Brief paper

Adaptive sinusoidal disturbance cancellation for unknown LTI systems despite input delay

Halil I. Basturk, Miroslav Krstic

A R T I C L E   I N F O

Article history:
Received 17 September 2014
Received in revised form
26 January 2015
Accepted 29 April 2015
Available online 2 June 2015

Keywords:
Adaptive control
Delay
Boundary control
Disturbance estimation

A B S T R A C T

Solutions exist for the problems of canceling sinusoidal disturbances and compensating input delays. In this paper, two problems are considered simultaneously and an adaptive controller is designed to cancel unknown sinusoidal disturbances forcing an unknown linear time-invariant system in controllable canonical form despite input delay. The design is based on three steps, (1) parametrization of the sinusoidal disturbance as the output of a known feedback system with an unknown output vector that depends on unknown disturbance parameters, (2) representation of the delay as a transport PDE, (3) design of the adaptive controller by using the backstepping boundary control technique for PDEs. It is proven that the equilibrium of the closed-loop system is stable and the state of the considered system converges to zero as $t \to \infty$ with perfect disturbance estimation. The effectiveness of the controller is illustrated with a simulation example of a second order system.

1. Introduction

The problem of canceling sinusoidal disturbances in dynamical systems is a fundamental control problem, with many applications such as active noise control (Bodson, Jensen, & Douglas, 2001), rotating mechanisms control (Gentili & Marconi, 2003), and marine vehicles (Basturk & Krstic, 2013b; Basturk, Rosenthal, & Krstic, 2013; Marconi, Isidori, & Serrani, 2002). The common method to approach this problem is the internal model principle for which a general solution is given in Francis and Wonham (1975) and Johnson (1971) in the case of linear systems. In the internal model approach, the disturbance is modeled as the output of a linear dynamic system which is called an exosystem.

The output regulation problem for minimum phase, uncertain nonlinear systems is solved in Serrani and Isidori (2000), Serrani, Isidori, and Marconi (2001), and extended for non-minimum phase plants in Marconi, Isidori, and Serrani (2004). Moreover, designs for nonlinear systems are proposed in Ding (2003), Marino and Santosoosso (2005), Marino and Tomei (2005) and Nikiforov (2001). The regulation of a linear time-varying system for unknown exosystem is considered in Marino and Tomei (2000). Disturbance cancellation and output regulation designs also exist for continuous-time LTI systems (Bobtsov & Pyrkin, 2009; Bodson & Douglas, 1997; Serrani, 2006; Zhang & Serrani, 2006) and discrete-time LTI systems (Guo & Bodson, 2009). Rejection algorithms are also given by state derivative feedback for both known (Basturk & Krstic, 2012a, 2013a) and unknown (Basturk & Krstic, 2012b, 2014) LTI systems.

On the other hand, the input delay causes significant stabilization problems in many dynamical systems including various chemical systems, hydraulically actuated systems, and combustion systems which may be driven by unknown sinusoidal disturbances. Therefore, the design of controllers for the stabilization of systems with delays continues to be an active research area. Controllers for both linear (Artstein, 1982; Bekiaris-Liberis & Krstic, 2010b; Fiagbedzi & Pearson, 1986; Jankovic, 2009, 2010; Kwon & Pearson, 1980; Mondie & Michiels, 2003; Olbrot, 1978; Zhong, 2006a,b) and nonlinear systems (Bresch-Pietri & Krstic, 2014; Krstic, 2008, 2010; Mazenc, Mondie, & Francisco, 2004; Mazenc & Niculescu, 2011) exist in the literature, many of which are based on predictor-like techniques. The solution for time varying delays in nonlinear systems is given in Bekiaris-Liberis and Krstic (2012). Input delays that depend on state in nonlinear systems are considered in Bekiaris-Liberis, Jankovic, and Krstic (2012). Moreover, adaptive control schemes for uncertain plants can be found in Evesque, Annaswamy, Niculescu, and Dowling (2003) and
Niculescu and Annaswamy (2003). The robustness of the standard adaptive backstepping technique with respect to time delay in input and unmodeled dynamics in the system is studied in Zhou, Wang, and Wen (2009). Adaptive controllers for unknown delays are given in Bekiaris-Liberis and Krstic (2010a) and Bresch-Pietri and Krstic (2010). The designs of trajectory tracking for uncertain linear systems with unknown input delay are given in Bresch-Pietri and Krstic (2009) and Bresch-Pietri, Chauvin, and Petits (2012). Controller designs for canceling unknown sinusoidal disturbances with input delay are considered in Bobtsov and Prykin (2008), Bobtsov, Kolyubin, and Prykin (2010) and Prykin et al. (2010a,b). However, in these references, the disturbance cancellation algorithms are given for systems whose parameters are assumed to be known. We present a method that does not require to know the actual value of the parameters of the system. This is the main contribution of the presented work and provides an important advantage for the application of the result.

