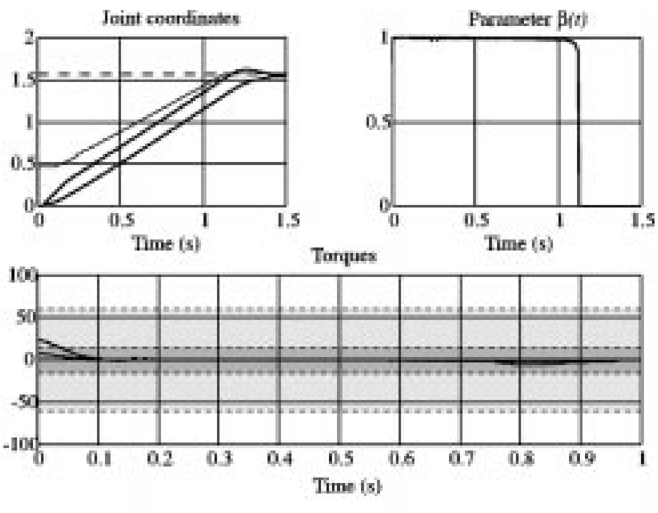

Fig. 6. Response with RG, torque constraints, and the constraint $|\theta_1 - \theta_2| \leq 0.2$ rad. The generated reference input is depicted (thin line) together with the joint trajectories (thick lines).

is taken into account by the RG, and the related simulated trajectories are depicted in Fig 6 with $r_1(t) = r_2(t) \equiv \frac{\pi}{4}$, $T = 0.001$ s. The slight chatter on the β and torque trajectories is caused by the approximations involved in the optimization procedure described in Section IV. The results described above were obtained on a 486 DX2/66 personal computer, using Matlab 4.2 and Simulink 1.3 with embedded C code. The CPU time required by the RG to select a single $\beta(t)$ ranged between 7 and 18 ms.

VI. Conclusion

For a broad class of nonlinear continuous-time systems and input/state hard constraints, this paper has addressed the RG problem, viz. the one of filtering the desired reference trajectory in such a way that a nonlinear primal compensated control system can operate in a stable way with satisfactory tracking performance and no constraint violation. The resulting computational burden turns out to be moderate and the related operations executable with current computing hardware. Alternatively, in some applications, the trajectory generated by the RG can be computed off-line and stored for subsequent task executions. Future developments of this research will be addressed toward numerical criteria for the determination of the constraint horizon and to an independent parameterization of the components of the reference.

References

[1] D. Q. Mayne and E. Polak, "Optimization based design and control," *Preprints 12th IFAC World Congress*, Sydney, Australia, 1993, vol. 3, pp. 129–138.

[2] H. J. Sussmann, E. D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2411–2424, 1994.

[3] D. Q. Mayne and H. Michalska, "Receding horizon control of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 814–824, 1990.

[4] S. S. Keerthi and E. G. Gilbert, "Optimal infinite-horizon feedback control laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations," *J. Opt. Theory Appl.*, vol. 57, pp. 265–293, 1988.

[5] T. H. Yang and E. Polak, "Moving horizon control of nonlinear systems with input saturation, disturbances and plant uncertainty," *Int. J. Contr.*, vol. 58, pp. 875–903, 1993.

[6] E. Mosca, *Optimal, Predictive, and Adaptive Control*. Englewood Cliffs, NJ: Prentice Hall, 1995.

[7] D. W. Clarke, "Advances in model-based predictive control," in *Advances in Model-Based Predictive Control*. New York: Oxford University Press, 1994, pp. 3–21.

[8] A. Bemporad and E. Mosca, "Constraint fulfilment in feedback control via predictive reference management," in *Proc. 3rd IEEE Conf. Control Applications*, Glasgow, U.K., 1994, pp. 1909–1914.

[9] A. Bemporad, A. Casavola, and E. Mosca, "Nonlinear control of constrained linear systems via predictive reference management," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 340–349, 1997.

[10] A. Bemporad and E. Mosca, "Fulfilling hard constraints in uncertain linear systems by reference managing," *Automatica*, to be published.

[11] P. Kapasouris, M. Athans, and G. Stein, "Design of feedback control systems for stable plants with saturating actuators," in *Proc. 27th IEEE Conf. Decision and Control*, 1988, pp. 469–479.

[12] P. Kapasouris, M. Athans, and G. Stein, "Design of feedback control systems for unstable plants with saturating actuators," in *Proc. IFAC Symp. Nonlinear Control System Design*, 1990, pp. 302–307.

[13] E. G. Gilbert, I. Kolmanovsky, and K. Tin Tan, "Discrete-time reference governors and the nonlinear control of systems with state and control constraints," *I. J. Robust Nonlinear Contr.*, pp. 487–504, 1995.

[14] E. G. Gilbert and I. Kolmanovsky, "Discrete-time reference governors for systems with state and control constraints and disturbance inputs," in *Proc. 34th IEEE Conf. on Decision and Control*, 1995, pp. 1189–1194.

[15] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*. New York: McGraw Hill, 1955.

[16] H. Asada and J.-J. E. Slotine, *Robot Analysis and Control*. New York: Wiley, 1986.

[17] R. Kelly and R. Salgado, "PD control with computed feedforward of robot manipulators: A design procedure," *IEEE Trans. Robotics Automat.*, vol. 10, pp. 566–571, 1994.

Geometric/Asymptotic Properties of Adaptive Nonlinear Systems with Partial Excitation

Zhong-Hua Li and Miroslav Krstić

***Abstract*—In this paper we continue the study of geometric/asymptotic properties of adaptive nonlinear systems. The long-standing question of whether the parameter estimates converge to *stabilizing* values—stabilizing if used in a *nonadaptive* controller—is addressed in the general set-point regulation case. The key quantifier of excitation in an adaptive system is the rank r of the regressor matrix at the resulting equilibrium. Our earlier paper showed that when either $r = 0$ or $r = p$ (where p is the number of uncertain parameters), the set of initial conditions leading to destabilizing estimates is of *measure zero*. Intuition suggests the same for the intermediate case $0 < r < p$ studied in this paper. We present a surprising result: the set of initial conditions leading to destabilizing estimates can have *positive measure*. We present results for the backstepping design with tuning functions; the same results can be established for other Lyapunov-based adaptive designs.**

***Index Terms*—Adaptive nonlinear control, invariant manifold, partial excitation.**

Manuscript received December 6, 1995; revised February 10, 1997. This work was supported in part by the National Science Foundation under Grant ECS-951011-8461 and by the Air Force Office of Scientific Research under Grant F496209610223.

Z.-H. Li is with the Department of Mechanical Engineering, University of Maryland, College Park, MD 20742 USA and the Department of Applied Mechanics and Engineering Sciences, University of California at San Diego, La Jolla, CA 92093-0411 USA.

M. Krstić is with the Department of Applied Mechanics and Engineering Sciences, University of California at San Diego, La Jolla, CA 92093-0411 USA (e-mail: krstic@ucsd.edu).

Publisher Item Identifier S 0018-9286(98)01390-7.

I. Introduction

In the absence of persistent excitation (PE), parameter convergence in adaptive feedback loops is a difficult problem. It is logical to expect that the case of "least" excitation (LE)—the regulation case with the regressor converging to zero—would be the most difficult. This case was recently approached in [8], where it was proved that: 1) the parameter estimates converge to constant values and 2) the set of initial conditions that lead to destabilizing nonadaptive controllers is of Lebesgue measure zero. Thus, we know that both extreme cases, PE and LE, guarantee that the parameter estimates *almost always* converge to stabilizing values. The case of *partial excitation*, initially expected to be a routine extension with the same conclusion, has, for almost two years, resisted our attempts, assisted by several colleagues (see the Acknowledgment). In this paper we reveal an entirely unexpected answer: *the solutions leading to destabilizing estimates can have positive measure.*

As in [8], our approach is based on exploring structures of invariant manifolds of adaptive equilibria. The general set-point regulation problem is considered with a regressor matrix of arbitrary rank at the resulting equilibrium. We focus our attention on the adaptive backstepping design with tuning functions [9], [10]. The same results can be established for other Lyapunov-based adaptive nonlinear designs [14], [5], [4], [9], [13]. At present, it is not clear if extensions to estimation-based designs ([12], [10] and references therein) would be straightforward.

