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Brief Paper

Adaptive Backstepping with Parameter Projection: Robustness and Asymptotic Performance*

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Key Words—Robust adaptive control; projection; tuning functions; backstepping.

Abstract—We employ parameter projection to improve robustness of the adaptive backstepping design with tuning functions. However, since projection is not compatible with the existing recursive procedure, we also propose a controller modification which enables the use of projection in backstepping designs. The following robustness properties are achieved. When the relative degree of the linear system equals that of the model, the closed loop signals are globally uniformly bounded. If this condition is not verified then the region of attraction of the closed loop system is shown to be inversely proportional to the "size" of the unmodeled dynamics. In both cases, the tracking error is shown to be proportional to the size of the uncertainties. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction and problem statement*

Transient performance of adaptive backstepping designs compares favorably to that of traditional adaptive controllers (cf. Krstic *et al.*, 1995) (Sections 10.4 and 10.5). However, the question of their robustness is still open and is a topic of intensive current studies (Ikhouane and Krstic, 1995; Li *et al.*, 1995; Polycarpou and Ioannou, 1993; Yao and Tomizuka, 1995; Zhang and Ioannou, 1995a–c; Jiang and Praly, 1996). In Ikhouane and Krstic (1995) we showed that, by adding a (sufficiently differentiable) "switching" σ-modification into the tuning functions and the update law, we achieve regional robustness properties with a region of attraction inversely proportional to the "size" of the unmodeled dynamics. (The robustness properties are regional—and not global as in the case of traditional robust adaptive schemes (Ioannou and Sun, 1995)—because the adaptive backstepping controller is *nonlinear*.) To achieve robustness in Ikhouane and Krstic (1995), we did not employ any type of dynamic normalization (which may come as a surprise to readers familiar with traditional robust adaptive control, Ioannou and Sun, 1995). At the same time, our proof is much simpler than that in Naik *et al.*, (1992) where dynamic normalization was shown *not* to be necessary in a certainty-equivalence scheme.

A natural question that Ikhouane and Krstic (1995) raises is: Can σ-modification be replaced by parameter projection? In traditional adaptive control these two tools can be used interchangeably. This, however, is not the case in the backstepping approach. Because of the recursive character of backstepping, replacing σ-modification by projection would require a projection operator which is sufficiently differentiable. It is a little appreciated technical finesse that the projection operator cannot be made a C^1 function of the vector field even though it can be made a smooth function of the parameter estimate. In this paper we propose a *new adaptive backstepping design with tuning functions* which circumvents this technical obstacle. We consider multiplicative unmodeled dynamics $\mu\Delta$. When the relative degree of the system with uncertainties equals that of the nominal model, the closed loop signals are *globally* uniformly bounded. If the relative degree of the true system is less than that of the model, then the region of attraction of the closed loop system is proportional to $1/\mu^p$. Unlike the σ-modification design in Ikhouane and Krstic (1995), the projection design insures *global* stability when Δ is proper. The mean square of the tracking error is shown to be proportional to the size of the uncertainties.

The paper is organized as follows. In Section 2 we present the design procedure. Section 3 deals with the stability and asymptotic performance analysis of the closed loop system when the transfer function Δ is improper. In Section 4 we address the case where Δ is proper. Section 5 illustrates our robust adaptive scheme on a non-minimum phase unstable second order example.

Problem statement. The control objective is to asymptotically track a reference signal $y_r(t)$ with the output y of the plant

$$y(t) = \frac{B(s)}{A(s)}(1 + \mu\Delta(s))u(t) + d(t), \qquad (1.1)$$

where $A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ and $B(s) = b_m s^m + \cdots + b_1 s + b_0$. The parameters a_i and b_i are unknown. Without loss of generality, we assume that $\mu \geq 0$.

Assumption 1.1. The plant is minimum phase, i.e., the polynomial $B(s)$ is Hurwitz. The plant order (n), relative degree ($\rho = n - m$) and sign of the high frequency gain ($sgn(b_m)$) are known.

Assumption 1.2. The reference signal $y_r(t)$ and its first ρ derivatives are known and bounded and, in addition, $y_r^{(\rho)}$ is piecewise continuous.

Assumption 1.3. The transfer function Δ is stable and its relative degree is no lower than $-\rho + 1$. The output disturbance $d(t)$ and its first derivative are uniformly bounded. Moreover, $\dot{d}$ is piecewise continuous.

Assumption 1.4. The unknown parameter vector $\theta = (b_m, \ldots, b_0, a_{n-1}, \ldots, a_0)^T$ lies in a known bounded convex set

$$\Pi_\theta = \{\hat{\theta} \in \mathscr{R}^{n+m+1} / \mathscr{P}(\hat{\theta}) \leq 0\}$$

where $\mathscr{P}$ is a convex smooth function. Furthermore, an upper bound ϱ_{max} on the the absolute value of the unknown parameter $\varrho = 1/b_m$ is known.

Notation. The following notation is used throughout the paper, unless otherwise stated.

- c: a generic positive constant independent of μ, d, $\dot{d}$ and the initial conditions.

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- g: a generic positive constant independent of μ, d, $\bar{d}$ and possibly depending on the initial conditions.