In this note, an adaptive controller is designed to estimate and cancel the unknown sinusoidal disturbances forcing general LTI systems in the controllable canonical form with input delay and unknown system parameters by full state feedback. The unknown disturbance is represented in a parameterized form by using the technique given in Nikiforov (2004). The essence of the approach for the compensation of the input delay is predictor feedback which has been shown in Krstic and Smyshlyaev (2008a) to be a form of backstepping boundary control for PDEs (Krstic & Smyshlyaev, 2008b). The results given in Krstic and Smyshlyaev (2008a) and Nikiforov (2004) allow us to reformulate the problem as an adaptive control problem for an uncertain PDE–ODE coupled system. Update laws for uncertain parameters are based on normalized Lyapunov-based tuning which is a similar approach given in Bresch-Pietri and Krstic (2009). Finally, it is proven that the equilibrium of the closed loop system is stable and the state converges to zero as $t \to \infty$ with perfect disturbance estimation.

In Section 2, the problem is introduced. The representation of unknown sinusoidal disturbances is given in Section 3. In Section 4, the main design is presented and the stability theorem is stated. In Section 5, the proof of the stability theorem is given. A simulation example is presented in Section 6.

2. Problem statement

We consider the single-input LTI system

$$\dot{X}(t) = A_0X(t) + B(y^TX(t) + \nu(t) + BU(t - D)), \quad (1)$$

where $A_0 = \begin{bmatrix} 0_{n-1} & 0 \end{bmatrix}, B = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix}, y = [a_1, \ldots, a_n]^T$, with $0_{n-1} = [0, \ldots, 0]$, the known input delay $D \in \mathbb{R}$, the state $X = [X_1, \ldots, X_n] \in \mathbb{R}^n$, the input $U \in \mathbb{R}$, and the unknown sinusoidal disturbance $\nu \in \mathbb{R}$ given by

$$\nu(t) = g \sin(\omega t + \phi), \quad (2)$$

where $\omega, g, \phi \in \mathbb{R}$.

The sinusoidal disturbance $\nu$ can be represented as the output of a linear system,

$$\dot{W}(t) = SW(t), \quad (3)$$

$$\nu(t) = h^T W(t), \quad (4)$$

where $W(t) \in \mathbb{R}^2, S = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, h^T = [1 \ 0]$. The matrix $S$ depends on the unknown frequency of the sinusoidal disturbance $\nu$, while the uncertainty of amplitude and phase is related to the unknown initial condition of $(3)$. The system parameters $\gamma$ and $b$ are unknown, the input delay $D$ is known. The disturbance $\nu(t)$ is not measured. The measurement of the state $X$ is available for feedback.

We make the following assumptions regarding the plant $(1)$ and the exosystem $(3)$–$(4)$:

**Assumption 1.** $\omega_{\text{max}} \geq \omega \geq \omega_{\text{min}} > 0$, where $\omega_{\text{max}}$ and $\omega_{\text{min}}$ are known.

**Assumption 2.** $g_{\text{max}} \geq |g|$ where $g_{\text{max}}$ is known.

**Assumption 3.** $a_i \in [a_{\text{min}}, a_{\text{max}}]$ for $i = 1, \ldots, n$. The bounds $a_{\text{min}}, a_{\text{max}}$ are known and $\bar{a} = \max\{|a_{\text{max}}|, |a_{\text{min}}|\}$.

**Assumption 4.** The sign of $b$ is known and $b_{\text{max}} \geq |b| \geq b_{\text{min}} > 0$ where $b_{\text{max}}, b_{\text{min}}$ are known.

The main aim is to design an adaptive controller to achieve the stability of the equilibrium of the closed loop system and the convergence of the state $X(t)$ to zero as $t \to \infty$ despite the input delay, unmeasured sinusoidal disturbance and uncertain system parameters.

3. Disturbance representation

The disturbance is parameterized by following (Nikiforov, 2004). Let $G \in \mathbb{R}^{2\times 2}$ be a Hurwitz matrix with distinct eigenvalues and let $(G, I)$ be a controllable pair. Since $(\theta^T, S)$ is observable and the spectra of $S$ and $G$ are disjoint the unique and nonsingular solution $M \in \mathbb{R}^{2\times 2}$ of the Sylvester equation

$$MS - GM = hh^T \quad (5)$$

is invertible (Chen, 1984). The change of coordinates $Z(t) = MW(t)$ transforms the exosystem $(3)$–$(4)$ into the form

$$\dot{Z}(t) = GZ(t) + Iv(t), \quad (6)$$

$$\nu(t) = \hat{\theta}^T \nu(t), \quad (7)$$

$$\hat{\nu}(t) = \hat{\theta}^T \nu(t), \quad (8)$$

where $\hat{\nu}_i = h_i^T M^{-1}, \hat{\theta}_i^T = h_i^T SM^{-1}$. Since there is a delay in the input, the disturbance $\nu(t)$ needs to be represented with a delayed $Z(t)$ to be able to design an adaptive controller. The following lemma establishes this result.

**Lemma 1.** The disturbance can be represented as

$$\nu(t) = \theta^T Z(t - D), \quad (9)$$

where

$$\theta = \cos(\omega_0)\hat{\theta}_0 + \sin(\omega_0)\hat{\theta}_1. \quad (10)$$

**Proof.** Using $(2)$, the necessary trigonometric calculations yield

$$\nu(t + D) = \cos(\omega_0)\nu(t) + \sin(\omega_0)\hat{\nu}(t). \quad (11)$$

Substituting $(7)$ and $(8)$ into $(11)$, we get $(9)$. \qed

Since $\nu(t)$ is not measured, the state $Z(t)$ cannot be used in the design. To overcome this problem, an observer is designed. The following lemma establishes the properties of the observer and the representation of the unknown disturbance.