II. Set-Point Regulation Using Tuning Function Design

Consider nonlinear systems transformable into the *strict-feedback form*

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \varphi_i(x_1, \cdots, x_i)^T\theta, \qquad i = 1, \cdots, n-1\\ \dot{x}_n &= u + \varphi_n(x)^T\theta\end{aligned} \tag{1}$$

where $\theta \in \mathbb{R}^p$ is the vector of unknown constant parameters and the elements of $F = [\varphi_1, \cdots, \varphi_n]$ are smooth nonlinear functions taking arguments in $\mathbb{R}^n$. In this paper we consider the problem of adaptive regulation of the output $y = x_1$ to a given set-point y_s. Starting with $x_1^e = y_s$, we solve the n equilibrium equations of (1) to get

$$\begin{aligned}x_1^e &= y_s\\ x_i^e &= -\varphi_{i-1}\left(x_1^e, \cdots, x_{i-1}^e\right)^T\theta, \qquad i = 2, \cdots, n.\end{aligned} \tag{2}$$

In [9], an adaptive controller was designed for (1) recursively using the expressions

$$z_i = x_i - \alpha_{i-1} \tag{3}$$

$$\begin{aligned}\alpha_i(\bar{x}_i, \hat{\theta}) = &-z_{i-1} - c_i z_i - w_i^T\hat{\theta} + \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}x_{k+1}\\ &+ \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma\tau_i + \sum_{k=2}^{i-1}\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\Gamma w_i z_k\end{aligned} \tag{4}$$

$$\tau_i(\bar{x}_i, \hat{\theta}) = \tau_{i-1} + w_i z_i \tag{5}$$

$$w_i(\bar{x}_i, \hat{\theta}) = \varphi_i - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}\varphi_k \tag{6}$$

where $\bar{x}_i = (x_1, \cdots, x_i)$, $i = 1, \cdots, n$, $z_0 = 0$, $\alpha_0 = y_s$, $\tau_0 = 0$, and $\Gamma = \Gamma^T > 0$. The control law is

$$u = \alpha_n(x, \hat{\theta}) \tag{7}$$

and the adaptive law is

$$\dot{\hat{\theta}} = \Gamma\tau_n(x, \hat{\theta}) = \Gamma W(z, \tilde{\theta})z \tag{8}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ is the parameter estimation error, $W(z, \tilde{\theta}) = [w_1, \cdots, w_n] = F(x)N(z, \tilde{\theta})^T$, and

$$N(z, \tilde{\theta}) = \begin{bmatrix} 1 & 0 & \cdots & 0\\ -\frac{\partial\alpha_1}{\partial x_1} & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ -\frac{\partial\alpha_{n-1}}{\partial x_1} & \cdots & -\frac{\partial\alpha_{n-1}}{\partial x_{n-1}} & 1\end{bmatrix}. \tag{9}$$

This adaptive controller results in a closed-loop system of the form

$$\dot{z} = A_z(z, \tilde{\theta})z + W(z, \tilde{\theta})^T\tilde{\theta} \tag{10}$$

$$\dot{\tilde{\theta}} = -\Gamma W(z, \tilde{\theta})z \tag{11}$$

where

$$A_z(z, \tilde{\theta}) = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0\\ -1 & -c_2 & 1+\sigma_{23} & \cdots & \sigma_{2n}\\ 0 & -1-\sigma_{23} & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 1+\sigma_{n-1,n}\\ 0 & -\sigma_{2n} & \cdots & -1-\sigma_{n-1,n} & -c_n\end{bmatrix} \tag{12}$$

and

$$\sigma_{jk}(z, \tilde{\theta}) = -\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma w_k. \tag{13}$$

Let us denote $F_e = F(x^e)$ and $r = \text{rank}\{F_e\}$. Then we have the following theorem.

Theorem 2.1 [9]: The closed-loop adaptive system (10) and (11) has a globally stable equilibrium $(z, \tilde{\theta}) = 0$. Furthermore, its state $(z(t), \tilde{\theta}(t))$ converges to the $(p-r)$-dimensional equilibrium manifold M given by

$$M = \left\{(z, \tilde{\theta}) \in \mathbb{R}^{n+p} | z = 0, F_e^T\tilde{\theta} = 0\right\}. \tag{14}$$

An important property of M is its dimension, $p - r$. Two extreme cases are as follows.

1) When $r = p$, i.e., $\dim\{M\} = 0$, M becomes the equilibrium point $z = 0$, $\tilde{\theta} = 0$. This equilibrium is globally asymptotically stable and the parameter estimate $\hat{\theta}(t)$ converges to its true value θ. This is the case of PE.
2) When $F_e = 0$, i.e., $\dim\{M\} = p$, M becomes the equilibrium manifold $z = 0$. The asymptotic properties of $\hat{\theta}(t)$ for this case were studied in [8]. In this case there is no guaranteed excitation.

Our objective here is to study the case between the above two extreme cases, i.e., the case of *partial* excitation: $0 < r < p$.

III. Asymptotic Constancy

The first difficulty in studying asymptotic properties of adaptive controllers is to prove that the parameter estimates converge to constant values. As we noted above, the two extreme cases, $r = 0$ and $r = p$, have been resolved in [8] and [9], respectively. The case of partial excitation, $0 < r < p$, is much harder than the two extreme cases. As will become clear from the proof of the next theorem, a major challenge is to show that $F_e^T\tilde{\theta}(t)$ not only converges to zero, but is also an $\mathcal{L}_2$ signal.

Theorem 3.1: Consider the adaptive system (10) and (11). There exists a constant vector $\hat{\theta}_\infty \in \mathbb{R}^p$ such that $\lim_{t\to\infty}\hat{\theta}(t) = \hat{\theta}_\infty$.

Proof: Consider the Lyapunov function $V_n = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1}\tilde{\theta}$. In view of (12), the derivative of V_n along (10) and (11) is $\dot{V}_n = -\sum_{k=1}^n c_k z_k^2 \le -c_0|z|^2$, where $c_0 = \min\{c_1, \cdots, c_n\}$. Then it follows that $z \in \mathcal{L}_2$. First, we show that $F_e^T\tilde{\theta} \in \mathcal{L}_2$. To do this, we use induction.

Consider the z_1-equation

$$\begin{aligned}\dot{z}_1 &= -c_1 z_1 + z_2 + \varphi_1(x_1)^T\tilde{\theta} \\ &= -c_1 z_1 + z_2 + \varphi_1(x_1^e)^T\tilde{\theta} + (\varphi_1(x_1) - \varphi_1(x_1^e))^T\tilde{\theta}. \end{aligned} \tag{15}$$

Since $z_1 = x_1 - x_1^e \in \mathcal{L}_2$, we have that $\varphi_1(x_1) - \varphi_1(x_1^e) \in \mathcal{L}_2$. In the following text, we use $[\mathcal{L}_2]$ as a generic expression for $\mathcal{L}_2$ terms. Since $z_2 \in \mathcal{L}_2$, (15) becomes

$$\dot{z}_1 = -c_1 z_1 + \varphi_1(x_1^e)^T\tilde{\theta} + [\mathcal{L}_2]. \tag{16}$$

On the other hand, we have

$$\begin{aligned}\varphi_1(x_1^e)^T\dot{\tilde{\theta}} &= -\varphi_1(x_1^e)^T\Gamma F(x)N^T z \\ &= -\varphi_1(x_1^e)^T\Gamma\varphi_1(x_1^e)z_1 - \varphi_1(x_1^e)^T\Gamma \\ &\quad\times (FN^T - \varphi_1(x_1^e)e_1^T)z \\ &= -\varphi_1(x_1^e)^T\Gamma\varphi_1(x_1^e)z_1 + [\mathcal{L}_2]\end{aligned} \tag{17}$$

due to $z \in \mathcal{L}_2$. Combining (16) with (17) we get

$$\begin{bmatrix}\dot{z}_1 \\ \varphi_1(x_1^e)^T\dot{\tilde{\theta}}\end{bmatrix} = \begin{bmatrix}-c_1 & 1 \\ -\varphi_1(x_1^e)^T\Gamma\varphi_1(x_1^e) & 0\end{bmatrix}\begin{bmatrix}z_1 \\ \varphi_1(x_1^e)^T\tilde{\theta}\end{bmatrix} + [\mathcal{L}_2]. \tag{18}$$

If $\varphi_1(x_1^e) \ne 0$, (18) is a stable linear time-invariant (LTI) system driven by an $\mathcal{L}_2$ signal, so that

$$\varphi_1(x_1^e)^T\tilde{\theta} \in \mathcal{L}_2 \tag{19}$$

(otherwise $\varphi_1(x_1^e)^T\tilde{\theta} = 0 \in \mathcal{L}_2$). This, in turn, means that $\varphi_1(x_1)^T\tilde{\theta} \in \mathcal{L}_2$, $\dot{z}_1 \in \mathcal{L}_2$, and $\dot{x}_1 \in \mathcal{L}_2$.