2. *Design procedure: tuning functions with damping*

2.1. *Ideal case*

For clarity, we first present the design procedure in the absence of uncertainties, that is for $\mu = 0$ and $d \equiv 0$. We represent the plant (1.1) in the observer canonical form:

$$\dot{x} = A_0 x + (k - a)x_1 + bu, \qquad y = x_1, \tag{2.1}$$

where

$$A_0 = \begin{pmatrix} -k & \begin{matrix} I_{n-1} \\ 0 \dots 0 \end{matrix} \end{pmatrix}, \qquad k = (k_1, \dots, k_n)^{\mathrm{T}},$$

$$a = (a_{n-1}, \dots, a_0)^{\mathrm{T}}, \quad b = (0_{(\rho-1)\times 1}, b_m, \dots, b_0)^{\mathrm{T}}. \tag{2.2}$$

By filtering u and y with two n-dimensional filters

$$\dot{\eta} = A_0 \eta + e_n y, \qquad \dot{\lambda} = A_0 \lambda + e_n u, \tag{2.3}$$

the state estimate is formed as $\hat{x} = B(A_0)\lambda - A(A_0)\eta$, and the estimation error is defined as $\varepsilon = x - \hat{x}$. We define the vectors v_j, Ξ, ξ, ω and $\bar{\omega}$ as

$$\begin{aligned} v_j &= A_0^j \lambda, \quad j = 0, \dots, m, \\ \Xi &= -[A_0^{n-1}\eta, \dots, \eta], \\ \xi &= -A_0^n \eta, \\ \omega &= [v_{m,2}, \dots, v_{0,2}, \Xi_{(2)} - y e_1^{\mathrm{T}}]^{\mathrm{T}}, \\ \bar{\omega} &= [0, v_{m-1,2}, \dots, v_{0,2}, \Xi_{(2)} - y e_1^{\mathrm{T}}]^{\mathrm{T}}. \end{aligned} \tag{2.4}$$

Table 1. Tuning functions design for linear systems

$$z_1 = y - y_r \tag{2.8}$$

$$z_i = v_{m,i} - \hat{\varrho} y_r^{(i-1)} - \alpha_{i-1}, \qquad i = 2, \dots, \rho \tag{2.9}$$

$$\alpha_1 = \hat{\varrho}\bar{\alpha}_1 \tag{2.10}$$

$$\bar{\alpha}_1 = -(c_1 + d_1)z_1 - \xi_2 - \bar{\omega}^{\mathrm{T}}\hat{\theta} \tag{2.11}$$

$$\alpha_2 = -\hat{b}_m z_1 - \left[c_2 + d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2\right] z_2 + \beta_2 + \Upsilon_2 \tag{2.12}$$

$$\alpha_i = -z_{i-1} - \left[c_i + d_i\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2\right] z_i + \beta_i + \Upsilon_i \tag{2.13}$$

$$\beta_i = \frac{\partial \alpha_{i-1}}{\partial y}(\xi_2 + \omega^{\mathrm{T}}\hat{\theta}) + \frac{\partial \alpha_{i-1}}{\partial \eta}(A_0\eta + e_n y) + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_r^{(j-1)}} y_r^{(j)} + k_i v_{m,1} + \sum_{j=1}^{m+i-1} \frac{\partial \alpha_{i-1}}{\partial \lambda_j}(-k_j\lambda_1 + \lambda_{j+1}) \tag{2.14}$$

$$\tau_1 = (\omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1)z_1 \tag{2.15}$$

$$\tau_i = \tau_{i-1} - \frac{\partial \alpha_{i-1}}{\partial y}\omega z_i, \quad i = 2, \dots, \rho \tag{2.16}$$

Adaptive control law:

$$u = \alpha_\rho - v_{m,\rho+1} + \hat{\varrho} y_r^{(\rho)} \tag{2.17}$$

The resulting error system is

$$\dot{z} = A_z(z,t)z + W_\varepsilon(z,t)\varepsilon_2 + W_\theta(z,t)^{\mathrm{T}}\tilde{\theta} - b_m(\dot{y}_r + \bar{\alpha}_1)e_1\tilde{\varrho}, \tag{2.18}$$

where the system matrix $A_z(z,t)$ is given by

$$A_z(z,t) = \begin{bmatrix} -c_1 - d_1 & \hat{b}_m & 0 & \cdots & \cdots & 0 \\ -\hat{b}_m & -c_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 & 1 + \sigma_{23} & \sigma_{24} & \cdots & \sigma_{2,\rho} \\ 0 & -1 - \sigma_{23} & \ddots & \ddots & \ddots & \vdots \\ \vdots & -\sigma_{24} & \ddots & \ddots & \ddots & \sigma_{\rho-2,\rho} \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 + \sigma_{\rho-1,\rho} \\ 0 & -\sigma_{2,\rho} & \cdots & -\sigma_{\rho-2,\rho} & -1 - \sigma_{\rho-1,\rho} & -c_\rho - d_\rho\left(\frac{\partial \alpha_{\rho-1}}{\partial y}\right)^2 \end{bmatrix} \tag{2.19}$$

The control law and the parameter update laws are designed in ρ steps (Krstic *et al.*, 1995, Section 10.2.1). This design is summarized in Table 1. The design in Krstic *et al.* (1995), which we refer as "tuning functions with cancellations", employs the following terms:

$$\Upsilon_2 = \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\tau_2 + \left(\dot{y}_r + \frac{\partial \alpha_1}{\partial \varrho}\right)\dot{\hat{\varrho}}$$

$$\Upsilon_i = \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\Gamma\tau_i - \sum_{j=2}^{i-1}\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{i-1}}{\partial y}z_j + \left(y_r^{(i-1)} + \frac{\partial \alpha_{i-1}}{\partial \hat{\varrho}}\right)\dot{\hat{\varrho}},$$

$$i = 3, \dots, \rho \tag{2.5}$$

and the parameter update laws:

$$\dot{\hat{\theta}} = \Gamma\tau_\rho = \sum_{j=1}^{\rho} w_j z_j, \tag{2.6}$$

$$\dot{\hat{\varrho}} = -\gamma\,\mathrm{sgn}(b_m)(\dot{y}_r + \bar{\alpha}_1)z_1 = w_0 z_1. \tag{2.7}$$

with $\sigma_{ij} = (\partial\alpha_{i-1}/\partial\hat{\theta})\Gamma(\partial\alpha_{j-1}/\partial y)\omega$, and $W_\varepsilon(z,t)$ and $W_\theta(z,t)$ are

$$W_\varepsilon(z,t) = \begin{bmatrix} 1 \\ -\frac{\partial \alpha_1}{\partial y} \\ \vdots \\ -\frac{\partial \alpha_{\rho-1}}{\partial y} \end{bmatrix} \in \mathbb{R}^\rho,$$

$$W_\theta(z,t)^{\mathrm{T}} = W_\varepsilon(z,t)\omega^{\mathrm{T}} - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1 e_1^{\mathrm{T}} \in \mathbb{R}^{\rho\times p}. \tag{2.20}$$