**Lemma 2.** The unknown disturbance $\nu(t)$ can be represented in the form

$$\nu(t) = \hat{\theta}^T Z_X(t - D) + \sum_{i=1}^n \beta_i^T \nu_i(t - D) + \theta_0^T \delta(t) \quad (12)$$

where $\theta_0 = e^{-\omega_0 t} \hat{\theta}_0, \beta_i = b_i \theta_i, \beta_i = a_i \hat{\theta}_i$ for $i = 1, \ldots, n$. 

The observer filters are given by
\begin{align}
\hat{x}_0(t) &= G(x_0(t) + N(t)), \\
\hat{x}_i(t) &= G\hat{x}_i(t) - I\hat{x}_i(t), \quad i = 1, \ldots, n \tag{13}
\end{align}
and
\begin{align}
\hat{z}_0(t) &= G\hat{z}_0(t) - IU(t - D) \tag{15}
\end{align}
with the matrices $N$ which is given by $N = \frac{1}{2}PB^1$, where the given $N$ is one of the many solutions of the following equation $NB = I$.

The estimation error $\delta \in R^d$ obeys the equation
\begin{align}
\dot{\delta}(t) &= G\delta(t). \tag{17}
\end{align}

**Proof.** Following (Nikiforov, 2004), the estimation error $\delta(t)$ is given by
\begin{align}
\delta(t) &= Z(t) - \left( Z_x(t) + \sum_{i=1}^n a_iZ_i(t) + b_i \right). \tag{18}
\end{align}
Using (18) and the fact that $\forall A_0 = 0$ and $NB = I$, the time derivative of (18) in view of (1), (13)–(15) yields (17). Representing $Z(t - D)$ by using (18) and substituting it into (9), we get (12).

The representation (12) established with Lemmas 1 and 2, allows us to represent a time-varying unknown sinusoidal disturbance $v(t)$ as a constant unknown vector multiplied by a known regressor which is delayed, plus an unknown exponentially decaying disturbance $\delta(t)$.

Defining the bound of the unknown vector $\hat{\gamma}$ is necessary for the update law. The parametric solution of (5) is given by
\begin{align}
m_1(\omega) &= -\frac{\sigma_m}{\sigma_m - \omega^2 + \omega^2 - \sigma_m}, \\
m_2(\omega) &= -\frac{\sigma_m}{\sigma_m + \omega^2 - \omega^2 - \sigma_m}, \\
m_3(\omega) &= -\frac{\sigma_m}{\sigma_m + \omega^2 - \omega^2 - \sigma_m}, \\
m_4(\omega) &= -\frac{\sigma_m}{\sigma_m - \omega^2 + \omega^2 - \sigma_m}, \\
G &= \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}, \\
I &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\end{align}
Then using (10) and $\theta^\top = \xi^\top M^{-1} - \theta^\top$, $\theta$ is written as
\begin{align}
\theta(\omega) &= \left[ \theta_1(\omega), \theta_2(\omega) \right] \tag{19}
\end{align}
where
\begin{align}
\theta_1(\omega) &= \frac{\sigma_m (\omega) - \sin(\omega) m_1(\omega)}{\det(M(\omega))}, \\
\theta_2(\omega) &= \frac{\sigma_m (\omega) + \sin(\omega) m_1(\omega)}{\det(M(\omega))}.
\end{align}
The extreme value of $\theta_1(\omega)$ and $\theta_2(\omega)$ for $\omega \in [\omega_{\min}, \omega_{\max}]$ are given by taking the first derivative with respect to $\omega$ and finding the roots. Define the sets $\Omega_i = \{ \omega \in R | \omega_{\min} \geq \omega \geq \omega_{\max} \}$ and $\delta(\omega) = 0$ for $i = 1, 2$.

The maximum value of $|\theta_1|$, $|\theta_2|$ can be defined as follows
\begin{align}
\bar{\theta} &= \left[ \max_{\omega \in \Omega_1} |\theta_1(\omega)|, \max_{\omega \in \Omega_2} |\theta_2(\omega)| \right]^T. \tag{20}
\end{align}

