Stochastic Averaging in Discrete Time and its Applications to Extremum Seeking

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Abstract—We investigate stochastic averaging theory for locally Lipschitz discrete-time nonlinear systems with stochastic perturbation and its applications to convergence analysis of discrete-time stochastic extremum seeking algorithms. Firstly, by defining two average systems (one is continuous time, the other is discrete time), we develop discrete-time stochastic averaging theorem for locally Lipschitz nonlinear systems with stochastic perturbation. Our results only need some simple and applicable conditions, which are easy to verify, and remove a significant restriction present in existing results: global Lipschitzness of the nonlinear vector field. Secondly, we provide a discrete-time stochastic extremum seeking algorithm for a static map, in which measurement noise is considered and an ergodic discrete-time stochastic process is used as the excitation signal. Finally, for discrete-time nonlinear dynamical systems, in which the output equilibrium map has an extremum, we present a discrete-time stochastic extremum seeking scheme and, with a singular perturbation reduction, we prove the stability of the reduced system. Compared with classical stochastic approximation methods, while the convergence that we prove is in a weaker sense, the conditions of the algorithm are easy to verify and no requirements (e.g., boundedness) are imposed on the algorithm itself.

Index Terms—Extremum seeking, stochastic averaging, stochastic perturbation.

I. INTRODUCTION

The averaging method is a powerful and elegant asymptotic analysis technique for nonlinear time-varying dynamical systems. Its basic idea is to approximate the original system (time-varying and periodic or almost periodic, or randomly perturbed) by a simpler (average) system (time-invariant, deterministic) or some approximating diffusion system (a stochastic system simpler than the original one). Averaging method has received intensive interests in the analysis of nonlinear dynamical systems ([2], [4], [7], [9], [22], [23], [26], [27], [23]), adaptive control or adaptive algorithms ([24], [29]), and optimization methods ([3], [5], [12], [30]).

Extremum seeking is a non-model based real-time optimization tool and also a method of adaptive control. Since the first proof of the convergence of extremum seeking [11], the research on extremum seeking has triggered considerable interest in the theoretical control community ([6], [16], [19], [31]–[34]) and in applied communities ([20], [21], [25]). According the choice of probing signals, the research on the extremum seeking method can be simply classified into two types: deterministic ES method ([1], [6], [19], [31]–[33]) and stochastic ES method ([14], [15], [18]). In the deterministic ES, periodic (sinusoidal) excitation signals are primarily used to probe the nonlinearity and estimate its gradient. The random trajectory is preferable in some source tasks where the orthogonality requirements on the elements of the periodic perturbation vector pose an implementation challenge for high dimensional systems. Thus there is merit in investigating the use of stochastic perturbations within the ES architecture ([15]).

In [15], we establish a framework of continuous-time stochastic extremum seeking algorithms by developing general stochastic averaging theory in continuous time. However, there exists a need to consider stochastic extremum seeking in discrete time due to computer implementation. Discrete-time extremum seeking with stochastic perturbation is investigated without measurement noise in [18], in which the convergence of the algorithm involves strong restrictions on the iteration process. In [31] and [32], discrete-time extremum seeking with sinusoidal perturbation is studied with measurement noise considered and the proof of the convergence is based on the classical idea of stochastic approximation method, in which the boundedness of iteration sequence is assumed to guarantee the convergence of the algorithm.

In this paper, we investigate stochastic averaging for a class of discrete-time locally Lipschitz nonlinear systems with stochastic perturbation and then present discrete-time stochastic extremum seeking algorithm. In the first part, we develop general discrete-time stochastic averaging theory by the following four steps: (i) we introduce two average systems: one is discrete-time average system, the other is continuous-time average system; (ii) by a time-scale transformation, we establish a general stochastic averaging principle between the continuous-time average system and the original system in the continuous-time form; (iii) With the help of the continuous-time average system, we establish stochastic averaging principle between the discrete-time average system and the original system; (iv) we establish some related stability theorems for the original system. To the best of our knowledge, this is the first work about discrete-time stochastic averaging for locally Lipschitz nonlinear systems.

In the second part, we investigate general discrete-time stochastic extremum seeking with stochastic perturbation and measurement noise. We supply discrete-time stochastic extremum seeking algorithm for a static map and analyze stochastic
extremum seeking scheme for nonlinear dynamical systems with output equilibrium map. With the help of our developed discrete-time stochastic averaging theory, we prove the convergence of the algorithms. Unlike in the continuous-time case [15], in this work we consider the measurement noise, which is assumed to be bounded and ergodic stochastic process. In the classical stochastic approximation method, boundedness condition or other restrictions are imposed on the iteration algorithm itself to achieve the convergence with probability one. In our stochastic discrete-time algorithm, the convergence condition is only imposed on the cost function or considered systems and is easy to verify, but as a consequence, we obtain a weaker form of convergence. Different from [16] in which unified frameworks are proposed for extremum seeking of general nonlinear plants based on a sampled-data control law, we use the averaging method to analyze the stability of estimation error systems and avoid to verify the decaying property with a K-L function of iteration sequence (the output sequence of extremum seeking controller), but we need justify the stability of average system.

The remainder of the paper is organized as follows. In Section II, we give problem formulation of discrete-time stochastic averaging. In Section III we establish our discrete-time stochastic averaging theorems, whose proofs are given in the Appendix. In Section IV we present stochastic extremum seeking algorithms for a static map. In Section V, we give the Appendix. In Section IV we present stochastic extremum seeking controller), but we need justify the stability of average system.

II. PROBLEM FORMULATION OF DISCRETE-TIME STOCHASTIC AVERAGING

Consider system

\[ X_{k+1} = X_k + \varepsilon f(X_k, Y_{k+1}), \quad k = 0, 1, 2, \ldots \]  

(1)

where \( X_k \in \mathbb{R}^n \) is the state, \( \{Y_k\} \subseteq \mathbb{R}^m \) is a stochastic perturbation sequence defined on a complete probability space \((\Omega, \mathcal{F}, P)\), where \( \Omega \) is the sample space, \( \mathcal{F} \) is the \( \sigma \)-field, and \( P \) is the probability measure. Let \( S_Y \subset \mathbb{R}^m \) be the living space of the perturbation process, \( \varepsilon \in (0, \varepsilon_0) \) is a small parameter for some fixed positive constant \( \varepsilon_0 \).

The following assumptions will be considered.

Assumption 1: The vector field \( f(x, y) \) is a continuous function of \((x, y)\), and for any \( x \in \mathbb{R}^n \), it is a bounded function of \( y \). Further it satisfies the locally Lipschitz condition in \( x \in \mathbb{R}^n \) uniformly in \( y \in S_Y \), i.e., for any compact subset \( D \subset \mathbb{R}^n \), there is a constant \( k_D \) such that for all \( x_1, x_2 \in D \) and all \( y \in S_Y \), \( |f(x_1, y) - f(x_2, y)| \leq k_D |x_1 - x_2| \).

Assumption 2: The perturbation process \( \{Y_k\} \) is ergodic with invariant distribution \( \mu \).

Under Assumption 2, we define two classes of average systems for system (1) as follows:

Discrete average system:

\[ \bar{X}_{k+1}^d = \bar{X}_k^d + \varepsilon f(\bar{X}_k^d), \quad k = 0, 1, \ldots \]  

(2)

Continuous average system:

\[ \frac{d\bar{X}^c(t)}{dt} = \bar{f}(\bar{X}^c(t)), \quad t \geq 0 \]  

(3)

where \( \bar{X}_0^d = \bar{X}^c(0) = X_0 \) and

\[ \bar{f}(x) \triangleq \int_{S_Y} f(x, y)\mu(dy) = \lim_{N \to +\infty} \frac{1}{N+1} \sum_{k=0}^{N} f(x, Y_{k+1}) \text{ a.s.} \]  

(4)

By Assumption 1, \( f(x, y) \) is bounded with respect to \( y \), thus \( y \to f(x, y) \) is \( \mu \)-integrable, so \( \bar{f} \) is well defined. Here the definition of the discrete average system is different from that in [29], where the average vector field is defined by \( \bar{f}(x) \triangleq E f(x, Y_{k+1}) \) (there, the perturbation process \( \{Y_{k+1}\} \) is assumed to be strict stationary). In this paper, we consider ergodic process as perturbation. It is easy to find discrete-time ergodic processes, e.g.,

- i.i.d random variable sequence;
- finite state irreducible and aperiodic Markov process;
- \( \{Y_i, i = 0, 1, \ldots\} \) where \( \{Y_i, t \geq 0\} \) is an Ornstein-Uhlenbeck (OU) process. In fact, for any continuous-time ergodic process \( \{Y_i, t \geq 0\} \), the subsequence \( \{Y_i, i = 0, 1, \ldots\} \) is a discrete-time ergodic process.