The role of the terms (2.5) is to eliminate $\dot{\hat{\theta}}$ and $\dot{\hat{\varrho}}$ from the error system (2.18). The cancelling tuning functions design (Krstic *et al.*, 1994) ensures global stability of the closed loop system and asymptotic tracking when there are no uncertainties (that is $\mu = 0$ and $d \equiv 0$). In the presence of unmodeled dynamics and bounded disturbances this scheme does not guarantee the stability of the closed loop. Thus we need to change the parameter update law and the control law. The simplest modification to think of would be to employ parameter projection in the tuning functions (2.15), (2.16) and in the update laws (2.6), (2.7). However, to be compatible with the recursive backstepping procedure (2.8)–(2.16), the projection operator would need to be $\rho - 2$ times differentiable. While the projection operator can be made smooth with respect to the parameter estimates, it (is little appreciated that it) cannot be made differentiable with respect to the vector field (see Krstic *et al.*, 1995, Appendix E). Thus we have to resort to a more extensive modification of the tuning function design, in which the recursive stabilizing functions do not explicitly incorporate the non-differentiable tuning functions.

Instead of equation (2.5) we choose:

$$\Upsilon_i = -z_i\left\{\sum_{k=1}^{i-1}\left[f_{ik}\left(\left\|\frac{\partial\alpha_{i-1}}{\partial\hat\theta}\right\|\|w_k\|\right)^2 + f_{k+1,i}\left(\left\|\frac{\partial\alpha_k}{\partial\hat\theta}\right\|\|w_i\|\right)^2\right] + f_{1i}\|w_0\|^2\left(y_r^{(i-1)} + \frac{\partial\alpha_{i-1}}{\partial\hat\varrho}\right)^2\right\},$$
$$i = 2, \ldots, \rho \tag{2.21}$$

and refer to the new design as "tuning functions with damping".

The error system becomes

$$\dot z = \bar A_z(z,t)z + W_\varepsilon(z,t)\varepsilon_2 + W_\theta(z,t)^{\mathrm T}\tilde\theta - b_m(\dot y_r + \bar\alpha_1)e_1\tilde\varrho + Q_\varrho(z,t)\dot{\hat\varrho} + Q_\theta(z,t)^{\mathrm T}\dot{\hat\theta}, \tag{2.22}$$

where the system matrix $\bar A_z(z,t)$ is given by

$$\bar A_z(z,t) = \begin{bmatrix} -c_1-\sigma'_{11} & \hat b_m & 0 & \cdots & \cdots & 0 \\ -\hat b_m & -c_2-\sigma'_{22} & 1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & -1 & -c_\rho-\sigma'_{\rho\rho} \end{bmatrix}, \tag{2.23}$$

$$\sigma'_{11} = d_1,$$

$$\sigma'_{ii} = d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 + \sum_{k=1}^{i-1}\left[f_{ik}\left(\left\|\frac{\partial\alpha_{i-1}}{\partial\hat\theta}\right\|\|w_k\|\right)^2 + f_{k+1,i}\left(\left\|\frac{\partial\alpha_k}{\partial\hat\theta}\right\|\|w_i\|\right)^2\right] + f_{1i}\|w_0\|^2\left(y_r^{(i-1)} + \frac{\partial\alpha_{i-1}}{\partial\hat\varrho}\right)^2 \tag{2.24}$$

and $Q_\varrho(z,t)$ and $Q_\theta(z,t)^{\mathrm T}$ are

$$Q_\varrho(z,t) = \begin{bmatrix} 0 \\ -\left(\dot y_r + \dfrac{\partial\alpha_1}{\partial\hat\varrho}\right) \\ \vdots \\ -\left(y_r^{(\rho-1)} + \dfrac{\partial\alpha_{\rho-1}}{\partial\hat\varrho}\right)\end{bmatrix},$$

$$Q_\theta(z,t)^{\mathrm T} = \begin{bmatrix} 0 \\ -\dfrac{\partial\alpha_1}{\partial\hat\theta} \\ \vdots \\ -\dfrac{\partial\alpha_{\rho-1}}{\partial\hat\theta}\end{bmatrix}. \tag{2.25}$$

The role of the damping terms (2.21) is to counteract the destabilizing effect of $\dot{\hat\theta}$ and $\dot{\hat\varrho}$ in the error system (2.22). We choose as update laws for the parameter estimates $\hat\varrho$ and $\hat\theta$

$$\dot{\hat\varrho} = \mathrm{Proj}_{\Pi_\varrho}(-\gamma\,\mathrm{sgn}(b_m)\,(\dot y_r + \bar\alpha_1)z_1),$$
$$\dot{\hat\theta} = \mathrm{Proj}_{\Pi_\theta}(\Gamma\tau_\rho). \tag{2.26}$$

