# Multi-Agent Deployment in 3-D via PDE Control

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Abstract—This paper introduces a methodology for modelling, analysis, and control design of a large-scale system of agents deployed in 3-D space. The agents' communication graph is a mesh-grid disk 2-D topology in polar coordinates. Treating the agents as a continuum, we model the agents' collective dynamics by complex-valued reaction-diffusion 2-D partial differential equations (PDEs) in polar coordinates, whose states represent the position coordinates of the agents. Due to the reaction term in the PDEs, the agents can achieve a rich family of 2-D deployment manifolds in 3-D space which correspond to the PDEs' equilibrium as determined by the boundary conditions. Unfortunately, many of these deployment surfaces are open-loop unstable. To stabilize them, a heretofore open and challenging problem of PDE stabilization by boundary control on a disk has been solved in this paper, using a new class of explicit backstepping kernels that involve the Poisson kernel. A dual observer, which is also explicit, allows to estimate the positions of all the agents, as needed in the leaders' feedback, by only measuring the position of their closest neighbors. Hence, an all-explicit control scheme is found which is distributed in the sense that each agent only needs local information. Closed-loop exponential stability in the  $L^2$ ,  $H^1$ , and  $H^2$  spaces is proved for both full state and output feedback designs. Numerical simulations illustrate the proposed approach for 3-D deployment of discrete agents.

*Index Terms*—Backstepping, boundary control, deployment, distributed parameter systems, multi-agent systems.

#### I. INTRODUCTION

**C** OOPERATIVE formation control for multi-agent system has seen a phenomenal growth in the past years due to a multitude of practical applications, such as UAV formation flying, mobile robot deployment, moving sensor networks, coordination of transportation vehicles, or satellite clusters [1], [2]. One of the intrinsic advantages of multi-agent systems is the ability to complete tasks, otherwise very difficult for a single individual, by collaborative work. Collaboration provides many advantages such as flexibility, fault tolerance, redundancy, or efficiency. Deployment configuration and formation control are key technologies if the agents are to perform complex coopera-

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tive tasks. The agents are often required to move to a designed position, on a given deployment manifold, to adequately fulfill detection, monitoring or sampling functions. Practical examples include applications such as rendezvous [3], large area exploration [4], [5], pattern formation [6], surveillance [7], [8], or distributed environmental monitoring and science imaging [9]. However, most designs found in the literature consider simpler 2-D deployments, and thus avoid to address the challenges of 3-D formations, even though real-life applications are rarely planar.

a) Related literature: A wealth of results on multi-agent cooperative control can be found in the literature [10]-[12]. Most works can be classified into two wide families. On the one hand, one can find designs based on traditional discrete models; borrowing tools from graph theory, matrix theory, spectrum analysis, or potential functions, they have already produced an impressive array of results [13]-[17]. More recently, a new continuum-based viewpoint has arisen, leaning heavily on partial differential equation (PDE) methods [18]-[23]. The discrete and continuous approaches are nevertheless essentially equivalent, given that semi-discrete partial difference equations (PdEs) are analogous to PDEs over graphs [24]. Since graph Laplacian control for state consensus [25] is formulated as a linear PdE which behaves like a heat equation over the graph, Laplacian control can be expressed as a linear heat equation discretized in space. The selected discretization scheme determines the underlying communication topology connecting different agents.

In [26], one can find how a distributed system and the discrete approximation of a PDE are connected in system-theoretic terms. In particular, the application of finite difference approximations results in the space variable(s) being mapped into the agent index(es), and the spatial derivatives being transformed into links between neighbors [11], [27].

In a discrete context, it is well-known that the consensus control algorithm plays a crucial role. Many formation controllers extend the consensus algorithm and analyze stability properties under the framework of graph Laplacian matrices. We next provide a small but relevant sample of the many results that can be found in the literature. For instance, a leader-follower distributed formation control and estimation architecture is proposed in [28]. A similar structure but with a time-varying reference state is considered in [29]. By tracking the team centroid, an observer-controller scheme is designed in [30]. Another approach, hinging on the use of potential functions and valid for nonholonomic mobile robots with curvature constraints, can be found in [31]. Potential functions have also been used to design decentralized controllers for large teams of robots [32]. Another work shows the design of distance-based formation controllers, in a quantized communication setting, by means of analyzing the spectral properties of the incidence matrix [33]. Other results include the design of a distributed controller based on optimal localized feedback gains for one-dimensional formations [34] or the development of a multilevel topology, gradientbased consensus controller for 2-D formation shapes [35].

In the continuum context, the agents index is modeled as a continuous variable, allowing the collective dynamics of a group of discrete agents to be formulated as a PDE. The possibility of applying PDE methods to solve multi-agent control problems opens up a wealth of new possibilities, many of them not yet explored. We next review some recent developments exploiting the connection between PDEs and multi-agent systems. For instance, in [36], a model reference adaptive control law based on PdEs is used to track desired deployment trajectories. [37] designs a feedforward controller for multi-agent deployment by using a flatness-based motion planning method for PDEs. Reaction-advection-diffusion PDE backstepping is used in [11] for leader-enabled deployment onto planar curves. Reference [38] combines extremum seeking and diffusion feedback for multi-agent deployment around a source. In [39], a large vehicular formation closed-loop dynamics are studied by means of PDE eigenvalue analysis. Similarly, hyperbolic PDE models of large vehicular platoons are used to design decentralized control laws in [27]. Hyperbolic models have also been used to analyze networks of oscillators in [40]. Another instance of PDE methods, which can be found in [41], is the use of smoothed particle hydrodynamics theory to address the problem of pattern generation by decentralized controllers.

b) Results and contributions of the paper: We present a PDE-based approach to deploy multi-agent system onto 3-D manifolds. The agents' collective dynamics are modeled by two advection-diffusion PDEs, whose states represent the agents' positions; the first PDE is complex-valued, with the x and y coordinates, respectively, representing the real and imaginary parts of the state, while the remaining PDE is real-valued and describes the z coordinate (height) evolution. The PDEs are defined on a disk, parameterized by polar coordinates  $(r, \theta)$ ; these correspond to discrete indexes of the agents when the PDEs are discretized via finite differences, which impose a fixed communication topology. The agents on the boundary (at the disk's edge) are selected as leaders; they drive the collective dynamics of the whole system (this idea was first introduced in [42]).