For the discrete average system (2), the solution can be obtained by iteration, thus the existence and uniqueness of the solution can be guaranteed by the local Lipschitzness of nonlinear vector field. For the continuous average system (3), \( f(x) \) is easy to be verified to be locally Lipschitz since \( f(x, y) \) is locally Lipschitz in \( x \). Thus, there exists a unique solution on \([0, \sigma_\infty)\), where \( \sigma_\infty \) is the explosion time. Thus we only need the following assumption.

Assumption 3: The continuous average system (3) has a solution on \([0, +\infty)\).

By (1), we have \( X_{k+1} = X_0 + \varepsilon \sum_{i=0}^{k} f(X_i, Y_{i+1}) \). We introduce a new time \( t_k = \varepsilon k \). Let \( m(t) = \max\{k : t_k \leq t\} \) and define \( X(t) \) as a piecewise constant version of \( X_k \), i.e., \( X(t) = X_k \) for \( t_k \leq t < t_{k+1} \), and \( Y(t) \) as a piecewise constant version of \( Y_n \), i.e., \( Y(t) = Y_k \) for \( t_k \leq t < t_{k+1} \). Then we can write (1) in the following form:

\[ X(t) = X_0 + \varepsilon \sum_{k=1}^{m(t)} f(X_{k-1}, Y_k) \]  

(5)

or as the continuous-time version

\[ X(t) = X_0 + \int_{0}^{t} f(X(s), Y(\varepsilon + s)) \, ds \]  

\[ - \int_{t_{m(t)}}^{t} f(X(s), Y(\varepsilon + s)) \, ds. \]  

(6)

Similarly, we can write the discrete average system (2) in the following continuous-time version:

\[ \bar{X}^d(t) = X_0 + \int_{0}^{t} \bar{f}(\bar{X}^d(s)) \, ds - \int_{t_{m(t)}}^{t} \bar{f}(\bar{X}^d(s)) \, ds \]  

(7)

and write the continuous average system (3) by

\[ \bar{X}^c(t) = X_0 + \int_{0}^{t} \bar{f}(\bar{X}^c(s)) \, ds \]  

(8)

where \( \bar{X}^d(t) \) is a piecewise constant version of \( \bar{X}_k^d \), i.e., \( \bar{X}^d(t) = X_k^d \) as \( t_k \leq t < t_{k+1} \). We now rewrite the
**Step 1.** \( |\bar{X}(t) - X(t)| \to 0, \) as \( \varepsilon \to 0 \)

**Step 2.** \( |\bar{X}(t) - \bar{X}(t)| \to 0, \) as \( \varepsilon \to 0 \)

**Step 3.** \( |\bar{X}(t) - X(t)| \to 0, \) as \( \varepsilon \to 0 \)

**Step 4.** \( |\bar{X}(t) - X(t)| \to 0, \) as \( \varepsilon \to 0 \)

Fig. 1. Main idea for the proof of discrete-time stochastic averaging.

**Continuous-time version** (6) of the original system (1) in the two forms

\[
X(t) = X_0 + \int_0^t \bar{f}(X(s)) ds - \int_0^t \bar{f}(X(s)) ds + R^{(1)}(t, X(\cdot), Y(\varepsilon + \cdot))
\]

\[
X(t) = X_0 + \int_0^t \bar{f}(X(s)) ds + R^{(2)}(t, X(\cdot), Y(\varepsilon + \cdot))
\]

**Proof:** See Appendix A.

**Lemma 1:** Consider continuous-time version (6) of the original system under Assumptions 1, 2 and 3. Then for any \( T > 0 \)

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} |X(t) - \bar{X}(t)| = 0 \ a.s.
\]

**Proof:** See Appendix B.

**Lemma 2:** Consider continuous-time version (6) of the original system under Assumptions 1 and 2. Then for any \( T > 0 \)

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} |X(t) - \bar{X}(t)| = 0 \ a.s.
\]

**Proof:** See Appendix C.

**A. Approximation and Stability Results With the Continuous Average System**

In this subsection, we present approximation results to the original system and the related stability results with the help of the continuous average system. First, we extend the finite-time approximation result in Lemma 1 to arbitrary long time intervals.

**Theorem 3:** Consider system (6) under Assumptions 1, 2 and 3. Then

i) for any \( \delta > 0 \)

\[
\lim_{\varepsilon \to 0} \inf \left\{ t \geq 0 : |X(t) - \bar{X}(t)| > \delta \right\} = +\infty \ a.s.
\]

ii) there exists a function \( T(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N} \) such that for any \( \delta > 0 \)

\[
\lim_{\varepsilon \to 0} P \left\{ \sup_{0 \leq t \leq T(\varepsilon)} |X(t) - \bar{X}(t)| > \delta \right\} = 0
\]

with \( \lim_{\varepsilon \to 0} T(\varepsilon) = +\infty. \)

**Proof:** See Appendix D for the result (15), and Appendix C for the result (16).
the original system as a perturbation of the continuous average system (3), and analyze weak stability properties by studying equilibrium stability of the continuous average system (3).

**Theorem 4:** Consider continuous-time version (6) of the original system under Assumptions 1, 2 and 3. Then if the equilibrium $\bar{X}(t) \equiv 0$ of the continuous average system (3) is exponentially stable, then it is weakly exponentially stable under random perturbation $R(2)(t, X(t), Y(t))$, i.e., there exist constants $r > 0, \gamma > 0$ and a function $T(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ such that for any initial condition $X(t_0) = X_0 = x \in \{ \bar{x} \in \mathbb{R}^n : \| \bar{x} \| < r \}$, and any $\delta > 0$, the solution of system (6) satisfies

$$\lim \inf_{\varepsilon \to 0} \{ t \geq 0 : |X(t)| < c_\varepsilon x e^{-\gamma t} + \delta, \forall t \in [0, T(\varepsilon)] \} = +\infty$$

**Proof:** See Appendix E.

**Remark 3.1:** By analyzing the weak stability of the continuous average system, we obtain the solution property of system (6). Here the term “weakly” is used because the properties in question involve $\lim_{\varepsilon \to 0}$ and are defined through the first exist time from a set. In [10], stability concepts that are similarly defined under random perturbations are introduced for a nonlinear system perturbed by a stochastic process.

**B. Approximation and Stability Results with Continuous-Time Version of the Discrete Average System**

Similarly, we can extend the finite-time approximation result in Lemma 2 to arbitrarily long time intervals. The following theorem can be proved by following the proof of Theorem 3.

**Theorem 5:** Consider continuous-time version (6) of the original system (1) under Assumptions 1 and 2. Then

i) for any $\delta > 0$

$$\lim \inf_{\varepsilon \to 0} \{ t \geq 0 : |X(t) - \bar{X}(t)| > \delta \} = +\infty \text{ a.s.}$$

ii) there exists a function $T(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ such that for any $\delta > 0$

$$\lim_{\varepsilon \to 0} P \left\{ \sup_{0 \leq t \leq T(\varepsilon)} |X(t) - \bar{X}(t)| > \delta \right\} = 0$$

with $\lim_{\varepsilon \to 0} T(\varepsilon) = +\infty$.