The projection operator is defined as in Krstic *et al.* (1995, p. 512)

$$\mathrm{Proj}_{\Pi_\theta}\{\tau\} = \begin{cases} \tau & \hat\theta\in\mathring{\Pi}_\theta \text{ or } \nabla_{\hat\theta}\mathcal{P}^{\mathrm T}\tau \le 0, \\ \left(I - c(\hat\theta)\Gamma\dfrac{\nabla_{\hat\theta}\mathcal{P}\nabla_{\hat\theta}\mathcal{P}^{\mathrm T}}{\nabla_{\hat\theta}\mathcal{P}^{\mathrm T}\Gamma\nabla_{\hat\theta}\mathcal{P}}\right)\tau & \\ & \hat\theta\in\Pi_{\theta,\varepsilon}\backslash\mathring{\Pi}_\theta \text{ and } \nabla_{\hat\theta}\mathcal{P}^{\mathrm T}\tau > 0, \end{cases} \tag{2.27}$$

$$c(\hat\theta) = \min\left\{1, \frac{\mathcal{P}(\hat\theta)}{\varepsilon}\right\}, \tag{2.28}$$

$$\Pi_{\theta,\varepsilon} = \{\hat\theta\in\mathcal{R}^{n+m+1}/\mathcal{P}(\hat\theta)\le\varepsilon\} \text{ for some positive real } \varepsilon. \tag{2.29}$$

This operator is locally Lipschitz but not C^1. The projection interval Π_ϱ is

$$\Pi_\varrho = \left[\mathrm{sgn}(b_m)\frac{1}{\sup_{\hat\theta\in\Pi_\theta}\|\hat\theta\|}, \mathrm{sgn}(b_m)\varrho_{\max}\right]. \tag{2.30}$$

The control u is chosen as in equation (2.17). With this choice, the derivative of the Lyapunov function

$$V_\rho \triangleq \sum_{j=1}^{\rho}\left(\frac{1}{2}z_j^2 + \frac{1}{d_j}\varepsilon^{\mathrm T}P_0\varepsilon\right) + \frac{|b_m|}{2\gamma}(\varrho-\hat\varrho)^2 + \frac{1}{2}(\theta-\hat\theta)^{\mathrm T}\Gamma^{-1}(\theta-\hat\theta) \tag{2.31}$$

satisfies

$$\dot V_\rho \le -\sum_{j=1}^{\rho}c_jz_j^2 - \frac{1}{2}\sum_{j=1}^{\rho}\frac{1}{d_j}\|\varepsilon\|^2 - \sum_{j=1}^{\rho}\frac{d_j}{2}\left(\frac{\partial\alpha_{j-1}}{\partial y}\right)^2z_j^2 - \sum_{j=2}^{\rho}z_j\frac{\partial\alpha_{j-1}}{\partial\hat\theta}\mathrm{Proj}_{\Pi_\theta}(\Gamma\tau_\rho)$$
$$- \sum_{j=2}^{\rho}z_j\left(y_r^{(j-1)} + \frac{\partial\alpha_{j-1}}{\partial\hat\varrho}\right)\mathrm{Proj}_{\Pi_\varrho}(w_0z_1)$$
$$- \sum_{j=2}^{\rho}z_j^2\left[f_{1j}\|w_0\|^2\left(y_r^{(j-1)} + \frac{\partial\alpha_{j-1}}{\partial\hat\varrho}\right)^2 + \sum_{k=1}^{j-1}f_{jk}\left(\left\|\frac{\partial\alpha_{j-1}}{\partial\hat\theta}\right\|\|w_k\|\right)^2 + f_{k+1,j}\left(\left\|\frac{\partial\alpha_k}{\partial\hat\theta}\right\|\|w_j\|\right)^2\right]. \tag{2.32}$$

The indefinite term $-\sum_{j=2}^{\rho}z_j\partial\alpha_{j-1}/\partial\hat\theta\,\mathrm{Proj}_{\Pi_\theta}(\Gamma\tau_\rho) - \sum_{j=2}^{\rho}z_j(y_r^{(j-1)} + \partial\alpha_{j-1}/\partial\hat\varrho\,\mathrm{Proj}_{\Pi_\varrho}(w_0z_1)$ reduces negativity of the derivative of V_ρ. The aim of the subsequent analysis is to quantify its effect. From property (ii) of Krstic *et al.* (1995, Lemma E1) and equation (2.6) it follows that

$$\|\mathrm{Proj}_{\Pi_\theta}(\Gamma\tau_\rho)\| \le \delta\sum_{k=1}^{\rho}\|w_k\||z_k|,$$
$$\|\mathrm{Proj}_{\Pi_\varrho}(w_0z_1)\| \le \|w_0\||z_1|, \tag{2.33}$$

where $\delta = \sqrt{\lambda_{\max}(\Gamma)/\lambda_{\min}(\Gamma)}$. With (2.33) and (2.32) we get

$$
\begin{aligned}
\dot V_\rho \le & -\sum_{j=1}^{\rho} \frac{c_j}{2} z_j^2 - \frac{1}{2}\sum_{j=1}^{\rho} \frac{1}{d_j}\|\varepsilon\|^2 - \sum_{j=1}^{\rho} \frac{d_j}{2}\left(\frac{\partial \alpha_{j-1}}{\partial y}\right)^2 z_j^2 \\
& + \sum_{j=2}^{\rho}\sum_{k=1}^{j-1}\left(\delta\left\|\frac{\partial \alpha_{j-1}}{\partial \hat\theta}\right\|\|w_k z_j\||z_k|\right. \\
& \left. - f_{jk}\left(\left\|\frac{\partial \alpha_{j-1}}{\partial \hat\theta}\right\|\|w_k z_j\|\right)^2 - \frac{c_k}{n_k} z_k^2\right) \\
& + \sum_{j=2}^{\rho}\sum_{k=1}^{j-1}\left(\delta\left\|\frac{\partial \alpha_{k}}{\partial \hat\theta}\right\|\|w_j z_j\||z_{k+1}|\right. \\
& \left. - f_{k+1,j}\left(\left\|\frac{\partial \alpha_{k}}{\partial \hat\theta}\right\|\|w_j z_j\|\right)^2 - \frac{c_{k+1}}{n_{k+1}} z_{k+1}^2\right) \\
& + \sum_{j=2}^{\rho}\left(\|w_0 z_j\|\left|y_r^{(j-1)} + \frac{\partial \alpha_{j-1}}{\partial \hat\varrho}\right||z_1|\right. \\
& \left. - f_{1j}\|w_0 z_j\|^2\left(y_r^{(j-1)} + \frac{\partial \alpha_{j-1}}{\partial \hat\varrho}\right)^2 - \frac{c_1}{n_1} z_1^2\right).
\end{aligned}
\tag{2.34}
$$

