INTERNATIONAL JOURNAL OF ADAPTIVE CONTROL AND SIGNAL PROCESSING
Int. J. Adapt. Control Signal Process. 2016; **30**:1080–1098
Published online 10 February 2015 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/acs.2542

Linear backstepping output feedback control for uncertain linear systems

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SUMMARY

This paper presents a variation on adaptive backstepping output feedback control design for uncertain minimum-phase linear systems. Unlike the traditional nonlinear design, the proposed control method is linear and Lyapunov-based without utilizing overparametrization, tuning functions, or nonlinear damping terms to address parameter estimation error. Local stability of the closed-loop system and trajectory tracking are guaranteed. If the system dimension equals to the relative degree, the global stabilization and asymptotic convergence are achieved. Copyright © 2015 John Wiley & Sons, Ltd.

Received 10 October 2014; Revised 23 December 2014; Accepted 7 January 2015

KEY WORDS: adaptive; backstepping; output feedback; uncertain

1. INTRODUCTION

For over 40 years, adaptive control research for uncertain linear systems has employed the core idea of certainty equivalence [1–5]. In the early 1990s, [6–9] have introduced the backstepping concept to adaptive control of linear systems, which outperforms certainty equivalence in transient performance.

Two sorts of backstepping output feedback control approaches for linear systems with unknown parameters are given in Chapter 10 of [9]. One is the tuning functions design, and the other is the modular design (y-passive and x-swapping). For the tuning functions scheme, a sole Lyapunov function including all states of the closed-loop system is utilized so that the stability analysis is clearly understandable. Its major weakness may lie in the complicated nonlinear adaptive control law, which incorporates tuning functions or nonlinear damping terms. The partial differential calculation of stabilizing functions with respect to parameter estimates becomes very complex, especially when the plant's relative degree is greater than or equal to three. To simplify the nonlinear control design in tuning functions method, [10] develops the dynamic surface control to avoid those sophisticated differentiation. But it needs to bring in some additional filters. As to the modular scheme, its controller is linear and much easier to design; however, its input-to-output stability and performance analysis involve time domain and Laplace domain and are neither simple nor straightforward. Motivated by above statements, a control scheme possessing both advantages of tuning functions and modular methodologies—a readily computable design and an insightful analysis—is desired.

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In this paper, a linear backstepping output feedback control for uncertain linear systems with minimum phase is developed. There is no nonlinear element like tuning functions nor nonlinear damping used in the design to deal with parameter estimation error. The analysis is Lyapunov-based and different from those for y-passive or x-swapping modular scheme. If some initial conditions of the error system are satisfied, we depend upon the adaptation gain of the update law to locally stabilize the closed-loop system and achieve trajectory tracking. If the linear systems have the property where its relative degree is identical with its system dimension, by applying a normalization to the update law, the local result is broaden to the global one. This is new to some extent in comparison with traditional adaptive backstepping control. We have to point out that the global result does not hold when the relative degree is less than the system dimension. This is because the unknown zero dynamics can not be immediately used in the normalized update law.

The rest of the paper is assembled as follows. We present the plant model and formulate the control problem in Section 2. The state estimator, the linear backstepping controller, and the parameter identifier designed in Sections 3–5, respectively. The stability analysis is provided in Section 6. A special case to achieve global stabilization is stated in Section 7 followed by the conclusion of the paper in Section 8.

2. SYSTEM DESCRIPTION

We consider a general class of single-input-single-output linear systems

$$y(s) = \frac{B(s)}{A(s)}u(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}u(s) \tag{1}$$

which can be represented as the following observer canonical form:

$$\begin{aligned} \dot{x} &= Ax - ay + \begin{bmatrix} 0_{(\rho-1)\times 1} \\ b \end{bmatrix} u \\ y &= x_1 \end{aligned} \tag{2}$$

where

$$A = \begin{bmatrix} 0 & & \\ \vdots & I_{n-1} & \\ 0 & \cdots & 0 \end{bmatrix}, \quad a = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix}, \quad b = \begin{bmatrix} b_m \\ \vdots \\ b_0 \end{bmatrix} \tag{3}$$

and $x = [x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n$ is the unmeasured state vector, $y \in \mathbb{R}$ is the measurable output, and $u \in \mathbb{R}$ is the input to design. $a_{n-1}, \cdots, a_0$ and $b_m, \cdots, b_0$ are unknown constant plant parameters and control coefficients, respectively.

To describe conveniently, the plant (2) can be rewritten compactly as

$$\begin{aligned} \dot{x} &= Ax + F(y,u)^T\theta \\ y &= e_1^T x \end{aligned} \tag{4}$$

where $p = m + 1 + n$-dimensional parameter vector θ is defined by

$$\theta = \begin{bmatrix} b \\ a \end{bmatrix} \tag{5}$$

DOI: 10.1002/acs

and

$$F(y,u)^T = \left[\begin{bmatrix} 0_{(\rho-1)\times(m+1)} \\ I_{m+1} \end{bmatrix} u, -I_n y \right] \tag{6}$$

e_i for $i = 1, 2, \cdots$ is the i-th coordinate vector in corresponding space.

Our control objective is to make output y asymptotically track a reference signal $y_r(t)$. Several traditional assumptions are given as follows.

Assumption 1
The reference signal $y_r(t)$ and its first ρ time derivatives ${y_r}^{(i)}(t)(i = 1, 2, \cdots, \rho)$ are known, bounded, and piecewise continuous.

Assumption 2
The plant is minimum phase, that is, the polynomial $B(s) = b_m s^m + \cdots + b_1 s + b_0$ is Hurwitz.

Assumption 3
The sign of the high-frequency gain b_m, *i.e.* $\mathrm{sgn}(b_m)$, is known. There exist two known constants $\underline{b}_m$ and $\bar{b}_m$ such that $0 < \underline{b}_m \leqslant |b_m| \leqslant \bar{b}_m$. In addition, there exists a convex compact set $\Theta \subset \mathbb{R}^{p-1}$ such that $\exists\bar{\theta}, \theta_0, |\theta^* - \theta_0| \leqslant \bar{\theta}$ for all $\theta^* \in \Theta$, where $\theta_0 \in \mathbb{R}^{p-1}$ is a known constant vector, $\bar{\theta} > 0$ is a known constant, and $\theta^* = [\theta_2, \theta_3, \cdots, \theta_n]^T \in \mathbb{R}^{p-1}$.

3. STATE ESTIMATION

In this section, the Kreisselmeier filters (K-filters) are brought in as follows to measure the states of system (2) .

$$\dot{\eta} = A_0\eta + e_n y \tag{7}$$

$$\dot{\lambda} = A_0\lambda + e_n u \tag{8}$$

$$\xi = -A_0^n \eta \tag{9}$$

$$\Xi = -[A_0^{n-1}\eta, \cdots, A_0\eta, \eta] \tag{10}$$

$$\upsilon_j = A_0^j \lambda, \quad j = 0, 1, \cdots, m \tag{11}$$

$$\Omega^T = [\upsilon_m, \cdots, \upsilon_1, \upsilon_0, \Xi] \tag{12}$$

where the vector $k = [k_1, k_2, \cdots, k_n]^T$ is chosen to let the matrix $A_0 = A - ke_1^T$ be Hurwitz, that is, $PA_0 + A_0^T P = -I$, $P = P^T > 0$. The unmeasurable state x is virtually estimated as $\hat{x} = \xi + \Omega^T\theta$, and obviously, the estimation error $\varepsilon = x - \hat{x}$ vanishes exponentially because

$$\dot{\varepsilon} = A_0\varepsilon \tag{13}$$

Then, we have

$$\begin{aligned} x &= \xi + \Omega^T\theta + \varepsilon && (14)\\ &= -A(A_0)\eta + B(A_0)\lambda + \varepsilon && (15) \end{aligned}$$

where $A(A_0) = A_0^n + \sum_{i=0}^{n-1} a_i A_0^i$, $B(A_0) = \sum_{i=0}^{m} b_i A_0^i$.

4. LINEAR BACKSTEPPING CONTROL

In this section, the adaptive backstepping output feedback control is employed to achieve our control objective. Focus on system

$$\dot{y} = \xi_2 + \omega^T\theta + \varepsilon_2 \tag{16}$$
$$= b_m\upsilon_{m,2} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 \tag{17}$$

$$\dot{\upsilon}_{m,i} = \upsilon_{m,i+1} - k_i\upsilon_{m,1}, \quad i = 2, 3, \cdots, \rho - 1 \tag{18}$$

$$\dot{\upsilon}_{m,\rho} = u + \upsilon_{m,\rho+1} - k_\rho\upsilon_{m,1} \tag{19}$$

where

$$\omega = [\upsilon_{m,2}, \upsilon_{m-1,2}, \cdots, \upsilon_{0,2}, \Xi_2 - ye_1^T]^T \tag{20}$$

$$\bar{\omega} = [0, \upsilon_{m-1,2}, \cdots, \upsilon_{0,2}, \Xi_2 - ye_1^T]^T \tag{21}$$

The backstepping recursive control scheme is presented in the succeeding text.