Similar as the last subsection, if the continuous-time version (7) of the discrete average system (2) has some stability property, we can obtain properties of the solution of the continuous-time version (6) of the original system (1). The following theorem can be proved by following the proof of Theorem 4.

**Theorem 6:** Consider continuous-time version (6) of the original system (1) under Assumptions 1 and 2. Then if for any $\varepsilon \in (0, \varepsilon_0)$, the equilibrium $\bar{X}(t) \equiv 0$ of continuous-time version (7) of the discrete average system (2) is exponentially stable, then it is weakly exponentially stable under random perturbation $R(1)(t, X(t), Y(t))$, i.e., there exist constants $r > 0, c_\varepsilon > 0, \gamma > 0$, and a function $T(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ such that for any initial condition $X_0 = X_0 = x \in \{ \bar{x} \in \mathbb{R}^n : \| \bar{x} \| < r \}$, and any $\delta > 0$, the solution of system (6) satisfies

$$\lim \inf_{\varepsilon \to 0} \{ t \geq 0 : |X(t)| > c_\varepsilon x e^{-\gamma t} + \delta \} = +\infty \text{ a.s.}$$

with $\lim_{\varepsilon \to 0} P \{ |X(t)| \leq c_\varepsilon x e^{-\gamma t} + \delta, \forall t \in [0, T(\varepsilon)] \} = 1$.

**Proof:** See Appendix F.

**Theorem 8:** Consider system (1) under Assumptions 1 and 2. Then we have

i) for any $\delta > 0$

$$\lim_{\varepsilon \to 0} \inf \{ k \in \mathbb{N} : |X_k - \bar{X}_k| > \delta \} = +\infty \text{ a.s.}$$

ii) for any $\delta > 0$ and any $N \in \mathbb{N}$

$$\lim_{\varepsilon \to 0} P \{ \sup_{0 \leq k \leq N} |X_k - \bar{X}_k| > \delta \} = 0.$$
\( R^{(3)}(X_k, Y_{k+1}) \), i.e., there exist constants \( r_\varepsilon > 0 \), \( c_\varepsilon > 0 \) and \( 0 < \gamma_\varepsilon < 1 \) such that for any initial condition \( X_0 = x \in \mathbb{R}^n : |x| < r_\varepsilon \), and any \( \delta > 0 \), the solution of system (1) satisfies

\[
\lim_{\varepsilon \to 0} \inf \{ k \in \mathbb{N} : |X_k| > c_\varepsilon |x| (\gamma_\varepsilon)^k + \delta \} = +\infty \text{ a.s.} \tag{27}
\]

\[
\lim_{\varepsilon \to 0} P \{ |X_k| \leq c_\varepsilon |x| (\gamma_\varepsilon)^k + \delta, \forall k = 0, 1, \ldots, [N/\varepsilon] \} = 1 \tag{28}
\]

where \( N \) is any natural number. If the equilibrium \( \hat{X}_k \equiv 0 \) of the discrete average system (2) is exponentially stable uniformly w.r.t. \( \varepsilon \in (0, \varepsilon_0) \), then the above constants \( r_\varepsilon, c_\varepsilon > 0 \) and \( \gamma_\varepsilon \) can be taken independent of \( \varepsilon \).

**Remark 3.3:** Similar to continuous-time stochastic averaging in [15], if the discrete average system has other properties (boundedness, attractivity, stability and asymptotic stability), we can obtain corresponding (boundedness, attractivity, stability and asymptotic stability) results for the original system.

**Remark 3.4:** The stability results in the above Theorems 4, 6, and 9 are local, but if the equilibrium of corresponding average systems is globally stable (asymptotic stability or exponentially stable), then it is globally weakly stable (asymptotical stable or exponentially stable, respectively).

**Remark 3.5:** In analyzing the similar kind of system as (1), average method is different from ordinary differential equation (ODE) method and weak convergence method of stochastic approximation ([5], [12], [17]). In ODE method (gain coefficients are changing with iteration steps) and weak convergence method (constant gain), regression function or cost function (i.e., \( f(x) \) in system (1)) acts as the nonlinear vector field of an ordinary differential equation, which is used to compare with the original system. In both methods, to analyze the convergence of the solution of system (1), there need some restrictions on both the growth rate of nonlinear vector fields and the algorithm itself (i.e., uniformly bounded), or the existence of some continuously differential function (Lyapunov function) satisfying some conditions. In some sense, these conditions (i.e., boundedness of iteration sequence) are not easy to verify. Average method is to use a new system (namely, average system) to approximate system (1) and thus makes it possible to obtain properties of the solution of system (1). From Theorems 8 and 9, we can see that the approximation conditions are easy to verify.

### IV. DISCRETE-TIME STOCHASTIC EXTREMUM SEEKING ALGORITHM FOR STATIC MAP

Consider the quadratic function

\[
\varphi(x) = \varphi + \frac{\varphi''}{2}(x - x^*)^2 \tag{29}
\]

where \( x^* \in \mathbb{R} \), \( \varphi^* \in \mathbb{R} \), and \( \varphi'' \in \mathbb{R} \) are unknown. Any \( C^2 \) function \( \varphi(\cdot) \) with an extremum at \( x = x^* \) and with \( \varphi'' \neq 0 \) can be locally approximated by (29). Without loss of generality, we assume that \( \varphi'' > 0 \). In this section, we design an algorithm to make \( |x_k - x^*| \) as small as possible, so that the output \( \varphi(x_k) \) is driven to its minimum \( \varphi^* \). The only available information is the output \( y = \varphi(x_k) \) with measurement noise.

Denote \( \hat{x}_k \) as the \( k \) step estimate of the unknown optimal input \( x^* \). Design iteration algorithm as

\[
\hat{x}_{k+1} = \hat{x}_k - \varepsilon \sin(v_{k+1}) y_{k+1}, \quad k = 0, 1, \ldots \tag{30}
\]

where \( y_{k+1} = \varphi(x_k) + W_{k+1} \) is the measurement output, \( \{v_k\} \) is an ergodic stochastic process with invariant distribution \( \mu \) and living space \( S_v \), and \( \{W_k\} \) is the measurement noise, which is assumed to be bounded with a bound \( M \) and ergodic with invariant distribution \( \nu \) and living space \( S_w \). \( \varepsilon \in (0, \varepsilon_0) \) is a positive small parameter for some constant \( \varepsilon_0 > 0 \). The perturbation process \( \{v_k\} \) is independent of the measurement noise process \( \{W_k\} \).

Define \( x_k = \hat{x}_k + a \sin(v_{k+1}) \), \( a > 0 \) and the estimation error \( \hat{x}_k = \hat{x}_k - x^* \). Then we have

\[
\hat{x}_{k+1} = \hat{x}_k - \varepsilon \sin(v_{k+1}) \\
\times \left( \varphi^* + \frac{\varphi''}{2}(\hat{x}_k + a \sin(v_{k+1}))^2 + W_{k+1} \right). \tag{31}
\]

To analyze properties of the solution of the error system (31), we will use stochastic averaging theory developed in the last section. First, to calculate the average system, we choose the excitation process \( \{v_k\} \) as a sequence of i.i.d. random variable with invariant distribution \( \mu(dy) = (1/\sqrt{2\pi\sigma}) e^{-y^2/(2\sigma^2)} dy \) \( \sigma > 0 \). We assume that the measurement noise process \( \{W_k\} \) is any bounded ergodic process.

