hat{x}_2 - z_n^3 \sum_{l=2}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\hat{x}_{l+1} &= -z_n^3 \sum_{l=2}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} z_{l+1} - z_n^3 \frac{\partial \alpha_{n-1}}{\partial y} z_2 \\ &\quad - z_n^3 \sum_{l=1}^{n-1}\sum_{k=1}^{l} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} z_k \alpha_{lk} \qquad \text{(cf. (9), (11))} \\ &= -\sum_{l=2}^{n-1} z_n^3 \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} z_{l+1} - z_n^3 \frac{\partial \alpha_{n-1}}{\partial y} z_2 \\ &\quad - \sum_{k=1}^{n-1}\sum_{l=k}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} \alpha_{lk} z_n^3 z_k \\ &\le \sum_{l=2}^{n-1}\left[\frac{3}{4}\left(\epsilon_6 \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_6^4} z_{l+1}^4\right] \\ &\quad + \frac{3}{4}\left(\epsilon_6 \frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_6^4} z_2^4 + \sum_{k=1}^{n-1} \\ &\quad \cdot \left[\frac{3}{4}\left(\epsilon_7 \sum_{l=k}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\alpha_{lk}\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_7^4} z_k^4\right] \\ &= z_n^4\left[\frac{3}{4}\sum_{l=2}^{n-1}\left(\epsilon_6 \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\right)^{4/3} + \frac{3}{4}\left(\epsilon_6 \frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} \right. \\ &\quad \left. + \frac{1}{4\epsilon_6^4} + \frac{3}{4}\sum_{k=1}^{n-1}\left(\epsilon_7 \sum_{l=k}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\alpha_{lk}\right)^{4/3}\right] \\ &\quad + \sum_{i=2}^{n-1} \frac{1}{4\epsilon_6^4} z_i^4 + \sum_{i=2}^{n-1} \frac{1}{4\epsilon_7^4} z_i^4 + \frac{1}{4\epsilon_7^4} y^4. \qquad \text{(B.4)} \end{aligned}$$

References

[1] T. Başar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach,* 2nd ed. Boston, MA: Birkhäuser, 1995.

[2] H. Deng and M. Krstić, "Stochastic nonlinear stabilization—Part I: A backstepping design," *Syst. Contr. Lett.,* vol. 32, pp. 143–150, 1997.

[3] ——, "Stochastic nonlinear stabilization—Part II: Inverse optimality," *Syst. Contr. Lett.,* vol. 32, pp. 151–159, 1997.

[4] P. Florchinger, "Lyapunov-like techniques for stochastic stability," *SIAM J. Contr. Optim.,* vol. 33, pp. 1151–1169, 1995.

[5] ——, "Global stabilization of cascade stochastic systems," in *Proc. 34th Conf. Decision and Control,* New Orleans, LA, 1995, pp. 2185–2186.

[6] ——, "A universal formula for the stabilization of control stochastic differential equations," *Stochastic Analysis and Appl.,* vol. 11, pp. 155–162, 1993.

[7] R. A. Freeman and P. V. Kokotović, *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques.* Boston, MA: Birkhäuser, 1996.

[8] G. Hardy, J. E. Littlewood, and G. Polya, *Inequalities,* 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 1989.

[9] U. G. Haussmann and W. Suo, "Singular optimal stochastic controls—I: Existence," *SIAM J. Contr. Optim.,* vol. 33, pp. 916–936, 1995.

[10] ——, "Singular optimal stochastic controls—II: Dynamic programming," *SIAM J. Contr. Optim.,* vol. 33, pp. 937–959, 1995.

[11] M. R. James, J. Baras, and R. J. Elliott, "Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems," *IEEE Trans. Automat. Contr.,* vol. 39, pp. 780–792, 1994.

[12] M. Jankovic, "Adaptive nonlinear output feedback tracking with a partial high-gain observer and backstepping," *IEEE Trans. Automat. Contr.,* vol. 42, pp. 106–113, Jan. 1997.

[13] Z. P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Math. Contr., Signals, Syst.,* vol. 7, pp. 95–120, 1995.

[14] H. K. Khalil, "Adaptive output feedback control of nonlinear systems represented by input–output models," *IEEE Trans. Automat. Contr.,* vol. 41, pp. 177–188, Feb. 1996.

[15] R. Z. Khas'minskii, *Stochastic Stability of Differential Equations.* Rockville, MD: S & N, 1980.

[16] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design.* New York: Wiley, 1995.

[17] H. J. Kushner, *Stochastic Stability and Control.* New York: Academic, 1967.

[18] X. Mao, *Stability of Stochastic Differential Equations with Respect to Semimartingales.* Longman, 1991.

[19] R. Marino and P. Tomei, *Nonlinear Control Design: Geometric, Adaptive, and Robust.* Englewood Cliffs, NJ: Prentice-Hall, 1995.

[20] H. Nagai, "Bellman equations of risk-sensitive control," *SIAM J. Contr. Optim.,* vol. 34, pp. 74–101, 1996.

[21] B. Øksendal, *Stochastic Differential Equations—An Introduction with Applications.* New York: Springer-Verlag, 1995.

[22] Z. Pan and T. Başar, "Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criterion," *SIAM J. Contr. Optim.,* to be published.

[23] L. Praly and Z. P. Jiang, "Stabilization by output feedback for systems with ISS inverse dynamics," *Syst. Contr. Lett.,* vol. 21, pp. 19–33, July 1993.

[24] T. Runolfsson, "The equivalence between infinite horizon control of stochastic systems with exponential-of-integral performance index and stochastic differential games," *IEEE Trans. Automat. Contr.,* vol. 39, pp. 1551–1563, 1994.

[25] E. D. Sontag, "A 'universal' construction of Artstein's theorem on nonlinear stabilization," *Syst. Contr. Lett.,* vol. 13, pp. 117–123, 1989.

[26] A. Teel and L. Praly, "Tools for semiglobal stabilization by partial state and output feedback," *SIAM J. Contr. Optim.,* vol. 33, pp. 1443–1488, Sept. 1995.