4. Main result-design and stability statement

The delayed signals are represented as the boundary of a transport PDE to prepare the system for the design. This idea for delay systems is given and discussed in Krstic and Symmslayev (2008a). Substituting (12) into (1) and representing the delayed signals as the boundary of a transport PDE, the ODE–PDE coupled system is written as
\begin{align}
\dot{X}(t) &= A_0X(t) + B\left( \gamma X(t) + \theta^\top \xi^\top (0, t) + bu(0, t) + \beta^\top \xi^U (0, t) + \sum_{i=1}^n \beta^\top i \xi^U (0, t) + \theta^\top \delta(t) \right), \tag{21}
\end{align}
\begin{align}
u_i(x, t) &= u_i(x, t), \\
u(D, t) &= U(t), \\
\xi^X(x, t) &= \xi^X(x, t), \\
\xi^D(x, t) &= Z_x(t), \\
\xi^U_n(x, t) &= \xi^U_n(x, t), \quad i = 1, \ldots, n \\
\xi^N(x, t) &= \xi^N(t), \\
\xi^U(t) &= \xi^U(t), \\
\xi^D(t) &= \xi^D(t), \\
\xi^U(t) &= \xi^U(t).
\end{align}
The estimation error $\hat{\delta}(t)$ is given by
\begin{align}
\hat{\delta}(t) &= G\hat{\delta}(t).
\end{align}
The solutions of the transport PDEs are given by $u(x, t) = U(t+x-D)$, $\xi^X(t) = \xi^X(t+x-D)$, $\xi^N(x, t) = \xi^N(t+x-D)$, $\xi^U(t) = \xi^U(t+x-D)$. The adaptive controller for the system (21)–(29) is given by
\begin{align}
U(t) &= \frac{1}{b} \left( \hat{\delta}(t)\right) Z_x(t) - \sum_{i=1}^n \hat{\beta}_i(t)Z_i(t) - \hat{\beta}_n(t)Z_u(t) \\
&+ (K - \gamma^\top) \int_0^0 e^{(A_0^\top + B^\top)^i(0-y)B} \xi(y, t)dy \\
&+ (K - \gamma^\top) e^{(A_0^\top + B^\top)^iU(X(t))}.
\end{align}
where the control gain $K \in R^{1 \times n}$ is chosen so that $(A_0^T + BK)$ is Hurwitz and the positive definite matrices $P$ and $M_C$ are solutions of the matrix equations
\begin{align}
(A_0^T + BK)^TP + P(A_0^T + BK) &= -Q, \\
G^TP_C + P_CG &= -Q_C.
\end{align}
with $\lambda_{\max}[Q_C] > 1 + \lambda_{\max}(c_0, c_\gamma) > 1 + \lambda_{\max}[P_C, G]$. $G^TP_C + P_CG + 2c_0\lambda_{\max}[N_I^T N] > 0$ where $c_0, c_\gamma > 0$ and
\begin{align}
\dot{\xi}(x, t) &= u_0(x, t) + \hat{\theta}^\top \xi^X(x, t) + \sum_{i=1}^n \hat{\beta}_i^\top \xi^N_i(x, t), \tag{33}
\end{align}
where
\begin{align}
u_0(x, t) &= \hat{\theta}^\top \xi^X(x, t), \tag{34}
\end{align}
The update laws are given by
\begin{align}
\dot{\theta}(t) &= \kappa_\theta \text{Proj}_{\theta}(1u(0)\tau_c(t)), \tag{35}
\dot{\gamma}(t) &= \kappa_\gamma \text{Proj}_{\gamma}(X(t)\tau_c(t)), \tag{36}
\dot{\hat{\beta}}(t) &= \kappa_{\hat{\beta}} \text{Proj}_{\hat{\beta}} \left\{ \xi^X(0)\tau_c(t) \right\}, \tag{37}
\dot{\hat{\beta}}(t) &= \kappa_{\hat{\beta}} \text{Proj}_{\hat{\beta}} \left\{ \xi^N_i(0)\tau_c(t) \right\}, \quad i = 1, \ldots, n \tag{38}
\end{align}
where $\kappa \in [0, \kappa^\star]$ (being chosen in the analysis) and
\begin{align}
\tau_c(t) &= \frac{\kappa^\star B^TPX(t)}{N_t(t)} - \frac{(K - \gamma^\top) \int_0^0 (1+x)e^{(A_0^\top + B^\top)^iU(X(t))} dx}{N_t(t)}.
\end{align}
with

\[ N_i(t) = 1 + X(t)^T P X(t) + \Xi_0(t)^T P_c \Xi_0(t) + \sum_{i=1}^n \Xi_i(t)^T P_c \Xi_i(t) \]

\[ + \int_0^t (1 + x) \left( c_{u, w}(x, t)^2 + c_\nu \xi_i(x, t)^T \xi_i(x, t) \right) dx 
\]

\[ + c_f \sum_{i=1}^n \xi_i(x, t)^T \xi_i(x, t) \right) dx \]

(41)

where \( c_{u, w} > \frac{1}{\min \{ P B B^T \} P \} \)

and the update

\[ w(x, t) = \zeta(x, t) - (K - \gamma T) \int_0^x e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)(x-y)} B \xi(x, t) dy \]

\[ - (K - \gamma T) e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)X(t)} \] \quad (42)

Using Assumptions 1–4 and (20) with \( \beta_u = b \theta, \beta_i = a \theta \) for \( i = 1, \ldots, n \), the sets of parameters are given by

\[ \Pi_0 = \{ b \in \mathbb{R} | b_{\min} > | b | > b_{\max} > 0 \} \] \quad (43)

\[ \Pi_{\gamma} = \left\{ \gamma \in [a, \ldots, a_1], a_i > a_{\min}, i = 1, \ldots, n \right\} \]

(44)

\[ \Pi_{\theta} = \left\{ \theta = \beta_1, \beta_2 \right\} \in \mathbb{R}^2 | b_{\max} > a_i > a_{\min}, j = 1, 2 \} \]

(45)

\[ \Pi_{\beta_i} = \left\{ \beta_i = \beta_1, \beta_2 \right\} \in \mathbb{R}^2 | b_{\max} > a_i > b_{\min}, i = 1, 2 \} \]