Thus, if we choose

$$
\begin{aligned}
& n_1 = 4(\rho - 1), \\
& n_k = 4(\rho - k) + 2, \quad k = 2, \ldots, \rho, \\
& f_{jk} \ge \frac{n_k}{4c_k}\delta^2, \quad j = 2, \ldots, \rho;\ 1 \le k \le j-1, \\
& f_{k+1,j} \ge \frac{n_{k+1}}{4c_{k+1}}\delta^2, \quad j = 2, \ldots, \rho;\ 1 \le k \le j-1, \\
& f_{1j} \ge \frac{n_1}{4c_1}, \quad j = 2, \ldots, \rho,
\end{aligned}
\tag{2.35}
$$

we obtain

$$
\dot V_\rho \le -\sum_{j=1}^{\rho} \frac{c_j}{2} z_j^2 - \frac{1}{2}\sum_{j=1}^{\rho} \frac{1}{d_j}\|\varepsilon\|^2 - \sum_{j=1}^{\rho} \frac{d_j}{2}\left(\frac{\partial \alpha_{j-1}}{\partial y}\right)^2 z_j^2. \tag{2.36}
$$

We now state the main result of this section.

Theorem 2.1 (Tuning functions with damping). All the signals in the closed-loop adaptive system consisting of the plant (1.1), the control law (2.17), the update laws (2.26), and the filters (2.3) are globally uniformly bounded, and asymptotic tracking is achieved:

$$
\lim_{t\to\infty} [y(t) - y_r(t)] = 0. \tag{2.37}
$$

Proof. Starting with equation (2.36), the rest of the proof is similar to Krstic *et al.* (1995, Theorem 10.6).

2.2. *Uncertain plant*

In the presence of uncertainties, the design procedure remains the same. However, additional terms appear in the right-hand side of equation (2.36) due to the presence of perturbations. We represent the plant (1.1) as

$$
\begin{aligned}
\dot x &= A_0 x + (k - a)x_1 + bu, \\
y &= (1 + \mu\Delta)x_1 + d.
\end{aligned}
\tag{2.38}
$$

Then the estimation error satisfies

$$
\dot\varepsilon = A_0\varepsilon + (a - k)(\mu\Delta x_1 + d). \tag{2.39}
$$

The derivative of the output is

$$
\dot y = x_2 - a_{n-1}y + \mu\Delta(s + a_{n-1})x_1 + \dot d + a_{n-1}d, \tag{2.40}
$$

so that the error system is

$$
\begin{aligned}
\dot z = & \bar A_z(z,t)z + W_\varepsilon(z,t)\varepsilon_2 + W_\theta(z,t)^{\mathrm T}\tilde\theta - b_m(\dot y_r + \bar\alpha_1)e_1\tilde\varrho \\
& + Q_\varrho(z,t)\dot{\hat\varrho} + Q_\theta(z,t)^{\mathrm T}\dot{\hat\theta} \\
& + W_\varepsilon(z,t)(\mu\Delta(s + a_{n-1})x_1 + \dot d + a_{n-1}d).
\end{aligned}
\tag{2.41}
$$

Combining equations (2.41), (2.40), (2.39) and (2.36) it follows that

$$
\begin{aligned}
\dot V_\rho \le & -\frac{c_1}{2}z_1^2 - \sum_{j=2}^{\rho}\frac{c_j}{2}z_j^2 - \sum_{j=2}^{\rho}\frac{d_j}{2}\left(\frac{\partial \alpha_{j-1}}{\partial y}\right)^2 z_j^2 \\
& + z_1(\mu\Delta(s + a_{n-1})x_1 + \dot d + a_{n-1}d) \\
& - \sum_{j=2}^{\rho} z_j \frac{\partial \alpha_{j-1}}{\partial y}(\mu\Delta(s + a_{n-1})x_1 + \dot d + a_{n-1}d) \\
& - \frac{1}{2}\sum_{j=1}^{\rho}\frac{1}{d_j}\|\varepsilon\|^2 + 2\sum_{j=1}^{\rho}\frac{1}{d_j}(a - k)^{\mathrm T}P_0\varepsilon(\mu\Delta x_1 + d).
\end{aligned}
\tag{2.42}
$$

The aim of the next section is to quantify the effect of the indefinite terms in the right-hand side of equation (2.42).

3. *Robustness properties with Δ improper*

In this section, we treat the most general case where Δ is improper, with a relative degree no lower than $-\rho + 1$. The stability analysis is carried out by using a similarity transformation to represent equation (2.38) as

$$
\begin{aligned}
\dot x_1 &= x_2 - a_{n-1}x_1, \\
&\ \vdots \\
\dot x_\rho &= c_b^{\mathrm T}\bar x - a_m x_1 + b_m u, \\
\dot\zeta &= A_b\zeta + b_b x_1, \\
y &= (1 + \mu\Delta)x_1 + d,
\end{aligned}
\tag{3.1}
$$

where $\bar x = (x_1, \ldots, x_\rho, \zeta^{\mathrm T})^{\mathrm T}$, $c_b \in \mathscr{R}^n$, and $b_b \in \mathscr{R}^m$. For stability analysis, we are interested in the deviation $\tilde\zeta = \zeta - \zeta_r$ which is governed by

$$
\dot{\tilde\zeta} = A_b\tilde\zeta + b_b x_{1r}, \tilde\zeta(0) = 0, \tag{3.2}
$$

where ζ_r is defined as

$$
\dot\zeta_r = A_b\zeta_r + b_b y_r, \zeta_r(0) = \zeta(0) \tag{3.3}
$$

and x_{1r} is defined as

$$
x_{1r} = x_1 - y_r. \tag{3.4}
$$

For the η variables we define analogously $\tilde\eta = \eta - \eta_r$

$$
\begin{aligned}
\dot{\tilde\eta} &= A_0\tilde\eta + e_n z_1, \tilde\eta(0) = 0, \\
\dot\eta_r &= A_0\eta_r + e_n y_r, \eta_r(0) = \eta(0).
\end{aligned}
\tag{3.5}
$$