By (4), we have

\[
\text{Ave} \left\{ \sin^i(v_{k+1}) \right\} \triangleq \int \sin^i(y) \mu(dy) = 0, \quad i = 1, 3 \tag{32}
\]

\[
\text{Ave} \left\{ \sin^2(v_{k+1}) \right\} \triangleq \int \sin^2(y) \mu(dy) = \frac{1}{2} - \frac{1}{2} e^{-2\sigma^2} \tag{33}
\]

\[
\text{Ave} \left\{ \sin(v_{k+1}) W_{k+1} \right\} \triangleq \int \sin(y) x \mu(dy) \times \nu(dx) = 0. \tag{34}
\]

Thus, we obtain the average system of the error system (31)

\[
\hat{x}_{k+1}^{\text{ave}} = \left( 1 - \frac{a \varphi''(1 - e^{-2\sigma^2})}{2} \right) \hat{x}_k^{\text{ave}}. \tag{35}
\]

Since \( \varphi'' > 0 \), there exists \( \varepsilon^* = 2/a \varphi''(1 - e^{-2\sigma^2}) \) such that the average system (35) is globally exponentially stable for \( \varepsilon \in (0, \varepsilon^*) \).

Thus by Theorem 9, Remark 2.1 and Remark 3.4, for the discrete-time stochastic extremum seeking algorithm in Fig. 2, we have the following theorem.

**Theorem 10:** Consider the static map (29) under the iteration algorithm (30). Then there exist constants \( c_\varepsilon > 0 \) and \( 0 < \gamma_\varepsilon < 1 \)
such that for any initial condition \( \tilde{x}_0 \in \mathbb{R} \) and any \( \delta > 0 \)
\[
\lim_{\varepsilon \to 0} \inf \{ k \in \mathbb{N} : |\tilde{x}_k| > c_\varepsilon |\tilde{x}_0(\gamma_\varepsilon)|^k + \delta \} = +\infty \text{ a.s.} \quad (36)
\]
\[
\lim_{\varepsilon \to 0} P \left\{ |\tilde{x}_k| \leq c_\varepsilon |\tilde{x}_0(\gamma_\varepsilon)|^k + \delta, \forall k = 0, 1, \ldots, [N/\varepsilon] \right\} = 1. \quad (37)
\]

These two results imply that the norm of the error vector \( \tilde{x}_k \)
exponentially converges, both almost surely and in probability, to below an arbitrarily small residual value \( \delta \) over an arbitrarily long time interval, which tends to infinity as \( \varepsilon \) goes to zero. To quantify the output convergence to the extremum, for any \( \varepsilon > 0 \), define a stopping time \( \tau_{2\varepsilon}^k = \inf\{k \in \mathbb{N} : |\tilde{x}_k| > c_\varepsilon |\tilde{x}_0(\gamma_\varepsilon)|^k + \delta \} \). Then by (36), we know that \( \lim_{\varepsilon \to 0} \tau_{2\varepsilon}^k = +\infty \), a.s. and
\[
|\tilde{x}_k| \leq c_\varepsilon |\tilde{x}_0(\gamma_\varepsilon)|^k + \delta, \forall k < \tau_{2\varepsilon}^k. \quad (38)
\]

Since \( y_{k+1} = \varphi(x^* + \tilde{x}_k + a \sin(v_{k+1})) + W_{k+1} \) and \( \varphi'(x^*) = 0 \), we have \( y_{k+1} - \varphi(x^*) = \varphi''(x^*)/2(\tilde{x}_k + a \sin(v_{k+1}))^2 + O((\tilde{x}_k + a \sin(v_{k+1}))^3) + W_{k+1} \). Thus by (38) and \( |W_{k+1}| \leq M \), it holds that \( |y_{k+1} - \varphi(x^*)| \leq O(a^2) + O(\delta^2) + C_\varepsilon |\tilde{x}_0(\gamma_\varepsilon)|^2k + M, \forall k \leq \tau_{2\varepsilon}^k \), for some positive constant \( C_\varepsilon \). Similarly, by (37)
\[
\lim_{\varepsilon \to 0} P \left\{ |y_{k+1} - \varphi(x^*)| \leq O(a^2) + O(\delta^2) + C_\varepsilon |\tilde{x}_0(\gamma_\varepsilon)|^2k, +M, \forall k = 0, 1, \ldots, [N/\varepsilon] \right\} = 1 \quad (39)
\]
which implies that the output can exponentially approach to the extremum \( \varphi(x^*) \) if \( a \) is chosen sufficiently small and the measurement noise can be ignored \( (M = 0) \). By (35), we can see that smaller \( \varepsilon \) is, slower the convergence rate of the average error system is. Thus, for this static map, the parameter \( \varepsilon \) is designed to consider the tradeoff of the convergence rate and the convergence precision.

**Remark 4.1:** As said above, the small parameter \( \varepsilon \) affects the convergence rate and the convergence precision of the stochastic ES algorithm. Generally, the faster convergence requires the larger \( \varepsilon \), while the better convergence precision requires the smaller \( \varepsilon \). Thus, in the practical applications, the tradeoff between the convergence rate and the convergence precision should be considered by the concrete optimization demands, in which some experiences can be useful.

**Remark 4.2:** As an optimization method, besides the different derivative estimation methods, there are some other differences between stochastic extremum seeking (SES) and stochastic approximation (SA) ([30], [32]). First, in the iteration, the gain coefficients in SA is often changing with the iteration step, but for SES, the gain coefficient is a small constant and denotes the amplitude of the excitation signal; Second, stochastic approximation may consider more kinds of measurement noise (i.e., martingale difference sequence, some kind of infinite correlated sequence), but here we assume the measurement noise as bounded ergodic stochastic sequence (the boundedness is to guarantee the existence of the integral in (4)); Third, to prove the convergence of the algorithm \( P(\lim_{k \to \infty} x_k = x^*) = 1 \), SA algorithm requires some restrictions on the cost function (regression function) or the iteration sequence, while the convergence conditions of SES algorithm are simple and easy to verify.

Fig. 3 displays the simulation results with \( \varphi^* = 1, \varphi_0^* = 1 \), \( x^* = 1 \), in the static map (29) and \( a = 0.8, \varepsilon = 0.002 \) in the parameter update law (30) and initial condition \( \tilde{x}_0 = 5 \). The probing signal \( \{v_k\} \) is taken as a sequence of i.i.d. Gaussian random variables with distribution \( N(0, 2^2) \) and the measurement noise \( \{W_k\} \) is taken as a sequence of truncated i.i.d. Gaussian random variables with distribution \( N(0, 0.2^2) \).

**V. DISCRETE-TIME STOCHASTIC EXTREMUM SEEKING FOR DYNAMIC SYSTEMS**

Consider a general nonlinear model
\[
x_{k+1} = f(x_k, u_k) \quad (40)
\]
\[
y_k = h(x_k), \quad k = 0, 1, 2, \ldots \quad (41)
\]
where \( x_k \in \mathbb{R}^n \) is the state, \( u_k \in \mathbb{R} \) is the input, \( y_k^0 \in \mathbb{R} \) is the nominal output, and \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \to \mathbb{R} \) are smooth functions. Suppose that we know a smooth control law \( u_k = \beta(x_k, \theta) \) parameterized by a scalar parameter \( \theta \). Then the closed-loop system \( x_{k+1} = f(x_k, \beta(x_k, \theta)) \) has equilibria parameterized by \( \theta \). We make the following assumptions about the closed-loop system.

**Assumption 4:** There exists a smooth function \( l : \mathbb{R} \to \mathbb{R}^n \) such that \( f(x_k, \beta(x_k, \theta), \theta) = x_k \) if and only if \( x_k = l(\theta) \).

**Assumption 5:** There exists \( \theta^* \in \mathbb{R} \) such that \( (h \circ l)'(\theta^*) = 0 \), and \( (h \circ l)'(\theta^*) < 0 \).

Thus, we assume that the output equilibrium map \( y = h(l(\theta)) \) has a local maximum at \( \theta = \theta^* \).

Our objective is to develop a feedback mechanism which makes the output equilibrium map \( y = h(l(\theta)) \) as close as possible to the maximum \( y^* = h(l(\theta^*)) \) but without requiring the knowledge of either \( \theta^* \) or the functions \( h \) and \( l \). The only available information is the output with measurement noise.