Coordinate transformation:

$$z_1 = y - y_r \tag{22}$$

$$z_i = \upsilon_{m,i} - \alpha_{i-1}, \quad i = 2, 3, \cdots, \rho \tag{23}$$

Stabilizing functions:

$$\alpha_1 = \frac{1}{\hat{b}_m}\left(-(c_1 + d_1)z_1 - \xi_2 - \bar{\omega}^T\hat{\theta} + \dot{y}_r\right) \tag{24}$$

$$\alpha_2 = -\hat{b}_m z_1 - \left(c_2 + d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2\right)z_2 + \beta_2 \tag{25}$$

$$\begin{aligned}\alpha_i &= -z_{i-1} - \left(c_i + d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\right)z_i + \beta_i \\ i &= 3, 4, \cdots, \rho\end{aligned} \tag{26}$$

$$\begin{aligned}\beta_i &= k_i\upsilon_{m,1} + \frac{\partial\alpha_{i-1}}{\partial y}\left(\xi_2 + \omega^T\hat{\theta}\right) + \frac{\partial\alpha_{i-1}}{\partial\eta}(A_0\eta + e_n y) \\ &\quad + \sum_{j=1}^{m+i-1}\frac{\partial\alpha_{i-1}}{\partial\lambda_j}(-k_j\lambda_1 + \lambda_{j+1}) + \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(j)}}y_r^{(j+1)} \\ i &= 2, 3, \cdots, \rho\end{aligned} \tag{27}$$

where $\hat{b}_m$ and $\hat{\theta}$ are estimates of b_m and θ, respectively, $\tilde{\theta} = \theta - \hat{\theta}$, and $c_i > 0$, $d_i > 0$ for $i = 1, 2, \cdots, \rho$ are design parameters.

Adaptive Control Law:

$$u = -\upsilon_{m,\rho+1} + \alpha_\rho \tag{28}$$

DOI: 10.1002/acs

Here, please note that there is no tuning function nor nonlinear damping term used in the previous control device to deal with parameter estimation error. It is clear α_i for $i = 1, 2, \cdots, \rho$ are linear in y, η, λ and $\bar{y}_r^{(\rho)}$ $\left(\text{where } \bar{y}_r^{(i)} = (y_r, \dot{y}_r, \cdots, y_r^{(i)})^T \text{ for } i = 1, 2, \cdots, \rho\right)$, and nonlinear only in $\hat{\theta}$. Thus, if we denote $\hat{A}(A_0) = A_0^n + \sum_{i=0}^{n-1} \hat{a}_i A_0^i$, $\hat{B}(A_0) = \sum_{i=0}^{m} \hat{b}_i A_0^i$ where $\hat{a}_i$ for $i = 0, 1, \cdots, n-1$ and $\hat{b}_i$ for $i = 0, 1, \cdots, m$ are estimates of a_i and b_i, respectively, through a recursive but straightforward calculation, we could show the following equalities.

$$z_2 = K_{2,y}(\hat{\theta})y + K_{2,\eta}(\hat{\theta})\eta + K_{2,\lambda}(\hat{\theta})\lambda + K_{2,y_r}(\hat{\theta})\bar{y}_r^{(1)} \tag{29}$$

$$z_3 = K_{3,y}(\hat{\theta})y + K_{3,\eta}(\hat{\theta})\eta + K_{3,\lambda}(\hat{\theta})\lambda + K_{3,y_r}(\hat{\theta})\bar{y}_r^{(2)} \tag{30}$$

$$\begin{aligned} z_{i+1} &= K_{i+1,y}(\hat{\theta})y + K_{i+1,\eta}(\hat{\theta})\eta + K_{i+1,\lambda}(\hat{\theta})\lambda + K_{i+1,y_r}(\hat{\theta})\bar{y}_r^{(i)} \\ i &= 3, 4, \cdots, \rho - 1 \end{aligned} \tag{31}$$

$$u = K_y(\hat{\theta})y + K_\eta(\hat{\theta})\eta + K_\lambda(\hat{\theta})\lambda + K_{y_r}(\hat{\theta})\bar{y}_r^{(\rho)} \tag{32}$$

where the explicit expressions of $K_{i,y}(\hat{\theta})$, $K_{i,\eta}(\hat{\theta})$, $K_{i,\lambda}(\hat{\theta})$, $K_{i,y_r}(\hat{\theta})\bar{y}_r^{(i-1)}$ for $i = 2, 3, \cdots, \rho - 1$ and $K_y(\hat{\theta})$, $K_\eta(\hat{\theta})$, $K_\lambda(\hat{\theta})$, $K_{y_r}(\hat{\theta})\bar{y}_r^{(\rho)}$ are given in Tables A.I and A.II of Appendix.

The previous design procedure results in the following closed-loop error system:

$$\dot{z} = A_z(\hat{\theta})z + W_\varepsilon(\hat{\theta})(\varepsilon_2 + \omega^T\tilde{\theta}) + Q(z,t)^T\dot{\hat{\theta}} \tag{33}$$

where $A_z(\hat{\theta})$, $W_\varepsilon(\hat{\theta})$, and $Q(z,t)^T$ are given in Table A.II of Appendix.

5. PARAMETER IDENTIFICATION

In this part, a Lyapunov-based identifier to estimate the unknown parameter vector θ is designed. The update law of $\hat{\theta}$ is chosen as follows:

$$\dot{\hat{\theta}} = \gamma_\theta \mathrm{Proj}_\Pi\{\tau_\theta\}, \quad \gamma_\theta > 0 \tag{34}$$

$$\tau_\theta = \omega W_\varepsilon(\hat{\theta})^T z \tag{35}$$

where $\mathrm{Proj}_\Pi\{\cdot\}$ is a smooth projection operator employed to guarantee that $\hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_p]^T \in \Pi$ with the set Π being defined as

$$\Pi = \left\{ \hat{\theta} \,\middle|\, \begin{array}{l} |\hat{b}_m \mathrm{sgn}(b_m) - \sigma_0| < \sigma_1, \quad \hat{b}_m = \hat{\theta}_1 \\ |\hat{\theta}^* - \theta_0| < \bar{\theta}, \quad \hat{\theta}^* = [\hat{\theta}_2, \hat{\theta}_3, ..., \hat{\theta}_p]^T \end{array} \right\} \tag{36}$$

and $\sigma_0 = (\underline{b}_m + \bar{b}_m)/2$, $\sigma_1 = \sigma_0 - \underline{b}_m$. Based on [9] and [11], we now derive a similar $\mathrm{Proj}_\Pi\{\cdot\}$ to [12–14]. Choose a smooth convex function $\mathcal{P}(\hat{\theta}) : \mathbb{R}^p \to \mathbb{R}$ as

$$\mathcal{P}(\hat{\theta}) = \left| \frac{\hat{b}_m \mathrm{sgn}(b_m) - \sigma_0}{\sigma_1} \right|^{\varsigma_2} + \left| \frac{\hat{\theta}^* - \theta_0}{\bar{\theta}} \right|^{\varsigma_2} - 1 + \varsigma_1 \tag{37}$$

DOI: 10.1002/acs

where $0 < \varsigma_1 < 1$ and $\varsigma_2 \geqslant 2$ are two real numbers. Given the function $\mathcal{P}(\hat{\theta})$, we now obtain a smooth projection operator in (34) as

$$\mathrm{Proj}_{\Pi}\{\tau_\theta\} = \begin{cases} \tau_\theta, & \mathcal{P}(\hat{\theta}) \leqslant 0 \quad \text{or} \quad \nabla_{\hat{\theta}}\mathcal{P}^T \tau_\theta \leqslant 0 \\ \left(I - \frac{\nabla_{\hat{\theta}}\mathcal{P}\nabla_{\hat{\theta}}\mathcal{P}^T}{\nabla_{\hat{\theta}}\mathcal{P}^T\nabla_{\hat{\theta}}\mathcal{P}}\right)\tau_\theta, & \text{if not} \end{cases} \tag{38}$$