Define the strictly proper and stable transfer functions Δ_1 and Δ_2 and the states v_1 and v_2 as

$$
\begin{aligned}
\Delta(s + a_{n-1}) &= \sum_{j=0}^{\rho}\varsigma_{1j}s^j + \Delta_1, \\
\Delta &= \sum_{j=0}^{\rho-1}\varsigma_{2j}s^j + \Delta_2, \\
\dot v_1 &= A_1 v_1 + b_{1v}x_{1r}, \\
\Delta_1 x_{1r} &= (1, 0, \ldots, 0)v_1 = v_{11}, \\
\dot v_2 &= A_2 v_2 + b_{2v}x_{1r}, \\
\Delta_2 x_{1r} &= (1, 0, \ldots, 0)v_2 = v_{21}.
\end{aligned}
\tag{3.6}
$$

The matrices A_1 and A_2 are Hurwitz since Δ is stable. We are now ready to introduce the augmented Lyapunov function V:

$$V = V_\rho + \frac{1}{k_\eta}\tilde{\eta}^T P_0 \tilde{\eta} + \frac{1}{k_\zeta}\tilde{\zeta}^T P_b \tilde{\zeta} + q_1 v_1^T P_1 v_1 + q_2 v_2^T P_2 v_2. \quad (3.7)$$

Note that V is a quadratic function $V = \chi^T P_\chi \chi$ of the vector

$$\chi = (z^T, \varepsilon^T, \tilde{\eta}^T, \tilde{\zeta}^T, v_1^T, v_2^T, \tilde{\theta}^T, \tilde{\varrho})^T. \quad (3.8)$$

It can be shown (Ikhouane and Krstic, 1995) that

$$\dot{V} \leq -\alpha\|X\|^2 + \beta + \mu^2 c((\Delta(s + a_{n-1})x_{1r})^2 + (\Delta x_{1r})^2), \quad (3.9)$$

where

$$\alpha = \min\left\{\frac{c_1}{4}, 2c_2, \ldots, 2c_\rho, \frac{\lambda_{\min}^{-1}(P_0)}{4}, \frac{\lambda_{\min}^{-1}(P_b)}{2}, \frac{\lambda_{\min}^{-1}(P_1)}{8}, \frac{\lambda_{\min}^{-1}(P_2)}{8}\right\},$$

$$\beta = c(\mu^2 + \dot{d}^2 + d^2), \quad (3.10)$$

$$X = (z^T, \varepsilon^T, \tilde{\eta}^T, \tilde{\zeta}^T, v_1^T, v_2^T)^T. \quad (3.11)$$

Using the fact that, due to the use of projection in the parameter update law, the parameter estimates are uniformly bounded, it follows:

$$\dot{V} \leq -\alpha' V + \beta' + \mu^2 c((\Delta(s + a_{n-1})x_{1r})^2 + (\Delta x_{1r})^2), \quad (3.12)$$

where

$$\frac{1}{\alpha'} = \frac{1}{\alpha}\max\left\{\frac{1}{2}, \sum_{j=1}^{\rho}\frac{1}{d_j}\lambda_{\max}(P_0), \frac{|b_m|}{2\gamma}, \lambda_{\max}(\Gamma^{-1}), \frac{\lambda_{\max}(P_0)}{k_\eta}, \frac{\lambda_{\max}(P_b)}{k_\zeta}, q_1\lambda_{\max}(P_1), q_2\lambda_{\max}(P_2)\right\}$$

$$\beta' = \beta + \frac{\alpha}{2}\left(\sup_{\hat{\theta}\in\Pi_{\theta}}\|\hat{\theta}\|^2 + (\varepsilon + \varrho_{\max})^2\right). \quad (3.13)$$

From this point on, the analysis continues exactly as in Ikhouane and Krstic (1995), leading to the following result.

Theorem 3.1. Consider the plant (1.1) subject to Assumptions (1.1)–(1.4) and the adaptive controller composed of the control law (2.17) and the parameter update laws (2.26). There exist positive constants μ^*, g and c independent of μ, $\dot{d}$ and d, and a positive integer m_1 independent of ρ such that for $0 \leq \mu < \mu^*$ and for

- $\|\lambda(0), \chi(0)\| \leq c/\mu^{1/m_1 4^{\rho-1}}$,
- $\|y_r\|_\infty + \|\dot{y}_r\|_\infty + \cdots + \|y_r^{(\rho)}\|_\infty \leq c/\mu^{1/m_1 4^{\rho-1}}$,
- $\|d\|_\infty \leq c/\mu^{1/m_1 4^{\rho}}$

we have

(i) All the signals of the closed loop are bounded;
(ii) The mean-square of the tracking error is proportional to the size of perturbations:

$$\int_t^{t+T}(y(\tau) - y_r(\tau))^2 d\tau \leq g + c\int_t^{t+T}(\mu^2 + \dot{d}(\tau)^2 + d(\tau)^2)d\tau, \quad \forall t, T \geq 0. \quad (3.14)$$