As discrete-time stochastic extremum seeking scheme in Fig. 4, we choose the parameter update law
\[
\hat{\theta}_{k+1} = \hat{\theta}_k + \epsilon \varrho \xi_k \quad (42)
\]
\[
\xi_{k+1} = \xi_k - \varepsilon w_1 \xi_k + \varepsilon w_1 (y_{k+1} - \zeta_k) \sin(v_{k+1}) \quad (43)
\]
\[
\zeta_{k+1} = \zeta_k - \varepsilon w_2 \xi_k + \varepsilon w_2 y_{k+1} \quad (44)
\]
\[
y_{k+1} = y_k^0 + W_{k+1} \quad (45)
\]
where \( \varrho > 0, w_1 > 0, w_2 > 0, \varepsilon > 0 \) are design parameters, \( \{v_k\} \) is assumed to be a sequence of i.i.d. Gaussian random variables with distribution \( \mu(dx) = (1/\sqrt{2\pi\sigma})e^{-(x^2/2\sigma^2)}dx \) \( (\sigma > 0) \), and \( W_k \overset{\Delta}{=} (-M) \lor Z_k \land M \) is measurement noise, where \( \{Z_k\} \) is a sequence of i.i.d. Gaussian random variable with distribution \( \nu(dx) = (1/\sqrt{2\pi\sigma_1})e^{-(x^2/2\sigma_1^2)}dx \) \( (\sigma_1 > 0) \). We assume that the probing signal \( \{v_k\} \) is independent of the measure noise process \( \{W_k\} \). It is easy to verify...
that \( \{W_k\} \) is a bounded and ergodic process with invariant distribution \( \nu_1(A) = \nu(A \wedge (-M, M)) + q_1 + q_2 \), for any \( A \subseteq \mathbb{R} \), where \( q_1 = \{ \nu((-\infty, +\infty)), \text{ if } M \in A \} \), and \( q_2 = 0 \), if \(-M \in A \).

Define \( \theta_k = \tilde{\theta}_k + a \sin(v_{k+1}) \). Then we obtain the closed-loop system as

\[
x_{k+1} = f\left(x_k, \beta \left(x_k, \tilde{\theta}_k + a \sin(v_{k+1})\right)\right)
\]

(46)

\[
\tilde{\theta}_{k+1} = \tilde{\theta}_k + \varepsilon \theta_k \xi_k
\]

(47)

\[
\xi_{k+1} = \xi_k - \varepsilon w_1 \xi_k + \varepsilon w_1 \left(b(x_k) - h(l(\theta^*)) - \xi_k\right) + W_{k+1}
\]

(48)

(49)

With the error variables \( \tilde{\theta}_k = \tilde{\theta}_k - \theta^* \) and \( \tilde{\xi}_k = \xi_k - h(l(\theta^*)) \), the closed-loop system is rewritten as

\[
x_{k+1} = f\left(x_k, \beta \left(x_k, \tilde{\theta}_k + a \sin(v_{k+1})\right)\right)
\]

(50)

\[
\tilde{\theta}_{k+1} = \tilde{\theta}_k + \varepsilon \theta_k \xi_k
\]

(51)

\[
\xi_{k+1} = \xi_k - \varepsilon w_1 \xi_k + \varepsilon w_1 \left(h(x_k) - h(l(\theta^*)) - \xi_k\right) + W_{k+1}
\]

(52)

(53)

We employ a singular perturbation reduction, freeze \( x_k \) in (50) at its quasi-steady state value as \( x_k = l(\theta^* + \tilde{\theta}_k + a \sin(v_{k+1})) \) and substitute it into (51)–(53), and then get the reduced system

\[
\tilde{\xi}_{k+1} = \xi_k - \varepsilon w_1 \xi_k + \varepsilon w_1 \left(\zeta \left(\tilde{\theta}_k + a \sin(v_{k+1})\right)\right) + W_{k+1}
\]

(54)

\[
\xi_{k+1} = \xi_k - \varepsilon w_1 \xi_k + \varepsilon w_1 \left(\zeta \left(\tilde{\theta}_k + a \sin(v_{k+1})\right)\right) + W_{k+1}
\]

(55)

\[
\tilde{\xi}_{k+1} = \xi_k - \varepsilon w_2 \tilde{\xi}_k + \varepsilon w_2 \left(\zeta \left(\tilde{\theta}_k + a \sin(v_{k+1})\right)\right) + W_{k+1}
\]

(56)

where

\[
\zeta(\tilde{\theta}_k + a \sin(v_{k+1})) = \frac{a}{\pi} h(l(\theta^* + \tilde{\theta}_k + a \sin(v_{k+1}))) - h(l(\theta^*)) \text{.}
\]

(57)

(58)

(59)

Now we use our stochastic averaging theorems to analyze the reduced system (54)–(56). According to (4), we obtain that the average system of (54)–(56) is

\[
\begin{align*}
\theta_{k+1}^{\text{ave}} - \hat{\theta}_k^{\text{ave}} & = w_1 \xi_{k+1}^{\text{ave}} + w_1 \int_{S_v} \zeta \left(\tilde{\theta}_k^{\text{ave}} + a \sin(y)\right) \sin(y) \mu(dy) \\
\xi_{k+1}^{\text{ave}} - \xi_k^{\text{ave}} & = w_2 \tilde{\xi}_{k+1}^{\text{ave}} + w_2 \int_{S_v} \zeta \left(\tilde{\theta}_k^{\text{ave}} + a \sin(y)\right) \mu(dy) \\
\tilde{\theta}_{k+1}^{\text{ave}} - \tilde{\theta}_k^{\text{ave}} & = -w_1 \xi_{k+1}^{\text{ave}} + w_1 \int_{S_v} \zeta \left(\tilde{\theta}_k^{\text{ave}} + a \sin(y)\right) \sin(y) \mu(dy)
\end{align*}
\]

(60)

where we use the following facts: \( \int_{S_v} x \nu_1(dx) = 0 \), \( \int_{S_v \times S_v} x \sin(y) \nu_1(dx) \mu(dy) = 0 \). Now, we determine the average equilibrium \( \hat{\theta}_{a,e}, \xi_{a,e}, \tilde{\xi}_{a,e} \) which satisfies

\[
\xi_{a,e} = 0
\]

(61)

\[
-w_1 \xi_{a,e} + w_1 \int_{S_v} \zeta \left(\hat{\theta}_{a,e} + a \sin(y)\right) \sin(y) \mu(dy) = 0
\]

(62)

\[
-w_2 \tilde{\xi}_{a,e} + w_2 \int_{S_v} \zeta \left(\hat{\theta}_{a,e} + a \sin(y)\right) \mu(dy) = 0.
\]

(63)

We assume that \( \hat{\theta}_{a,e} \) has the form

\[
\hat{\theta}_{a,e} = b_1 a + b_2 a^2 + O(a^3)
\]

(64)

By (57) and (58), define

\[
\zeta(x) = \frac{\zeta''(0)}{2} x^2 + \frac{\zeta'''(0)}{3!} x^3 + O(x^4)
\]

(65)

Then substituting (64) and (65) into (62) and noticing that \( S_v = \mathbb{R} \), we have

\[
\int_{-\infty}^{+\infty} \left[\frac{\zeta''(0)}{2} (b_1 a + b_2 a^2 + O(a^3) + a \sin(y)) \right]^2 dy
\]

(66)

where the following facts are used: \( \left(1/\sqrt{2\pi} \sigma \right) \int_{-\infty}^{+\infty} x^2 e^{-x^2/(2\sigma^2)} dx = 1 \), \( \int_{-\infty}^{+\infty} x^3 e^{-x^2/(2\sigma^2)} dx = 0 \), \( \int_{-\infty}^{+\infty} x^4 e^{-x^2/(2\sigma^2)} dx = \frac{1}{4} \pi \sigma^4 \).