Next, we bring in the dynamic equation for the reference signal η^r based on (7)

$$\dot{\eta}^r = A_0\eta^r + e_n y_r \tag{39}$$

so that the error state $\tilde{\eta} = \eta - \eta^r$ is governed by

$$\dot{\tilde{\eta}} = A_0\tilde{\eta} + e_n z_1 \tag{40}$$

We also introduce the m-dimensional zero dynamics of (2)

$$\zeta = Tx \tag{41}$$

and its reference signal ζ^r, where

$$T = \left[A_b^\rho e_1, \cdots, A_b e_1, I_m\right] \tag{42}$$

$$A_b = \begin{bmatrix} -\frac{b_{m-1}}{b_m} & & & \\ \vdots & & I_{m-1} & \\ -\frac{b_0}{b_m} & 0 & \cdots & 0 \end{bmatrix} \tag{43}$$

With the help of the readily verifiable identities

$$T\begin{bmatrix} 0 \\ b \end{bmatrix} = 0, \quad TA = A_b T + TA^\rho \begin{bmatrix} 0 \\ \frac{b}{b_m} \end{bmatrix} e_1^T \tag{44}$$

we obtain the dynamic equations governing ζ and ζ^r as

$$\dot{\zeta} = A_b\zeta + b_b y \tag{45}$$

$$\dot{\zeta}^r = A_b\zeta^r + b_b y_r \tag{46}$$

where

$$b_b = T\left(A^\rho \begin{bmatrix} 0 \\ \frac{b}{b_m} \end{bmatrix} - a\right) \tag{47}$$

Thus, the error state $\tilde{\zeta} = \zeta - \zeta^r$ is defined by

$$\dot{\tilde{\zeta}} = A_b\tilde{\zeta} + b_b z_1 \tag{48}$$

Under Assumption 2, we can see A_b is Hurwitz, that is, $P_b A_b + A_b^T P_b = -I$, $P_b = P_b^T > 0$.

Based on (8), we have

$$\lambda_i(s) = \frac{s^{i-1} + k_1 s^{i-2} + \cdots + k_{i-1}}{K(s)} u(s), \quad i = 1, 2, \cdots, n \tag{49}$$

where

$$K(s) = s^n + k_1 s^{n-1} + k_2 s^{n-2} + \cdots + k_{n-1}s + k_n \tag{50}$$

By multiplying both sides of (15) with T, recalling (41), we obtain

$$TB(A_0)\lambda = \tilde{\zeta} + \zeta^r + TA(A_0)(\tilde{\eta} + \eta^r) - T\varepsilon \tag{51}$$

It is not hard to prove that

$$TB(A_0) = \left[\, T_m \;\; 0_{m\times\rho} \,\right] \tag{52}$$

where $T_m \in \mathbb{R}^{m\times m}$. It is easy to show that T_m is nonsingular if and only if $B(s)$ in (1) and $K(s)$ in (50) are co-prime. Consequently, under this condition, if we denote $\bar{\lambda}_i = (\lambda_1, \cdots, \lambda_i)$ for $i = 1, \cdots, n$, there is no trouble revealing that $\bar{\lambda}_m$ is a smooth function of $\tilde{\eta}$, $\tilde{\zeta}$, and ε. Furthermore, multiplying the identity (15) by e_1^T from the left and noting that $e_1^T B(A_0) = [*, \cdots, *, 1, 0, \cdots, 0]$, where $*$ denotes entries that can have any values and 1 is the $(m+1)$-st entry, one can see that λ_{m+1} is a smooth function of z_1, $\tilde{\eta}$, $\tilde{\zeta}$, and ε. Exploiting $\upsilon_{i,j} = [*, \cdots, *, 1]\begin{bmatrix}\lambda_1\\ \vdots \\ \lambda_{i+j}\end{bmatrix}$, where $\lambda_k \triangleq 0$ for $k > n$, and (22)–(23) implying $\upsilon_{m,i} = z_i + \alpha_{i-1}(y, \eta, \bar{\lambda}_{m+i-1}, \bar{y}_r^{(i-1)}, \hat{\theta})$ for $i = 2, \cdots, \rho$, we could depict λ as a smooth linear function of z, $\tilde{\eta}$, $\tilde{\zeta}$, and ε such that

$$\begin{aligned}\lambda = {} & H_z(\hat{\theta})z + H_\eta(\hat{\theta})\tilde{\eta} + H_\zeta(\hat{\theta})\tilde{\zeta} + H_\varepsilon(\hat{\theta})\varepsilon \\ & + H_\eta(\hat{\theta})\eta^r + H_\zeta(\hat{\theta})\zeta^r + H_{y_r}(\hat{\theta})\bar{y}_r^{(\rho-1)}\end{aligned} \tag{53}$$

where $H_z(\hat{\theta}) \in \mathbb{R}^{n\times\rho}$, $H_\eta(\hat{\theta}) \in \mathbb{R}^{n\times n}$, $H_\zeta(\hat{\theta}) \in \mathbb{R}^{n\times m}$, $H_{y_r}(\hat{\theta}) \in \mathbb{R}^{n\times\rho}$, $H_\varepsilon(\hat{\theta}) \in \mathbb{R}^{n\times n}$.

Remark 1
Based on the previous analysis, one can notice that (53) is a one-to-one and smooth function if and only if $B(s)$ and $K(s)$ are co-prime.

Based on the projection algorithm, $\hat{\theta}$ is bounded for any $\hat{\theta}(0) \in \Pi$. We can also derive boundedness of ε, $\bar{y}_r^{(\rho)}$, η^r, ζ^r, and θ from (13) and Assumptions 1–3. As a result, if $K(s)$ and $B(s)$ are assumed to satisfy the co-prime condition, according to Young's inequality, the parameter estimator (34)–(38) has the following property

$$\begin{aligned}|\dot{\hat{\theta}}_i| &= \gamma_\theta \left|e_i^T \mathrm{Proj}_\Pi\{\tau_\theta\}\right| \leqslant \gamma_\theta \left|e_i^T \tau_\theta\right| \\ &\leqslant \gamma_\theta \left|e_i^T \omega W_\varepsilon(\hat{\theta})^T z\right| \\ &\leqslant \gamma_\theta M_\theta \left(|z| + |z|^2 + |\tilde{\eta}|^2 + |\tilde{\zeta}|^2\right) \\ i &= 1, 2, \cdots, p\end{aligned} \tag{54}$$

where $M_\theta > 0$ is a large constant to some extent.

6. STABILITY ANALYSIS

In this section, the system stability is summarized in a main theorem.

DOI: 10.1002/acs

Theorem 1
Consider the closed-loop system consisting of the plant (2), the K-filters (7)–(12), the backstepping controller (22)–(28), and the parameter identifier (34)–(38). Let Assumptions 1–3 hold, and let $B(s)$ and $K(s)$ be co-prime. There exists a constant $M > 0$ such that if the initial state satisfies the condition

$$|z(0)|^2 + |\tilde{\eta}(0)|^2 + |\tilde{\xi}(0)|^2 + |\tilde{\theta}(0)|^2 + |\varepsilon(0)|^2 \leqslant M \tag{55}$$

then the following results hold:

(i) All the signals of the closed-loop system are uniformly bounded.
(ii) The asymptotic tracking is achieved, that is, $\lim_{t\to\infty} z_1 = \lim_{t\to\infty}(y - y_r) = 0$.

Remark 2
Based on the equalities (15), (22), (29)–(31), $\tilde{\eta} = \eta - \eta^r$, $\tilde{\xi} = Tx - \zeta^r$, and $\tilde{\theta} = \theta - \hat{\theta}$, the initial condition of error state (55) can be checked based on the initial conditions of the states of the actual plant, filters, and parameter estimator, namely, $x(0), \lambda(0), \eta(0), \hat{\theta}(0)$.