4. *Robustness properties with Δ proper*

In this section we suppose that Δ is proper. Similar to Section 3, we introduce an augmented Lyapunov function as in equation (3.7). Define the state v_3 as

$$\dot{v}_3 = A_3 v_3 + b_3 z_1,$$

$$\left(\frac{1}{1+\mu\Delta} - \frac{1}{1+\mu\varsigma_{20}}\right)z_1 = (1, 0, \ldots, 0)v_3 = v_{31}. \quad (4.1)$$

Note that the proper transfer function $1/(1+\mu\Delta)$ is stable and the term $1/(1+\mu\varsigma_{20})$ is well defined for sufficiently small μ. We introduce the final Lyapunov function for our closed loop system as

$$V = V_\rho + \frac{1}{k_\eta}\tilde{\eta}P_0\tilde{\eta} + \frac{1}{k_\zeta}\tilde{\zeta}P_b\tilde{\zeta} + q_1 v_1^T P_1 v_1 + q_2 v_2^T P_2 v_2 + q_3 v_3^T P_3 v_3. \quad (4.2)$$

Observe that V is a quadratic function $V = \chi^T P_\chi \chi$ of the vector

$$\chi = (z^T, \varepsilon^T, \tilde{\eta}^T, \tilde{\zeta}^T, v_1^T, v_2^T, v_3^T, \tilde{\varrho}, \tilde{\theta}^T)^T. \quad (4.3)$$

Similar to Section 3, we obtain

$$\dot{V} \leq -\alpha\|X\|^2 + \beta + \mu^2 c((\Delta(s + a_{n-1})x_{1r})^2 + (\Delta x_{1r})^2), \quad (4.4)$$

where

$$X = (z^T, \varepsilon^T, \tilde{\eta}^T, \tilde{\zeta}^T, v_1^T, v_2^T, v_3^T)^T. \quad (4.5)$$

We now need a bound on Δx_{1r} and $\Delta(s + a_{n-1})x_{1r}$. In Ikhouane and Krstic (1995) we proved that

$$\Delta\tilde{x}_1 = v_{21} + \varsigma_{20}(z_1 - \mu v_{21} - \mu\Delta y_r - d), \quad (4.6)$$

$$\Delta(s + a_{n-1})\tilde{x}_1 = v_{11} + \varsigma_{10}(z_1 - \mu v_{21} - \mu\Delta y_r - d) + \varsigma_{11}\dot{x}_{1r}, \quad (4.7)$$

$$\dot{x}_{1r} = h\varepsilon + \mu h v_{11} + h\lambda_{m+2} + h\tilde{\eta} + h\tilde{\zeta} + hz_1 + hv_{31} + \kappa, \quad (4.8)$$

$$\lambda_{m+2} = z_2 + \alpha_1 + h\varepsilon + h\lambda_{m+2} + h\tilde{\eta} + h\tilde{\zeta} + \kappa, \quad (4.9)$$

where h is a generic constant scalar or vector of appropriate dimensions independent of d, $\dot{d}$, and μ; κ is a generic bounded function of time which depends only on y_r and its first ρ derivatives. From equation (4.6) it is clear that Δx_{1r} is *linearly* bounded in $\|X\|$. To show that $\Delta(s + a_{n-1})x_{1r}$ is also linearly bounded in $\|X\|$, we note from equations (4.7) and (4.8) that it is enough to show that λ_{m+2} is linearly bounded in $\|X\|$. Since we are using projection, $\hat{\theta}$ and $\hat{\varrho}$ are bounded, so α_1 is *linearly* bounded in $\|X\|$, and, according to equation (4.9), so is λ_{m+2}. Thus

$$|\Delta(s + a_{n-1})x_{1r}| \leq c(\|X\| + \|y_r\|_\infty + \|\dot{y}_r\|_\infty + \|d\|_\infty), \quad (4.10)$$

$$|\Delta x_{1r}| \leq c(\|\chi\| + \|y_r\|_\infty + \|\dot{y}_r\|_\infty + \|d\|_\infty). \quad (4.11)$$

With equations (4.10), (4.11), and (4.4), and using the boundedness of $\tilde{\theta}$ and $\tilde{\varrho}$ due to projection, we get

$$\dot{V} \leq -\frac{\alpha}{2}\|X\|^2 + 2\beta \leq -\frac{\alpha'}{2}V + 2\beta' \quad (4.12)$$

which implies that V is *globally* uniformly bounded. The boundedness of the control u and the vectors λ and x is shown as in Krstic *et al.* (1995, Section 10.2.2). The asymptotic performance is derived as in Theorem 3.1. We now state the main result of this section

Theorem 4.1. Consider the plant (1.1) subject to Assumptions 1.1–1.4 and the adaptive controller composed of the control law (2.17) and the parameter update laws (2.26). If Δ is proper then there exist positive constants μ^*, c and g independent of μ, d and $\dot{d}$ such that for any $0 \leq \mu < \mu^*$, we have

(i) All the signals of the closed loop are *globally* bounded;
(ii) The mean-square of the tracking error is proportional to the size of perturbations:

$$\int_t^{t+T}(y(\tau) - y_r(\tau))^2 d\tau \leq g + c\int_t^{t+T}(\mu^2 + \dot{d}(\tau)^2 + d(\tau)^2)d\tau, \quad \forall t, T \geq 0. \quad (4.13)$$

The linear bounds (4.10) and (4.11) represent a key difference between the present paper and Ikhouane and Krstic (1995) for proper perturbations. In Ikhouane and Krstic (1995) the bounds were cubic because, without parameter projection, $\tilde{\theta}$ and $\tilde{\varrho}$ cannot be bounded independently, and thus, α_1 cannot be bounded linearly in $\|X\|$ (a cubic bound in $\|\chi\|$ was used in Ikhouane and Krstic (1995). Thus, the robustness result for proper Δ in this paper is global, whereas in Ikhouane and Krstic (1995), it was only regional.