By modeling the multi-agent system as a continuum, the achievable deployment profiles are found from all possible equilibria of the model PDE equations. Since these equilibria can be analytically expressed by explicit formulae, we find a wide variety of deployment manifolds; in particular, it is found that a *layer* of agents can be deployed on any periodic, square-integrable planar curve fixed at a designated height.

From the underlying PDE it is clear that many of these equilibrium profiles are potentially open-loop unstable. To address this problem, a heretofore open and challenging problem of PDE stabilization by boundary control on a disk has been solved in this paper, using a new class of explicit backstepping kernels that involves the Poisson kernel. The basis of the design is the backstepping method for PDE boundary control. At its inception, backstepping was developed for 1-D parabolic equations [43]; it has been since extended to many other equations and domain shapes [44]. In particular, past results include a design for a complex-valued PDE (the Ginzburg–Landau equation) at the exterior of a disk [45] and an output-feedback controller for an annular geometry [46]. However, the inclusion of the disk's origin presents a major design challenge due to the appearance of singular terms in the PDE; this is addressed by applying a series of transformations that allows us solve the resulting kernel equations. In addition, we show  $L^2$ ,  $H^1$ , and  $H^2$  closed-loop stability despite these potential singularities, thus guaranteeing a good system behavior. When discretized, our control law enables the leader agents to drive the followers asymptotically into the desired deployment manifold.

To reduce the communication needs of our design, we also formulate an observer to estimate the positions of all the agents, which are required in the leaders' feedback law. Since the observer is driven exclusively by their neighbors' information, only local communication is needed for all agents, both leaders and followers. In addition, the controller and observer have explicitly computable gains. Thus, our design results in an allexplicit feedback law which is distributed in the sense that each agent only needs local information.

c) Organization: Section II introduces the agents' model and the explicit deployment profiles. Section III focuses on the design of a novel boundary control law for a disk-shaped domain.  $H^2$  closed-loop exponential stability is stated and proven in Section IV. Next, Section V introduces a boundary observer and proves the stability of the closed-loop system when the observer estimates are used in the control laws. We connect the PDE design with the agents' distributed control law by means of discretization in Section VI, along with a simulation study that supports the theoretical results. We conclude with some remarks in Section VII.

#### II. 3-D DEPLOYMENT

The following section models multi-agent deployment as a PDE problem. Considering the agents' communication topology as a continuum, we reformulate the discrete partial difference equations (PdEs) describing the agents' dynamics as a set of continuous complex-valued reaction-diffusion PDEs. This allows us to obtain all possible deployment profiles by solving for the PDEs' equilibrium. Using this idea we determine several deployment profiles of interest.

## A. Agents' Model

We consider that the communication structure for the agents is given by an undirected graph G(V, E) on a mesh-grid disk with  $M \times N$  nodes, as shown in Fig. 1. The pair  $(i, j) \in V$ represents a node, but also an agent located at the node. If  $((i, j), (i', j')) \in E$ , then (i, j) and (i', j') are neighbors, i.e., the respective agents at the nodes can share states of information; we abbreviate this relation by denoting  $(i, j) \sim (i', j')$ . The agents at the boundary (outermost layer) are selected as leaders, following the concept of boundary actuation. The rest are follower agents. Thus, for  $j = 1, 2, \ldots, N$ , the leaders are denoted by (M, j) and the followers by (i, j), with i = $1, 2, \ldots, M - 1$ .

Let  $x_{ij}(t)$ ,  $y_{ij}(t)$ , and  $z_{ij}(t)$  denote agent (i, j)'s position coordinates at time t in 3-D space over the time-invariant communication graph in Fig. 1. It is known [24] that Laplacian control (consensus law [1]) coincides with the continuous Laplace operator; thus, consensus dynamics can be modeled as a PDE containing a Laplace operator. Based on this idea, we describe the dynamics of the agents by using a set of PDEs. For that,



Fig. 1. Graph defining the communication relationship among agents. (i, j) denotes an agent located in the *i*th ray layer and *j*th angle layer. The agents at the outermost layer are leaders. Each follower except at the innermost layer has four neighbors.

we first map the graph G to a closed disk  $\overline{B}(0, R) = \{(r, \theta) : 0 \le r \le R, 0 \le \theta < 2\pi\}$ . As  $M, N \to \infty$ , the discrete graph approaches<sup>1</sup>  $\overline{B}(0, R)$ , namely,  $(i, j) \to (r_i, \theta_j) \to (r, \theta)$ . Letting now  $u(t, r, \theta) = x(t, r, \theta) + jy(t, r, \theta)$  and  $z(t, r, \theta)$  denote, respectively, the horizontal and vertical coordinates of agent  $(r, \theta)$ , the agent dynamics is expressed by the following two 2-D PDEs:

$$u_t(t,r,\theta) = \frac{\varepsilon}{r} \left( r u_r(t,r,\theta) \right)_r + \frac{\varepsilon}{r_1^2} u_{\theta\theta}(t,r,\theta) + \lambda u(t,r,\theta) \quad (1)$$

$$z_t(t,r,\theta) = \frac{1}{r} \left( r z_r(t,r,\theta) \right)_r + \frac{1}{r^2} z_{\theta\theta}(t,r,\theta) + \mu z(t,r,\theta)$$
(2)

for  $(t, r, \theta) \in \mathbb{R}^+ \times \overline{B}(0, R)$ ,  $u, \varepsilon, \lambda \in \mathbb{C}$ ,  $z, \mu \in \mathbb{R}$ , and  $\operatorname{Re}(\varepsilon), \mu > 0$ , where  $\operatorname{Re}(\cdot)$  denotes the real part. The first two terms of (1) and (2) are the Laplacian written in polar coordinates, and the third term is a linear reaction term, which would model the effect of the state of a given agent on the agent itself. The boundary conditions are the leaders' position, given by

$$u(t, R, \theta) = U(t, \theta) \tag{3}$$

$$z(t, R, \theta) = Z(t, \theta).$$
(4)

Since u and z are uncoupled, they can be analyzed separately. Furthermore, given that (2) is a particular case of (1), in the sequel we only analyze the dynamics of u in detail; the results for z can be derived in a similar fashion.