Proof
First of all, a nonnegative Lyapunov candidate function is constructed to encompass major states of the closed-loop system as follows:

$$V = \frac{1}{2}z^T z + \frac{1}{k_\eta}\tilde{\eta}^T P \tilde{\eta} + \frac{1}{k_\xi}\tilde{\xi}^T P_b \tilde{\xi} + \frac{1}{2\gamma_\theta}\tilde{\theta}^T\tilde{\theta} + \sum_{i=1}^{\rho}\frac{1}{4d_i}\varepsilon^T P \varepsilon \tag{56}$$

where $k_\eta > 0$ and $k_\xi > 0$. Taking time derivative of it along (13), (33), (34), (40), and (48), we obtain

$$\begin{aligned}\dot{V} &\leqslant -\sum_{i=1}^{\rho} c_i z_i^2 + \sum_{i=1}^{\rho}\frac{1}{4d_i}\varepsilon_2^2 + z^T W_\varepsilon(\hat{\theta})\omega^T\tilde{\theta} \\ &\quad + z^T Q(z,t)^T\dot{\hat{\theta}} - \frac{1}{k_\eta}|\tilde{\eta}|^2 + \frac{2}{k_\eta}\tilde{\eta}^T P e_n z_1 \\ &\quad - \frac{1}{k_\xi}|\tilde{\xi}|^2 + \frac{2}{k_\xi}\tilde{\xi}^T P_b b_b z_1 - \frac{1}{\gamma_\theta}\tilde{\theta}^T\dot{\hat{\theta}} - \sum_{i=1}^{\rho}\frac{1}{4d_i}|\varepsilon|^2 \\ &\leqslant -c_0|z|^2 + \frac{1}{4d_0}|\varepsilon_2|^2 + z^T Q(z,t)^T\dot{\hat{\theta}} - \frac{1}{2k_\eta}|\tilde{\eta}|^2 + \frac{2}{k_\eta}|Pe_n|^2 z_1^2 \\ &\quad - \frac{1}{2k_\xi}|\tilde{\xi}|^2 + \frac{2}{k_\xi}|P_b b_b|^2 z_1^2 - \frac{1}{4d_0}|\varepsilon|^2 - \frac{1}{\gamma_\theta}\tilde{\theta}^T(\dot{\hat{\theta}} - \gamma_\theta\tau_\theta)\end{aligned} \tag{57}$$

where $c_0 = \min_{i=1,2,\cdots,\rho} c_i$, $d_0 = \left(\sum_{i=1}^{\rho}\frac{1}{d_i}\right)^{-1}$. If we choose $k_\eta \geqslant \frac{4|Pe_n|^2}{c_0}$ and $k_\xi \geqslant \frac{4|p_b b_b|^2}{c_0}$, we have

$$\dot{V} \leqslant -\left(c_0 - \frac{2}{k_\eta}|Pe_n|^2 - \frac{2}{k_\xi}|P_b b_b|^2\right)|z|^2 - \frac{1}{2k_\eta}|\tilde{\eta}|^2 - \frac{1}{2k_\xi}|\tilde{\xi}|^2 + z^T Q(z,t)^T\dot{\hat{\theta}} \tag{58}$$

From (23), (29)–(31), and (53), it is easy to show that $-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}$ for $i = 2, 3, \cdots, \rho$, $j = 1, 2, \cdots, p$ are linear functions of z, $\tilde{\eta}$, $\tilde{\xi}$, η^r, ζ^r, $\bar{y}_r^{(\rho-1)}$, and ε such that

DOI: 10.1002/acs

$$
\begin{aligned}
-\alpha_{i-1} &= z_i - \upsilon_{m,i} \\
&= K_{i,y}(\hat{\theta})y + K_{i,\eta}(\hat{\theta})\eta + K_{i,\lambda}(\hat{\theta})\lambda + K_{i,y_r}(\hat{\theta})\bar{y}_r^{(i-1)} - \upsilon_{m,i} \\
-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j} &= \frac{\partial K_{i,y}(\hat{\theta})}{\partial\hat{\theta}_j}y + \frac{\partial K_{i,\eta}(\hat{\theta})}{\partial\hat{\theta}_j}\eta + \frac{\partial K_{i,\lambda}(\hat{\theta})}{\partial\hat{\theta}_j}\lambda + \frac{\partial K_{i,y_r}(\hat{\theta})}{\partial\hat{\theta}_j}\bar{y}_r^{(i-1)} \\
&= \frac{\partial K_{i,y}(\hat{\theta})}{\partial\hat{\theta}_j}(z_1 + y_r) + \frac{\partial K_{i,\eta}(\hat{\theta})}{\partial\hat{\theta}_j}(\tilde{\eta} + \eta^r) \\
&\quad + \frac{\partial K_{i,\lambda}(\hat{\theta})}{\partial\hat{\theta}_j}\Big(H_z(\hat{\theta})z + H_\eta(\hat{\theta})\tilde{\eta} + H_\eta(\hat{\theta})\eta^r + H_\zeta(\hat{\theta})\tilde{\zeta} + H_\zeta(\hat{\theta})\zeta^r \\
&\quad + H_{y_r}(\hat{\theta})\bar{y}_r^{(\rho-1)} + H_\varepsilon(\hat{\theta})\varepsilon\Big) + \frac{\partial K_{i,y_r}(\hat{\theta})}{\partial\hat{\theta}_j}\bar{y}_r^{(i-1)} \\
&= \left[\frac{\partial K_{i,y}(\hat{\theta})}{\partial\hat{\theta}_j}e_1^T + \frac{\partial K_{i,\lambda}(\hat{\theta})}{\partial\hat{\theta}_j}H_z(\hat{\theta})\right]z \\
&\quad + \left[\frac{\partial K_{i,\eta}(\hat{\theta})}{\partial\hat{\theta}_j} + \frac{\partial K_{i,\lambda}(\hat{\theta})}{\partial\hat{\theta}_j}H_\eta(\hat{\theta})\right]\tilde{\eta} + \left[\frac{\partial K_{i,\eta}(\hat{\theta})}{\partial\hat{\theta}_j} + \frac{\partial K_{i,\lambda}(\hat{\theta})}{\partial\hat{\theta}_j}H_\eta(\hat{\theta})\right]\eta^r \\
&\quad + \frac{\partial K_{i,\lambda}(\hat{\theta})}{\partial\hat{\theta}_j}H_\zeta(\hat{\theta})\tilde{\zeta} + \frac{\partial K_{i,\lambda}(\hat{\theta})}{\partial\hat{\theta}_j}H_\zeta(\hat{\theta})\zeta^r + \frac{\partial K_{i,\lambda}(\hat{\theta})}{\partial\hat{\theta}_j}H_\varepsilon(\hat{\theta})\varepsilon \\
&\quad + \frac{\partial K_{i,y}(\hat{\theta})}{\partial\hat{\theta}_j}y_r + \frac{\partial K_{i,\lambda}(\hat{\theta})}{\partial\hat{\theta}_j}H_{y_r}(\hat{\theta})\bar{y}_r^{(\rho-1)} + \frac{\partial K_{i,y_r}(\hat{\theta})}{\partial\hat{\theta}_j}\bar{y}_r^{(i-1)} \\
i &= 2, 3, \cdots, \rho, \quad j = 1, 2, \cdots, p
\end{aligned}
\tag{59}
$$

If we denote $V_0 = |z|^2 + |\tilde{\eta}|^2 + |\tilde{\zeta}|^2$, combining (54) with (59), we have

$$
\begin{aligned}
z^T Q(z,t)^T \dot{\hat{\theta}} &= -\sum_{i=2}^{\rho}\sum_{j=1}^{p} z_i \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j \\
&\leqslant \gamma_\theta M_1(V_0 + V_0^2)
\end{aligned}
\tag{60}
$$

where $M_1 > 0$ is a large constant to some extent. Then, (58) becomes

$$
\dot{V} \leqslant -M_2 V_0 + \gamma_\theta M_1 V_0 + \gamma_\theta M_1 V_0^2 \tag{61}
$$

where $M_2 = \min\left\{c_0 - \frac{2}{k_\eta}|Pe_n|^2 - \frac{2}{k_\zeta}|P_b b_b|^2, \frac{1}{2k_\eta}, \frac{1}{2k_\zeta}\right\}$. Furthermore, we obtain

$$
\begin{aligned}
\dot{V} &\leqslant -M_2 V_0 + \gamma_\theta M_1 V_0 + \gamma_\theta M_1 V_0^2 + \left(\frac{1}{2\gamma_\theta}\tilde{\theta}^T\tilde{\theta} + \frac{1}{4d_0}\varepsilon^T P\varepsilon\right)V_0 \\
&\leqslant -M_2 V_0 + \gamma_\theta M_1 V_0 + \gamma_\theta M_1 V_0^2 + (V - M_3 V_0)V_0 \\
&\leqslant -(M_2 - \gamma_\theta M_1 - V)V_0 + (\gamma_\theta M_1 - M_3)V_0^2
\end{aligned}
\tag{62}
$$

where $M_3 = \min\left\{\frac{1}{2}, \frac{\lambda_{\min}(P)}{k_\eta}, \frac{\lambda_{\min}(P_b)}{k_\zeta}\right\}$. If we choose $\gamma_\theta \leqslant \min\left\{\frac{M_2}{M_1}, \frac{M_3}{M_1}\right\}$ and restrict initial conditions so that $V(0) \leqslant M_2 - \gamma_\theta M_1$, we obtain

$$
\dot{V} \leqslant -M_4 V_0 - M_5 V_0^2 \tag{63}
$$

where $M_4 = M_2 - \gamma_\theta M_1 - V$ and $M_5 = M_3 - \gamma_\theta M_1$. As a consequence, signals z, $\tilde{\eta}$, and $\tilde{\zeta}$ are bounded. Based on a very analogous procedure to Chapter 10 of [9], there is no difficulty showing that remaining signals including η, λ, u, and x are bounded. Thus, Theorem 1(i) has been proved.

