Brief paper

Stabilization and robustness analysis for a chain of exponential integrators using strict Lyapunov functions

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Abstract

We provide a new method for building strict Lyapunov functions for two dimensional chains of exponential integrators, using nested exponential functions. One challenge is that the right sides of the systems saturate, so they are not completely controllable. The strictness of the Lyapunov functions is key to proving input-to-state stability (or ISS) properties with respect to additive uncertainty on the controls. We show how a large class of tracking problems for nonlinear systems with positivity constraints on the states can be solved using our theory.

1. Introduction

This paper continues our search (begun in Malisoff & Mazenc, 2009; Mazenc & Malisoff, 2006, 2010; Mazenc, Malisoff, & Bernard, 2009) for new constructions of strict Lyapunov functions that can be used to prove stability and robustness properties for nonlinear control systems. In some cases, stabilization problems can be solved with the help of nonstrict Lyapunov functions, which are proper and positive definite functions whose time derivatives are nonpositive along all solutions of the closed loop system. By proper and positive definiteness of a function $V$, we mean that $V$ is zero at the equilibrium, positive at all other states, and satisfies $V(Z) \to \infty$ as $|Z| \to \infty$ or as $Z$ approaches the boundary of the state space. However, nonstrict Lyapunov functions by themselves are insufficient to solve asymptotic stabilization problems, since they do not ensure convergence to the equilibrium. Instead, one often uses nonstrict Lyapunov functions in conjunction with LaSalle invariance or a Matrosov approach; see Khalil (2002) and Matrosov (1962).

However, even if one uses LaSalle invariance or standard Matrosov approaches, there is usually no guarantee of robustness, e.g., with respect to control or model uncertainty. This helped motivate the ‘strictification’ approach from Malisoff and Mazenc (2009), which converts nonstrict Lyapunov functions into strict ones. A strict Lyapunov function is a proper and positive definite function whose time derivative is negative along all trajectories of the closed loop system at all points outside the equilibrium. Strict Lyapunov functions allow us to robustify controls, e.g., to prove robustness in the key sense of ISS; see Khalil (2002).

To see how this ‘strictification’ approach can be done in the special case of time invariant nonlinear control affine systems of the form $\dot{z} = f(z) + g(z)u(z)$ with state space $\mathbb{R}^n$ for any $n$, assume that we found a control $u(z)$ such that the closed loop system is globally asymptotically stable to the origin, and that we have a strict Lyapunov function $v$ for the closed loop system such that $-\dot{v}(z)$ is a proper and positive definite function, or equivalently, there is a class $\mathcal{K}_\infty$ function $\alpha$ such that $\dot{v}(z) \leq -\alpha(|z|)$ holds along all trajectories of the closed loop system; see Khalil (2002). Assume that there are actuator errors, which we model by $\dot{z} = f(z) + g(z)(u(z) + \delta)$ where $\delta$ is an unknown measurable essentially bounded function. Then, by a simple application of the triangle inequality that uses the control affinity, the closed loop system $\dot{z} = f(z) + g(z)(u^*(z) + \delta)$ is input-to-state stable when we use the augmented controller $u^*(z) = u(z) - (\nabla v(z)g(z))^\top$; see Sections 5 and 8 for more on robustifying controls.

However, to implement the control $u^*$, one needs an explicit closed form formula for the gradient $\nabla v$ of the strict Lyapunov function in the formula for $u^*$, and this was another motivation...
for the strictification approach, but there are other motivations. For instance, having strict Lyapunov functions in closed form often makes it possible to find explicit formulas for the comparison functions in ISS estimates, and strict Lyapunov function constructions also make it possible to quantify the effects of input or state delays, and to perform backstepping.

In this paper, we use a Matrosov type strictification to prove a stabilization result for a controlled chain of exponential integrators in which the right sides of both equations saturate, so the system is not completely controllable. While the system is time invariant, many interesting tracking problems for time varying bilinear systems can be transformed into chains of exponential integrators that are covered by our theory, including cases where there are positivity constraints on the state values and delays; see Sections 7–8. Our strict Lyapunov function for the closed loop system allows us to prove robustness under uncertainty and delays, including ISS with respect to additive uncertainties.

loop system allows us to prove robustness under uncertainty and delays, including ISS with respect to additive uncertainties on the control for cases where the system is not control affine; see Section 5. Our strictification uses just one auxiliary function, but the choices of the functions needed for the strictification are not obvious, which makes our work novel and interesting. We believe that our new approach for overcoming these challenges has the potential for many other applications to higher dimensional systems that are also not completely controllable.

2. Chain of exponential integrators model

We begin by studying systems of the form

\[
\begin{align*}
\dot{x} &= 1 - e^u \\
\dot{y} &= D^*(1 - e^u)
\end{align*}
\] (1)

having the state space \(\mathbb{R}^2\), where \(u\) is the control and \(D^*\) is a positive constant (but see below for cases where the 1’s in (1) are replaced by more general nonlinear functions \(M_1\) and \(M_2\), Sections 5–6 for cases with uncertainties and time delays, and Sections 7–8 for ways to transform many interesting systems with positivity constraints on the states into the form (1)). The global stabilization of (1) is nontrivial. By contrast, the local stabilization is easy, since any linear feedback \(u = k_x x + k_y y\) for suitable constant \(k_x\)’s will do. For global stabilization, we have to contend with the fact that the right sides of both equations in (1) saturate on the upper end at 1, so the system is not completely controllable. However, the system is not exponentially unstable in open loop. Instead, it is polynomially unstable, e.g., \(x(t) = x_0 + t(1 - e^0)\) when \(u = 0\).

Attempting stabilization with backstepping is fruitless, because backstepping would require both positive and negative values of \(e^u\), so backstepping is too aggressive. There are valuable results in the literature on bounded backstepping that lead to controls that satisfy input constraints, including more general situations where the dynamics are time varying or have delays; see Mazenc and Bowong (2004) and Mazenc and Malisoff (2015). However, these results either do not ensure ISS, or do not provide the strict Lyapunov functions for (1) that are essential for what follows.

On the other hand, we can build controls that stabilize the equilibrium \((0, 0)\) of (1), and corresponding nonstrict Lyapunov functions for the closed loop system, on \(\mathbb{R}^2\). For instance, we can use the following nonstrict Lyapunov function from Malisoff and Krstic (2015):

\[
V(x, y) = x + \exp(-x) - 1 + \frac{1}{D^*} (\exp(y) - y - 1).
\] (2)

In fact, the time derivative of (2) along all trajectories of (1) in closed loop with the control

\[
u = y - x
\] (3)

satisfies

\[
\dot{V} = -\exp(-x)(\exp(y) - 1)^2
\] (4)

and then stability of the closed loop system follows from LaSalle’s theorem. However, (2) is not a strict Lyapunov function, since its time derivative (4) is zero at all points where \(y = 0\), so (2) is not amenable to the robustification objective we discussed in the introduction.