We then assume that at step $i-1$, we have proved that $x_k - x_k^e \in \mathcal{L}_2$, $\varphi_k(\bar{x}_k) - \varphi_k(\bar{x}_k^e) \in \mathcal{L}_2$, $\varphi_k(\bar{x}_k^e)^T\tilde{\theta} \in \mathcal{L}_2$, $\varphi_k(\bar{x}_k)^T\tilde{\theta} \in \mathcal{L}_2$, $\dot{z}_k \in \mathcal{L}_2$, $\dot{x}_k \in \mathcal{L}_2$, $k = 1, \cdots, i-1$. Now consider the z_i-equation

$$\begin{aligned}\dot{z}_i &= -c_i z_i - z_{i-1} + z_{i+1} + \sum_{k=i+1}^n \sigma_{ik}z_k - \sum_{k=2}^{i-1}\sigma_{ki}z_k + w_i^T\tilde{\theta} \\ &= -c_i z_i + \varphi_i(\bar{x}_i)^T\tilde{\theta} - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}\underbrace{\varphi_k(\bar{x}_k)^T\tilde{\theta}}_{\in\mathcal{L}_2(\text{step } i-1)} + [\mathcal{L}_2] \\ &= -c_i z_i + \varphi_i(\bar{x}_i^e)^T\tilde{\theta} + (\varphi_i(\bar{x}_i) - \varphi_i(\bar{x}_i^e))^T\tilde{\theta} + [\mathcal{L}_2]\end{aligned} \tag{20}$$

in which we have used $z \in \mathcal{L}_2$ and $w_i = \varphi_i(\bar{x}_i) - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}\varphi_k(\bar{x}_k)$. Since

$$\begin{aligned}x_i &= \underbrace{\dot{x}_{i-1}}_{\in\mathcal{L}_2} - \varphi_{i-1}(\bar{x}_{i-1})^T\theta \\ &= \underbrace{-\varphi_{i-1}(\bar{x}_{i-1}^e)^T\theta}_{x_i^e} - \underbrace{(\varphi_{i-1}(\bar{x}_{i-1}) - \varphi_{i-1}(\bar{x}_{i-1}^e))^T\theta}_{\in\mathcal{L}_2(\text{step } i-1)} \\ &\quad + [\mathcal{L}_2] \\ &= x_i^e + [\mathcal{L}_2]\end{aligned} \tag{21}$$

combining the assumption $x_k - x_k^e \in \mathcal{L}_2, k = 1, \cdots, i-1$, it follows that

$$\varphi_i(\bar{x}_i) - \varphi_i(\bar{x}_i^e) = \varphi_i(\bar{x}_i^e + [\mathcal{L}_2]) - \varphi_i(\bar{x}_i^e) \in \mathcal{L}_2. \tag{22}$$

Therefore, we have

$$\dot{z}_i = -c_i z_i + \varphi_i(\bar{x}_i^e)^T\tilde{\theta} + [\mathcal{L}_2]. \tag{23}$$

On the other hand

$$\begin{aligned}\varphi_i(\bar{x}_i^e)^T\dot{\tilde{\theta}} &= -\varphi_i(\bar{x}_i^e)^T\Gamma F(x)N^T z \\ &= -\varphi_i(\bar{x}_i^e)^T\Gamma\varphi_i(\bar{x}_i^e)z_i - \varphi_i(\bar{x}_i^e)^T\Gamma \\ &\quad\times (FN^T - \varphi_i(\bar{x}_i^e)e_i^T)z \\ &= -\varphi_i(\bar{x}_i^e)^T\Gamma\varphi_i(\bar{x}_i^e)z_i + [\mathcal{L}_2].\end{aligned} \tag{24}$$

Combining (23) with (24), we get

$$\begin{bmatrix}\dot{z}_i \\ \varphi_i(\bar{x}_i^e)^T\dot{\tilde{\theta}}\end{bmatrix} = \begin{bmatrix}-c_i & 1 \\ -\varphi_i(\bar{x}_i^e)^T\Gamma\varphi_i(\bar{x}_i^e) & 0\end{bmatrix}\begin{bmatrix}z_i \\ \varphi_i(\bar{x}_i^e)^T\tilde{\theta}\end{bmatrix} + [\mathcal{L}_2]. \tag{25}$$

If $\varphi_i(\bar{x}_i^e) \ne 0$, (25) is a stable LTI system driven by an $\mathcal{L}_2$ signal so that

$$\varphi_i(\bar{x}_i^e)^T\tilde{\theta} \in \mathcal{L}_2 \tag{26}$$

(otherwise $\varphi_i(\bar{x}_i^e)^T\tilde{\theta} = 0 \in \mathcal{L}_2$). This, in turn, means that $\varphi_i(\bar{x}_i)^T\tilde{\theta} \in \mathcal{L}_2$, $\dot{z}_i \in \mathcal{L}_2$. Also, note that

$$\dot{x}_i = \underbrace{\dot{z}_i}_{\in\mathcal{L}_2} - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_k}\underbrace{\dot{z}_k}_{\in\mathcal{L}_2} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\underbrace{\dot{\hat{\theta}}}_{\in\mathcal{L}_2} \in \mathcal{L}_2. \tag{27}$$

We conclude from the above induction that $x_i - x_i^e \in \mathcal{L}_2$, $\varphi_i(\bar{x}_i) - \varphi_i(\bar{x}_i^e) \in \mathcal{L}_2$, $\varphi_i(\bar{x}_i^e)^T\tilde{\theta} \in \mathcal{L}_2$, $\varphi_i(\bar{x}_i)^T\tilde{\theta} \in \mathcal{L}_2$, $\dot{z}_i \in \mathcal{L}_2$, and $\dot{x}_i \in \mathcal{L}_2$ for $i = 1, \cdots, n$. Thus

$$\begin{bmatrix}\varphi_1(x_1^e)^T \\ \varphi_2(\bar{x}_2^e)^T \\ \vdots \\ \varphi_n(\bar{x}_n^e)\end{bmatrix}\tilde{\theta} = F_e^T\tilde{\theta} \in \mathcal{L}_2. \tag{28}$$

Now we finish the proof of the theorem. Let $r = \text{rank}\{F_e^T\}$ and define

$\bar{P} = p \times r$ matrix of basis vectors of Range$\{F_e\}$
$\bar{Q} = r \times p$ matrix of basis vectors of Null$\{(\Gamma F_e)^T\}$.