(46)

The standard projection operators are given by

\[ \text{Proj}_{\Pi_{\theta}} [\tau_\theta(t)] = \begin{cases} 0, & \text{sign}(b) \hat{b} = b_{\max} \text{ and } \tau_\theta > 0 \\ 0, & \text{sign}(b) \hat{b} = b_{\min} \text{ and } \tau_\theta < 0 \end{cases} \]

(48)

where \( \tau_\theta(t) = u(0) \tau_\theta(t) \) and

\[ \text{Proj}_{\Pi_{\beta_i}} [\tilde{\theta}_i(t)] = \begin{cases} 0, & \tilde{\theta}_i = \tilde{\beta}_{i+1} \text{ and } f_i \tau_\theta(t) > 0 \\ \tilde{\beta}_i, & \tilde{\theta}_i = \tilde{\beta}_i \text{ and } f_i \tau_\theta(t) < 0 \end{cases} \]

(49)

where \( f_k = X(t), \xi^0(0), \xi^0(0), \xi^0(0), k = \gamma, \theta, \beta, \hat{\beta}, \hat{\beta} \) respectively.

For \( \tilde{\beta}_i \) represents the estimate of the parameter and \( \hat{\beta}_i, \hat{\beta}_i \) are the maximum and minimum bounds of the parameter, respectively.

The inverse of the transformation (42) is given by

\[ u(x, t) = \frac{1}{b} \left( w(x, t) - \hat{\beta}_i^T \xi_i(x, t) - \sum_{i=1}^n \tilde{\beta}_j^T \xi_j(x, t) \right) \]

\[ - \hat{\beta}_i^T \xi_i(x, t) + (K - \gamma T) \int_0^x e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)(x-y)} B \xi(x, t) dy \]

\[ \times B w(y, t) dy + (K - \gamma T) e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)X(t)} \] \quad (50)

Using (18), (24), (26) and (28), Eq. (50) is written as

\[ u(x, t) = \zeta_0(x, t) - \frac{1}{b} \tilde{\beta}_i^T \xi_i(x, t) \]

\[ - \sum_{i=1}^n \tilde{\beta}_j^T \xi_j(x, t) + (K - \gamma T) \]

(51)

where

\[ \zeta_0(x, t) = \frac{1}{b} \left( w(x, t) - \left( \hat{\beta}_i^T - \frac{1}{b} \tilde{\beta}_i^T \right) \xi_i(x, t) \right) \]

\[ - \sum_{i=1}^n \left( \tilde{\beta}_j^T - \frac{1}{b} \tilde{\beta}_j^T \right) \xi_j(x, t) + (K - \gamma T) \]

5. Stability proof

The transformed system is given by

\[ \dot{X}(t) = \left( A_0 + B K \right) X(t) + B \left( \hat{\gamma}^T X(t) + \hat{\gamma}^T e(x) \right) \]

\[ + \sum_{i=1}^n \hat{\beta}_i^T \xi_i(0) + \hat{\beta}_0^T \xi(0) \]

\[ + \frac{1}{b} w(0) \]

\[ + B \xi(0) + \hat{\theta}^T \delta(t) \] \quad (56)

where

\[ w(x, t) = w_i(x, t) + \hat{\beta}_b p_i(x, t) + \hat{\gamma} p_{\gamma} (x, t) \]

\[ + \sum_{i=1}^n \hat{\beta}_{i+1} p_{i+1}(x, t) - \hat{\beta}_0 q_0(x, t) \]

\[ - \hat{\beta}_i^T \xi_i(x, t) + (K - \gamma T) e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)X(t)} \]

\[ \times B w(y, t) dy + (K - \gamma T) e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)X(t)} \] \quad (50)

(51)

where

\[ p_b = u(x, t) - (K - \gamma T) \int_0^x e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)(x-y)} B w(y, t) dy \]

\[ p_{\gamma} = \frac{1}{b} \left( \dot{X}(t) - \dot{X} \right) e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)B} \]

\[ - B (K - \gamma T) \int_0^x e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)(x-y)} B \xi(x, t) dy \] \quad (60)

\[ p_{\theta} = \xi_0(x, t) - \int_0^x \xi_0(y, t) B e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)(x-y)} (K - \gamma T) T dy \]

\[ p_{\beta_i} = \xi_0(x, t) - \int_0^x \xi_0(y, t) B e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)(x-y)} (K - \gamma T) T dy \]

\[ p_{\beta_0} = \xi_0(x, t) - \int_0^x \xi_0(y, t) B e^{(\hat{\Delta}_0 + \hat{\beta}_0^T)(x-y)} (K - \gamma T) T dy \]

(52)
\[ p_{\hat{p}_u} = \xi^U(x, t) - \int_0^\gamma \xi^U(y, t) B^T e^{(A_0 + B_0 y)^T (x - y)} (K - \hat{\nu}_u) \gamma^T \, dy , \]  
(63)

\[ q_b = u(0)(K - \hat{\nu}_u) e^{(A_0 + B_0) y^T} B, \]  
(64)

\[ q_v = X(t)(K - \hat{\nu}_u) e^{(A_0 + B_0) y^T} B, \]  
(65)

\[ q_\theta = \xi^X(0)(K - \hat{\nu}_u) e^{(A_0 + B_0) y^T} B, \]  
(66)

\[ q_{\hat{p}_u} = \xi^X(0)(K - \hat{\nu}_u) e^{(A_0 + B_0) y^T} B. \]  
(67)

Before the main proof of theorem, we state the following lemma.