The system (1) naturally arises in the study of systems with positivity constraints on the states, such as

\[
\begin{align*}
\dot{X} &= (1 - Y)X \\
\dot{Y} &= (D^* - D)Y
\end{align*}
\] (5)

on the state space \((0, \infty)^2\), by setting \(x(t) = \ln(X(t)/X_0), y(t) = \ln(Y(t))\), and \(u(t) = \ln(D(t)/D^*)\) for any constant \(X_0 > 0\) (but see Section 7 for much more general systems with positivity constraints). In fact, the preceding argument proves the following, by setting \(u = y - x\):

**Theorem 1.** For all constants \(X_0 > 0\), the system (5), in closed loop with the positive valued control \(D = D^* Y X_0 / X\), has the positively invariant set \((0, \infty)^2\) and is globally asymptotically stable to \((1, 1)\). \(\square\)

Before providing our general theory, we illustrate Theorem 1 using simulations for (5) with the control from Theorem 1. We choose \(X_0 = 1\) and \(D^* = 5\), with the initial condition \((X(0), Y(0)) = (e^1, -e^2)\), which corresponds to the initial condition \((x(0), y(0)) = (3, -2)\) in the transformed variables on \(\mathbb{R}^2\). In Fig. 1, we plot the corresponding trajectories for \(X(t), Y(t)\) and \(D(t)\) as solid lines, with the set point levels given as dotted lines. Our simulations illustrate our controller’s ability to ensure asymptotic convergence of the state vector toward the equilibrium \((1, 1)\).

To examine the overshoots in Fig. 1, we also provide a phase portrait in Fig. 2 that shows four trajectories of the system in closed loop with \(D = D^* Y X_0 / X\), superimposed by six level curves of our nonstrict Lyapunov function \(V_1\) from (2). We choose \(X_0 = 1\) and \(D^* = 5\). Clockwise from the top in Fig. 2, the solid lines show the solutions \((X(t), Y(t))\) on the time interval \([0, 2.5]\) and the initial conditions \((e^1, e^1), (e^1, e^2), (e^2, e^2)\), and \((e^2, -e^2)\), respectively. The dotted lines in Fig. 2 show level curves of the nonstrict Lyapunov function \(V_1\) from (2) with \(x(t) = \ln(X(t)/X_0)\) and \(y(t) = \ln(Y(t))\), with \(V_1 = L\) for the following values of \(L\): 0.01, 0.1, 0.3, 0.5, 0.75, and 1.45. The figure shows convergence of the solutions towards \((1, 1)\) and the crossings through the level curves of \(V_1\). See Section 7 for large classes of systems with positivity constraints, delays, and uncertainties and other changes of variables that are covered by our theory. Such results require strict Lyapunov functions. This motivates the next section, which transforms (2) into a strict Lyapunov function for a suitable generalization of (1).

3. Main strict Lyapunov function construction

We construct an explicit strict Lyapunov function for a broad class of systems of the form

\[
\begin{align*}
\dot{x} &= M_1(x, y) - e^u \\
\dot{y} &= D^*(M_2(x, y) - e^u)
\end{align*}
\] (6)

(but see below for many other systems with positive state constraints). The results in this section cover the system (1) in closed loop with (3) on \(\mathbb{R}^2\), which plays a key role in our robustness and delays analysis in Sections 5–8. We again allow \(D^*\) to be any positive constant, and we assume that \(M_1\) and \(M_2\) are any locally Lipschitz functions that admit constants \(m_1 \ge 0\) and \(m_2 \ge 0\) such that
Convergence of solutions

Fig. 2.

hold for all \((x, y) \in \mathbb{R}^2\), where \(\text{sgn}\) is defined by \(\text{sgn}(p) = 1\) (resp., \(-1\)) when \(p \geq 0\) (resp., \(p < 0\)). The preceding conditions allow many interesting cases where \(\mathcal{M}_1\) and \(\mathcal{M}_2\) can be unbounded. For instance, they allow \(\mathcal{M}_1(x, y) = 1 - \text{sat}(x)|y|\) and \(\mathcal{M}_2 = 1 - \max(0, y^p)\) for any odd integer \(p \geq 3\), or \(\mathcal{M}_1(x, y) = 1 - k_1|x|/(1 + x^2)\) and \(\mathcal{M}_2(x, y) = 1 - k_2y(e^{p - 1} - 1)^2/(1 + y^2)\) for any constants \(k_1 \geq 0\) and \(k_2 \geq 0\), where the saturation \text{sat} is defined by \(\text{sat}(p) = p\) (resp., \(\text{sgn}(p)\)) for all \(p \in [-1, 1]\) (resp., \(p \in \mathbb{R}\setminus[-1, 1]\)).

Conditions C1–C2 ensure that the time derivative of our nonstrict Lyapunov function \(V_1\) from (2) satisfies

\[
\dot{V}_1 = -e^{-y}e^{e^y - 1} + (1 - e^{-y})(\mathcal{M}_1(x, y) - 1) + (1/D^\ast)(e^{p - 1} - 1)D^\ast(\mathcal{M}_2(x, y) - 1)
\]

\[
\leq -e^{-y}e^{e^y - 1}^2
\]

(7)

along all trajectories of (6) when we use our control \(u = y - x\), and so is also a nonstrict Lyapunov function for (6) on \(\mathbb{R}^2\). This can be used to prove that (6) is GAS to 0 on \(\mathbb{R}^2\), by LaSalle invariance. However, \(V_1\) is not a strict Lyapunov function (6), because we could have \(1 - e^{-y})(\mathcal{M}_1(x, y) - 1) = 0\) for some nonzero values of \(x\), which occurs in the special case where \(\mathcal{M}_1(x, y) = 1\) for all \((x, y) \in \mathbb{R}^2\). Hence, we transform the nonstrict Lyapunov function \(V_1\) from (2) into a strict Lyapunov function for (6) on \(\mathbb{R}^2\), which plays a key role in our robustness and delays analysis below. For the special case where \(\mathcal{M}_1 = \mathcal{M}_2 = 1\), our strict Lyapunov function will have the form

\[
V_2(x, y) = \mathcal{H}(V_1(x, y)) + V_2(x, y)
\]

(8)

for a suitable \(C^1\) function \(\mathcal{H}\), with the auxiliary function \(V_2(x, y) = -xy\). However, it is not obvious how to choose \(\mathcal{H}\). See Section 4 for the advantages of the new construction we are about to give over the approach in Malisoff and Mazenc (2009). We prove the following, which implies that we can choose \(\mathcal{H}(v) = \mathcal{L}(e^{e^y - 1})\) for a suitable constant \(\mathcal{L} > 0\) when \(\mathcal{M}_1 = \mathcal{M}_2 = 1\) (but see Remark 1 for a different construction, and see Corollary 1 for the special case of the following construction where \(\mathcal{M}_1 = \mathcal{M}_2 = 1\)):

**Theorem 2.** Let the locally Lipschitz functions \(\mathcal{M}_1\) and \(\mathcal{M}_2\) satisfy C1–C4 on \(\mathbb{R}^2\) for some constants \(\bar{m}_1 \geq 0\) and \(\bar{m}_2 \geq 0\). Let \(P\) and \(D^\ast\) be any positive constants such that

\[
P < D^\ast \left(1 - \frac{1}{e}\right),
\]

(9)

and \(\mathcal{L}\) be any constant satisfying \(\mathcal{L} \geq \max\{A, B\}\), where

\[
A = \frac{12D^\ast}{P}\max\left\{\frac{4}{3}, e^{1/D^\ast}\right\}\max\left\{1, D^\ast\right\}
\]

and

\[
B = \max\left\{\frac{8(D^\ast)^2e^{1/D^\ast}}{P}, 11.1D^\ast, 24(D^\ast)^3e^{1/(2D^\ast)}\right\}.
\]