# B. Deployment Profiles

Since the agents' dynamics are given by (1)–(4), final deployment formations correspond to the equilibrium profiles  $(\bar{u}, \bar{z})$  of the PDEs, which are given by

$$\frac{1}{r}(r\bar{u}_r(r,\theta))_r + \frac{1}{r^2}\bar{u}_{\theta\theta}(r,\theta) + \frac{\lambda}{\varepsilon}\bar{u}(r,\theta) = 0$$
(5)

$$\frac{1}{r}\left(r\bar{z}_r(r,\theta)\right)_r + \frac{1}{r^2}\bar{z}_{\theta\theta}(r,\theta) + \mu\bar{z}(r,\theta) = 0 \tag{6}$$

<sup>1</sup>The approximating error between the PDEs and the discrete agent system is related to both the discretization method and the number of the agents. This is discussed in Section VI.

with designed boundary actuation

$$\bar{u}(R,\theta) = f(\theta), \quad \bar{z}(R,\theta) = C$$
 (7)

where the boundary of  $\bar{z}$  has been set as a constant C for simplicity. Equations (5)–(7) characterize all achievable 3-D deployments; their explicit solution is given by

$$\bar{u}(r,\theta) = \sum_{n=-\infty}^{+\infty} C_n J_n \left(\sqrt{\frac{\lambda}{\varepsilon}}r\right) e^{jn\theta}$$
(8)  
where  $C_n = \frac{1}{2\pi J_n \left(\sqrt{\frac{\lambda}{\varepsilon}}R\right)} \int_{-\pi}^{\pi} f(\omega) e^{-jn\omega} d\omega,$   
 $\bar{z}(r,\theta) = C \frac{J_0 \left(\sqrt{\mu}r\right)}{J_0 \left(\sqrt{\mu}R\right)}$ (9)

where  $J_n$  is the *n*th-order Bessel function of the first kind.

By changing the position of the leaders (boundary actuators) we can obtain a wide variety of combinations  $\bar{x}$  and  $\bar{y}$  from  $\bar{u} = \bar{x} + j\bar{y}$  enabling rich and interesting deployment manifolds. In particular, the system can provide any formation curve for a given height for a desired layer  $r_0 \leq R$  of follower agents. For example, if one wants to deploy the agents at height l onto a curve defined by  $h(\theta) = \bar{x}(\theta) + j\bar{y}(\theta)$ , this is achievable as follows. First, determine the constant C that achieves the curve at height l and layer  $r_0$  of agents from  $C = l(J_0(\sqrt{\mu}R)/J_0(\sqrt{\mu}r_0))$ . Then, substitute  $r_0$  into (8) and invert the series to obtain the coefficients  $C_n$  as a function of  $h(\theta)$ . Finally, letting r = R, one obtains the boundary input as

$$f(\theta) = \bar{x}(R,\theta) + j\bar{y}(R,\theta) = \sum_{n=-\infty}^{+\infty} \varphi_n \mathbf{J}_n\left(\sqrt{\frac{\lambda}{\varepsilon}}R\right) e^{jn\theta} \quad (10)$$

where

$$\varphi_n = \frac{\int_{-\pi}^{\pi} h(\omega) \mathrm{e}^{-\mathrm{j}n\omega} d\omega}{2\pi \mathrm{J}_n\left(\sqrt{\frac{\lambda}{\varepsilon}} r_0\right)}.$$
(11)

Once the deployment curve at a given height and layer is fixed, the remaining agents will follow a family of continuum curves that extend the shape of the reference curve  $h(\theta)$  to a smooth manifold in three-dimensional space. To illustrate the procedure, we next show several examples.

Fig. 2 depicts eight possible deployment profiles under the action of boundary actuations determined by this procedure. One could also imagine the transitions between these deployment patterns. For instance, at the beginning [Fig. 2(a)], all the agents are gathered on a disk at height zero. Then, the leaders move to z = 1 but still remain with a circle shape, while the follower agents separate and arrive at different heights [Fig. 2(b)]. Next, the agents cycle through a series of patterns: asteroid, quadrifolium, deltoid, 3-petal polar rose, 3-petal epicycle and 4-petal epicycle, as shown in Fig. 2(c) to Fig. 2(h). The periodic property of the Bessel function  $J_n$  makes the leaders' formation profile shape to repeatedly appear at different heights for the follower agents, in particular there are six layers of self-similar deployment patterns at different heights when  $\lambda/\varepsilon = 400$  and  $\mu = 15$ .



Fig. 2. Agent deployment manifolds with the parameters  $\lambda/\varepsilon = 400 + j0$  and  $\mu = 15$ . (a) Circular shape  $u = \exp(j\theta)$  at height z = 0. (b) Circular shape  $u = \exp(j\theta)$  at height z = 1. (c) Asteroid shape  $u = \exp(j\theta) + (1/3)\exp(-j3\theta)$  at height z = 1. (d) Quadrifolium shape  $u = \exp(j\theta) + \exp(-j3\theta)$  at height z = 1. (e) Deltoid shape  $u = \exp(j\theta) + (1/2)\exp(-j2\theta)$  at height z = 1. (f) 3-petal polar rose shape  $u = \exp(j\theta) + \exp(-j(\pi - 2\theta))$  at height z = 1. (g) 3-petal epicycloid shape  $u = \exp(j\theta) - (1/4)\exp(-j4\theta)$  at height z = 1. (h) 4-petal epicycloid shape  $u = \exp(j\theta) - (1/5)\exp(j5\theta)$  at height z = 1.



Fig. 3. Agent deployment manifolds with positive imaginary part of  $\lambda/\varepsilon$ , with  $z(R,\theta) = 1$  and  $\mu = 15$ . (a) Polar rose shape  $u = \exp(j\theta) + \exp(-j5\theta)$  with  $\lambda/\varepsilon = 100 + j60$ . (b) 3-petal rose shape  $u = \exp(j\theta) + \exp(-j(\pi - 2\theta))$  with  $\lambda/\varepsilon = 100 + j60$ . (c) 10-petal epicycloid shape  $u = \exp(j\theta) - (1/11) \exp(j11\theta)$  with  $\lambda/\varepsilon = 200 + j40$ . (d) 4-petal epicycloid shape  $u = \exp(j\theta) - (1/5) \exp(j5\theta)$  with  $\lambda/\varepsilon = 100 + j20$ .