5. *Simulation results*

Consider the following relative degree one unstable and non-minimum phase plant

$$y(t) = \frac{1}{s(s-a)}(1-\mu s)u(t) + d(t), \tag{5.1}$$

where the parameter $a = 1$ is unknown and μ is a positive scalar. The design procedure in Krstic *et al.* (1994) does not apply directly because of the presence of the unstable zero. Instead we can consider (5.1) to be relative degree two and $(1-\mu s)$ to be the unmodeled dynamics. We apply both the robust adaptive scheme from this paper and the scheme presented in Krstic *et al.* (1994). The design parameters and the reference signal are

$$\gamma = 0.5,\quad k_1 = k_2 = 4,\quad c_1 = c_2 = 1,\quad d_1 = d_2 = 0.1,\quad \varepsilon = 0.3,$$

$$\begin{aligned} y_r(t) &= \frac{1}{(s+1)\,(s+2)}\,r(t), \\ r(t) &= 5\sin(0.5t), \\ \Pi_a &= [0, 2]. \end{aligned} \tag{5.2}$$

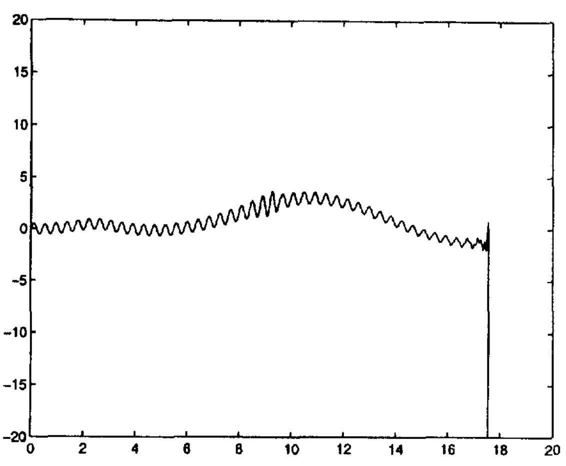

Fig. 1. Tracking error vs. time (without projection).

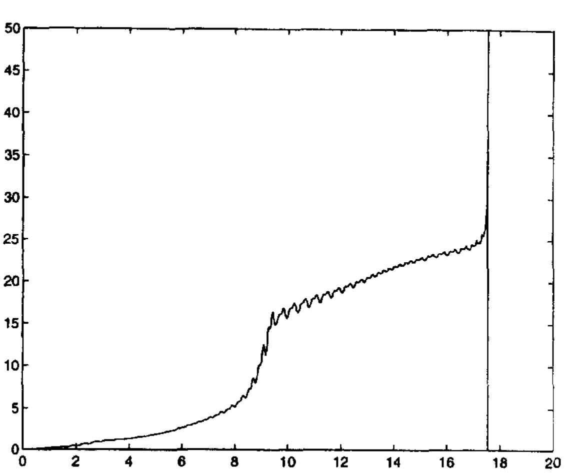

Fig. 2. Parameter estimate vs. time (without projection).

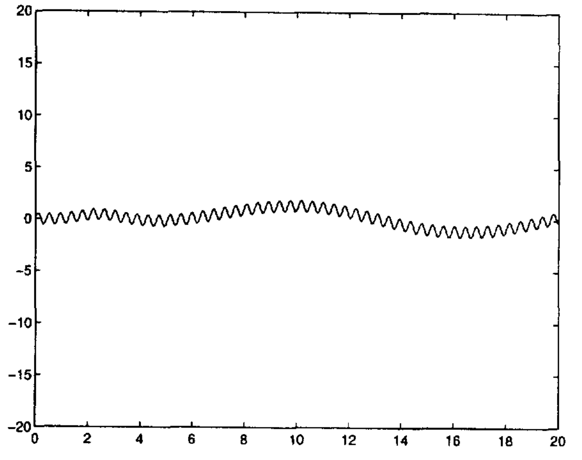

Fig. 3. Tracking error vs. time (with projection).

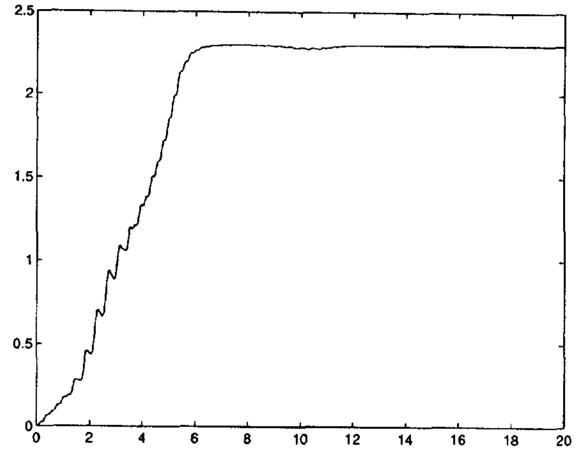

Fig. 4. Parameter estimate vs. time (with projection).

The control law is given by

$$u = -\beta_2 - z_1 + \frac{\partial\alpha_1}{\partial y}\omega^{\mathrm{T}} + \frac{\partial\alpha_1}{\partial\hat{\theta}}\mathrm{Proj}_{\Pi_a}\{\tau_2\} - c_2 z_2 - d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_2. \tag{5.3}$$

Note that in the particular case of relative degree two systems, instead of using damping, we may include $\dot{\hat{\theta}}$ and $\dot{\hat{\varrho}}$ in the control law. The stability proof is very similar to that of Section 3.

Figures 1–4 show the behavior of the closed loop system subject to the output disturbance $d(t) = 0.5\sin(15t)$ and $\mu = 0.04$. The closed loop system is unstable when the projection is not used in the parameter update law. This instability is removed with our projection design.

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