(10)

Choose the nonnegative constant

\[
L^\ast = \frac{e^2}{2} \left(\bar{m}_2D^\ast e + \overline{m}_1\frac{1}{(1 - e^{-y})^2}\max\left\{1, 2(D^\ast)^2e^{1/D^\ast}\right\}\right).
\]

(11)

Then

\[
V_2(x, y) = V_2(x, y) + L^\ast(e^{2\mathcal{L}^\ast(x, y)} - 1), \quad \text{where}
\]

\[
V_2(x, y) = \mathcal{L}(e^{\mathcal{L}^\ast(x, y)} - 1) - xy \quad \text{and}
\]

\[
V_1(x, y) = x + e^{x - 1} + (1/D^\ast)(e^{y - 1} - 1)
\]

(12)
satisfies \( V_4(x, y) \geq \frac{1}{2} V_1(x, y) \) for all \((x, y) \in \mathbb{R}^2\)
and
\[
V_4(x, y) \leq -\frac{1}{2}(\bar{L}^2 V_{y_1}(x, y) - x^2) + D^* x(1 - e^{-x})
\]
(13)
along all trajectories of (6) in closed loop with (3), so \( V_4 \) is a strict Lyapunov function for (3) and (6) on \( \mathbb{R}^2 \).

**Proof.** It is convenient to first study the case where \( M_1 = M_2 = 1 \), and then to study the additional terms that arise for more general functions \( M_1 \) and \( M_2 \).

1st Part: Case Where \( M_1 = M_2 = 1 \).

In that case, the time derivative of the auxiliary function \( V_2(x, y) = -x y \) along all trajectories of the closed loop system satisfies
\[
V_2 = -(1 - e^y) y - D^* x(1 - e^{-x}) + x D^*(e^{-x} + x^2)
\leq \left\{ \left[ (1 - e^y) y + D^* e^{-x} x(1 - e^{-x}) \right] - D^* x(1 - e^{-x}). \right\}
(14)

Next, we find upper bounds for the terms in curly braces in (14).

By our condition (9) on \( P \), we can fix a constant \( M > 1 \) such that
\( P < (D^*/M)(1 - e^{-M}) \), and then
\[
|(1 - e^y) y| \leq (M/(1 - e^{-M}))(1 - e^y)^2 \leq (D^*/P)(1 - e^y)^2
\]
for all \( y \geq -M \), where the first inequality follows because \( Q(y) = (e^y - 1)/y \) is increasing, so \( Q(-M) \leq |Q(y)| \) for all \( y \geq -M \). On the other hand if \( y < -M \), then we get
\[
|(1 - e^y) y| \leq \frac{(1 - e^y)^2 |y|}{1 - e^{-M}} \leq (D^*/P)(1 - e^y)^2 |y|,
\]
where the last inequality used the fact that
\[
\frac{D^*}{P} \geq \frac{1}{1 - e^{-1}} \geq \frac{1}{1 - e^{-M}}
\]
which follows because \( M > 1 \). By the previous two cases,
\[
|(1 - e^y) y| \leq (D^*/P)(1 - e^y)^2 (1 + |y|)
\]
(17)
for all \( y \in \mathbb{R} \), which is our desired bound on \( |1 - e^y|y \).

To upper bound \( D^* e^{-x} x(1 - e^{-x}) \) from (14), first note that since
\( Q(x) = (e^x - 1)/x \) is an increasing function, we have \( P \leq D^*(1 - e^{-M})/M \leq D^*(e^x - 1)/x \) for all nonzero \( x \geq -M \). Hence, we can use the triangle inequality to get
\[
|x D^* e^{-x}(1 - e^{-x})| \leq \frac{P}{2} x^2 e^{-x} + \frac{1}{2P}(D^*)^2 e^{-x}(1 - e^x)^2
\leq \frac{D^*}{2} x(1 - e^{-x}) + \frac{1}{2P}(D^*)^2 e^{-x}(1 - e^x)^2
\]
for all \( x \geq -M \). On the other hand, if \( x < -M \), then since \( P \leq (0, D^*) \) and \( M > 1 \), we get
\[
\frac{P}{2} x^2 e^{-x} \leq \frac{P}{2} |x e^x| \leq \frac{P}{2} |x| e^{-M} \leq \frac{P}{2} |x| (e^M - 1)
\]
\[
\leq \frac{D^*}{2} x(1 - e^{-x}),
\]
so in this case, the triangle inequality gives
\[
|x D^* e^{-x}(1 - e^{-x})| = |x D^* e^{-x} e^{-2x}(1 - e^{-x})|
\leq \frac{P}{2} x^2 e^{-x} + \frac{1}{2P}(D^*)^2 e^{-x}(1 - e^x)^2
\leq \frac{D^*}{2} x(1 - e^{-x}) + \frac{1}{2P}(D^*)^2 e^{-x}(1 - e^x)^2
\]
(18)

Combining the preceding two cases gives
\[
D^* e^{-x} x(1 - e^{-x}) \leq \frac{D^*}{2} x(1 - e^{-x})
+ \frac{(D^*)^2}{2P}(e^{-x} + x)(1 - e^x)^2
\]
(20)

for all \((x, y) \in \mathbb{R}^2 \). Then (14), (17), and (21) give
\[
\dot{V}_2 \leq -\frac{D^*}{2} x(1 - e^{-x})
+ (1 - e^y)^2 \left( \frac{(D^*)^2}{2P}(e^{-x} + x)(1 - e^x)^2 \right)
+ \frac{D^*}{2} x(1 - e^{-x})
+ \frac{(D^*)^2}{2P}(e^{-x} + x)(1 - e^x)^2
\]
(22)
along all trajectories of (1). Next set \( R(s) = \bar{L}^e \). We will presently show that our lower bounds (10) on \( \bar{L} \) imply that
\[
\frac{1}{2} \bar{L} R(V_1(x, y)) e^{-x}
\geq \frac{(D^*/P)(1 + |y|)}{2} + \frac{(D^*)^2}{2P}(e^{-x} + x)^2
\]
(23)
and
\[
V_1(x, y) R \left( \frac{1}{2} V_1(x, y) \right) \geq 4 |xy|
\]
(24)
hold for all \((x, y) \in \mathbb{R}^2 \). Assuming for now that (23)–(24) hold, we show that the conclusions of the theorem hold when \( M_1 = M_2 = 1 \).

To check (13), notice that along all trajectories of (1), our decay estimates (4) and (22) on \( V_1 \) and \( V_2 \) give
\[
\dot{V}_3(x, y) = R(V_1(x, y)) \dot{V}_1(x, y) + \dot{V}_2(x, y)
\leq -\frac{R(V_1(x, y))}{2} e^{-x}(1 - e^x)^2
\]
\[
- \frac{D^*}{2} x(1 - e^{-x}) + (1 - e^y)^2 \left\{ \frac{(D^*)^2}{2P}(e^{-x} + x)^2 \right\}
+ \frac{D^*/P}{2} (1 + |y| - \frac{R(V_1(x, y))}{2} e^{-x})
\]
(25)
and (23) implies that the quantity in curly braces in (25) is negative, so this gives (13). Also, \( R \) is bounded below by \( A \geq 12(D^*/P)(4/3) \). Hence, (24) and the fact that \( R \) is increasing give
\[
\int_0^{V_1(x, y)} R(t) dt \geq \int_{V_1(x, y)/2}^{V_1(x, y)} R(t) dt
\geq V_1(x, y) R(V_1(x, y)/2)/2
\]
and therefore also
\[
V_3(x, y) \geq \frac{1}{2} \int_0^{V_1(x, y)} R(t) dt - xy
\]
\[
+ \frac{1}{4} V_1(x, y) R \left( \frac{1}{2} V_1(x, y) \right) \geq \frac{1}{2} V_1(x, y)
\]
(26)
for all \((x, y) \in \mathbb{R}^2 \), which implies the proper and positive definiteness of \( V_3 \). Therefore, since \( V_4 \leq V_3 \) everywhere, it remains to check that (23)–(24) for \((x, y) \in \mathbb{R}^2 \).