Comparing the coefficients of the powers of \( a \) on the right-hand and left-hand sides of (66), we have \( b_1 = 0, b_2 = -\left(\zeta'''(0)/3 - 4 e^{-2a^2} + e^{-8a^2}\right)/24 \zeta''(0) \). Thus, by (64), we have

\[
\hat{\theta}_{a,e} = -\frac{\zeta'''(0)}{24 \zeta''(0)} \left(3 - 4 e^{-2a^2} + e^{-8a^2}\right) / (1 - e^{-2a^2})^2 + O(a^3).
\]

(67)

From this equation, together with (63), we have

\[
\tilde{\xi}_{a,e} = \int_{-\infty}^{+\infty} \left(\frac{\zeta''(0)}{2} (b_2 a^2 + O(a^3) + a \sin(y)) \right) e^{-y^2/(2\pi \sigma)} dy
\]

(68)

where

\[
\int_{-\infty}^{+\infty} \left(\frac{\zeta''(0)}{2} (b_2 a^2 + O(a^3) + a \sin(y)) \right) e^{-y^2/(2\pi \sigma)} dy = \int_{-\infty}^{+\infty} \frac{\zeta''(0)}{4} (b_2 a^2 + O(a^3) + a \sin(y)) e^{-y^2/(2\pi \sigma)} dy
\]

and

\[
\int_{-\infty}^{+\infty} \frac{\zeta''(0)}{4} (b_2 a^2 + O(a^3) + a \sin(y)) e^{-y^2/(2\pi \sigma)} dy = \frac{1}{4} \pi \sigma^4 \]

(69)
we have
\[
\frac{\sin(\theta)}{\sin(\theta)} \leq r_1 - \delta,
\]
where $r_1$ is an arbitrary fixed value. Then, for any $\epsilon > 0$, we have
\[
\int_{-\infty}^{\infty} \frac{1}{2} - \frac{1}{2} e^{-2\sigma^2} - 2r_1 \sin(\theta) y^2 e^{-\gamma_1^2 \theta^2} d\theta > 0.
\]
By substituting (72) into (71) we get
\[
det(\lambda I - J^{a^*}) = (\lambda - 1 + \epsilon w_2) (\lambda - 1) + \epsilon w_1 (\lambda - 1 - \epsilon^2 \rho J^{a^*}_{21}).
\]
With Taylor expansion and by integrating the integral, we get
\[
\int_{-\infty}^{\infty} \frac{1}{2} - \frac{1}{2} e^{-2\sigma^2} - 2r_1 \sin(\theta) y^2 e^{-\gamma_1^2 \theta^2} d\theta > 0.
\]
By substituting (72) into (71) we get
\[
det(\lambda I - J^{a^*}) = (\lambda - 1 + \epsilon w_2) (\lambda - 1 - \Pi_1) (\lambda - 1 - \Pi_2).
\]
where
\[
\Pi_1 = \frac{1}{2} - \frac{1}{2} e^{-2\sigma^2} - 2r_1 \sin(\theta) y^2 e^{-\gamma_1^2 \theta^2} + O(a^2).
\]
\[
\Pi_2 = \frac{1}{2} - \frac{1}{2} e^{-2\sigma^2} - 2r_1 \sin(\theta) y^2 e^{-\gamma_1^2 \theta^2} + O(a^2).
\]
Since $\theta > 0$, we have sufficiently small $a$, $\Pi_2 > 0$. Thus, we have $\lambda_1 > 0$, such that for $\epsilon \in (0, \epsilon_1)$, the eigenvalues of the Jacobian matrix of the average system (60) are in the unit ball, and thus the equilibrium of the average system is exponentially stable. Then according to Theorem 9, we have the following result for stochastic extremum seeking algorithm in Fig. 4.

**Theorem 11:** Consider the reduced system (54)–(56) under Assumption 5. Then there exists a constant $a^* > 0$ such that for any $0 < a < a^*$ there exist constants $r_2 > 0$, $c_2 > 0$ and $0 < \gamma_1 < 1$ such that for any initial condition $|\Delta_0| < r_2$, and any $\epsilon > 0$,
\[
\lim_{\epsilon \to 0} \{k \in \mathbb{N} : |\Delta_k| > c_2 |\Delta_0|^2 (\gamma_1^2 - \delta) \} = +\infty, \text{ a.s.}
\]
\[
\lim_{\epsilon \to 0} P \left\{ |\Delta_k| \leq c_1 |\Delta_0|^2 (\gamma_1^2 - \delta), \forall k = 0, 1, \ldots, [N/\epsilon] \right\} = 1
\]

**Remark 5.1:** For stochastic ES scheme for dynamical systems with output equilibrium map, we focus on the stability of the reduced system. Different from the deterministic ES case (periodic probing signal), the closed-loop system (50)–(53) has two perturbations (small parameter $\epsilon$ and stochastic perturbation $\{v_k\}$) and thus generally, there is no equilibrium solution or periodic solution. So we can not analyze properties of the solution of the closed-loop system by general singular perturbation methods for both deterministic systems (9) and stochastic systems (28). But for the reduced system (parameter estimation error system when the state is at its quasi-steady state value), we can analyze properties of the solution by our
VI. CONCLUSION

In this paper, we develop discrete-time stochastic averaging theory and apply it to analyze the convergence of our proposed stochastic discrete-time extremum seeking algorithms. Our results of stochastic averaging extend the existing discrete-time averaging theorems for globally Lipschitz systems to locally Lipschitz systems. Compared with other stochastic optimization methods, e.g., stochastic approximation, simulated annealing method and genetic algorithm, the convergence conditions of discrete-time stochastic extremum seeking algorithm are easier to verify and clearer. Compared with continuous-time stochastic extremum seeking, in the discrete-time case, we consider the bounded measurement noise. In our results, we can only prove the weaker convergence than the convergence with probability one of classical stochastic approximation. Better convergence of algorithms and improved algorithms are our future work directions. For dynamical systems, we only focus on the stability of parameter estimation error system at the quasi-steady state value (the reduced system). For the whole closed-loop system with extremum seeking controller, we will investigate the proper singular perturbation method in future work.

APPENDIX

A. Proof of Lemma 1: Approximation in Finite-Time Interval

With the Continuous Average System

Fix $T > 0$ and define $M' = \sup_{0 \leq t \leq T} |\hat{X}(t)|$. Since $(\hat{X}(t), t \geq 0)$ is continuous and $[0, T]$ is a compact set, we have that $M' < +\infty$. Denote $M = M' + 1$. For any $\varepsilon \in (0, \varepsilon_0)$, define a stopping time $\tau_{\varepsilon}$ by

$$ \tau_{\varepsilon} = \inf \{ t \geq 0 : |X(t)| > M \}. $$

(A1)

By the definition of $M$ (noting that $|x| = |X_0| = |\hat{X}(0)| \leq M'$), we know that $0 < \tau_{\varepsilon} < +\infty$. If $\tau_{\varepsilon} < +\infty$, then by the definition of $\tau_{\varepsilon}$, we know that for any $s < \tau_{\varepsilon}$, $|X(s)| \leq M$. By Assumption 1, we know that there exists a positive constant $C_M$ such that for any $|x| \leq M$ and any $y$, we have $|f(x, y)| \leq C_M$. And thus by (1), we know that

$$ M \leq |X(\tau_{\varepsilon})| \leq M + \varepsilon C_M \leq M + \varepsilon_0 C_M. $$

(A2)

Denote $M = M + \varepsilon_0 C_M$. By Assumption 1 again, we know that there exists a positive constant $C_M$ such that for any $|x| \leq M$ and any $y$, we have $|f(x, y)| \leq C_M$. It follows by (4) that for any $|x| \leq M$, $|f(x)| \leq C_M$.