Fig. 3 illustrates other possible deployments by choosing a positive imaginary part of  $\lambda/\varepsilon$ ; increasing this value "twists" the deployment profile while simultaneously shrinking the deployment manifold. The larger the imaginary part is, the smaller the bottom of the deployment becomes. This would allow users to implement different 3-D formations, such as the twist vase of Fig. 3(c), by tuning the value of the imaginary part of  $\lambda/\varepsilon$ .

The deployment profiles corresponding to the equilibrium of (1)–(4) are potentially open-loop unstable, particularly for large values  $\lambda$  and  $\mu$ . Namely, the agents would not converge to the desired formation from their initial positions unless they started exactly at the equilibrium. Given that large values of  $\lambda$  and  $\mu$  are required for most of the desired deployments, such as those shown in Figs. 2 and 3, it is necessary to design stabilizing feedback laws for the leaders. In the next section, we focus on such stabilizing control designs.

# III. BOUNDARY CONTROL DESIGN

To stabilize the equilibrium profiles of (1)–(4), we design a feedback law by using a backstepping-based method.

First, to eliminate angular dependence, we expand the system state (1) and the boundary control (3) as a Fourier series:

$$u(t,r,\theta) = \sum_{\substack{n=-\infty\\\infty}}^{\infty} u_n(t,r) \mathrm{e}^{\mathrm{j}n\theta},$$
 (12)

$$U(t,\theta) = \sum_{n=-\infty}^{\infty} U_n(t) e^{jn\theta}$$
(13)

where the coefficients  $u_n$  and  $U_n$ , for  $n \in \mathbb{Z}$ , are obtained from

$$(t,r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t,r,\psi) e^{-jn\psi} d\psi$$
(14)

$$U_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(t,\psi) \mathrm{e}^{-\mathrm{j}n\psi} d\psi.$$
(15)

Each coefficient  $u_n(t, r)$  verifies the following uncoupled PDE:

$$u_{nt} = \frac{\varepsilon}{r} (r u_{nr})_r - n^2 \frac{\varepsilon}{r^2} u_n + \lambda u_n \tag{16}$$

evolving in  $(t,r) \in \mathbb{R}^+ \times [0,R]$ , with boundary conditions  $u_n(t,R) = U_n(t)$ . Thus, we can independently design each  $U_n$  to stabilize  $u_n$ ; later, (15) can be used to assemble all the  $U_n$ 's to find U.

# A. Backstepping Transformation

 $u_n$ 

Following [43], our approach to design  $U_n(t)$  is to seek a transformation to map (16) into the following stable target system:

$$w_{nt} = \frac{\varepsilon}{r} (rw_{nr})_r - n^2 \frac{\varepsilon}{r^2} w_n \tag{17}$$

with boundary conditions  $w_n(t, R) = 0$ . Define the transformation as follows:

$$w_n(t,r) = u_n(t,r) - \int_0^t K_n(r,\rho)u_n(t,\rho)d\rho$$
(18)

where the kernel  $K_n(r, \rho)$  defined on  $\mathcal{T} = \{(r, \rho) : 0 \le \rho \le r \le R\}$ . By substituting the transformation into the target system and performing differentiation and integration by parts (see [43]), one arrives at a hyperbolic PDE for the kernel

$$K_{nrr} + \frac{K_{nr}}{r} - K_{n\rho\rho} + \frac{K_{n\rho}}{\rho} - \frac{K_n}{\rho^2} - n^2 \left(\frac{1}{r^2} - \frac{1}{\rho^2}\right) K_n = \frac{\lambda}{\varepsilon} K_n \quad (19)$$

with boundary conditions

$$K_n(r,0) = 0, \quad K_n(r,r) = -\frac{\lambda}{2\varepsilon}r.$$
 (20)

#### B. Kernel Function

While (19) is similar to the kernel PDE found in [43], there are extra terms which are singular at  $r, \rho = 0$ . These terms make (19) not amenable to previously developed solution methods for backstepping kernels that can be found in the literature. However, by using a series of transformations it is possible to obtain the explicit solution of (19); for brevity, we just enumerate the transformations and give the final result. First, let  $K_n(r,\rho) = \prod_n (r,\rho)\rho(\rho/r)^{|n|}$ . Next, set  $\prod_n (r,\rho) = \Phi(x)$  where  $x = ((\lambda/\epsilon)(r^2 - \rho^2))^{1/2}$ , and finally define  $\Psi(x) = x\Phi(x)$ . By applying the transformations, we end up with

$$x^{2}\Psi'' + x\Psi' - (1+x^{2})\Psi = 0$$
(21)

with boundary condition  $\Psi(0) = -(\lambda/2\varepsilon)$ , which is Bessel's modified differential equation of order 1. Solving (21) and undoing the transformations, we reach an explicit expression for the kernel

$$K_n(r,\rho) = -\rho\left(\frac{\rho}{r}\right)^{|n|} \frac{\lambda}{\varepsilon} \frac{I_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2 - \rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2 - \rho^2)}}$$
(22)

where  $I_1$  is the first-order modified Bessel function of the firstkind. Applying (12) and summing the Fourier series, we can obtain transformation (18) in the angular space

$$w = \sum_{n=-\infty}^{\infty} w_n(t,r) e^{jn\theta}$$
$$= \sum_{n=-\infty}^{\infty} u_n(t,r) e^{jn\theta} - \sum_{n=-\infty}^{\infty} \int_0^r K_n(r,\rho) u_n(t,\rho) e^{jn\theta} d\rho$$
$$= u(t,r,\theta) + \int_0^r \int_{-\pi}^{\pi} K(r,\rho,\theta-\psi) u(t,\rho,\psi) d\psi d\rho \qquad (23)$$

where we have used the Convolution Theorem for Fourier series; then a formula analog to (12) can be used to obtain  $K(r, \rho, \theta - \psi)$  from the  $K_n$ 's by summing them. This yields

$$K(r,\rho,\theta-\psi) = -\frac{\lambda}{\varepsilon}\rho \frac{I_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}P(r,\rho,\theta-\psi) \quad (24)$$

where  $P(r, \rho, \theta - \psi)$  is the Poisson kernel:

$$P(r, \rho, \theta - \psi) = \sum_{n=-\infty}^{+\infty} \frac{1}{2\pi} \left(\frac{\rho}{r}\right)^{|n|} e^{jn(\theta - \psi)}$$
$$= \frac{1}{2\pi} \frac{1 - \frac{\rho^2}{r^2}}{1 + \frac{\rho^2}{r^2} - 2\frac{\rho}{r}\cos(\theta - \psi)}.$$
 (25)

From (23), we see that u is mapped into w, which verifies

$$w_t = \frac{\varepsilon}{r} \left( r w_r \right)_r + \frac{\varepsilon}{r^2} w_{\theta\theta}, \qquad (26)$$

$$w(t, R, \theta) = 0. \tag{27}$$

Note that setting r = R in (23) and applying the boundary conditions (3) and (27) we obtain the control law U as a feedback law in the state u.