Fix \((x, y) \in \mathbb{R}^2 \). To check (23), it suffices to note that
\[
\frac{3I}{4} \exp((1/D^*) e^y - y - 1) \geq 5.56(D^*/P)(1 + |y|),
\]
(27)
\[
\exp(x + e^{-x} - 1) \geq e^{-x} \quad \text{and}
\]
\[
\frac{I}{4} \exp(x + e^{-x} - 1) \geq \frac{(D^*)^2}{P}(e^{-3x} + 1)
\]
(29)
all hold (by multiplying the left sides of (27)–(28)). We now check (27) and (29). To check (27), we consider three cases. Case 1: If \( |y| \leq 1 \), then the right side of (27) is bounded above by 12(D^*/P), so (27) follows because (10) gives \( I \geq A \geq 16(D^*/P) \). Case 2: If
\[ y > 1, \text{ then } e^y - 1.7y - 1 \geq 0, \text{ so the left side of (27) is bounded below by} \]
\[
\frac{31}{4} \exp \left( \frac{0.7y}{D^*} \right) \geq \frac{31}{4} \left( 1 + \frac{0.7|y|}{D^*} \right),
\]  
so in this case, (27) holds because (10) gives
\[ \bar{L} \geq \frac{1}{3} \left( \frac{1}{1.43D^*} \right). \]
\[ \text{Case 3: If } y < -1, \text{ then the left side of (27) is bounded below by} \]
\[
\frac{31}{4} \exp(-1/D^*) \exp((1/D^*)|y|) \geq \frac{31}{4} \exp(-1/D^*) \left( 1 + |y|/D^* \right),
\]  
so in this case (27) follows because (10) also gives
\[ \bar{L} \geq \frac{1}{3} \left( \frac{1}{1.43D^*} \right). \]
\[
\text{This proves (27), and (28) holds because } e^{-x} \geq 0. \text{ Therefore, to conclude that (23) holds, it remains to check (29). To this end, first notice that } e^{-x+1} \geq \exp(-(1+0.75)|x|) \geq -3x \text{ for all } x \in \mathbb{R}, \text{ so the left side of (29) is bounded below by} \]
\[
\frac{31}{8} e^{-3x} \exp \left( \frac{3}{4} - \ln(4) \right) + \frac{1}{8} \geq \frac{31}{8} e^{-3x} + \frac{1}{8},
\]  
so (29) holds because our conditions (10) on A and B also give
\[ \bar{L} \geq B \geq 8(D^*)^2 x^2 \exp(1/2). \text{ Since (27)--(29) all hold, we conclude that (23) is satisfied.} \]
\[ \text{To check (24), we consider two cases. Case 1: } |y| \leq |x|. \text{ In this case, it suffices to check that} \]
\[ \bar{L}(x + e^{-x} - 1) \exp \left( \frac{1}{2} (x + e^{-x} - 1) \right) \geq 4x^2. \]
\[
\text{However, we can check numerically that } Q(x) = (x + e^{-x} - 1) \exp(0.5(x + e^{-x} - 1))/x^2 \text{ is bounded below by 0.4 over } \mathbb{R} \setminus \{0\}. \text{ So (35) follows because (10) also gives } \bar{L} \geq A \geq 16(D^*)^2 \geq 19 \geq 4/0.4. \text{ Case 2: } \bar{L} \geq |x|. \text{ In this case, it suffices to check that} \]
\[ \bar{L} \left( e^{-(y - 1)} \right) \exp \left( \frac{1}{2D^*} (e^{-(y - 1)} - 1) \right) \geq 4y^2, \]
\[ \text{which we check by considering three subcases. If } y \in [-1, 1], \text{ then since we can numerically check that } |y| = (e^y - y - 1)/y^2 \text{ is bounded below by 0.36, condition (36) holds for all } y \in [-1, 1] \text{ because (10) gives} \]
\[ \bar{L} \geq \frac{B \geq 4D^*}{0.36} \approx 11.1D^*.
\]  
If \( y > 1 \), then \( e^y - y - 1 \geq \frac{1}{2}y^2 \), so (36) holds because (10) gives
\[ \bar{L} \geq B \geq 8D^*. \text{ If } y < -1, \text{ then } e^y - y - 1 \geq \max(1/|e|, |y|) \geq 1. \text{ Also, } e^y/p^2 \geq 1.84 \text{ on } (0, \infty). \text{ Therefore, in this subcase, the left side of (36) is bounded below by} \]
\[
\bar{L} \geq \frac{1}{2D^*} \left( (y - 1) \right) \exp \left( \frac{1}{2D^*} (y - 1) \right) \geq \frac{1}{2D^*} \left( 1.84 \right) \left( \frac{1}{2D^*} \right) y^2 
\]  
\[ \geq 0.166 \left( \frac{1}{D^*} \right) \exp \left( -\frac{1}{2D^*} y^2 \right), \]
so (36) holds in this case because (10) gives
\[ \bar{L} \geq B \geq 24(D^*)^3 \exp \left( \frac{1}{2D^*} \right). \]
\[
\text{Hence, (24) holds in all cases, so the theorem follows when } M_1 = M_2 = 1.
\]
\[
\text{2nd Part: General Case Where } M_1 \text{ and } M_2 \text{ Satisfy C1--C4.} \]
\[ \text{We next indicate the additional arguments needed for the general case of locally Lipschitz functions } M_1 \text{ and } M_2 \text{ satisfying C1--C4.} \]
\text{Since the } 1\text{'s in (1) are replaced by the } M_1\text{'s, we must add the terms}
\[ -x(D^*)(M_2(x, y) - 1) - y(M_1(x, y) - 1) \]
\text{to the time derivative of } V_2, \text{ where } V_2(x, y) = -xy \text{ as before. Hence, we can use our decay estimate (7) to check that the theorem will follow once we show that}
\[ 2L^* e^{2V_1(x, y)} e^{-x} (e^x - 1)^2 \geq \left| xD^*(M_2(x, y) - 1) \right| + y(M_1(x, y) - 1) \]
\[
\text{holds for all } (x, y) \in \mathbb{R}^2.
\]
\[ \text{To check this property for } L^*, \text{ we first build a constant } J_1 \geq 0 \text{ such that} \]
\[ J_1 \exp(2(x + e^{-x} - 1))e^{-x} \geq \sqrt{m_2}|x| \]
\[
\text{for all } x \in \mathbb{R}. \text{ We consider two cases. Case 1: } x \geq 0. \text{ In this case, we have} \]
\[ \exp(2(x + e^{-x} - 1))e^{-x} \geq 4e^{-2} \geq |x|^{-2}, \text{ so it suffices to have} \]
\[ J_1 \geq e^2 \sqrt{m_2} \text{ in this case. Case 2: } x < 0. \text{ In that case, we use} \]
\[ \exp(2(x + e^{-x} - 1))e^{-x} \geq \exp(x + 2(e^{-x} - 1)) \geq \exp(|x|) \geq |x|, \]
\[ \text{so in this case it suffices to have } J_1 \geq \sqrt{m_2}. \text{ Hence, } J_1 = e^2 \sqrt{m_2} \text{ satisfies our requirements in both cases.} \]
\[
\text{Recalling Condition C4, it follows that}
\[ 2L^* e^{2V_1(x, y)} e^{-x} (1 - e^y)^2 \geq \left| xD^*(M_2(x, y) - 1) \right| \]
\[
\text{holds for all } (x, y) \in \mathbb{R}^2, \text{ if we choose } L^* = J_1D^* \sqrt{m_2}/2, \text{ since then the left side of (43) is} \]
\[ 2L^* e^{2V_1(x, y)} e^{-x} (1 - e^y)^2 \geq \left| xD^*(M_2(x, y) - 1) \right| \]
\[
\text{for all } (x, y) \in \mathbb{R}^2 \text{ and such that } L^* \geq L^* + L^* \]
\[
\text{To find } L^*, \text{ we first build a constant } J_2 \geq 0 \text{ such that } \]
\[ \exp((1/D^*)((e^y - y - 1) - (1 - e^y)^2) \geq m_1 \text{ for all } y \in \mathbb{R}, \text{ by considering three cases. Case 1: } y \geq 0. \text{ For that case, it suffices to require that} \]
\[ J_2 \geq m_1, \text{ since } e^y - 1 \geq y \text{ for all } y \geq 0. \text{ Case 2: } y \in [-1, 0]. \text{ For that case, it suffices to choose } J_2 \text{ such that} \]
\[ J_2 \geq m_1 \text{ sup}|y|^2/(1 - e^y)^2 : y \in [-1, 0] | = m_1(1 - e^{-1})^{-2}. \text{ Case 3: } y < -1. \text{ In that case, we have} \]
\[ \exp((1/D^*)((e^y - y - 1) - (1 - e^y)^2) \text{ exp}(-1/D^*) \geq \left| xD^*(M_2(x, y) - 1) \right| \]
\[
\text{so we require that } J_2 \geq 2m_1 \text{ exp}((1/D^*) \geq (1 - e^{-1})^{-2}. \text{ Hence, we can choose} \]
\[ J_2 = m_1 (1 - e^{-1}) \text{ max} \left\{ 1, \left| (D^*)^2 e^{1/D^*} \right| \right\} \]
\[
\text{to satisfy our requirements on } J_2 \text{ in all three cases.} \]
By Condition C3 and the fact that \( x + e^{-x} - 1 \geq x - 1 \) for all \( x \in \mathbb{R} \), we conclude that the choice \( L^* = J_2 e/2 \) satisfies