**Lemma 3.** There exists a constant \( M_\epsilon \) such that the following bounds hold

\[ \hat{b}(t) \int_0^\gamma (1 + x) w(x, t) p_b(x, t) dx \leq \kappa_\epsilon c_w M_\epsilon \left[ |X(t)|^2 + |\xi^X(0)|^2 + \max_{i=1}^n |\xi^u_i(0)|^2 + \sum_{i=1}^n |\xi^v_i(t)|^2 \right], \]  
(69)

\[ \hat{\theta}(t) \int_0^\gamma (1 + x) w(x, t) p_b(x, t) dx + \sum_{i=1}^n \hat{\beta}_i(t) \int_0^\gamma (1 + x) w(x, t) p_{\hat{p}_u}(x, t) dx \leq \max_{\kappa_\beta, \kappa_\theta, \kappa_\beta, \kappa_\theta} c_w M_\epsilon \left[ |X(t)|^2 + |\xi^X(0)|^2 + \sum_{i=1}^n |\xi^a_i(0)|^2 + \sum_{i=1}^n |\xi^u_i(t)|^2 + |\hat{\theta}(t)|^2 \right], \]  
(70)

where

\[ M_\epsilon = \frac{3M_d}{\beta_{\min} \max_{\gamma \in \gamma_T} \left[ \frac{1}{b_{\min}} \left( \overline{b} + \frac{1}{b_{\max}} \overline{b} \right)^T \left( \overline{b} + \frac{1}{b_{\min}} \overline{b} \right) \right] \left( \overline{b} + \frac{1}{b_{\max}} \overline{b} \right)^T \left( \overline{b} + \frac{1}{b_{\min}} \overline{b} \right) \lambda_{\max} [PBB^T P], \]  
(73)

\[ M_\delta = \max_{\gamma \in \gamma_T} \left( 1 + D(1 + (2\gamma^T M_\delta B^T D) \gamma^T \gamma) \right), \]  
(74)

\[ M_{\kappa_\beta} = \min \{ 1, \lambda_{\min} \{ P \}, \lambda_{\min} \{ P_c \}, \lambda_{\min} \{ P_u \}, c_w, c_\theta \}, \]  
(75)

and using Assumptions 1 and 2, the fact that \( Z(t) = MW(t) \) and the parametric solution of \( M \), the bound for \( Z(t) \) is given by

\[ |Z(t)|^2 \leq \overline{Z}^T \overline{Z} = 2\overline{a}_{\min}(\overline{a}_{\min}, \overline{a}_{\max}) \lambda_{\max} \{ M_T M \}. \]  
(76)

**Proof.** It is possible to show the bounds by using the Cauchy-Schwarz and Young’s inequalities, the change of coordinates for the double integrals, and (18), (51).

**Proof of Theorem.** Consider the following Lyapunov function

\[ V(t) = \ln(N(t)) + \frac{c_w}{\kappa_b} \vartheta + \frac{c_w}{\kappa_\gamma} \overline{\gamma}^T \overline{\gamma} + \frac{c_w}{\kappa_\theta} \overline{\theta}^T \overline{\theta} + \epsilon_\delta \delta^T \delta, \]  
(77)

where \( \epsilon_\delta > \frac{1}{\lambda_{\min} \{ Q_c \}} \left( 4 \max \{ \kappa_\beta, \kappa_\gamma, \kappa_\beta, \kappa_\theta \} c_w M_\epsilon + \lambda_{\max} \{ PBB^T P \} \right) \). Substituting (35)–(39), using Lemma 3 and Young’s inequality for cross terms, the time derivative of (77) is given by

\[ \dot{V} \leq -(M_d \overline{M}_\epsilon - 4c_w \kappa^* M_\epsilon) \frac{N(t)}{\max_{t \in [0, T]} \{ N(t) \}} \left[ |X(t)|^2 + |\xi^X(0)|^2 + \sum_{i=1}^n |\xi^u_i(t)|^2 \right] + \frac{n}{\kappa_\gamma} \left[ |\xi^X(0)|^2 + \sum_{i=1}^n |\xi^a_i(0)|^2 + |w(t)|^2 + |\xi^X(t)|^2 \right] + \sum_{i=1}^n |\xi^u_i(t)|^2 + |w(t)|^2 \right) - M_\delta |\hat{\theta}(t)|^2 \]  
(78)

where

\[ M_d = \frac{c_w}{\kappa_\gamma} \kappa^* M_\epsilon - \lambda_{\max} \{ PBB^T P \}, \]  
(79)

\[ M_\delta = \min \left\{ \lambda_{\min} \{ Q \} - 2 \lambda_{\max} \{ P_C G \} \right\} \]  
(80)

By choosing

\[ \kappa^* = \frac{M_d}{5c_w M_\epsilon}, \]  
(81)

and \( (\kappa_\beta, \kappa_\gamma, \kappa_\beta, \kappa_\theta) \in [0, \kappa^*] \), we obtain \( \dot{V}(t) \leq 0 \) and \( V(t) \leq V(0) \).