From Theorem 2.1, we have that $F_e^T\tilde{\theta}(t) \to 0$, so $\bar{P}^T\tilde{\theta}(t) \to 0$, i.e., $\bar{P}^T\hat{\theta}(t) \to \text{const}$. On the other hand, noting (8) we have

$$\begin{aligned}\bar{Q}^T\dot{\hat{\theta}} &= \bar{Q}^T\Gamma FN^T z \\ &= \bar{Q}^T(\Gamma F_e N^T z + \Gamma(F - F_e)N^T z).\end{aligned} \tag{29}$$

Since $x - x^e \in \mathcal{L}_2$, we note that $F(x) - F(x^e) \in \mathcal{L}_2$. Recalling that $z \in \mathcal{L}_2$ and $\bar{Q}^T\Gamma F_e = 0$, we conclude from (29) that $\bar{Q}^T\dot{\hat{\theta}} \in \mathcal{L}_1$. We now write $Q^T\hat{\theta}(t)$ as

$$\bar{Q}^T\hat{\theta}(t) = \bar{Q}^T\hat{\theta}(0) + \int_0^t \bar{Q}^T\dot{\hat{\theta}}(\tau)\,d\tau. \tag{30}$$

The fact that $\bar{Q}^T\dot{\hat{\theta}} \in \mathcal{L}_1$ assures us that $\bar{Q}^T\hat{\theta}(t) \to \text{const}$. That is, we have that $[\bar{P}, \bar{Q}]^T\hat{\theta}(t) \to \text{const}$. Since $\Gamma = \Gamma^T > 0$, one easily proves that $[\bar{P}, \bar{Q}]^T$ is invertible. Thus $\hat{\theta}(t) \to \text{const}$. □

IV. Classification of Equilibria and Invariant Manifolds

Since each solution of the adaptive system converges to an equilibrium point on M, it is first of interest to determine which of the equilibria on M are stable and which are unstable. Let us for further notational convenience rewrite the system (10) and (11) as

$$\dot{z} = \mathcal{A}_z(z, \tilde{\vartheta})z + \mathcal{W}(z, \tilde{\vartheta})^T\tilde{\vartheta} \tag{31}$$

$$\dot{\tilde{\vartheta}} = -\Gamma\mathcal{W}(z, \tilde{\vartheta})z \tag{32}$$

and denote $\Gamma = \gamma\Gamma_0$, where $\gamma = \lambda_{\max}(\Gamma)$. Then we have the following lemma.

Lemma 4.1: System (31) and (32) is transformed into

$$\dot{z} = A_z(z,\tilde{\theta}_1,\tilde{\theta}_2)z + W_1(z,\tilde{\theta}_1,\tilde{\theta}_2)^T\tilde{\theta}_1 + W_2(z,\tilde{\theta}_1,\tilde{\theta}_2)^T\tilde{\theta}_2 \quad (33)$$
$$\dot{\tilde{\theta}}_1 = -\gamma W_1(z,\tilde{\theta}_1,\tilde{\theta}_2)z \quad (34)$$
$$\dot{\tilde{\theta}}_2 = -\gamma W_2(z,\tilde{\theta}_1,\tilde{\theta}_2)z \quad (35)$$

using the transformation

$$\begin{aligned}\tilde{\theta} &= \begin{bmatrix}\tilde{\theta}_1\\ \tilde{\theta}_2\end{bmatrix} = T\tilde{\vartheta}\\ T &= \begin{bmatrix}(P^TP)^{-1/2}P^T\\ (Q^TQ)^{-1/2}Q^T\end{bmatrix}\Gamma_0^{-1/2}\\ T^{-1} &= \Gamma_0^{1/2}[P(P^TP)^{-1/2} \quad Q(Q^TQ)^{-1/2}]\\ P &= \text{matrix of basis vectors of Range}\{\Gamma_0^{1/2}F_e\}\\ Q &= \text{matrix of basis vectors of Null}\{(\Gamma_0^{1/2}F_e)^T\}.\end{aligned} \quad (36)$$

Furthermore, $\text{rank}\{W_1(0,0,\tilde{\theta}_2)\} = r$ and $W_2(0,0,\tilde{\theta}_2) = 0$ for all $\tilde{\theta}_2 \in \mathbb{R}^{p-r}$.

This decomposition clearly separates the part of the parameter error vector which is guaranteed to converge to a zero vector from the part which converges to a possibly nonzero constant vector. To see this, recall that $F_e^T\tilde{\vartheta}(t) \to 0$ which implies that $F_e^T\Gamma_0^{1/2}P(P^TP)^{-1/2}\tilde{\theta}_1(t) \to 0$. Since $\text{rank}\{F_e^T\Gamma_0^{1/2}P(P^TP)^{-1/2}\} = \deg\{\tilde{\theta}_1\}$, then $\tilde{\theta}_1(t) \to 0$.

Proof of Lemma 4.1: Applying the transformation (36), system (31) and (32) becomes

$$\begin{aligned}\dot{z} &= \mathcal{A}_z(z,\tilde{\vartheta})z + \mathcal{W}(z,\tilde{\vartheta})^T\tilde{\vartheta}\\ &= \mathcal{A}_z(z,T^{-1}\tilde{\theta})z + NF^TT^{-1}T\tilde{\vartheta}\\ &= A_z(z,\tilde{\theta}_1,\tilde{\theta}_2)z\\ &\quad + NF^T\Gamma_0^{1/2}[P(P^TP)^{-1/2} \quad Q(Q^TQ)^{-1/2}]\tilde{\theta}\\ &= A_z(z,\tilde{\theta}_1,\tilde{\theta}_2)z + W_1(z,\tilde{\theta}_1,\tilde{\theta}_2)^T\tilde{\theta}_1 + W_2(z,\tilde{\theta}_1,\tilde{\theta}_2)^T\tilde{\theta}_2\end{aligned} \quad (37)$$

$$\begin{aligned}\dot{\tilde{\theta}} &= \begin{bmatrix}\dot{\tilde{\theta}}_1\\ \dot{\tilde{\theta}}_2\end{bmatrix} = -\gamma\begin{bmatrix}(P^TP)^{-1/2}P^T\\ (Q^TQ)^{-1/2}Q^T\end{bmatrix}\Gamma_0^{1/2}FN^Tz\\ &= \begin{bmatrix}-\gamma W_1(z,\tilde{\theta}_1,\tilde{\theta}_2)z\\ -\gamma W_2(z,\tilde{\theta}_1,\tilde{\theta}_2)z\end{bmatrix}\end{aligned} \quad (38)$$

where

$$W_1(z,\tilde{\theta}_1,\tilde{\theta}_2) = (P^TP)^{-1/2}P^T\Gamma_0^{1/2}FN^T \quad (39)$$
$$W_2(z,\tilde{\theta}_1,\tilde{\theta}_2) = (Q^TQ)^{-1/2}Q^T\Gamma_0^{1/2}FN^T. \quad (40)$$

Using Lemma A.1 in the Appendix, we can see that $\tilde{\theta}_1 = 0$ and $z = 0$ imply $x = x_e$. Thus (39) and (40) yield

$$W_1(0,0,\tilde{\theta}_2) = (P^TP)^{-1/2}P^T\Gamma_0^{1/2}F_eN^T\big|_{z,\tilde{\theta}_1=0} \quad (41)$$
$$W_2(0,0,\tilde{\theta}_2) = (Q^TQ)^{-1/2}Q^T\Gamma_0^{1/2}F_eN^T\big|_{z,\tilde{\theta}_1=0}. \quad (42)$$

Since $\text{rank}\{(P^TP)^{-1/2}P^T\Gamma_0^{1/2}F_e\} = r \le n$ and $\text{rank}\{N\} = n$ for all $\tilde{\theta}_2$, then $\text{rank}\{W_1(0,0,\tilde{\theta}_2)\} \equiv r$. On the other hand, since $Q^T\Gamma_0^{1/2}F_e = 0$, then $W_2(0,0,\tilde{\theta}_2) \equiv 0$. □

Since all the solutions converge to the $\tilde{\theta}_2$-subspace, that is, the manifold

$$M = \{(z,\tilde{\theta}_1,\tilde{\theta}_2) \in \mathbb{R}^{n+p} \mid z = 0, \tilde{\theta}_1 = 0\} \quad (43)$$

we study stability of equilibria on M as a function of $\tilde{\theta}$. Let us denote (44), as shown at the bottom of the page, where

$$A_0 = \begin{bmatrix}-c_1 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1\\ & & -1 & -c_n\end{bmatrix} \quad (45)$$

$$\Sigma(\tilde{\theta}_2) = \begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ 0 & 0 & \bar{\sigma}_{23} & \cdots & \bar{\sigma}_{2n}\\ 0 & -\bar{\sigma}_{23} & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & \bar{\sigma}_{n-1,n}\\ 0 & -\bar{\sigma}_{2n} & \cdots & -\bar{\sigma}_{n-1,n} & 0\end{bmatrix} \quad (46)$$

$$\bar{\sigma}_{jk}(\tilde{\theta}_2) = -\frac{\partial\alpha_{j-1}}{\partial\hat{\vartheta}}\Gamma_0 w_k\bigg|_{\substack{z=0\\ \tilde{\theta}_1=0}} \quad k = 1,\cdots,n. \quad (47)$$

Theorem 4.1: Consider the closed-loop adaptive system (33)–(35). The equilibrium $(z,\tilde{\theta}_1,\tilde{\theta}_2) = (0,0,\tilde{\theta}_2^e)$ is

1) globally stable if all the eigenvalues of $A_e(\tilde{\theta}_2^e)$ have negative real parts;
2) unstable if at least one eigenvalue of $A_e(\tilde{\theta}_2^e)$ has positive real part.