One goes back from w to the original system u by the inverse mapping

$$u(t,r,\theta) = w(t,r,\theta) + \int_{0}^{r} \int_{-\pi}^{\pi} L(r,\rho,\theta,\psi)w(t,\rho,\psi)d\psi\,d\rho \quad (28)$$

whose kernel (the inverse kernel) is found analogously to K:

$$L(r,\rho,\theta-\psi) = -\rho \frac{\lambda}{\varepsilon} \frac{\mathbf{J}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}} P(r,\rho,\theta-\psi).$$
(29)

Let now  $\overline{U}(\theta) = \overline{u}(r,\theta)|_{r=R}$  be the steady state of the leader agents obtained from the desired deployment profile  $\overline{u}(r,\theta)$ . Then, we obtain the leaders' feedback laws as

$$U(t,\theta) = \overline{U}(\theta) - \mathcal{A}\{\overline{u}\}(\theta) + \mathcal{A}\{u\}(t,\theta)$$
(30)

where  $\mathcal{A}\{\cdot\}$  denotes the following operator acting on u, which is found by setting r = R in (24):

$$\mathcal{A}\{u\}(t,\theta) = -\frac{\lambda}{\varepsilon} \int_{0}^{R} \rho \frac{\mathrm{I}_{1} \left[ \sqrt{\frac{\lambda}{\varepsilon} (R^{2} - \rho^{2})} \right]}{\sqrt{\frac{\lambda}{\varepsilon} (R^{2} - \rho^{2})}} \\ \times \left( \int_{-\pi}^{\pi} P(R,\rho,\theta - \psi) u(t,\rho,\psi) d\psi \right) d\rho. \quad (31)$$

The leaders' control law (30) contains a state feedback part (the third term) which stabilizes the deployment and an openloop part (the first two terms), which guides all the agents to the target positions and can be pre-computed–prior to deployment–from  $\bar{u}$ . Note that (30) implies that the user only needs to inform the leaders about the desired deployment.

Following a similar procedure, the leaders' control law for the height coordinate is:

$$Z(t,\theta) = \bar{z}(R,\theta) - \mu \int_{0}^{R} \rho \frac{I_1 \left[ \sqrt{\mu(R^2 - \rho^2)} \right]}{\sqrt{\mu(R^2 - \rho^2)}} \times \left( \int_{-\pi}^{\pi} P(R,\rho,\theta - \psi) (z(t,\rho,\psi) - \bar{z}(\rho,\psi)) d\psi \right) d\rho.$$
(32)

# IV. $H^2$ Stability

In this section, we investigate the stability of the closed-loop system. In what follows, let  $\|\cdot\|_{L^2}$ ,  $\|\cdot\|_{H^1}$ , and  $\|\cdot\|_{H^2}$  denote, respectively, the usual  $L^2$ ,  $H^1$  and  $H^2$  norms on the disk. Since the domain is two-dimensional, to have continuity of the state variables an  $H^2$  stability result is required [47].

Based on the shape of transformations (23) and (28) we first formulate a result that states the equivalence of norms of the original and target variables, thus allowing us to do norm computations in the simpler target system (26).

*Proposition 1:* Let  $g(r, \theta)$  be related to  $f(r, \theta)$  as follows:

$$g(r,\theta) = f(r,\theta) + \int_{0}^{r} \int_{-\pi}^{\pi} \rho F(r,\rho) P(r,\rho,\theta-\psi) f(\rho,\psi) \, d\psi \, d\rho$$
(33)

where  $F(r, \rho) \in C^2(\mathcal{T})$ . Then  $\|g\|_{L^2} \leq \hat{C}_0 \|f\|_{L^2}$ ,  $\|g\|_{H^1} \leq \hat{C}_1 \|f\|_{H^1}$ , and  $\|g\|_{H^2} \leq \hat{C}_2 \|f\|_{H^2}$ , where the constants  $\hat{C}_i$  depend only on R and  $F(r, \rho)$ .

The proof of Proposition 1 is outlined in the Appendix. We next prove stability of the target system.

Proposition 2: Consider the system (26), (27) with initial conditions  $w_0(r, \theta)$  which satisfies  $w_0 \in H^2$  and  $w_0(R, \theta) = 0$ . Then  $w \in C([0, \infty), H^2)$  and the equilibrium  $w(t, r, \theta) \equiv 0$  is exponentially stable in the  $L^2$ ,  $H^1$ , and  $H^2$  norms, i.e., there exist  $D_0$ ,  $D_1$ ,  $D_2$ ,  $\alpha_0$ ,  $\alpha_1$ ,  $\alpha_2 > 0$ , such that

$$\|w(t,\cdot)\|_{L^2} \le D_0 \mathrm{e}^{-\alpha_0 t} \|w_0\|_{L^2},\tag{34}$$

$$\|w(t,\cdot)\|_{H^1} \le D_1 \mathrm{e}^{-\alpha_1 t} \|w_0\|_{H^1},\tag{35}$$

$$\|w(t,\cdot)\|_{H^2} \le D_2 \mathrm{e}^{-\alpha_2 t} \|w_0\|_{H^2}.$$
(36)

*Proof:* The well-posedness result is standard, see for instance the book by Brezis [47]. For the norm estimates, take first

$$V_1 = \frac{1}{2} \|w(t, \cdot)\|_{L^2}^2 = \frac{1}{2} \int_0^R \int_{-\pi}^{\pi} |w(t, r, \theta)|^2 r \, d\theta \, dr.$$
(37)