By applying the LaSalle-Yoshizawa theorem to (63), it follows such that $z(t) \to 0$ as $t \to \infty$, which indicates that $\lim_{t\to\infty}(y - y_r) = 0$. Thus Theorem 1(ii) has also been proved.

Before ending this proof, we give a short comparison about performance between our method and conventional backstepping tuning functions design. Based on Chapter 10.3 of [9], our method has the identical nonadaptive performance with the tuning functions design, in which an increase of design parameters c_0 and d_0 results in a performance improvement for the closed-loop error system $(z, \tilde{\eta}, \tilde{\zeta})$. As to the adaptive case, Chapter 10.4 tells us that the adaptation gain γ_θ in tuning functions design provides an additional degree of freedom with which the performance can be improved. However, from (62)–(63), to make the whole closed-loop system stable, γ_θ in our method should meet some conditions and cannot be chosen arbitrarily, and the initial state $V(0)$ should also satisfy some restriction concerning the design parameters. □

Example 1

We illustrate the proposed scheme with a numerical example in which $n = 3$, $\rho = 2$, and $m = 1$. We also apply the traditional tuning functions design to it and make a brief comparison with our method. Consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 - a_2 x_1 \\ \dot{x}_2 &= x_3 - 2x_1 + u \\ \dot{x}_3 &= u \\ y &= x_1 \end{aligned} \tag{64}$$

where the full state x is unmeasurable except for the output y. Parameter $a_2 = -3$ is unknown. It is easy to check that $B(s) = b_1 s + b_0$ is Hurwitz while $A(s) = s^3 + a_2 s^2 + a_1 s + a_0$ is unstable.

The information $\theta_0 = -2$, $\bar{\theta} = 1.5$, $y_r = \sin t$ with its first two derivatives is known to designers. The initial states are all set as 0, while the initial parameter estimate is $\hat{a}_2(0) = -2$. The design parameters are chosen as follows: $k_1 = 6, k_2 = 12, k_3 = 8$, $c_1 = c_2 = 1$, and $d_1 = d_2 = 0.1$, $\varsigma_1 = 0.01$, $\varsigma_2 = 60$. The adaptation gain in our method is designed as $\gamma_\theta = 0.1$ while in tuning functions design is chosen as $\gamma_\theta = 0.3$. The simulation results are shown in Figures 1 and 2. All the signals including control input and parameter estimates are bounded, and output trajectory tracking

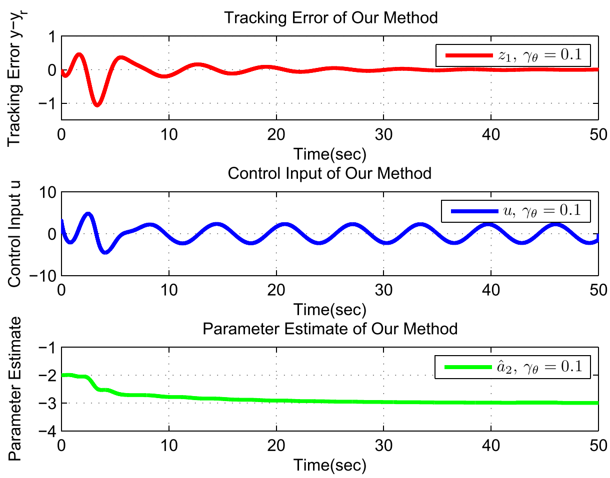

Figure 1. Our method.

DOI: 10.1002/acs

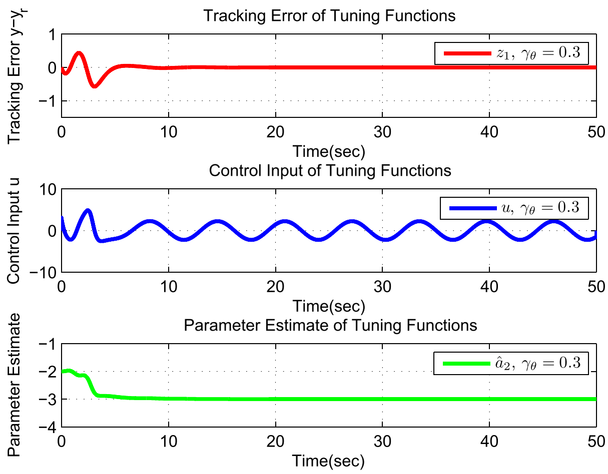

Figure 2. Tuning functions design.

is achieved. Make a comparison between Figures 1 and 2, it is not hard to find that by increasing adaptation gain γ_θ, tuning functions design outperforms mildly our method in terms of performance.

7. GLOBAL STABILIZATION FOR A SPECIAL CASE

In this section, we consider a class of linear systems whose relative degree equals to its system dimension. By a homologous but subtle different design, we achieve global stabilization of the closed-loop error systems. Now the following systems are taken into account:

$$\begin{aligned} \dot{x} &= Ax - ay + bu \\ y &= x_1 \end{aligned} \tag{65}$$

where

$$A = \begin{bmatrix} 0 & & \\ \vdots & I_{n-1} & \\ 0 & \cdots & 0 \end{bmatrix}, \quad a = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} \tag{66}$$

It is obvious that (65) is a special case of (2) with $b_m = b_{m-1} = \cdots = b_1 = 0$, which means the relative degree is identical with the system dimension ($\rho = n$, $m = 0$). As far as we know, a few of linear plants in practice do own this kind of structure. Similar to (4), we rewrite (65) compactly as

$$\begin{aligned} \dot{x} &= Ax + F(y,u)^T \theta \\ y &= e_1^T x \end{aligned} \tag{67}$$

where $p = n + 1$-dimensional parameter vector θ is defined by

$$\theta = [b_0, a_{n-1}, \cdots, a_0]^T \tag{68}$$

and

$$F(y,u)^T = \left[\begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix} u, \; -I_n y \right] \in \mathbb{R}^{n \times p} \tag{69}$$

To achieve the same control objective in Section 1, we need Assumptions 1 and 4 which is a counterpart of Assumption 3.

Assumption 4
The sign of the high-frequency gain b_0, that is, $\mathrm{sgn}(b_0)$, is known, and there exist two known finite constants $\underline{b}_0$ and $\bar{b}_0$ such that $0 < \underline{b}_0 \leqslant |b_0| \leqslant \bar{b}_0$. Moreover, there exists a convex compact set $\mathcal{A} \subset \mathbb{R}^n$ such that $\exists \bar{a}, a^*, |a - a^*| \leqslant \bar{a}$ for all $a \in \mathcal{A}$, where $a^* \in \mathbb{R}^n$ is a known constant vector and $\bar{a} > 0$ is a known finite constant.

The analogous K-filters to (7)–(12) are introduced as follows to measure the states of the system (65) with exponential rate of convergence:

$$\dot{\eta} = A_0 \eta + e_n y \tag{70}$$

$$\dot{\lambda} = A_0 \lambda + e_n u \tag{71}$$

$$\xi = -A_0^n \eta \tag{72}$$

$$\Xi = -[A_0^{n-1}\eta, \cdots, A_0\eta, \eta] \tag{73}$$

$$\Omega^T = [\lambda, \Xi] \tag{74}$$

And we obtain

$$\begin{aligned} x &= \xi + \Omega^T \theta + \varepsilon \\ &= -A(A_0)\eta + b_0 \lambda + \varepsilon \end{aligned} \tag{75}$$

Aiming at system

$$\dot{y} = \xi_2 + \omega^T \theta + \varepsilon_2 \tag{76}$$

$$= b_0 \lambda_2 + \xi_2 + \bar{\omega}^T \theta + \varepsilon_2 \tag{77}$$

$$\dot{\lambda}_i = \lambda_{i+1} - k_i \lambda_1, \quad i = 2, 3, \cdots, n-1 \tag{78}$$

$$\dot{\lambda}_n = u - k_n \lambda_1 \tag{79}$$

where

$$\omega = [\lambda_2, \Xi_2 - y e_1^T]^T \tag{80}$$

$$\bar{\omega} = [0, \Xi_2 - y e_1^T]^T \tag{81}$$

in parallel with (22)–(28), the backstepping regular control scheme is depicted in the succeeding text.