Proof: Using Lemma A.2 from the conference version of this paper [11] (omitted here for space limitations), we transform system (33)–(35) into the form

$$\begin{aligned}\begin{bmatrix}\dot{z}\\ \dot{\tilde{\theta}}_1\end{bmatrix} &= A_e(\tilde{\theta}_2^e)\begin{bmatrix}z\\ \tilde{\theta}_1\end{bmatrix} + G(z,\tilde{\theta}_1,\bar{\theta}_2)\\ \dot{\bar{\theta}}_2 &= H(z,\tilde{\theta}_1,\bar{\theta}_2)\end{aligned} \quad (48)$$

where $\bar{\theta}_2 = \tilde{\theta}_2 - \tilde{\theta}_2^e$, and $G(z,\tilde{\theta}_1,\bar{\theta}_2)$ and $H(z,\tilde{\theta}_1,\bar{\theta}_2)$ satisfy

$$\begin{aligned}G(0,0,\bar{\theta}_2) &= 0, & \frac{\partial G(0,0,0)}{\partial z} &= 0\\ \frac{\partial G(0,0,0)}{\partial\tilde{\theta}_1} &= 0, & \frac{\partial G(0,0,\bar{\theta}_2)}{\partial\bar{\theta}_2} &= 0\\ H(0,0,\bar{\theta}_2) &= 0, & \frac{\partial H(0,0,\bar{\theta}_2)}{\partial z} &= 0\\ \frac{\partial H(0,0,\bar{\theta}_2)}{\partial\tilde{\theta}_1} &= 0, & \frac{\partial H(0,0,\bar{\theta}_2)}{\partial\bar{\theta}_2} &= 0.\end{aligned} \quad (49)$$

We first prove the stability part, assuming that all the eigenvalues of $A_e(\tilde{\theta}_2^e)$ have negative real parts. Since the equilibrium manifold $[z^T,\tilde{\theta}_1^T]^T = h(\bar{\theta}_2) = 0$ is invariant and $\frac{\partial h(0)}{\partial\bar{\theta}_2} = 0$, then $[z^T,\tilde{\theta}_1^T]^T = 0$ is a center manifold. The reduced system of (48)

$$\dot{\bar{\theta}}_2 = H(0,0,\bar{\theta}_2) = 0 \quad (50)$$

is stable. By the center manifold theorem (reduction principle) [2, Th. 2, p. 21], the equilibrium $(z,\tilde{\theta}_1,\bar{\theta}_2) = (0,0,0)$ of the full system (48) is stable. The stability property is global because Theorem 2.1 guarantees global boundedness.

The instability part, when at least one of the eigenvalues of $A_e(\tilde{\theta}_2^e)$ has positive real part, is immediate from the linearization theorem by noting that the linearization of (48) around the equilibrium $(z,\tilde{\theta}_1,\bar{\theta}_2) = (0,0,0)$ is

$$\begin{bmatrix}\delta\dot{z}\\ \delta\dot{\tilde{\theta}}_1\\ \delta\dot{\bar{\theta}}_2\end{bmatrix} = \begin{bmatrix}A_e(\tilde{\theta}_2^e) & \mathbf{0}\\ \mathbf{0} & 0\end{bmatrix}\begin{bmatrix}\delta z\\ \delta\tilde{\theta}_1\\ \delta\bar{\theta}_2\end{bmatrix}. \quad (51)$$

□

$$A_e(\tilde{\theta}_2) = \begin{bmatrix}A_0 + \gamma\Sigma(\tilde{\theta}_2) + \frac{\partial W_2(0,0,\tilde{\theta}_2)^T\tilde{\theta}_2}{\partial z} & W_1^T(0,0,\tilde{\theta}_2) + \frac{\partial W_2(0,0,\tilde{\theta}_2)^T\tilde{\theta}_2}{\partial\tilde{\theta}_1}\\ -\gamma W_1(0,0,\tilde{\theta}_2) & 0\end{bmatrix} \quad (44)$$

Theorem 4.1 does not cover the case where, in addition to eigenvalues with negative real parts, there are also eigenvalues of $A_e(\tilde{\theta}_2^e)$ with zero real parts.

With Theorem 4.1, we can determine whether a given equilibrium point on M is stable or unstable. Next we determine which parts of M are attractive and which are repulsive. Let us categorize the equilibria on M into the following four sets:

$$\mathcal{S}^s = \bigcap_{i=1}^{n} \{(z, \tilde{\theta}_1, \tilde{\theta}_2) \in M \mid \mathrm{Re}\, \lambda_i(A_e(\tilde{\theta}_2)) < 0\} \tag{52}$$

$$\mathcal{S}^u = \bigcap_{i=1}^{n} \{(z, \tilde{\theta}_1, \tilde{\theta}_2) \in M \mid \mathrm{Re}\, \lambda_i(A_e(\tilde{\theta}_2)) > 0\} \tag{53}$$

$$\mathcal{S}^c = \bigcup_{i=1}^{n} \{(z, \tilde{\theta}_1, \tilde{\theta}_2) \in M \mid \mathrm{Re}\, \lambda_i(A_e(\tilde{\theta}_2)) = 0\} \tag{54}$$

$$\mathcal{S}^{su} = M \backslash (\mathcal{S}^s \cup \mathcal{S}^u \cup \mathcal{S}^c). \tag{55}$$

The set $\mathcal{S}^s$ is, by Theorem 4.1, a set of stable equilibria. The set $\mathcal{S}^u \cup \mathcal{S}^{su}$ is the set of unstable equilibria from the second part of Theorem 4.1. The set $\mathcal{S}^c$ is a set of equilibria at which at least one of the eigenvalues of $A_e(\tilde{\theta}_2)$ has zero real part. In [8] it was proved that $\mathcal{S}^c$ has Lebesgue measure zero in M.

Now our analysis proceeds along the lines in [8, Sec. 5]. Let us consider an equilibrium point $X^e = (z, \tilde{\theta}_1, \tilde{\theta}_2) = (0, 0, \tilde{\theta}_2^e) \in M \backslash \mathcal{S}^c$. By the center manifold theorem, [3, Th. 3.2.1], there exist local invariant manifolds $W^s_{\mathrm{loc}}(X^e)$ (stable), $W^u_{\mathrm{loc}}(X^e)$ (unstable), and $W^c_{\mathrm{loc}}(X^e)$ (center). Denote $\zeta = [z^T, \tilde{\theta}_1^T]^T$. By [1, Th. 2.7.2],[1] the flow of (48) is *topologically equivalent*[2] to the flow of the system

$$\dot{\zeta}^s = -\zeta^s, \qquad \zeta^s \in W^s_{\mathrm{loc}}(X^e) \tag{56}$$

$$\dot{\zeta}^u = \zeta^u, \qquad \zeta^u \in W^u_{\mathrm{loc}}(X^e) \tag{57}$$

$$\dot{\bar{\theta}}_2 = 0, \qquad \bar{\theta}_2 \in W^c_{\mathrm{loc}}(X^e). \tag{58}$$

While $\dim\{W^c_{\mathrm{loc}}(X^e)\} = p - r$, the dimensions of $W^s_{\mathrm{loc}}(X^e)$ and $W^u_{\mathrm{loc}}(X^e)$ are as follows.