\[
|y|, M_1(x, y) - 1| \leq \tilde{m}_1 y^2 \\
\leq J_2 \exp((1/D^*)(e^y - y - 1)) (1 - e')^2 \\
\leq 2J_2 e V^*_N(y, x) e^{-x} (1 - e')^2
\]

(48)

for all \((x, y) \in \mathbb{R}^2 \), which gives (45). Theorem 2 now follows because our choice (11) satisfies \( L^* \geq L^*_N + \tilde{L}_1^* \). \( \square \)

Our proof of Theorem 2 shows that in the special case where \( M_1 = M_2 = 1 \), the function \( V_3 \) defined in (12) is a strict Lyapunov function for (1) in closed loop with our control \( u = y - x \) that satisfies the same decay condition as \( V_4 \). Due to the importance of the \( M_1 = M_2 = 1 \) case, we state this special case separately as the following corollary of Theorem 2:

**Corollary 1.** Let \( P \) and \( D^* \) be positive constants satisfying (9) and \( \tilde{L} \) be any constant satisfying \( \tilde{L} \geq \max(A, B) \), where \( A \) and \( B \) are defined by (10). Then the function \( V_3 \) defined in (12) satisfies \( V_3(x, y) \geq \frac{1}{2} V_1(x, y) \) for all \((x, y) \in \mathbb{R}^2 \) and

\[
V_3(x, y) \leq -\frac{1}{2} \left( \tilde{L} e^{V_3(x, y) - x} (1 - e')^2 + D^* x (1 - e^{-x}) \right)
\]

(49)

along all trajectories of (1) in closed loop with (3), so \( V_3 \) is a strict Lyapunov function for (1) and (3) on \( \mathbb{R}^2 \). \( \square \)

**Remark 1.** Our preliminary conference version of this work (i.e., Malisoff and Krstic (2015)) provides a different strict Lyapunov function for (1) in closed loop with (3) on \( \mathbb{R}^2 \), of the form

\[
V_3(x, y) = \int_0^{V_1(x, y)} \mathcal{R}(s) ds - xy,
\]

(50)

where \( \mathcal{R} : [0, \infty) \to [1, \infty) \) is any continuous increasing function satisfying

\[
\mathcal{R}(s) \geq \tilde{c} (1 + D^*)^2 e^{2s + D^* - 1},
\]

\[
\mathcal{R}(s) \geq \frac{4D^*}{P} (e^{s_0} + 1) \quad \text{and}
\]

\[
\mathcal{R}(s) \geq 4(D^*/P) (e^{s_0 + 1} + 1) (1 + e^2(D^* + 1))
\]

(51)

for all \( s \geq 0, P \) is any positive constant satisfying (9), and the constant \( \tilde{c} \geq 2.12128 \) is arbitrary. For instance, we can choose \( \mathcal{R} \) to be the sum of the right sides of (51). The preceding \( V_3 \) construction may have smaller coefficients than the one in (12), but it has larger powers in the exponentials and is more complicated than the one in Theorem 2. Due to space constraints, we do not prove here that the preceding alternative construction gives a strict Lyapunov function. However, we believe that our simpler choice in (12) may be better suited for the robustness and delays analyses below, which were not included in Malisoff and Krstic (2015). Also, Malisoff and Krstic (2015) did not consider the large class of systems with positivity constraints that we study below, nor did it cover choices of the functions \( M_1 \) and \( M_2 \) that are different from 1, so we believe that the present work adds considerable value relative to Malisoff and Krstic (2015).

**Remark 2.** Notice that

\[
\hat{y}(x, y) = \tilde{L} \exp(V_1(x, y) - x) (1 - e')^2 + D^* x (1 - e^{-x})
\]

(52)

is proper and positive definite. Hence, \( V_3 \) is proper and positive definite along all trajectories of (1) in closed loop with (3) by (13). In Sections 5 and 8, we use this properness to design augmented feedbacks that enjoy key ISS properties with respect to additive uncertainty on controls.