From the transformation (42) and its inverse (50), by using (18) and (34), the inequalities between the states are written as

\[ \|w(t)\|^2 \leq r_1 \|u_{\hat{p}_u}(t)\|^2 + r_2 \|\xi^X(t)\|^2 \]  
(82)

\[ + r_3 \sum_{i=1}^n |\xi^u_i(t)|^2 + r_4 |X(t)|^2, \]  
(83)

\[ \|u_{\hat{p}_u}(t)\|^2 \leq s_1 \|w(t)\|^2 + s_2 \|\xi^X(t)\|^2 \]  
(84)

where \( r_1, r_2, s_1, s_2 \) are sufficiently large positive constants. From (77) and (84), it follows that

\[ \overline{b}^2 + \overline{\gamma}^T \overline{\gamma} + \overline{\theta}^T \overline{\theta} + \sum_{i=1}^n \beta_i^2 \overline{\beta}_i + \overline{\beta}_u^2 \overline{\beta}_u + \overline{\delta}^T \overline{\delta} \leq \left( \frac{(n + 4)\kappa^*}{c_w} + \frac{\epsilon_\delta}{\lambda_{\min} \{ P_C \}} \right) \overline{V}(t), \]  
(85)
\[ |X(t)|^2 \leq \frac{1}{\lambda_{\min}[P]} \left( e^{\nu(t)} - 1 \right) \]  

(86)

\[ |\xi_0(t)|^2 \leq \frac{1}{\lambda_{\min}[P_c]} \left( e^{\nu(t)} - 1 \right), \]  

(87)

\[ |\xi_i(t)|^2 \leq \frac{1}{\lambda_{\min}[P_c]} \left( e^{\nu(t)} - 1 \right), \quad i = 1, \ldots, n \]  

(88)

\[ \|\xi^T(t)\|^2 \leq \frac{1}{c_0} \left( e^{\nu(t)} - 1 \right), \]  

(89)

\[ \|\xi^n(t)\|^2 \leq \frac{1}{c_r} \left( e^{\nu(t)} - 1 \right), \quad i = 1, \ldots, n \]  

(90)

\[ \|u_{\phi_k}(t)\|^2 \leq \left( \frac{s_1}{c_u} + \frac{s_2}{c_0} + \frac{ns_3}{c_r} + \frac{s_4}{\lambda_{\min}[P]} \right) \left( e^{\nu(t)} - 1 \right). \]  

(91)

From the definition \( \mathcal{Y}(t) \), it follows that
\[ \mathcal{Y}(t) \leq R \left( e^{\nu(t)} - 1 \right), \]  

(92)

where \( R = \left( \frac{s_1}{c_u} + \frac{2s_2}{c_0} + \frac{n(s_3+1)}{c_r} + \frac{s_4+1}{\lambda_{\min}[P]} + \frac{n+1}{\lambda_{\min}[P]} \right) \). Using the fact \( \ln(1 + c_0) \leq c_c \) for \( c_c > -1 \) and \( (83) \), we obtain
\[ V(0) \leq \rho \mathcal{Y}(0) \]  

(93)

with \( \rho = \left( \frac{s_1}{c_u} + \frac{s_2}{c_0} + \frac{ns_3}{c_r} + \frac{s_4}{\lambda_{\min}[P]} \right) + c_0 + nc_c + \lambda_{\min}[P] \). Therefore we obtain the stability result in theorem. We now prove the convergence of the state \( X(t) \). From (82), we obtain the uniform boundedness of \( |X(t)|, \|\xi^T(t)\|, \|\xi^n(t)\|, |||u(t)||, ||\hat{u}(t)||, ||\hat{\phi}(t)||, ||\hat{u}(t)||, ||\xi_0(t)||, ||\xi_i(t)|| \) and \( ||\xi(t)|| \) from (6), (16), (18), it follows that \( |\xi(t)| \) and \( |\xi_0(t)| \) are bounded. From (84), it is obtained that \( ||u_{\phi_k}(t)|| \) is also bounded in time. From (30), we get uniformly boundedness of \( U(t) \). Thus we get that \( u(0, t) \) is uniformly bounded for \( t \geq D \). Finally, with (56), we obtain that \( dX(t)^2/\,dt \) is uniformly bounded for \( t \geq D \). Since \( |X(t)| \) is square integrable, from (82), we conclude from Barbalat’s lemma (Liu & Krstić, 2001) that \( X \to 0 \) as \( t \to \infty \). Moreover, from (82) it follows that \( ||w(t)||, ||\xi^T(t)||, ||\xi^n(t)|| \) are square integrable. From (84), we obtain the square integrability of \( ||u_{\phi_k}(t)|| \). In addition to this, from (16) and (82), it follows that \( ||\xi_0(t)|| \) and \( ||\xi_i(t)|| \) are also square integrable. Therefore, using (30), the square integrability of \( b(t)U(t) + \hat{\phi}_u(t)\xi_0(t) \) is obtained. Furthermore, the solution to
\[ d\left( \hat{b}(t)U(t) + \hat{\phi}_u(t)\xi_0(t) \right)^2/\,dt \]  

(94)

The signals \( \hat{b}(t), \hat{v}(t), \hat{\phi}(t), \hat{\phi}_u(t), \hat{\phi}_v(t) \) are uniformly bounded for \( t \geq 0 \) according to (35)–(39). From (18), it follows that \( \|\xi(t)\| \) is bounded. Consequently, from (50), the boundedness of \( ||u(t)|| \) is obtained. Moreover, by using the uniform boundedness of \( X(t), \xi(t), \xi_0(t) \) and the parameter estimations, we obtain the uniform boundedness of \( \frac{d(\hat{b}(t)U(t) + \hat{\phi}_u(t)\xi_0(t))}{\,dt} \).