Coordinate transformation:

$$z_1 = y - y_r \tag{82}$$

$$z_i = \lambda_i - \alpha_{i-1}, \quad i = 2, 3, \cdots, n \tag{83}$$

Stabilizing functions:

$$\alpha_1 = \frac{1}{\hat{b}_0} \left(-(c_1 + d_1) z_1 - \xi_2 - \bar{\omega}^T \hat{\theta} + \dot{y}_r \right) \tag{84}$$

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$$\alpha_2 = -\hat{b}_0 z_1 - \left(c_2 + d_2 \left(\frac{\partial \alpha_1}{\partial y}\right)^2\right) z_2 + \beta_2 \tag{85}$$

$$\begin{aligned}\beta_2 = {} & k_2\lambda_1 + \frac{\partial \alpha_1}{\partial y}(\xi_2 + \omega^T\hat{\theta}) + \frac{\partial \alpha_1}{\partial \eta}(A_0\eta + e_n y) \\ & + \sum_{j=0}^{1} \frac{\partial \alpha_1}{\partial y_r^{(j)}} y_r^{(j+1)}\end{aligned} \tag{86}$$

$$\alpha_i = -z_{i-1} - \left(c_i + d_i \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2\right) z_i + \beta_i \tag{87}$$

$$\begin{aligned}\beta_i = {} & k_i\lambda_1 + \frac{\partial \alpha_{i-1}}{\partial y}(\xi_2 + \omega^T\hat{\theta}) + \frac{\partial \alpha_{i-1}}{\partial \eta}(A_0\eta + e_n y) \\ & + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \lambda_j}(-k_j\lambda_1 + \lambda_{j+1}) + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_r^{(j)}} y_r^{(j+1)} \\ i = {} & 3, 4, \cdots, n\end{aligned} \tag{88}$$

where $\hat{b}_0$ and $\hat{\theta}$ are estimates of b_0 and θ, respectively, $\tilde{\theta} = \theta - \hat{\theta}$, and $c_i > 0$, $d_i > 0$ for $i = 1, 2, \cdots, n$ are design parameters.

Adaptive control Law:

$$u = \alpha_n \tag{89}$$

Then, we obtain a closed-loop error system

$$\dot{z} = A_z(\hat{\theta})z + W_\varepsilon(\hat{\theta})(\varepsilon_2 + \omega^T\tilde{\theta}) + Q(z,t)^T\dot{\hat{\theta}} \tag{90}$$

where $A_z(\hat{\theta})$, $W_\varepsilon(\hat{\theta})$ and $Q(z,t)^T$ are almost the same as those given in Table A.II of Appendix just with $\rho = n, m = 0$.

Next, the comparable parameter identifier with (34)–(38) is designed as follows:

$$\dot{\hat{\theta}} = \gamma_\theta \mathrm{Proj}_\Pi\{\tau_\theta\}, \quad \gamma_\theta > 0 \tag{91}$$

$$\tau_\theta = \frac{\omega W_\varepsilon(\hat{\theta})^T z}{1 + \frac{1}{2}z^T z + \frac{1}{k_\eta}\tilde{\eta}^T P \tilde{\eta}}, \quad k_\eta > 0 \tag{92}$$

where $\mathrm{Proj}_\Pi\{\cdot\}$ is a smooth projection operator employed to guarantee that $\hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_p]^T \in \Pi$ with the set Π being defined as

$$\Pi = \left\{ \hat{\theta} \;\middle|\; \begin{array}{l} |\hat{b}_0 \mathrm{sgn}(b_0) - \sigma_0| < \sigma_1, \quad \hat{b}_0 = \hat{\theta}_1 \\ |\hat{a} - a^*| < \bar{a}, \quad \hat{a} = [\hat{\theta}_2, \hat{\theta}_3, ..., \hat{\theta}_p]^T \end{array} \right\} \tag{93}$$

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and $\sigma_0 = (\underline{b}_0 + \bar{b}_0)/2$, $\sigma_1 = \sigma_0 - \underline{b}_0$. Choose a smooth convex function $\mathcal{P}(\hat{\theta}) : \mathbb{R}^p \to \mathbb{R}$ as

$$\mathcal{P}(\hat{\theta}) = \left| \frac{\hat{b}_0 \mathrm{sgn}(b_0) - \sigma_0}{\sigma_1} \right|^{\varsigma_2} + \left| \frac{\hat{a} - a^*}{\bar{a}} \right|^{\varsigma_2} - 1 + \varsigma_1 \tag{94}$$

where $0 < \varsigma_1 < 1$ and $\varsigma_2 \geqslant 2$ are two real numbers. Given the function $\mathcal{P}(\hat{\theta})$, we now obtain a smooth projection operator in (91) as

$$\mathrm{Proj}_{\Pi}\{\tau_\theta\} = \left\{ \begin{array}{l} \tau_\theta, \quad \mathcal{P}(\hat{\theta}) \leqslant 0 \quad \text{or} \quad \nabla_{\hat{\theta}} \mathcal{P}^T \tau_\theta \leqslant 0 \\ \left(I - \frac{\nabla_{\hat{\theta}} \mathcal{P} \nabla_{\hat{\theta}} \mathcal{P}^T}{\nabla_{\hat{\theta}} \mathcal{P}^T \nabla_{\hat{\theta}} \mathcal{P}} \right) \tau_\theta, \text{ if not} \end{array} \right\} \tag{95}$$

Remark 3

A short comparison of identifier (91)–(92) with (34)–(35) will tell us that there is an additional normalization appearing in the denominator of the parameter update law (92) when the relative degree is equal to system dimension. One will see this normalized update law (92) is very helpful for a global stabilization for later derivation. Nevertheless, this technique cannot be directly extended to the situation when the relative degree is less than the system dimension. In that case, to achieve the global stabilization of the closed-loop system, one should employ the remaining m-dimensional zero dynamics $\tilde{\zeta}$ in (48), which is just for analysis in Section 6 on the design of the normalized update law (35) such that $\tau_\theta = \frac{\omega W_\varepsilon(\hat{\theta})^T z}{1 + \frac{1}{2} z^T z + \frac{1}{k_\eta} \tilde{\eta}^T P \tilde{\eta} + \frac{1}{k_\zeta} \tilde{\xi}^T P_b \tilde{\xi}}$. This is apparently impossible because of unmeasured ζ and unknown b.

Then, we bring in $\tilde{\eta}$ and η^r, which are fully the same with (39) and (40). According to (53), we have

$$\lambda = H_z^*(\hat{\theta}) z + H_\eta^*(\hat{\theta}) \tilde{\eta} + H_\eta^*(\hat{\theta}) \eta^r + H_{y_r}^*(\hat{\theta}) \bar{y}_r^{(n-1)} + H_\varepsilon^*(\hat{\theta}) \varepsilon \tag{96}$$

where $H_z^*(\hat{\theta}) \in \mathbb{R}^{n \times n}$, $H_\eta^*(\hat{\theta}) \in \mathbb{R}^{n \times n}$, $H_{y_r}^*(\hat{\theta}) \in \mathbb{R}^{n \times n}$, $H_\varepsilon^*(\hat{\theta}) \in \mathbb{R}^{n \times n}$. Here, please note that there is no unmeasurable zero dynamics $\tilde{\zeta}$ in (96) because $\rho = n$, $m = 0$. Then, the parameter estimator (91)-(95) has the following property:

$$\begin{aligned} |\dot{\hat{\theta}}_i| &= \gamma_\theta \left| e_i^T \mathrm{Proj}_{\Pi}\{\tau_\theta\} \right| \leqslant \gamma_\theta \left| e_i^T \tau_\theta \right| \\ &\leqslant \frac{\gamma_\theta}{V^*} \left| e_i^T \omega W_\varepsilon(\hat{\theta})^T z \right| \\ &\leqslant \frac{\gamma_\theta}{V^*} M_\theta^* \left(|z| + |z|^2 + |\tilde{\eta}|^2 \right) \\ i &= 1, 2, \cdots, p \end{aligned} \tag{97}$$

where $V^* = 1 + \frac{1}{2} z^T z + \frac{1}{k_\eta} \tilde{\eta}^T P \tilde{\eta}$ and $M_\theta^* > 0$ is a large constant to some extent. Next, the system stability for the situation where relative degree is identical with system dimension is summarized in another main theorem.

Theorem 2

Consider the closed-loop system consisting of the plant (65), the K-filters (70)–(74), the backstepping controller (82)–(89), and the parameter identifier (91)–(95). Under Assumptions 1 and 4, the following results hold.

(i) All the signals of the closed-loop system are globally bounded.
(ii) The asymptotic tracking is achieved, that is, $\lim_{t \to \infty} z_1 = \lim_{t \to \infty} (y - y_r) = 0$.