From (6) and (8), we have that, for any $t \geq 0$

$$ X(t) - \hat{X}(t) = \int_0^t \left[ f(X(s), Y(\varepsilon + s)) - f(\hat{X}(s), Y(\varepsilon + s)) \right] ds $$

$$ + \int_0^t \left[ f(\hat{X}(s), Y(\varepsilon + s)) - \hat{f}(\hat{X}(s)) \right] ds $$

$$ - \int_{t_{m(t)}}^t f(X(s), Y(\varepsilon + s)) ds. $$

(A3)

By Assumption 1 and the definition of $\hat{f}$, there exists a positive constant $K_M$ such that for any $x_1, x_2$ in the subset $D_M := \{ x \in \mathbb{R}^n : |x| \leq M \}$ of $\mathbb{R}^n$, and any $y \in \mathbb{R}^m$, we have

$$ |f(x_1, y) - f(x_2, y)| \leq K_M |x_1 - x_2| $$

(A4)

$$ |\hat{f}(x_1) - \hat{f}(x_2)| \leq K_M |x_1 - x_2|. $$

(A5)

By (A3)–(A5), we have that if $t \leq \tau_{\varepsilon} \wedge T$, then

$$ |X(t) - \hat{X}(t)| \leq K_M \int_0^t |X(s) - \hat{X}(s)| ds $$

$$ + \int_0^t \left[ f(\hat{X}(s), Y(\varepsilon + s)) - \hat{f}(\hat{X}(s)) \right] ds $$

$$ + \int_{t_{m(t)}}^t f(X(s), Y(\varepsilon + s)) ds. $$

(A6)

Define

$$ \Delta_t = |X(t) - \hat{X}(t)| $$

(A7)

$$ \alpha(\varepsilon) = \sup_{0 \leq t \leq T} \int_0^t \left[ f(\hat{X}(s), Y(\varepsilon + s)) - \hat{f}(\hat{X}(s)) \right] ds $$

(A8)

$$ \beta(\varepsilon) = \sup_{0 \leq t \leq \tau_{\varepsilon} \wedge T} \int_{t_{m(t)}}^t f(X(s), Y(\varepsilon + s)) ds. $$

(A9)

Then by (A6) and Gronwall’s inequality, we have

$$ \sup_{0 \leq t \leq \tau_{\varepsilon} \wedge T} \Delta_t \leq (\alpha(\varepsilon) + \beta(\varepsilon)) e^{K_M T}. $$

(A10)

Since for any $t \geq 0$, we have $t - t_{m(t)} \leq \varepsilon$, and thus $\beta(\varepsilon) \leq C_M \varepsilon$. Hence

$$ \lim_{\varepsilon \to 0} \beta(\varepsilon) = 0. $$

(A11)

In the following, we prove that $\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0$ a.s., i.e.,

$$ \lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \left| \int_0^t \left[ f(\hat{X}(s), Y(\varepsilon + s)) - \hat{f}(\hat{X}(s)) \right] ds \right| = 0 $$

a.s. For any $n \in \mathbb{N}$, define a function $\hat{X}^n(s), s \geq 0$, by

$$ \hat{X}^n(s) = \sum_{k=0}^{\infty} X_c \left( \frac{k}{n} \right) I_{k \leq s < k + 1/n}. $$

(A12)

Then for any $n \in \mathbb{N}$, we have

$$ \sup_{0 \leq s \leq T} |\hat{X}^n(s)| \leq \sup_{0 \leq s \leq T} |\hat{X}(s)| = M' < M. $$

(A13)

By (A4), (A5), (A12) and (A13), we obtain that

$$ \sup_{0 \leq t \leq T} \left| \int_0^t \left[ f(\hat{X}(s), Y(\varepsilon + s)) - \hat{f}(\hat{X}(s)) \right] ds \right| $$

$$ \leq \sup_{0 \leq t \leq T} \int_0^t \left| f(\hat{X}(s), Y(\varepsilon + s)) - \hat{f}(\hat{X}(s)) \right| ds $$

$$ + \sup_{0 \leq t \leq T} \left| \int_0^t \left( f(\hat{X}^n(s), Y(\varepsilon + s)) - \hat{f}(\hat{X}^n(s)) \right) ds \right|. $$

(A14)
\[
+ \sup_{0 \leq t \leq T} \int_{0}^{t} \left| \hat{f} (X^n(s)) - \hat{f} (X^c(s)) \right| \, ds \\
\leq 2K_M T \sup_{0 \leq t \leq T} \left| \hat{X}^c(s) - \hat{X}^n(s) \right| \\
+ \sup_{0 \leq t \leq T} \int_{0}^{t} \left( f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right) \, ds .
\]

(A14)

Next, we focus on the second term on the right-hand side of (A14). We have

\[
\sup_{0 \leq t \leq T} \int_{0}^{t} \left( f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right) \, ds \\
= \sup_{0 \leq t \leq T} \int_{0}^{t} \left( f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right) \\
\times \sum_{k=0}^{\infty} I \left\{ \|s - \frac{(k+1)}{n} \| < \varepsilon \right\} \, ds \\
\leq \sup_{0 \leq t \leq T} \sum_{k=0}^{n(\lfloor t \rfloor + 1)} \int_{\frac{k}{n} \wedge t}^{\frac{(k+1)}{n} \wedge t} \left( f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right) \, ds
\]

(A15)

For fixed \( n \) and \( k \) with \( \frac{k}{n} \leq \frac{t}{\varepsilon} \), we have

\[
\sup_{0 \leq t \leq T} \int_{\frac{k}{n} \wedge t}^{\frac{(k+1)}{n} \wedge t} \left( f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right) \, ds \\
= \sup_{0 \leq t \leq T} \int_{0}^{t} \left( f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right) \, ds
\]

(A16)

For the second term on the right-hand side of (A16), we have

\[
\int_{0}^{t} \left( f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right) \, ds \\
= \varepsilon \sum_{i=0}^{\lfloor t/\varepsilon \rfloor - 1} \left( f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right) \\
= \varepsilon \left[ \frac{1}{\lfloor t/\varepsilon \rfloor} \sum_{i=0}^{\lfloor t/\varepsilon \rfloor - 1} f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right]
\]

(A17)

Then by (A17), the Birkhoff’s ergodic theorem and [13, Problem 5.3.2], we obtain that

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} t_{m(t)} \left( f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right) ds = 0 \text{ a.s.}
\]

which together with (A15) and (A16) implies that for any \( n \in \mathbb{N} \)

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} t_{m(t)} \left( f (X^n(s), \varepsilon + s) - \hat{f} (X^n(s)) \right) ds = 0 \text{ a.s.} \quad \text{(A18)}
\]

Thus by (A14), (A18) and \( \lim_{n \to \infty} \sup_{0 \leq s \leq T} |X^c(s) - X^n(s)| = 0 \), we obtain \( \lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} t_{m(t)} \left( f (X^n(s), \varepsilon + s) - f (X(s)) \right) ds = 0 \text{ a.s.} \), i.e.,

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} |X(t) - X^c(t)| = 0 \text{ a.s.} \quad \text{(A19)}
\]

By (A7), (A10), (A11) and (A19), we have

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} |X(t) - X^c(t)| = 0 \text{ a.s.} \quad \text{(A20)}
\]
By the definition of $M'$ and (A20), we have
\[
\limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq \tau_{\varepsilon}^d} |X(t)| \\
\leq \limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq \tau_{\varepsilon}^d} |X(t) - \bar{X}(t)| + \sup_{0 \leq t \leq \tau_{\varepsilon}^d} |\bar{X}(t)| \\
\leq \limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq \tau_{\varepsilon}^d} |X(t) - \bar{X}(t)| + M' = M' < M \text{ as}. \quad (A21)
\]

By (A2) and (A21), we obtain that, for almost every $\omega \in \Omega$, there exists an $\varepsilon_0(\omega)$ such that for any $0 < \varepsilon < \varepsilon_0(\omega)$:
\[
\tau_{\varepsilon}(\omega) > T. \quad (A22)
\]

Thus by (A20) and (A22), we obtain that $\limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq T} |X(t) - \bar{X}(t)| = 0$ a.s. Hence (13) holds. The proof is completed.