4. Remarks on Matrosov strictification approach

Our strict Lyapunov function construction (12) is beyond the scope of the Matrosov construction in Malisoff and Mazenc (2009, Chap. 5). To see why, recall that Malisoff and Mazenc (2009) constructs global strict Lyapunov functions, assuming that one knows a nonstrict Lyapunov function \( V_1 \) such that

\[
\nabla V_1(Z) f(Z) \leq -r_1(Z) h_1^2(Z) - \cdots - r_m(Z) h_m^2(Z)
\]

(53)

holds along all trajectories of the system

\[
\dot{Z} = f(Z),
\]

(54)

for suitable functions \( r_i \) and \( h_i \) and some constant \( m \); for simplicity, we omit the additional Lie derivative conditions on the \( h_i \)'s that are also required in Malisoff and Mazenc (2009). Then the strict Lyapunov functions from Malisoff and Mazenc (2009) have the form

\[
V_{x+1}(Z) = \sum_{i=1}^{N} \Omega_i(k_i(V_i(Z)) + V_i(Z))
\]

(55)

for suitable functions \( \Omega_i \) and \( k_i \) with the auxiliary functions

\[
V_i(Z) = -\sum_{i=1}^{m} \int_{-h_i(Z)}^{h_i(Z)} h_i(Z) h_i(Z), \quad i = 2, \ldots, N
\]

(56)

(56) (using standard Lie derivative notation).

It is natural to try to apply the preceding construction to (1) in closed loop with (3) and \( Z = (x, y) \), the nonstrict decay condition (4), \( m = 1 \), and \( h_1(Z) = e^y - 1 \). If we wish to have only one auxiliary function \( V_2 \), then we take \( N = 2 \), and (56) becomes

\[
V_2(Z) = D^* e^y (e^{y') (1 - e^{-x}) (1 - e^{-x}),
\]

(57)

which is much more complicated than the auxiliary function \( V_2(Z) = -xy \) that we used in Theorem 2. Moreover, Malisoff and Mazenc (2009) does not build \( \Omega_i \) and \( k_i \) explicitly in general. In this way, the Matrosov strictification results in Malisoff and Mazenc (2009) are not amenable to the ISS robustification process we discuss next.

5. Using strictness to certify ISS

We next use our strict Lyapunov function constructions to prove a novel control redesign result for ensuring ISS with respect to additive uncertainty on the control \( u \); see Remarks 3 and 5 for other control redesign results, such as the \( u^d \) construction for control affine systems that we discussed in the introduction. For simplicity, we assume that \( M_1 = M_2 = 1 \) in (6), but we can prove analogs in more complex cases that are covered by Theorem 2.

**Theorem 3.** Let \( D^* > 0 \) be constants that satisfy the requirements from Theorem 2. Then

\[
\begin{cases}
\dot{x} = 1 - e^y \\
\dot{y} = D^* (1 - e^y) (x + e^y - 1 + \frac{1}{D^*}(e^y - 1))
\end{cases}
\]

(58)

in closed loop with

\[
\dot{u}_p(x, y) = y - x (1 + e^{y - x}) + \frac{i e^{y - x}}{D^*} (e^{y - x}) \exp \left( x + e^{y - x} - 1 + \frac{1}{D^*}(e^y - 1) \right)
\]

(59)

is input-to-state stable with respect to the set of all measurable essentially bounded functions \( \delta : [0, \infty) \to \mathbb{R} \). \( \square \)
Proof. Let $\delta : [0, \infty) \to \mathbb{R}$ be measurable and essentially bounded, and define $V_5$ as in (12). Then, since $u_p(x, y) = y - x + e^{x^3/2} \leq \exp(V_1(x, y))$, for all $(x, y) \in \mathbb{R}^2$, we obtain

$$V_3(x, y) \leq -\frac{\exp(V_1(x, y))}{2} e^{-x^3/2} - D^x(1 - e^{-x^3/2})$$

along all trajectories of (58)–(59), by (13), where $e$ is a control to ensure ISS since (62) is globally asymptotically stable to 0 under the perturbations $\delta$ and the augmented control (59) as before, but we can extend the arguments of this section to the more general system (6); see below for an analysis of other types of delays. Let us take $Z = (x, y)$ as before and the function $f(Z, v, \Delta) = (1 - \exp(\sigma)), D^s(1 - \exp(u_p(Z + \Delta)))^T$. Then we can rewrite (64) as

$$\tilde{z}(t) = f(Z(t), v(Z(t - t_m)), \delta(t))$$

with the perturbations $\delta$ and the augmented control (59) as before, but we can extend the arguments of this section to the more general system (6); see below for an analysis of other types of delays. Let us take $Z = (x, y)$ as before and the function $f(Z, v, \Delta) = (1 - \exp(\sigma)), D^s(1 - \exp(u_p(Z + \Delta)))^T$. Then we can rewrite (64) as

$$\tilde{z}(t) = f(Z(t), v(Z(t - t_m)), \delta(t))$$

for all $Z \in \mathbb{R}^2$ and $\Delta \in \mathbb{R}^2$.

6. Robustness of ISS to delays in interconnection

We next consider cases with delays in the interconnection term, which (for the special case where our functions $M$, in (6) are identically 1) produces the model

$$\begin{cases}
\dot{x}(t) = f(x, y, Z) \\
\dot{y}(t) = D^s(1 - \exp(u_p(Z + \Delta)))^T
\end{cases}$$

(64)

with the perturbations $\delta$ and the augmented control (59) as before, but we can extend the arguments of this section to the more general system (6); see below for an analysis of other types of delays. Let us take $Z = (x, y)$ as before and the function $f(Z, v, \Delta) = (1 - \exp(\sigma)), D^s(1 - \exp(u_p(Z + \Delta)))^T$. Then we can rewrite (64) as

$$\tilde{z}(t) = f(Z(t), v(Z(t - t_m)), \delta(t))$$

for all $Z \in \mathbb{R}^2$ and $\Delta \in \mathbb{R}^2$.

Then Teel (1998, Theorem 3) provides an upper bound $\tilde{t}^2 > 0$ such that (64) is ISS with gain $\gamma = \alpha_3^{-1} \circ \alpha_2 \circ \gamma$ for all constant delays $\Delta \in [0, \tilde{t}^2]$, with a nonzero offset and a nonzero restriction. Roughly speaking, nonzero offset and nonzero restriction means that the standard ISS estimate from Khalil (2002) holds except with an added constant on the right side of the usual ISS estimate (called an offset), and that this estimate with the added constant is only guaranteed to hold if the initial states and perturbations are in certain neighborhoods of zero (called restrictions); see Teel (1998, Definition 1). Also, Teel (1998) shows that as $\tilde{t}^2$ approaches zero, the offset converges to zero and the restriction becomes arbitrarily large. (Although Teel, 1998, Section III models the delay as occurring in the feedback control, Teel, 1998, Theorem 3 also covers cases like ours, where the delay is elsewhere in the system.) Due to space constraints, we will not compute $\tilde{t}^2$ here. However, our ISS Lyapunov function $V_3$ makes it possible to obtain explicit estimates of the magnitude of the delay that can be handled without destroying stability properties. This is an additional advantage of our robustified controller (59) and our ISS Lyapunov function $V_3$ for ensuring ISS like properties under small delays, which would not have been possible if we had relied on nonstrict Lyapunov functions.