1) If $X^e \in \mathcal{S}^s$, then $\dim\{W^s_{\mathrm{loc}}(X^e)\} = n + r$ and $\dim\{W^u_{\mathrm{loc}}(X^e)\} = 0$.
2) If $X^e \in \mathcal{S}^u$, then $\dim\{W^s_{\mathrm{loc}}(X^e)\} = 0$ and $\dim\{W^u_{\mathrm{loc}}(X^e)\} = n + r$.
3) If $X^e \in \mathcal{S}^{su}$, then $0 < \dim\{W^s_{\mathrm{loc}}(X^e)\}, \dim\{W^u_{\mathrm{loc}}(X^e)\} < n + r$.

Only solutions along stable invariant manifolds can converge to points in $M \backslash \mathcal{S}^c$. These solutions are described by the sets

$$U^s = \bigcup_{t \le 0} \phi_t\left(W^s_{\mathrm{loc}}(S^s)\right) \tag{59}$$

$$U^u = \bigcup_{t \le 0} \phi_t\left(W^s_{\mathrm{loc}}(S^{su})\right) \tag{60}$$

where $\phi_t(\cdot)$ is the flow generated by (48). The remaining solutions, those converging to $\mathcal{S}^c$, belong to the set denoted by U^c. We point out that $U^s \cup U^u \cup U^c = \mathbb{R}^{n+p}$. It was proved in [8] that U^u and U^c have measure zero in $\mathbb{R}^{n+p}$.

Theorem 4.2 [8]: Consider the adaptive system (10) and (11). Solutions starting from *almost all* initial conditions $(z(0), \tilde{\theta}(0)) \in \mathbb{R}^{n+p}$ converge to the set of stable equilibria $\mathcal{S}^s$. The set of initial conditions that generate solutions converging to either $\mathcal{S}^{su}$ or $\mathcal{S}^c$ has Lebesgue measure zero in $\mathbb{R}^{n+p}$. No solutions converge to $\mathcal{S}^u$.

[1] For a detailed proof, see [7, Th. 4.1].

[2] Two flows, $\phi_t(x)$ and $\psi_t(x)$, are said to be topologically (C^0) equivalent if there exists a homeomorphism h, taking orbits of φ_t onto those of ψ_t, preserving their orientation.

V. Stability of Nonadaptive Controllers

Now we address the main question of this paper: does the adaptive controller "converge" to a stabilizing nonadaptive (constant) controller?

- In the case $r = p$, the parameter estimates converge to the actual parameter values, so the answer to this question is affirmative.
- In the case $r = 0$, the answer provided in [8] was affirmative except for a set of initial conditions $(z(0), \tilde{\theta}(0))$ of measure zero in $\mathbb{R}^{n+p}$.

It is natural to expect that in the case $0 < r < p$ considered here the measure of initial conditions that lead to destabilizing controllers remains zero. The fact that this is not so is the main result of this paper. In this section we show that the set of initial conditions that lead to destabilizing controllers may have *positive measure*.

Let us consider (10) with $\gamma = 0$ (which means $\Gamma = 0$) and $\tilde{\theta}_1 = 0$. Recalling from the definition (13) that $\sigma_{jk}(z, \tilde{\theta})$ has Γ as a factor, in view of (12), we conclude that $A_z(z, \tilde{\theta})|_{\Gamma=0} \equiv A_0$. Therefore, (33) with $\gamma = 0$ and $\tilde{\theta}_1 = 0$, becomes

$$\begin{aligned}\dot{z} &= \left(A_z(z, \tilde{\theta}_1, \tilde{\theta}_2)z + W_1^T\tilde{\theta}_1 + W_2^T\tilde{\theta}_2\right)\Big|_{\gamma, \tilde{\theta}_1 = 0} \\ &= A_0 z + W_2(z, 0, \tilde{\theta}_2)^T\tilde{\theta}_2\Big|_{\gamma=0}.\end{aligned} \tag{61}$$

The linearization of (61) around the equilibrium $z = 0$ is

$$\delta\dot{z} = A_l(\tilde{\theta}_2)\,\delta z \tag{62}$$

where

$$A_l(\tilde{\theta}_2) = A_0 + \left.\frac{\partial W_2(0, 0, \tilde{\theta}_2)^T\tilde{\theta}_2}{\partial z}\right|_{\gamma=0}. \tag{63}$$

Similar to (52)–(55), we introduce

$$\Lambda^s = \bigcap_{i=1}^{n} \{(z, \tilde{\theta}_1, \tilde{\theta}_2) \in M \mid \mathrm{Re}\, \lambda_i(A_l(\tilde{\theta}_2)) < 0\} \tag{64}$$

$$\Lambda^u = \bigcap_{i=1}^{n} \{(z, \tilde{\theta}_1, \tilde{\theta}_2) \in M \mid \mathrm{Re}\, \lambda_i(A_l(\tilde{\theta}_2)) > 0\} \tag{65}$$

$$\Lambda^c = \bigcup_{i=1}^{n} \{(z, \tilde{\theta}_1, \tilde{\theta}_2) \in M \mid \mathrm{Re}\, \lambda_i(A_l(\tilde{\theta}_2)) = 0\} \tag{66}$$

$$\Lambda^{su} = M \backslash (\Lambda^s \cup \Lambda^u \cup \Lambda^c). \tag{67}$$

The values of $\tilde{\theta}_2$ in Λ^s correspond to nonadaptive controllers which are (locally asymptotically) stabilizing. The values of $\tilde{\theta}_2$ in $\Lambda^{su} \cup \Lambda^u$ are destabilizing. Our goal is to see whether all, or almost all, solutions converge to Λ^s. Since Theorem 4.2 establishes that almost all solutions converge to $\mathcal{S}^s$, our task is to determine whether $\mathcal{S}^s \subset \Lambda^s$. This translates into the question whether Hurwitzness of $A_e(\tilde{\theta}_2)$ implies Hurwitzness of $A_l(\tilde{\theta}_2)$.

Before we proceed, we recall that

$$r = 0\text{:}\quad A_e(\tilde{\theta}) = A_l(\tilde{\theta}) \tag{68}$$

$$r = p\text{:}\quad A_e = \begin{bmatrix} A_l + \gamma\Sigma & W(0,0)^T \\ -\gamma W(0,0) & 0 \end{bmatrix}, \qquad A_l = A_0. \tag{69}$$

Since $W(0,0)$ has full row rank and Σ is skew-symmetric, using LaSalle's theorem [6], it is easy to show that A_e in (69) is Hurwitz. Thus, for both $r = 0$ and $r = p$, the stability properties of $A_e(\tilde{\theta})$ and $A_l(\tilde{\theta})$ coincide.

This is not the case for $0 < r < p$. Rewrite (44) as

$$\begin{aligned} A_e(\tilde{\theta}_2) \\ = \begin{bmatrix} A_l(\tilde{\theta}_2) + \gamma(\Sigma(\tilde{\theta}_2) + \Delta(\tilde{\theta}_2)) & W_1^T(0,0,\tilde{\theta}_2) + \frac{\partial W_2(0,0,\tilde{\theta}_2)^T\tilde{\theta}_2}{\partial\tilde{\theta}_1} \\ -\gamma W_1(0,0,\tilde{\theta}_2) & 0 \end{bmatrix}\end{aligned} \tag{70}$$

where

$$\gamma\Delta(\tilde{\theta}_2,\gamma) = \frac{\partial W_2(0,0,\tilde{\theta}_2)^T\tilde{\theta}_2}{\partial z} - \left.\frac{\partial W_2(0,0,\tilde{\theta}_2)^T\tilde{\theta}_2}{\partial z}\right|_{\gamma=0}. \tag{71}$$

The relationship between the stability properties of $A_e(\tilde{\theta}_2)$ and $A_l(\tilde{\theta}_2)$ is complicated, and no simple conclusions can be drawn. In fact, it is conceivable that for some $\tilde{\theta}$, $A_e(\tilde{\theta}_2)$ would be Hurwitz while $A_l(\tilde{\theta}_2)$ is not Hurwitz. Let us define the set of equilibria corresponding to that situation

$$\mathcal{R}^s = \mathcal{S}^s \bigcap \left(\Lambda^u \bigcup \Lambda^{su}\right) \tag{72}$$

and denote by $\mathcal{U}^s$ the set of solutions converging to $\mathcal{R}^s$. If $\mathcal{R}^s$ were to have nonzero measure in the $(p-r)$-dimensional manifold M, $\mathcal{U}^s$ would have nonzero measure in $\mathbb{R}^{n+p}$ because the stable invariant manifolds of equilibria in $\mathcal{R}^s$ are $(n+r)$-dimensional. In this case the adaptive controller would be converging to destabilizing nonadaptive controllers from a set of initial conditions of positive measure! The next example illustrates this possibility. We point out in advance that there is nothing unusual about the system in the example—it is a second-order *linear* plant.