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Proof

Correspondingly, a nonnegative function is constructed to encompass major states of the closed-loop system as follows:

$$V = \ln\left(1 + \frac{1}{2}z^T z + \frac{1}{k_\eta}\tilde{\eta}^T P\tilde{\eta}\right) + \frac{1}{2\gamma_\theta}\tilde{\theta}^T\tilde{\theta} + \sum_{i=1}^{n}\frac{1}{2d_i}\varepsilon^T P\varepsilon \tag{98}$$

Taking time derivative of it, we obtain

$$\begin{aligned}
\dot{V} \leqslant{} & \frac{1}{V^*}\left[-\sum_{i=1}^{n} c_i z_i^2 + \sum_{i=1}^{n}\frac{1}{4d_i}\varepsilon_2^2 + z^T W_\varepsilon(\hat{\theta})\omega^T\tilde{\theta}\right. \\
& \left. + z^T Q(z,t)^T\dot{\hat{\theta}} - \frac{1}{k_\eta}|\tilde{\eta}|^2 + \frac{2}{k_\eta}\tilde{\eta}^T P e_n z_1\right] \\
& - \frac{1}{\gamma_\theta}\tilde{\theta}^T\dot{\hat{\theta}} - \sum_{i=1}^{n}\frac{1}{2d_i}|\varepsilon|^2 \\
\leqslant{} & \frac{1}{V^*}\left[-c_0|z|^2 - \frac{1}{2k_\eta}|\tilde{\eta}|^2 + \frac{2}{k_\eta}|Pe_n|^2 z_1^2\right. \\
& \left. + z^T Q(z,t)^T\dot{\hat{\theta}}\right] - \frac{1}{4d_0}|\varepsilon|^2 \\
& - \frac{1}{\gamma_\theta}\tilde{\theta}^T(\dot{\hat{\theta}} - \gamma_\theta\tau_\theta)
\end{aligned} \tag{99}$$

where $c_0 = \min_{i=1,2,\cdots,n} c_i$, $d_0 = \left(\sum_{i=1}^{n}\frac{1}{d_i}\right)^{-1}$ and $V^* = 1 + \frac{1}{2}z^T z + \frac{1}{k_\eta}\tilde{\eta}^T P\tilde{\eta}$ has already been given in (97).

Like (59), it is easy to show that $-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}$ for $i = 2, 3, \cdots, n$, $j = 1, 2, \cdots, p$ are linear functions of z, $\tilde{\eta}$, η^r, $\bar{y}_r^{(n-1)}$, and ε. Thus, by applying inequalities $0 < \frac{x^2}{1+x^2} < 1$ and $0 < \frac{|x|}{1+x^2} < 1$ to bound the cubic and biquadratic terms with only quadratic ones like [15], we have

$$\begin{aligned}
z^T Q(z,t)^T\dot{\hat{\theta}} &= -\sum_{i=2}^{n}\sum_{j=1}^{p} z_i\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j \\
&\leqslant \gamma_\theta M_6(|z|^2 + |\tilde{\eta}|^2)
\end{aligned} \tag{100}$$

where $M_6 > 0$ is a large constant to some extent. Thus, (99) becomes

$$\dot{V} \leqslant \frac{1}{V^*}\left[-c_0|z|^2 - \frac{1}{2k_\eta}|\tilde{\eta}|^2 + \frac{2}{k_\eta}|Pe_n|^2 z_1^2 + \gamma_\theta M_6(|z|^2 + |\tilde{\eta}|^2)\right] \tag{101}$$

If adaptation gains k_η and γ_θ are chosen to satisfy $k_\eta \geqslant \frac{2}{c_0}|Pe_n|^2$ and $\gamma_\theta \leqslant \frac{1}{2}\min\left\{\frac{1}{M_6}(c_0 - \frac{2}{k_\eta}|Pe_n|^2), \frac{1}{2k_\eta M_6}\right\}$, it is obvious that we can make $\dot{V}$ nonpositive such that

$$\dot{V} \leqslant -\frac{M_7}{V^*}(|z|^2 + |\tilde{\eta}|^2) \tag{102}$$

where $M_7 > 0$. Thus, signals z and $\tilde{\eta}$ are bounded. Based on a very analogous process to Chapter 10 of [9], there is no difficulty in showing that remaining signals including η, λ, u, and x are bounded. Thus, Theorem 2(i) has been proved.

DOI: 10.1002/acs

By applying the LaSalle–Yoshizawa theorem to (102), it follows such that $z(t) \to 0$ as $t \to \infty$, which indicates that $\lim_{t\to\infty}(y - y_r) = 0$. Thus, Theorem 2(ii) has also been proved.

Furthermore, we would like to give a brief discussion about the boundedness of the closed-loop system state. From (102), we have $V(t) \leqslant V(0), \forall t \geqslant 0$. Lyapunov function (98) implies

$$|z|^2 \leqslant 2(e^V - 1), \quad |\tilde{\eta}|^2 \leqslant \frac{k_\eta}{\lambda_{\min}(P)}(e^V - 1) \tag{103}$$

Based on the inequality $\ln(1 + x) \leqslant x, \forall x \geqslant 0$, we have

$$V \leqslant \frac{1}{2}|z|^2 + \frac{1}{k_\eta}\lambda_{\max}(P)|\tilde{\eta}|^2 + \frac{1}{\gamma_\theta}|\tilde{\theta}|^2 + \frac{1}{2d_0}\lambda_{\max}(P)|\varepsilon|^2 \tag{104}$$

Thus, there is no difficulty in obtaining

$$\begin{aligned} |z|^2 + |\tilde{\eta}|^2 \leqslant & \left(2 + \frac{k_\eta}{\lambda_{\min}(P)}\right) \\ & \times \left(e^{\left(\frac{1}{2}|z(0)|^2 + \frac{\lambda_{\max}(P)}{k_\eta}|\tilde{\eta}(0)|^2 + \frac{1}{2\gamma_\theta}|\tilde{\theta}(0)|^2 + \frac{\lambda_{\max}(P)}{2d_0}|\varepsilon(0)|^2\right)} - 1\right) \end{aligned} \tag{105}$$

□

Example 2

In this section, we illustrate the global control scheme with the example that is identical to that in Chapter 10.2.4 of [9] and briefly compare our method with the traditional tuning functions design. Consider the unstable relative-degree-three plant

$$\begin{aligned} \dot{x}_1 &= x_2 - a_2 y \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \\ y &= x_1 \end{aligned} \tag{106}$$

where the full state x is unmeasurable and the plant parameter $a_2 = -3$ is assumed to be unknown.

The information $a^* = -2$, $\bar{a} = 3$, $y_r = \sin t$ with its first three derivatives is known. The initial states are all set as 0. The design parameters are chosen as follows: $k_1 = 3, k_2 = 2, k_3 = 1$, $c_1 = c_2 = c_3 = 3.5$, $d_1 = d_2 = d_3 = 0.1$, $\varsigma_1 = 0.01$, $\varsigma_2 = 60$, and $k_\eta = 30$. The adaptation gain is designed as $\gamma_\theta = 0.3$. The results are shown in Figures 3 and 4. It is obvious that with the same

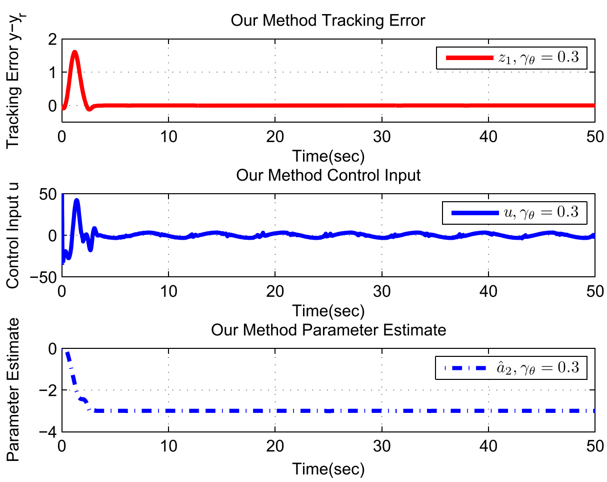

Figure 3. Our method.

DOI: 10.1002/acs

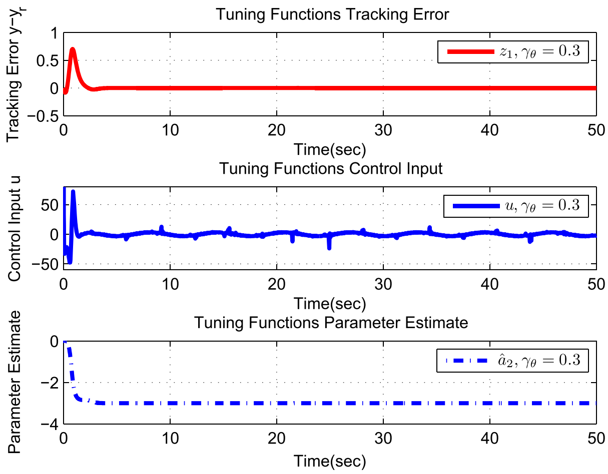

Figure 4. Tuning functions design.

design parameters, tuning functions design is a little better than our method as far as the performance is considered.