7. Systems with positivity constraints

While stated for time invariant chains of exponential integrators [1], our work also applies to a broad scope of tracking problems for time varying systems of the form

$$\begin{cases}
\dot{X} = G(t, X, Y)X \\
\dot{Y} = H(t, X, Y, D)Y
\end{cases}$$

(65)

Recall that $\mathcal{X}$ is the set of all continuous strictly increasing functions $\sigma : [0, \infty) \to [0, \infty]$ that are $0$ at $0$, which includes saturating functions such as $\sigma(t) = t/(1 + t)$ and the identity function $\sigma(t) = t$. Also, $\mathcal{X}_{\infty}$ is the set of all unbounded class $X$ functions.
having the state space \((0, \infty)^3\), where \(G\) and \(H\) are \(C^2\) functions that admit the affine forms
\[
G(t, X, Y) = G^t(t, X) + G^r(t, X)\sigma(Y) \quad \text{and} \quad H(t, X, Y, D) = H^d(t, X, Y) + H^r(t, X, Y)D
\]
for some known \(C^1\) function \(\sigma \in \mathcal{K}\) and known functions \(G^t, G^r, H^d, H^r\) that are uniformly locally bounded and uniformly locally Lipschitz in \(t\) on \((0, \infty)^2\), where the superscripts \(d\) and \(s\) indicate the drift and slope terms in the affine structure, respectively (but see below for more general cases with delays).\(^4\) Such systems arise in the study of predator–prey interactions; see Section 8, which also uses the strictness of our Lyapunov function \(V_2\) and control redesign to prove ISS under additive uncertainty on the control \(D\). We assume:

**Assumption 1.** The functions \(G^t\) and \(H^s\) are nowhere zero on \([0, \infty) \times (0, \infty)^2\), and (65) admits a bounded \(C^2\) reference trajectory \((X_t, Y_t)\) such that

\[
1 > G(t, X, Y) - G(t, X_t, Y_t) \tag{66}
\]
holds for all \(t \geq 0\) and \((X, Y) \in (0, \infty)^2\), and such that \(\inf_{t \geq 0} |G^t(t, X_t, Y_t)| > 0\).

However, we can replace the upper bound 1 in (66) by any positive constant \(\mathcal{M}_s\), by scaling time by \(1/\mathcal{M}_s\), which is equivalent to dividing \(G\) through by \(\mathcal{M}_s\). For simplicity, we take \(\mathcal{M}_s = 1\). However, we do not impose sign restrictions on \(X, Y, G^t, G^r, H^d, H^r\), nor do we require \(G\) to be bounded; see Section 8 for applications. Our condition \(\inf_{t \geq 0} |G^t(t, X_t, Y_t)| > 0\) is used to rule out cases such as

\[
\lim_{t \to \infty} |G^t(t, X_t, Y_t)| = 0.
\]

The control objective is to choose \(D\) such that \((X, Y)\) tracks \((X_t, Y_t)\), i.e., for the tracking error \((X - X_t, Y - Y_t)\) to converge to 0 for all initial states.

To rewrite (65) in the form (1), we use the new changes of variables \(x(t) = \ln(X(t)/X_t(t))\) and \(y(t) = \ln(1 - G(t, X(t), Y(t)) + G(t, X_t(t), Y_t(t)))\). Since the slope terms \(G^t\) and \(H^d\) are never 0 and \(\sigma'(r) > 0\) for all \(r > 0\), we can find a function \(D(t, X, Y)\) such that along all trajectories of (65) on \((0, \infty)^2\), we have

\[
\dot{x}(t) = \frac{X_t(t)}{X(t)} \quad \text{and} \quad \dot{y}(t) = \frac{-1}{1 - G(t, X(t), Y(t)) + X_t(t)/X(t)} \left[ G_t(t, X, Y) + G_X(t, X, Y)Y + \sigma'(Y) \right]\frac{H^d(t, X, Y)}{X(t)}
\]

and

\[
\dot{y}(t) = \frac{-1}{1 - G(t, X(t), Y(t)) + X_t(t)/X(t)} \left[ \frac{G_t(t, X, Y)}{X(t)} + \sigma'(Y) \right] \frac{H^r(t, X, Y)}{X(t)}
\]

Hence, we obtain (1) with the choice \(D^* = 1\). Therefore, our theory from the previous sections ensures that the \((x, y)\) dynamics are globally asymptotically stable to 0 on \(\mathbb{R}^2\). One can now use the fact that \(\sigma \in \mathcal{K}\) and the boundedness of \(Y_t\) to show that this implies that our tracking objective is realized. The details are as follows.

---

\(^3\) By uniform local boundedness and uniform local Lipschitzness of \(H^d\) in \(t\), we mean that there exist a function \(\gamma : \mathcal{K} \to [0, \infty)\) and a constant \(c > 0\) such that \(|H^d(t, X, Y)| \leq c + \gamma(X)(Y)\) on \([0, \infty) \times (0, \infty)^2\) and such that \(\frac{d}{dt}(t, X_t, Y_t) - H^d(t, X_t, Y_t) \leq c + \gamma(X)(X_t - \bar{X}_t, Y_t - \bar{Y}_t)\) hold for all \(t \geq 0\), all \(X_0, X_t, Y_t\), and \(Y_t \in (0, X),\) and all \(R > 0\), and similarly for \(G^t, G^r, H^d, H^r\).

\(^4\) A continuous function \(\beta : [0, \infty) \times [0, \infty) \to [0, \infty)\) is defined to be of class \(\mathcal{K}\) on \([0, \infty)^2\) if \(\beta(\cdot, s) \in \mathcal{K}\) for all \(s \geq 0\), the function \(\beta(t, s)\) is in \(\mathcal{K}\) and \(\beta(t, s)\) is nonincreasing and satisfies \(\beta(t, s) \to 0\) as \(s \to \infty\).
coefficient for lynxes. It is reasonable for $D$ to take both positive and nonpositive values. By scaling time, we can assume that $r = 1$.

The system (70) is not of the chain of exponential integrators form (1). However, if we choose $(X, Y) = (H, L)$ and $(X_r, Y_r) \in (0, \infty)^2$ to be an equilibrium of (70), then all requirements from Section 7 are satisfied (with $\sigma(Y) = Y, G(t, X, Y) = 1 - X/k - aY/(c + X), G^2(t, X) = 1 - X/k, G^2(t, X) = -q/(c + X), H(t, X, Y, D) = abX/(c + X) - D, H^2(t, X, Y) = abX/(c + X),$ and $H^2(t, X, Y) = -1$), so (70) is covered by our theory. Due to space constraints, we will not construct $D$ here, and instead turn to a tracking problem (but see Remark 5 for ways to prove ISS for dynamics such as (70), with respect to additive uncertainties on $D$, which can represent model uncertainty or actuator errors).

8.2. Tracking example

Consider the system

$$\begin{align*}
\dot{X} &= (1 - A(t)Y)X \\
\dot{Y} &= (D^* + C(X, Y) - D)Y
\end{align*}$$ (71)

for any constant $D^* > 0$, where $A : \mathbb{R} \to (0, \infty)$ is any nondecreasing $C^1$ function, $C$ is locally Lipschitz and nonnegative valued, and the state space is $(0, \infty)^2$. The model (71) can represent a predator (with population $Y$) consuming its prey (which has population $X$) with control $D$, but see Remark 5-7 for models with additional mortality rates, uncertainties, and gestation delays. The objective is for $(X(t), Y(t))$ to track $(X_r, 1/A(t))$ where $X_r > 0$ is any constant, i.e., the dynamics for the tracking error are UGAS to 0.