Example 5.1: Let us consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1\theta_2\\ \dot{x}_2 &= u + \theta_1.\end{aligned} \tag{73}$$

After the tuning functions design from Section II is applied, the resulting error system is

$$\begin{aligned}\dot{z} &= \begin{bmatrix}-c_1 & 1\\ -1 & -c_2\end{bmatrix} z + \begin{bmatrix}0\\1\end{bmatrix}\tilde{\theta}_1 + \begin{bmatrix} z_1\\ (c_1+\hat{\theta}_2)z_1\end{bmatrix}\tilde{\theta}_2\\ \dot{\tilde{\theta}}_1 &= -\gamma[0 \quad 1]z\\ \dot{\tilde{\theta}}_2 &= -\gamma[z_1 \quad (c_1+\hat{\theta}_2)z_1]z\end{aligned} \tag{74}$$

where $z_1 = x_1, z_2 = x_2 + (c_1+\hat{\theta}_2)z_1$. Therefore

$$A_l(\tilde{\theta}_2) = \begin{bmatrix} -c_1+\tilde{\theta}_2 & 1\\ -1+(c_1+\hat{\theta}_2)\tilde{\theta}_2 & -c_2\end{bmatrix} \tag{75}$$

$$A_e(\tilde{\theta}_2) = \begin{bmatrix} -c_1+\tilde{\theta}_2 & 1 & 0\\ -1+(c_1+\hat{\theta}_2)\tilde{\theta}_2 & -c_2 & 1\\ 0 & -\gamma & 0\end{bmatrix}. \tag{76}$$

For carefully "engineered" values $c_1 = c_2 = 2$, $\gamma = 6$, and $\theta_2 = 3$, the characteristic polynomials of $A_l(\tilde{\theta}_2)$ and $A_e(\tilde{\theta}_2)$ are

$$p_l(s) = s^2 + (4-\tilde{\theta}_2)s + (\tilde{\theta}_2)^2 - 7\tilde{\theta}_2 + 5 \tag{77}$$

$$p_e(s) = s^3 + (4-\tilde{\theta}_2)s^2 + ((\tilde{\theta}_2)^2 - 7\tilde{\theta}_2 + 11)s + 12 - 6\tilde{\theta}_2. \tag{78}$$

We calculate Λ^α and $\mathcal{S}^\alpha$, $\alpha = \{s, u, su\}$ as follows:

$$\begin{aligned}\Lambda^s &= \{(z,\tilde{\theta}_1,\tilde{\theta}_2) \in M \mid \tilde{\theta}_2 \in (-\infty, 0.8074)\}\\ \Lambda^{su} &= \{(z,\tilde{\theta}_1,\tilde{\theta}_2) \in M \mid \tilde{\theta}_2 \in (0.8074, 6.1926)\}\\ \Lambda^u &= \{(z,\tilde{\theta}_1,\tilde{\theta}_2) \in M \mid \tilde{\theta}_2 \in (6.1926, +\infty)\}\end{aligned} \tag{79}$$

and

$$\begin{aligned}\mathcal{S}^s &= \{(z,\tilde{\theta}_1,\tilde{\theta}_2) \in M \mid \tilde{\theta}_2 \in (-\infty, 2)\}\\ \mathcal{S}^{su} &= \{(z,\tilde{\theta}_1,\tilde{\theta}_2) \in M \mid \tilde{\theta}_2 \in (2, 6.8791)\}\\ \mathcal{S}^u &= \{(z,\tilde{\theta}_1,\tilde{\theta}_2) \in M \mid \tilde{\theta}_2 \in (6.8791, +\infty)\}.\end{aligned} \tag{80}$$

In light of (79) and (80), we have

$$\begin{aligned}\mathcal{R}^s &= \mathcal{S}^s \bigcap \left(\Lambda^u \bigcup \Lambda^{su}\right)\\ &= \{(z,\tilde{\theta}_1,\tilde{\theta}_2) \in M \mid \tilde{\theta}_2 \in (0.8074, 2)\}.\end{aligned} \tag{81}$$

The set $\mathcal{R}^s$ is an interval $\tilde{\theta}_2 \in (0.8074, 2)$ with positive measure in M. Each point on this interval has a three-dimensional stable invariant manifold. Therefore, the set $\mathcal{U}^s$ of initial conditions leading to destabilizing estimates has positive measure in $\mathbb{R}^4$.

Another set of interest is

$$\begin{aligned}\mathcal{R}^{su} &= \mathcal{S}^{su} \bigcap \Lambda^u\\ &= \{(z,\tilde{\theta}_1,\tilde{\theta}_2) \in M \mid \tilde{\theta}_2 \in (6.1926, 6.8791)\}.\end{aligned} \tag{82}$$

Along their stable invariant manifolds, the equilibria in $\mathcal{R}^{su}$ attract some solutions denoted by $\mathcal{U}^{su}$. Since $\mathcal{R}^{su} \subset \Lambda^u$, these solutions result in parameter estimates such that *all* of the eigenvalues of the linearized nonadaptive system are unstable. This is different from [8], where no solutions could converge to such "completely destabilizing" parameter estimates. However, $\mathcal{U}^{su}$ has measure zero in $\mathbb{R}^4$. □

APPENDIX

Lemma A.1: Let $F_e^T\tilde{\theta} = 0$. Then $z = 0$ if and only if $x = x_e$.

Proof: We start by noting that $z_1 = 0$ iff $x_1 = x_1^e$. Assume that $z_k = 0$ iff $x_k = x_k^e$, $k = 1, \cdots, i-1$. Recalling that

$$\alpha_k = -z_{k-1} - c_k z_k - \varphi_k(\bar{x}_k)^T\hat{\theta} + \sum_{j=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_j} \times (x_{j+1} + \varphi_j(\bar{x}_j)^T\hat{\theta}) + \frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\Gamma\tau_k - \sum_{j=2}^{k-1}\sigma_{j,k}z_k \tag{A1}$$

we have

$$\begin{aligned} z_i &= x_i - \alpha_{i-1} = x_i - x_i^e - \varphi_{i-1}(\bar{x}_{i-1}^e)^T\theta - \alpha_{i-1}\\ &= x_i - x_i^e - \underbrace{\varphi_{i-1}(\bar{x}_{i-1}^e)^T\tilde{\theta}}_{0}\\ &\quad + \underbrace{\left(\varphi_{i-1}(\bar{x}_{i-1}) - \varphi_{i-1}(\bar{x}_{i-1}^e)\right)}_{0}{}^T\hat{\theta}\\ &\quad - \sum_{k=1}^{i-2}\frac{\partial\alpha_{i-2}}{\partial x_k}\underbrace{\left(\varphi_k(\bar{x}_k) - \varphi_k(\bar{x}_k^e)\right)}_{0}{}^T\hat{\theta}\\ &\quad - \sum_{k=1}^{i-2}\frac{\partial\alpha_{i-2}}{\partial x_k}\underbrace{\left(x_{k+1} - x_{k+1}^e\right)}_{0} + \underbrace{z_{i-2}}_{0} + c_{i-1}\underbrace{z_{i-1}}_{0}\\ &\quad - \frac{\partial\alpha_{i-2}}{\partial\hat{\theta}}\Gamma\underbrace{\tau_{i-1}}_{0} + \sum_{k=2}^{i-2}\sigma_{k,i-1}\underbrace{z_{i-1}}_{0}\\ &= x_i - x_i^e.\end{aligned} \tag{A2}$$

By induction, we conclude that $z = 0$ iff $x = x^e$. □

ACKNOWLEDGMENT

The authors would like to thank D. Ayeles, R. Freeman, P. Kokotovic, L. Praly, and R. Sepulchre for discussions and their interest in this problem.