Then

١

$$\dot{V}_{1} = \int_{0}^{R} \int_{-\pi}^{\pi} \frac{ww_{t}^{*} + w^{*}w_{t}}{2} r \, d\theta \, dr$$

$$= \varepsilon_{R} \int_{0}^{R} \int_{-\pi}^{\pi} \operatorname{Re} \{w^{*}(rw_{r})_{r} + \frac{1}{r}w^{*}w_{\theta\theta}\} \, d\theta \, dr$$

$$= -\varepsilon_{R} \int_{0}^{R} \int_{-\pi}^{\pi} \left(|w_{r}|^{2} + \frac{1}{r^{2}}|w_{\theta}|^{2}\right) r \, d\theta \, dr \qquad (38)$$

where  $w^*$  denotes the conjugate of w and  $\varepsilon_R$  is the real part of the complex coefficient  $\varepsilon$ . To proceed we need Poincare's inequality in polar coordinates, whose proof we skip

$$\int_{0}^{R} \int_{-\pi}^{\pi} |w|^2 r \, d\theta \, dr \le 4R^2 \int_{0}^{R} \int_{-\pi}^{\pi} |w_r|^2 r \, d\theta \, dr.$$
(39)

Then,

$$\dot{V}_1 \le -\frac{\varepsilon_R}{4R^2} \int_0^R \int_{-\pi}^{\pi} |w|^2 r \, d\theta \, dr = -\alpha_0 V_1$$
 (40)

thus reaching the  $L^2$  result. Take now

$$V_2 = V_1 + \frac{1}{2} \|w_r(t,\cdot)\|_{L^2}^2 + \frac{1}{2} \left\|\frac{w_\theta}{r}(t,\cdot)\right\|_{L^2}^2$$
(41)

which equivalent to the  $H^1$  norm. Then

$$\begin{split} \dot{V}_{2} &= \dot{V}_{1} + \int_{0}^{R} \int_{-\pi}^{\pi} \operatorname{Re} \left\{ w_{r}^{*} w_{rt} + \frac{1}{r^{2}} w_{\theta}^{*} w_{\theta t} \right\} r \, d\theta \, dr \\ &= \dot{V}_{1} - \varepsilon_{R} \int_{0}^{R} \int_{-\pi}^{\pi} \left| \frac{1}{r} (rw_{r})_{r} + \frac{1}{r^{2}} w_{\theta \theta} \right|^{2} r \, d\theta \, dr \\ &\leq -\varepsilon_{R} \int_{0}^{R} \int_{-\pi}^{\pi} \left( |w_{r}|^{2} + \frac{1}{r^{2}} |w_{\theta}|^{2} \right) r \, d\theta \, dr \\ &\leq -\frac{\varepsilon_{R}}{4R^{2}} V_{1} - \frac{\varepsilon_{R}}{2} \int_{0}^{R} \int_{-\pi}^{\pi} \left( |w_{r}|^{2} + \frac{1}{r^{2}} |w_{\theta}|^{2} \right) r \, d\theta \, dr \\ &\leq -\alpha_{1} V_{2}. \end{split}$$
(42)

For the  $H^2$  norm estimate, denote  $s = \varepsilon \bigtriangleup w = (\varepsilon/r)(rw_r)_r + (\varepsilon/r^2)w_{\theta\theta} = w_t$ . The equation verified by s is  $s_t = \varepsilon \bigtriangleup s$  with  $s(t, R, \theta) = 0$ . Taking

$$V_3 = V_1 + V_2 + \|s(t, \cdot)\|_{L^2}^2$$
(43)

it is obvious from the previous developments that  $\dot{V}_3 \leq -\alpha_2 V_3$ . Since  $V_3$  is equivalent (given the null boundary condition of w) to the  $H^2$  norm, the proposition is proved.

Combining Propositions 1 and 2 and applying them to u minus the desired equilibrium profile  $\bar{u}$ , we obtain the following theorem stating that the leaders' feedback law achieves exponential stability of the deployment profile in  $H^2$  norm (this result also holds for z).

Theorem 1: Consider the system (1) and (3) with control law (30) and initial condition  $u_0(r, \theta)$ . Then there exist D and  $\alpha > 0$  such that if  $u_0 \in H^2$  and satisfies the compatibility condition

$$u_0(R,\theta) = \bar{U}(\theta) - \mathcal{A}\{\bar{u}\}(\theta) + \mathcal{A}\{u\}(t_0,\theta)$$
(44)

then  $u \in \mathcal{C}([0,\infty), H^2)$  and  $||u(t,\cdot)||_{H^2} \leq De^{-\alpha t} ||u_0||_{H^2}$ .

The  $H^2$  stability result guarantees the continuity of the agent system, in the sense that neighbors in terms of the network topology remain neighbors in geometric space; this is important in practice to avoid the agents going out of communications' range.

#### V. OBSERVER DESIGN

Feedback law (30) assumes that leaders know all the agents' positions at all times, which is not realistic. We now design an observer to estimate these positions using a measurement at the boundary, specifically, the derivative  $u_r(t, R, \theta)$ . We pose the following observer:

$$\hat{u}_t(t,r,\theta) = \frac{\varepsilon}{r} (r\hat{u}_r)_r + \frac{\varepsilon}{r^2} \hat{u}_{\theta\theta} + \lambda \hat{u} + T(t,R,r,\theta), \quad (45)$$
$$\hat{u}(t,R,\theta) = U(t,\theta) + q_{10}(u_r(t,R,\theta) - \hat{u}_r(t,R,\theta)) \quad (46)$$

where T is an output injection operator given by

$$T(t, R, r, \theta) = \int_{-\pi}^{\pi} q_1(R, r, \theta - \psi) \left( u_r(t, R, \psi) - \hat{u}_r(t, R, \psi) \right) d\psi$$
(47)

where  $q_1$  is the observer kernel gain,  $\hat{u}(t, r, \theta)$  is the estimated state, and U is the applied control. To find the observer kernel  $q_1(R, r, \theta)$  and the value  $q_{10}$  that guarantee convergence of  $\hat{u}$  to u, we introduce the error variable  $\tilde{u} = u - \hat{u}$ , which verifies

$$\begin{split} \tilde{u}_t(t,r,\theta) &= \frac{\varepsilon}{r} \, (r\tilde{u}_r)_r + \frac{\varepsilon}{r^2} \tilde{u}_{\theta\theta} + \lambda \tilde{u} \\ &- \int_{-\pi}^{\pi} q_1(R,r,\theta-\psi) \tilde{u}_r(t,R,\psi) d\psi, \\ \tilde{u}(t,R,\theta) &= -q_{10} \tilde{u}_r(t,R,\theta). \end{split}$$
(48)