8. CONCLUSION

In this paper, we develop an adaptive backstepping control scheme for minimum-phase linear systems to achieve trajectory tracking although full state is unmeasured and plant parameters are unknown. The output feedback design is linear and brief, and the stability analysis is Lyapunov-based and intuitive. In the case where the relative degree is less than the system dimension, a local stabilization result is achieved. In the situation where the relative degree is the same as the system dimension, a global stabilization result is obtained. To our best knowledge, Lyapunov-based stabilization by adaptation gain and normalization of update law rather than utilizing tuning functions and nonlinear damping terms is new for the backstepping approach.

APPENDIX

Table A.I.

$$K_{2,y}(\hat{\theta}) = \frac{1}{\hat{b}_m}(c_1 + d_1 - \hat{a}_{n-1}), \quad K_{2,\eta}(\hat{\theta}) = -\frac{1}{\hat{b}_m} e_2^T \hat{A}(A_0) \tag{A.1}$$

$$K_{2,\lambda}(\hat{\theta}) = \frac{1}{\hat{b}_m} e_2^T \hat{B}(A_0), \quad K_{2,y_r}(\hat{\theta})\bar{y}_r^{(1)} = -\frac{1}{\hat{b}_m}(c_1 + d_1) y_r - \frac{1}{\hat{b}_m}\dot{y}_r \tag{A.2}$$

$$K_{3,y}(\hat{\theta}) = \hat{b}_m + \left(c_2 + d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2\right) K_{2,y}(\hat{\theta}) + \frac{\partial \alpha_1}{\partial y}\hat{a}_{n-1} - \frac{\partial \alpha_1}{\partial \eta} e_n \tag{A.3}$$

$$K_{3,\eta}(\hat{\theta}) = \left(c_2 + d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2\right) K_{2,\eta}(\hat{\theta}) + \frac{\partial \alpha_1}{\partial y} e_2^T \hat{A}(A_0) - \frac{\partial \alpha_1}{\partial \eta} A_0 \tag{A.4}$$

$$K_{3,\lambda}(\hat{\theta}) = \left(c_2 + d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2\right) K_{2,\lambda}(\hat{\theta}) + e_2^T A_0^{m+1} - \frac{\partial \alpha_1}{\partial y} e_2^T \hat{B}(A_0) - \sum_{j=1}^{m+1} \frac{\partial \alpha_1}{\partial \lambda} e_j e_j^T A_0 \tag{A.5}$$

$$K_{3,y_r}(\hat{\theta})\bar{y}_r^{(2)} = -\hat{b}_m y_r + \left(c_2 + d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2\right) K_{2,y_r}(\hat{\theta})\bar{y}_r^{(1)} - \sum_{j=0}^{1} \frac{\partial \alpha_1}{\partial y_r^{(j)}} y_r^{(j+1)} \tag{A.6}$$

DOI: 10.1002/acs

Table A.I. *Continued.*

$$K_{i+1,y}(\hat{\theta}) = K_{i-1,y}(\hat{\theta}) + \left(c_i + d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\right)K_{i,y}(\hat{\theta}) + \frac{\partial\alpha_{i-1}}{\partial y}\hat{a}_{n-1} - \frac{\partial\alpha_{i-1}}{\partial\eta}e_n \tag{A.7}$$

$$K_{i+1,\eta}(\hat{\theta}) = K_{i-1,\eta}(\hat{\theta}) + \left(c_i + d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\right)K_{i,\eta}(\hat{\theta}) + \frac{\partial\alpha_{i-1}}{\partial y}e_2^T\hat{A}(A_0) - \frac{\partial\alpha_{i-1}}{\partial\eta}A_0 \tag{A.8}$$

$$K_{i+1,\lambda}(\hat{\theta}) = K_{i-1,\lambda}(\hat{\theta}) + \left(c_i + d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\right)K_{i,\lambda}(\hat{\theta}) + e_i^T A_0^{m+1} - \frac{\partial\alpha_{i-1}}{\partial y}e_2^T\hat{B}(A_0) - \sum_{j=1}^{m+i-1}\frac{\partial\alpha_{i-1}}{\partial\lambda}e_j e_j^T A_0 \tag{A.9}$$

$$\begin{aligned} K_{i+1,y_r}(\hat{\theta})\bar{y}_r^{(i)} &= K_{i-1,y_r}(\hat{\theta})\bar{y}_r^{(i-2)} + \left(c_i + d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\right)K_{i,y_r}(\hat{\theta})\bar{y}_r^{(i-1)} - \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(j)}}y_r^{(j+1)} \\ i &= 3,4,\cdots,\rho-1 \end{aligned} \tag{A.10}$$

Table A.II.

$$K_y(\hat{\theta}) = -\left[K_{\rho-1,y}(\hat{\theta}) + \left(c_\rho + d_\rho\left(\frac{\partial\alpha_{\rho-1}}{\partial y}\right)^2\right)K_{\rho,y}(\hat{\theta}) + \frac{\partial\alpha_{\rho-1}}{\partial y}\hat{a}_{n-1} - \frac{\partial\alpha_{\rho-1}}{\partial\eta}e_n\right] \tag{A.11}$$

$$K_\eta(\hat{\theta}) = -\left[K_{\rho-1,\eta}(\hat{\theta}) + \left(c_\rho + d_\rho\left(\frac{\partial\alpha_{\rho-1}}{\partial y}\right)^2\right)K_{\rho,\eta}(\hat{\theta}) + \frac{\partial\alpha_{\rho-1}}{\partial y}e_2^T\hat{A}(A_0) - \frac{\partial\alpha_{\rho-1}}{\partial\eta}A_0\right] \tag{A.12}$$

$$\begin{aligned} K_\lambda(\hat{\theta}) = -\Bigg[&K_{\rho-1,\lambda}(\hat{\theta}) + \left(c_\rho + d_\rho\left(\frac{\partial\alpha_{\rho-1}}{\partial y}\right)^2\right)K_{\rho,\lambda}(\hat{\theta}) + e_\rho^T A_0^{m+1} - \frac{\partial\alpha_{\rho-1}}{\partial y}e_2^T\hat{B}(A_0) \\ &- \sum_{j=1}^{m+\rho-1}\frac{\partial\alpha_{\rho-1}}{\partial\lambda}e_j e_j^T A_0\Bigg] \end{aligned} \tag{A.13}$$

$$K_{y_r}(\hat{\theta})\bar{y}_r^{(\rho)} = -K_{\rho-1,y_r}(\hat{\theta})\bar{y}_r^{(\rho-2)} - \left(c_\rho + d_\rho\left(\frac{\partial\alpha_{\rho-1}}{\partial y}\right)^2\right)K_{\rho,y_r}(\hat{\theta})\bar{y}_r^{(\rho-1)} + \sum_{j=0}^{\rho-1}\frac{\partial\alpha_{\rho-1}}{\partial y_r^{(j)}}y_r^{(j+1)} \tag{A.14}$$

$$A_z(\hat{\theta}) = \begin{bmatrix} -(c_1+d_1) & \hat{b}_m & 0 & \cdots & 0 \\ -\hat{b}_m & -\left(c_2 + d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2\right) & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & -\left(c_\rho + d_\rho\left(\frac{\partial\alpha_{\rho-1}}{\partial y}\right)^2\right) \end{bmatrix} \tag{A.15}$$

$$W_\varepsilon(\hat{\theta}) = \begin{bmatrix} 1 \\ -\frac{\partial\alpha_1}{\partial y} \\ \vdots \\ -\frac{\partial\alpha_{\rho-1}}{\partial y} \end{bmatrix}, \quad Q(z,t)^T = \begin{bmatrix} 0 \\ -\frac{\partial\alpha_1}{\partial\hat{\theta}} \\ \vdots \\ -\frac{\partial\alpha_{\rho-1}}{\partial\hat{\theta}} \end{bmatrix} \tag{A.16}$$

DOI: 10.1002/acs

ACKNOWLEDGEMENTS

This work obtained supports from the China Scholarship Council (CSC) and National Natural Science Foundation of China (NSFC: 61134007, 61320106009, 61104048).

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DOI: 10.1002/acs