REFERENCES

[1] D. K. Arrowsmith and C. M. Place, *An Introduction to Dynamical Systems*. Cambridge, U.K.: Cambridge Univ. Press, 1990.
[2] J. Carr, *Applications of Centre Manifold Theory*. New York: Springer-Verlag, 1981.
[3] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. New York: Springer-Verlag, 1983.

[4] Z. P. Jiang and L. Praly, "Iterative designs of adaptive controllers for systems with nonlinear integrators," in *Proc. 30th IEEE Conf. Decision and Control*, Brighton, U.K., Dec. 1991, pp. 2482–2487.

[5] I. Kanellakopoulos, P. V. Kokotović, and A. S. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 1241–1253, 1991.

[6] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 1996.

[7] U. Kirchgraber and K. J. Palmer, *Geometry in the Neighborhood of Invariant Manifolds of Maps and Flows and Linearization*. Longman, 1990.

[8] M. Krstić, "Invariant manifolds and asymptotic properties of adaptive nonlinear stabilizers," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 817–829, 1996.

[9] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, "Adaptive nonlinear control without overparametrization," *Syst. Contr. Lett.*, vol. 19, pp. 177–185, 1992.

[10] ——, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.

[11] Z. H. Li and M. Krstić, "Geometric/asymptotic properties of adaptive nonlinear systems with partial excitation," in *Proc. 35th IEEE Conf. Decision and Control*, Kobe, Japan, Dec. 1996, pp. 4683–4688.

[12] L. Praly, G. Bastin, J.-B. Pomet, and Z. P. Jiang, "Adaptive stabilization of nonlinear systems," in *Foundations of Adaptive Control*, P. V. Kokotović, Ed. Berlin, Germany: Springer-Verlag, 1991, pp. 347–434.

[13] L. Praly, "Adaptive regulation: Lyapunov design with a growth condition," *Int. J. Adaptive Contr. Signal Processing*, vol. 6, pp. 329–351, 1992.

[14] D. Taylor, P. V. Kokotović, R. Marino, and I. Kanellakopoulos, "Adaptive regulation of nonlinear systems with unmodeled dynamics," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 405–412, 1989.

Robustness Analysis of Polynomials with Polynomial Parameter Dependency Using Bernstein Expansion

M. Zettler and J. Garloff

***Abstract*— This paper considers the robust stability verification of polynomials with coefficients depending polynomially on parameters varying in given intervals. Two algorithms are presented. Both rely on the expansion of a multivariate polynomial into Bernstein polynomials. The first one is an improvement of the so-called Bernstein algorithm and checks the Hurwitz determinant for positivity over the parameter set. The second one is based on the analysis of the value set of the family of polynomials and profits from the convex hull property of the Bernstein polynomials. Numerical results to real-world control problems are presented showing the efficiency of both algorithms.**

***Index Terms*— Bernstein polynomials, polynomial parameter dependency, robust Hurwitz stability.**

I. Introduction

A standard approach to robustness analysis of linear dynamic systems is to examine the characteristic polynomial in the presence of parametric uncertainties. So far, most attention has been paid to the case of affine and multiaffine parameter dependency of the coefficients of the characteristic polynomial; see, e.g., [2], [6], [20], and the references therein. However, these cases do not cover most real-life problems. Therefore, we are concerned here with the far more general case of *polynomial* dependency.

Manuscript received February 26, 1997. This work was supported by the Ministry of Science and Research Baden-Württemberg.

The authors are with the Fachhochschule Konstanz, Fachbereich Informatik, D-78405 Konstanz, Germany (e-mail: garloff@fh-konstanz.de).

Publisher Item Identifier S 0018-9286(98)01388-9.

The Robust Stability Problem: Let the parameter set Q be an l-dimensional box, i.e., $Q = [\underline{q}_1, \overline{q}_1] \times \cdots \times [\underline{q}_l, \overline{q}_l]$, and let a family of polynomials be given by

$$p(s, \mathbf{q}) = a_0(\mathbf{q})s^m + \cdots + a_{m-1}(\mathbf{q})s + a_m(\mathbf{q}) \tag{1}$$

where the coefficients are depending polynomially on parameters q_i, $i = 1, \cdots, l$, $\mathbf{q} = (q_1, \cdots, q_l)$, i.e., for $k = 0, \cdots, m$

$$a_k(\mathbf{q}) = \sum_{i_1, \cdots, i_l = 0}^{d} a^{(k)}_{i_1 \cdots i_l} q_1^{i_1} \cdots q_l^{i_l}. \tag{2}$$

Question: Is the family of polynomials (robustly) stable for Q, i.e., are the polynomials $p(\mathbf{q})$ stable for all $\mathbf{q} \in Q$?

Here stability is meant in the sense of Hurwitz or asymptotical stability, i.e., we want to show that $p(s, \mathbf{q}) \neq 0$ for all $s \in \mathbf{C}$ with $\mathrm{Re}\, s \geq 0$, $\mathbf{q} \in Q$. To avoid dropping in degree, we assume for simplicity throughout this paper that $a_0(\mathbf{q}) > 0$ for all $\mathbf{q} \in Q$.

Unfortunately, most of the methods known from literature, e.g., [4], [5], [14]–[18], [23], [32], and [34]–[36], can only treat problems with polynomial dependency with only a few parameters and/or polynomials of lower degree. The genetic algorithm [25] appears to be an exception. However, this algorithm seems to be not fully tested for large control problems and gives no guarantee for finding the global solution. In Example 4 in Section V, we present an example in which this algorithm fails to give the correct solution.

A possible approach is to consider the Hurwitz determinant associated with the family of polynomials, e.g., [14], [16], [18], [23], [34], and [35]. In principle, by space and time limitations this approach is restricted to problems with a moderate number of parameters and to lower degree polynomials. The first algorithm which we present in Section III adopts this approach and is based on the expansion of the Hurwitz determinant into Bernstein polynomials. This leads to a fast algorithm. Focusing on larger control problems we develop then in Section IV a second algorithm which avoids the blowing up of the problem caused by using the Hurwitz determinant. The underlying idea of the algorithm is to watch for zero crossing over the imaginary axis by inspecting the so-called value set. Here we profit again from the convex hull property of the Bernstein expansion.

The results of this paper are presented in greater detail in the report [37] which is available upon request. We note that the approach the first algorithm is based on can be applied to other stability regions as well as to matrix stability using the determinantal criteria listed in [30], cf. [6, Ch. 17], often at the expense of an increase of dimensionality, however. For the related problem of Schur stability and the problem of computing the stability margin see [28].

II. Bernstein Expansion

For compactness, we define a *multi-index* I as an ordered l-tupel of nonnegative integers $(i_1, \cdots, i_l)$. We will use multi-indexes, e.g., to shorten power products; for $\mathbf{x} = (x_1, \cdots, x_l) \in \mathbf{R}^l$ we set $\mathbf{x}^I = x_1^{i_1} x_2^{i_2} \cdot \cdots \cdot x_l^{i_l}$. For simplicity, we sometimes suppress the brackets in the notation of multi-indexes. We write $I \leq N$ if $N = (n_1, \cdots, n_l)$ and if $0 \leq i_k \leq n_k$, $k = 1, \cdots, l$. Further, let $S = \{I : I \leq N\}$. Then we can write an l-variate polynomial p in the form

$$p(\mathbf{x}) = \sum_{I \in S} a_I \mathbf{x}^I, \qquad \mathbf{x} \in \mathbf{R}^l \tag{3}$$