Proceeding as in Section III, we use Fourier series expansion to represent  $\tilde{u}$ ,  $q_1$ , and  $\tilde{u}_r(R)$  with Fourier coefficients  $\tilde{u}_n$ ,  $q_{n1}$ ,  $\tilde{u}_{nr}(R)$ , respectively. From the convolution theorem, the Fourier coefficients of T can be expressed from  $q_{n1}$  as products

$$T_n(t, R, r) = 2\pi q_{n1}(R, r)\tilde{u}_{nr}(t, R).$$
(49)

Consequently, we get a set of uncoupled equations for  $\tilde{u}_n$ :

$$\tilde{u}_{nt}(t,r) = \frac{\varepsilon}{r} \left( r \tilde{u}_{nr} \right)_r (t,r) - \frac{\varepsilon n^2}{r^2} \tilde{u}_n(t,r) + \lambda \tilde{u}_n(t,r) - 2\pi q_{n1}(R,r) \tilde{u}_{nr}(t,R),$$
  
$$\tilde{u}_n(t,R) = -q_{10} \tilde{u}_{nr}(t,R).$$
(50)

We transform system (50) to a target system  $\tilde{w}_n$  by using the mapping

$$\tilde{u}_n(t,r) = \tilde{w}_n(t,r) - \int_r^R Q_n(r,\rho)\tilde{w}_n(t,\rho)\,d\rho \qquad (51)$$

where the kernel  $Q_n(r, \rho)$  is defined on  $\mathcal{T}$  and the desired target system verifies

$$\tilde{w}_{nt}(t,r) = \frac{\varepsilon}{r} \left( r \tilde{w}_{nr} \right)_r - \frac{\varepsilon n^2}{r^2} \tilde{w}_n,$$
  
$$\tilde{w}_n(t,R) = 0.$$
 (52)

Analogous to the controller design, we get the observer kernel equation

$$Q_{nrr} + \frac{Q_{nr}}{r} - Q_{n\rho\rho} + \frac{Q_{n\rho}}{\rho} - \frac{Q_{n\rho}}{\rho^2} - n^2 \left(\frac{1}{r^2} - \frac{1}{\rho^2}\right) Q_n$$
$$= -\frac{\lambda}{\varepsilon} Q_n, \tag{53}$$

$$Q_n(r,r) = Q_n(R,R) + \frac{\lambda(R-r)}{2\varepsilon}$$
(54)

and the boundary conditions determine the observer gains

$$\varepsilon Q_n(r,R) = 2\pi q_{n1}, \quad q_{10} = 0.$$
 (55)

Even though (55) has one less boundary condition than (19), this does not affect the kernel-solving process. Following the same steps of Section III, we obtain:

$$q_1(R,r,\theta) = -\lambda R \frac{\mathbf{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2 - r^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2 - r^2)}} P(R,r,\theta).$$
(56)

It is of interest to note that Q is similar to the inverse transformation kernel (29); this is due to the duality between the observer and control design procedures. Substituting (56) into (45), we get the observer in explicit form:

$$\hat{u}_{t}(t,r,\theta) = \frac{\varepsilon}{r} (r\hat{u}_{r})_{r} + \frac{\varepsilon}{r^{2}} \hat{u}_{\theta\theta} + \lambda \hat{u} - \int_{-\pi}^{\pi} \lambda R \frac{I_{1} \left[ \sqrt{\frac{\lambda}{\varepsilon} (R^{2} - r^{2})} \right]}{\sqrt{\frac{\lambda}{\varepsilon} (R^{2} - r^{2})}} P(R,r,\theta - \psi) \tilde{u}_{r}(t,R,\psi) d\psi, \hat{u}(t,R,\theta) = U(t,\theta).$$
(57)

#### A. Output Feedback Stability

If we use the observer estimates from (57) in the leaders' control law (30), we obtain an output feedback controller, namely

$$U(t,\theta) = \bar{U}(t,\theta) - \mathcal{A}\{\bar{u}\}(\theta) + \mathcal{A}\{\hat{u}\}(t,\theta).$$
(58)

We now analyze the stability of the output-feedback closedloop system in the  $H^2$  Sobolev space, by studying the augmented system  $(u, \hat{u})$ . These two variables are equivalent to  $(\hat{u}, \tilde{u})$ , which are in turn related to the target variables  $(\hat{w}, \tilde{w})$ by the following two mappings:

$$\hat{v}(t,r,\theta) = \hat{u}(t,r,\theta) - \int_{r-\pi}^{R} \int_{-\pi}^{\pi} K(r,\rho,\theta,\psi) \hat{u}(t,\rho,\psi) \, d\psi \, d\rho, \quad (59)$$

$$\tilde{u}(t,r,\theta) = \tilde{w}(t,r,\theta) - \int_{r-\pi}^{R} \int_{-\pi}^{\pi} Q(r,\rho,\theta,\psi) \tilde{w}(t,\rho,\psi) d\psi \, d\rho.$$
(60)

If we apply (59), (60), we get the  $(\hat{w}, \tilde{w})$  equations:

$$\hat{w}_t(t,r,\theta) = \frac{\varepsilon}{r} (r\hat{w}_r)_r + \frac{\varepsilon}{r^2} \hat{w}_{\theta\theta} + \int_{-\pi}^{\pi} \bar{F}(R,r,\theta,\phi) \tilde{w}_r(R,\phi) d\phi \qquad (61)$$

$$\tilde{w}_t(t,r,\theta) = \frac{\varepsilon}{r} (r\tilde{w}_r)_r + \frac{\varepsilon}{r^2} \tilde{w}_{\theta\theta}$$
(62)

$$\hat{w}(t, R, \theta) = \tilde{w}(t, R, \theta) = 0$$
(63)

where a new operator  $\overline{F}$  appears due to the combination of the controller and observer transformations. This operator is defined as

$$\bar{F}(R, r, \theta, \phi) = q_1(R, r, \theta - \phi) - \int_0^r \int_{-\pi}^{\pi} K(r, \rho, \theta - \psi) q_1(R, \rho, \psi - \phi) \, d\rho \, d\psi.$$
(64)

While  $H^2$  stability of the origin for  $\tilde{w}$  follows by Proposition 2, the stability of the cascade system is slightly more involved. We obtain the next result.