B. Proof of Lemma 2: Approximation for Finite-Time Interval With the Discrete Average System

By Lemma 1, we need only to prove that
\[
\limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq T} |\bar{X}(t) - X(t)| = 0. \quad (A23)
\]

Let $M', M, M, M, C_M, C_M, K_M$ be defined in the above proof of Lemma 1. For any $\varepsilon \in (0, \varepsilon_0)$, define a time $\tau_{\varepsilon}^d$ by
\[
\tau_{\varepsilon}^d = \inf \left\{ t \geq 0 : |\bar{X}(t)| > M \right\}. \quad (A24)
\]

By the definition of $M$ (noting that $|x| = |\bar{X}(0)| = |\bar{X}(0)| \leq M'$), we know that $0 < \tau_{\varepsilon}^d \leq \infty$. If $\tau_{\varepsilon}^d < \infty$, then by the definition of $\tau_{\varepsilon}^d$, we know that for any $0 < \varepsilon < \tau_{\varepsilon}^d, |\bar{X}(s)| \leq M$. By (2), we know that
\[
M \leq |\bar{X}(\tau_{\varepsilon}^d)| \leq M + \varepsilon C_M \leq M + \varepsilon_0 C_M. \quad (A25)
\]

Noting that if $t \leq \tau_{\varepsilon}^d \wedge T$, then
\[
|\bar{X}(t)| \leq M, \quad |\bar{X}(t)| \leq M \quad (A26)
\]

and for any $x_1, x_2$ in the subset $D_M := \{ x \in \mathbb{R}^n : |x| \leq M \}$ of $\mathbb{R}^n$, we have
\[
|\bar{f}(x_1) - \bar{f}(x_2)| \leq K_M |x_1 - x_2| \quad (A27)
\]

By (7) and (8), we have
\[
\bar{X}(t) - \bar{X}(s) = \int_0^t \left( \bar{f}(\bar{X}(s)) - \bar{f}(\bar{X}(s)) \right) ds \quad (A28)
\]

Then by (A26)–(A28) and the fact that $t - t_{m(t)} \leq \varepsilon$, we obtain that for any $0 \leq t \leq \tau_{\varepsilon}^d \wedge T$
\[
|\bar{X}(t) - \bar{X}(s)| \leq K_M \int_0^t |\bar{X}(s)| ds + C_M \varepsilon. \quad (A29)
\]

By (A29) and the Gronwall’s inequality, we get $\sup_{0 \leq t \leq \tau_{\varepsilon}^d \wedge T} |\bar{X}(t) - \bar{X}(s)| \leq C_M \varepsilon \exp(K_M T)$, which implies that
\[
\limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq \tau_{\varepsilon}^d \wedge T} |\bar{X}(t) - \bar{X}(s)| = 0. \quad (A30)
\]

By the definition of $M'$ and (A30), we have
\[
\limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq \tau_{\varepsilon}^d \wedge T} |\bar{X}(t) - \bar{X}(s)| \leq C_M \varepsilon \exp(K_M T) \quad (A31)
\]

By (A25) and (A31), we obtain that there exists an $\varepsilon_0$ such that for any $0 < \varepsilon < \varepsilon_0$
\[
\tau_{\varepsilon}^d > T. \quad (A32)
\]

Thus by (A30) and (A32), we obtain that $\limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq T} |\bar{X}(t) - \bar{X}(s)| = 0$. Hence (A23) holds. The proof is completed.

C. Proof of Approximation Results (15) of Theorem 3: Approximation for any Long Time With the Continuous Average System

Now we prove that for any $\delta > 0$
\[
\liminf_{\varepsilon \to 0} \left\{ t \geq 0 : |X(t) - \bar{X}(t)| > \delta \right\} = +\infty \text{ a.s.} \quad (A33)
\]

Define
\[
\Omega' = \left\{ \omega : \limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq T} |X(t, \omega) - \bar{X}(t)| = 0, \forall T \in \mathbb{N} \right\} \quad (A34)
\]

where $X(t, \omega)(= X(t))$ only makes the dependence on the sample clear. Then by Lemma 1, we have
\[
P(\Omega') = 1. \quad (A35)
\]

Let $\delta > 0$. For $\varepsilon \in (0, \varepsilon_0)$, define a stopping time $\tau_{\varepsilon}^d$ by $\tau_{\varepsilon} = \inf \left\{ t \geq 0 : |X(t) - \bar{X}(t)| > \delta \right\}$. By the fact that $\bar{X}_0 - X_0 = 0$, and the right continuity of the sample paths of $(X(t) - \bar{X}(t), t \geq 0)$, we know that $0 < \tau_{\varepsilon}^d \leq +\infty$, and if $\tau_{\varepsilon}^d < +\infty$, then
\[
|X(\tau_{\varepsilon}^d) - \bar{X}(\tau_{\varepsilon}^d)| \geq \delta. \quad (A36)
\]

For any $\omega \in \Omega'$, by (A34) and (A36), we get that for any $T \in \mathbb{N}$, there exists an $\varepsilon_0(\omega, \delta, T) > 0$ such that for any $0 < \varepsilon < \varepsilon_0(\omega, \delta, T)$, $\tau_{\varepsilon}^d(\omega) > T$, which implies that
\[
\lim_{\varepsilon \to 0} \tau_{\varepsilon}^d(\omega) = +\infty. \quad (A37)
\]

Thus it follows from (A35) and (A37) that $\lim_{\varepsilon \to 0} \tau_{\varepsilon}^d = +\infty$ a.s. The proof is completed.

D. Proof of Approximation Results (16) of Theorem 3

The proof is similar to the proof (Appendices C and D) of the continuous-time averaging results in [15] by replacing $X_t^e$ and $\bar{X}_t$ with $X(t)$ and $\bar{X}(t)$, respectively. The only difference lies in that $\bar{X}(t) - \bar{X}(t)$ is right continuous with respect to $t$, while both $X_t^e$ and $\bar{X}_t$ in [15] are continuous.
E. Proof of Theorem 4: The Stability of the Continuous-Time Version (6) of the Original Systems With the Continuous Average System

Since the equilibrium \( \bar{X}(t) \equiv 0 \) of the continuous average system (3) is exponentially stable, there exist constants \( r > 0, c > 0 \) and \( \gamma > 0 \) such that for any \( |x| < r, |\dot{X}(t)| < c|x|e^{-\gamma t}, \forall t > 0 \). Thus for any \( \delta > 0 \), we have \( \{X(t) > c|x|e^{-\gamma t} + \delta\} \subseteq \{|X(t) - \bar{X}(t)| > \delta\} \), which together with Theorem 3 implies that

\[
\liminf_{\varepsilon \to 0} \{ \varepsilon \Delta \geq 0 \} = \liminf_{\varepsilon \to 0} \{ t \geq 0 : |X(t) - \bar{X}(t)| > \delta \} = +\infty \text{ a.s.}
\]

Hence (17) holds.

Let \( T(\varepsilon) \) be defined in Theorem 3. Thus \( \lim_{\varepsilon \to 0} T_\varepsilon = +\infty \).

F. Proof of Lemma 7

By Lemma 2 and the time scale transform, we get

\[
\limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq T(\varepsilon)} |X(t) - \bar{X}(t)| = \limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq T(\varepsilon)} |X(t) - \bar{X}(t)| = 0
\]

a.s. Hence (24) holds. The proof is completed.

G. Proof of Theorem 8

i) Noticing that \( |N/\varepsilon| \geq N \) for \( \varepsilon \leq 1 \). Then by Lemma 7, we know that for any natural number \( N \)

\[
\varepsilon \to 0 \sup_{0 \leq k \leq N} |X_k - \bar{X_k}| = 0 \text{ a.s. (A38)}
\]

By (A38) and following the proof of Theorem 3(i), we can prove (25).

ii) By Lemma 7, we know that for any natural number \( N, \sup_{0 \leq k \leq N} |X_k - \bar{X_k}| \) converges to 0 a.s., and thus it converges to 0 in probability, i.e., (26) holds.

REFERENCES

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