Then, from Barbalat’s lemma (Liu & Krstić, 2001) that \( \frac{\hat{b}(t)U(t) + \hat{\phi}_u(t)\xi_0(t)}{\,dt} \to 0 \) as \( t \to \infty \).

By using (14), (13), (17) and (24)–(27) and the fact that \( X \to 0 \) as \( t \to \infty \), it is concluded that \( \xi_0(t), \xi(t), \delta(t) \to 0 \) as \( t \to \infty \). Therefore, considering (21) and the fact that \( \frac{d(\hat{b}(t)U(t) + \hat{\phi}_u(t)\xi_0(t))}{\,dt} \to 0 \) as \( t \to \infty \), it is obtained that
\[ \frac{1}{b(t-D)} \hat{b}(t-D)\xi_0(t-D) \to \frac{1}{b}v(t) \]  

as \( t \to \infty \). 

6. Simulation results

We illustrate the performance of our controller with a second-order system with \( \gamma = 0.5 \), \( b = 1 \), the unknown disturbance \( v(t) = 0.3 \sin(2\pi t + \pi)/5 \), the known delay \( D = 1.8 \), and initial conditions \( x(0) = [0, 0]^T \). It is assumed that \( a_{\max} = 1, a_{\min} = -1, b_{\max} = 2, b_{\min} = 0.2, \|g\|_{\max} = 0.6, \omega_{\max} = 1 \) and \( \omega_{\min} = 0.001 \). According to the considered bounds on parameters, \( k^* \) is calculated as 0.0013. The control gain \( K \) is chosen such that the eigenvalues of \( A_0 + BK \) are \( -4, -5 \) and \( c_0 = c_r = 0.01 \). We set all the update gains to 0.00125. Finally, the controllable pair \((G, L)\) is chosen as \( l = [0, 1]^T \). From Figs. 1 and 2, one can observe that \( x(t) \) and \( \frac{1}{b(t-D)} \hat{b}(t-D)\xi_0(t-D) \) converge to zero as Theorem 1 predicts.

7. Conclusions

The problem of disturbance cancellation for unknown linear systems with input delay is considered. The problem is converted to an adaptive control problem by representing the disturbance as a constant unknown vector multiplied by a known regressor plus an exponentially decaying disturbance. The delay is represented as
a transport PDE. Using the certainty equivalence principle and the backstepping boundary control procedure for PDEs, an adaptive controller is designed and it is shown that the equilibrium of the closed loop system is stable. Moreover, it is proven that the state $X(t)$ converges to zero as $t \to \infty$ with perfect disturbance estimation. The effectiveness of the controller is demonstrated with a numerical example.

References


Jankovic, M. (2010). Recursive predictor design for state and output feedback controllers for linear time delay systems. Automatica, 46.


Krstic, M., & Smyslavey, A. (2008b). Boundary control of PDEs: a course on backstepping designs. SIAM.


Fig. 2. The disturbance estimation error for the simulation example.


Halil I. Basturk received his B.S. and M.S. degrees in Mechanical Engineering from Bogazici University in 2006 and 2008, and the Ph.D. degree in Mechanical and Aerospace Engineering from the University of California at San Diego in 2013. Since 2014 he has been Assistant Professor of Mechanical Engineering at Bogazici University, Istanbul, Turkey. His research interests include disturbance estimation/cancellation, adaptive control, acceleration feedback, and control of delay systems.

Miroslav Krstic holds the Alsopch endowed chair and is the founding director of the Cymer Center for Control Systems and Dynamics at UC San Diego. He also serves as Associate Vice Chancellor for Research at UCSD. As a graduate student, Krstic won the UC Santa Barbara best dissertation award and student best paper awards at CDC and ACC. Krstic is a Fellow of IEEE, IFAC, SIAM, ASME, and IET (UK), and a Distinguished Visiting Fellow of the Royal Academy of Engineering. He has received the PECASE, NSF Career, and ONR Young Investigator awards, the Axelby and Schuck paper prizes, the Chestnut textbook prize, and the first UCSD Research Award given to an engineer. Krstic has given over 25 keynote lectures including IEEE CDC and many IFAC conferences, and is the 2015 ASME Nyquist lecturer. Krstic has held the Springer Visiting Professorship at UC Berkeley. He serves as Senior Editor in IEEE Transactions on Automatic Control and Automatica, as editor of two Springer book series, and has served as Vice President for Technical Activities of the IEEE Control Systems Society and as chair of the IEEE CSS Fellow Committee. Krstic has coauthored ten books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.