We use the changes of variables $x(t) = \ln(X(t)/X_r)$ and $y(t) = \ln(A(t)Y(t))$, which correspond to choosing $\sigma(Y) = Y$ in the preceding section. Then

$$\begin{align*}
\dot{x}(t) &= \dot{X}(t)/X(t) = 1 - A(t)Y(t) = 1 - e^{\lambda(t)} \\
\dot{y}(t) &= \frac{\dot{A}(t)Y(t) + A(t)\dot{Y}(t)}{A(t)Y(t)} = \frac{\dot{Y}(t)}{A(t)} + \frac{\dot{A}(t)}{A(t)} \\
&= D^* + C(X(t), Y(t)) - D + \frac{\dot{A}(t)}{A(t)} \\
&= D^*(1 - e^{\lambda(t)}),
\end{align*}$$ (72)

if we pick $u$ and $D$ such that

$$u(t) = \ln\left(\frac{1}{D^*} (D - C(X(t), Y(t)) - \frac{\dot{A}(t)}{A(t)})\right)$$ (73)

holds for all $t \geq 0$. Since the new dynamics agree with (1), our tracking objectives will be met if we choose $u(t) = y(t) - x(t)$, i.e., the positive valued function

$$D(t) = \frac{D^*X(t)A(t)Y(t)}{X(t)} + \frac{\dot{A}(t)}{A(t)} + C(X(t), Y(t)),$$ (74)

which satisfies $D(t) - (\dot{A}(t)/A(t)) - C(X(t), Y(t)) \to D^*$ as $t \to \infty$. The strict Lyapunov function constructions we gave above apply to (71) as well.

Remark 5. Even though (65), (70), and (71) have positivity constraints on the states, we can use our strict Lyapunov function $V_2$ and the control redesign approach from Remark 3 to redesign the controls for (65), (70), and (71) to prove ISS results with respect to piecewise continuous additive uncertainties $\delta(t)$ on the controls, which can represent actuator or modeling uncertainties. To emphasize the main points, we only show how this can be done for (71), but analogous arguments apply for (65) and (70). Consider

$$\begin{align*}
\dot{X} &= (1 - A(t)Y)X \\
\dot{Y} &= (D^* + C(X, Y) - D - D_{add} + \delta(t))Y
\end{align*}$$ (75)

where $D^*, C$, and $A$ satisfy the same requirements that they did in (71), $D$ is the control in (71), and the additional control component $D_{add}$ will be derived. Then similar calculations to the ones we used to derive (72) give

$$\begin{align*}
\dot{x} &= 1 - e^{\lambda(t)} \\
y &= D^*(1 - e^{\lambda(t)}) - D_{add} + \delta(t)
\end{align*}$$ (76)

where $x(t) = \ln(X(t)/X_r)$ and $y(t) = \ln(A(t)Y(t))$. Choose $V_2$ from Theorem 2 and

$$D_{add}(t, x, y) = \partial V_3(x, y)/\partial y = \frac{\partial V_3(x, y)/\partial y}{\partial V_3(x, y)/\partial y} \delta(t) = \exp(V_1(\ln(X/X_r), \ln(A(t)Y)))$$

where $\dot{V}_1 \leq -\frac{1}{2} \dot{g}(x, y) - \frac{\partial V_3(x, y)/\partial y}{\partial V_3(x, y)/\partial y} \dot{y}(t) + \frac{\partial V_3(x, y)/\partial y}{\partial V_3(x, y)/\partial y} \delta(t)$

$$\leq -\frac{1}{2} \dot{g}(x, y) + \frac{1}{2} \delta^2(t)$$ (78)

along all trajectories of (75). Hence, $V_1$ is an ISS Lyapunov function for (76) on $\mathbb{R}^2$, which gives ISS for the tracking dynamics for (75). This feedback design would not have been possible if we had used the nonstrict Lyapunov function $V_1$.

Remark 6. For any constants $m_1$ and $m_2$ in $[0, 1)$, we can change variables to cover more general chains of exponential integrators of the form

$$\begin{align*}
\dot{x} &= 1 - m_1 - e^{\lambda(t)} \\
y &= D^*(1 - m_2 - e^{\lambda(t)})
\end{align*}$$ (79)

which could be used to represent additional constant mortality rates (where the $m_i$'s could represent mortality from famine, illegal hunting, preying by other predators, and other factors). This is because the new variables $(x_{new}, y_{new}) = (x/(1 - m_1), y - \ln(1 - m_2))$, the new feedback $u_{new} = u - \ln(1 - m_2)$, and the new constant $D_{new} = D^*(1 - m_2)$ satisfy

$$\begin{align*}
\dot{x}_{new} &= 1 - e^{\lambda_{new}} \\
y_{new} &= D_{new}^*(1 - e^{\lambda_{new}})
\end{align*}$$ (80)

which has the form (1). Hence, we can assume without loss of generality that the $m_i$'s are zero.

Remark 7. Incorporating constant gestation delays $\tau > 0$ in the predator dynamics in (71) would produce time delay systems such as

$$\begin{align*}
\dot{x}(t) &= (1 - A(t)Y(t))X(t) \\
y(t) &= D^*(Y(t) - \tau + C(X(t), Y(t)) - D)Y(t)
\end{align*}$$ (81)

on the state space $(0, \infty)^2$, but this would not prevent transforming the model into the chain of exponential integrators (1). To see why, we assume for simplicity that $A(t) = 1$ and $C(X, Y) = 0$ for all $t \geq 0$ and $(X, Y) \in (0, \infty)^2$, but similar arguments apply to more general choices of the functions $A$ and $C$. Then the same changes of variables $x(t) = \ln(X(t)/X_r)$ and $y(t) = \ln(Y(t))$ produce

$$\begin{align*}
\dot{x}(t) &= 1 - e^{\lambda(t)} \\
y(t) &= D^*\frac{Y(t) - \tau}{Y(t)} - D = D^*(1 - e^{\lambda(t)} - \lambda(t))
\end{align*}$$ (82)

when we choose the control

$$D = D^*\frac{X(t)Y(t)}{X(t)} + \frac{Y(t) - \tau}{Y(t) - 1}.$$ (83)
so we again obtain (1). Hence, our change of variables transformed a time delayed system (81) that has the state space $\mathbb{R}^2$ into the new system (1) with state space $\mathbb{R}^2$ that has no delays. Moreover, we do not require any upper bound on the delay $\tau$, which sets our work apart from the emulation approach to time delay compensation, which generally imposes maximum allowable delays (but which makes our work complementary to the prediction approach for linear systems which also allows arbitrarily long delays); see Karafyllis, Malisoff, Mazenc, and Pepe (2015) and Krstic (2009). This is yet another novel and interesting feature of our analysis.

9. Conclusions

Constructing strict Lyapunov functions is a challenging problem that has important implications for stability and robustness analysis for nonlinear systems. We proved stabilization results for two dimensional chains of exponential integrators, by converting a nonstrict Lyapunov function into a strict one. This made it possible to prove ISS with respect to additive uncertainty on a robustified control. This differed from the known robustification methods because the control in the chain of exponential integrators enters in a nonaffine way, and therefore was beyond the scope of the usual $L_2V$ feedback redesign for control affine systems. The conversion made it possible to prove stability properties under delays. We applied our results to broad classes of systems with positivity constraints on the states, including population models with gestation delays and uncertainties.

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References


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