Proposition 3: Consider the system (61)–(63) with initial conditions  $\hat{w}_0(r,\theta)$  and  $\tilde{w}_0(r,\theta)$ . Then there exist  $D, \alpha > 0$  such that if  $\hat{w}_0, \tilde{w}_0 \in H^2$  and  $\hat{w}_0(R,\theta) = 0, \tilde{w}_0(R,\theta) = 0$ , then  $\hat{w}, \tilde{w} \in \mathcal{C}([0,\infty), H^2)$  and  $\|\hat{w}(t,\cdot)\|_{H^2} + \|\tilde{w}(t,\cdot)\|_{H^2} \leq De^{-\alpha t}(\|\hat{w}_0\|_{H^2} + \|\tilde{w}_0\|_{H^2}).$ 

*Proof:* Due to the presence of  $\tilde{w}_r(R)$  in (61), one needs the  $H^1$  norm of  $\tilde{w}$  to deduce the  $L^2$  stability of  $\hat{w}$ , so we begin with the  $H^1$  analysis. Take

$$V_{1} = \frac{1}{2} \|\hat{w}\|_{L^{2}}^{2} + \frac{1}{2} \|\tilde{w}\|_{L^{2}}^{2} + \frac{1}{2} \|\hat{w}_{r}\|_{L^{2}}^{2} + \frac{1}{2} \left\|\frac{\hat{w}_{\theta}}{r^{2}}\right\|_{L^{2}}^{2} + \frac{A_{1}}{2} \|\tilde{w}_{r}\|_{L^{2}}^{2} + \frac{A_{1}}{2} \left\|\frac{\tilde{w}_{\theta}}{r^{2}}\right\|_{L^{2}}^{2}$$
(65)

where  $V_1$  is an equivalent definition of  $\|\hat{w}\|_{H^1} + \|\tilde{w}\|_{H^1}$ . By operating and integrating by parts, we find

$$\dot{V}_{1} = -\varepsilon_{R} \int_{0}^{R} \int_{-\pi}^{\pi} \left( |\tilde{w}_{r}|^{2} + \frac{1}{r^{2}} |\tilde{w}_{\theta}|^{2} \right) r \, d\theta \, dr$$

$$-\varepsilon_{R} \int_{0}^{R} \int_{-\pi}^{\pi} \left( |\hat{w}_{r}|^{2} + \frac{1}{r^{2}} |\hat{w}_{\theta}|^{2} \right) r \, d\theta \, dr$$

$$-A_{1} \varepsilon_{R} \int_{0}^{R} \int_{-\pi}^{\pi} |\Delta \tilde{w}|^{2} r \, d\theta \, dr$$

$$-\varepsilon_{R} \int_{0}^{R} \int_{-\pi}^{\pi} |\Delta \hat{w}|^{2} r \, d\theta \, dr$$

$$+ \operatorname{Re} \left\{ \int_{0}^{R} \int_{-\pi}^{\pi} (\hat{w}^{*}(r,\theta) - \Delta \hat{w}^{*}(r,\theta)) r \right\}$$

$$\times \int_{-\pi}^{\pi} \bar{F}(R, r, \theta, \phi) \tilde{w}_{r}(R, \phi) \, d\phi \, d\theta \, dr \right\}$$
(66)

By using the Cauchy–Schwarz and Young inequalities, and Lemma 8 in the Appendix, we obtain

$$\begin{split} \dot{V}_{1} &\leq -\varepsilon_{R} \|D\tilde{w}\|_{L^{2}}^{2} - \varepsilon_{R} \|D\hat{w}\|_{L^{2}}^{2} - A_{1}\varepsilon_{R} \|\Delta\tilde{w}\|_{L^{2}}^{2} \\ &\quad -\varepsilon_{R} \|\Delta\tilde{w}\|_{L^{2}}^{2} \\ &\quad + \frac{1}{2\gamma_{1}} \int_{0}^{R} \int_{-\pi}^{\pi} \left( |\hat{w}|^{2} + \Delta\hat{w}|^{2} \right) r \, d\theta \, dr \\ &\quad + \frac{\gamma_{1}}{2} \int_{0}^{R} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \bar{F}(R, r, \theta, \phi) \tilde{w}_{r}(R, \phi) d\phi \right|^{2} r \, d\theta \, dr \\ &\leq -\varepsilon_{R} \|D\tilde{w}\|_{L^{2}}^{2} - \varepsilon_{R} \|D\hat{w}\|_{L^{2}}^{2} - A_{1}\varepsilon_{R} \|\Delta\tilde{w}\|_{L^{2}}^{2} \\ &\quad - \varepsilon_{R} \|\Delta\hat{w}\|_{L^{2}}^{2} + \frac{1}{2\gamma_{1}} \left( \|\hat{w}\|_{L^{2}}^{2} + \|\Delta\hat{w}\|_{L^{2}}^{2} \right) \\ &\quad + \gamma_{1}C_{5} \int_{0}^{R} \int_{-\pi}^{\pi} |\tilde{w}_{r}(R, \theta)|^{2} \, d\theta \, dr \end{split}$$
(67)

where D denotes the gradient operator, and

$$\gamma_1 = \frac{1+4R^2}{\varepsilon_R}, \quad A_1 = \frac{2C_5}{\varepsilon_R^2}(1+4R^2).$$
 (68)

On the other hand, given the boundary condition of  $\tilde{w}$ , we have the following result:

$$\int_{-\pi}^{\pi} |\tilde{w}_r(R,\theta)|^2 d\theta = \|\Delta \tilde{w}\|_{L^2}^2 - \|D^2 \tilde{w}\|_{L^2}^2$$
(69)

which combined with Poincare's inequality allows to rewrite (67) as

$$\dot{V}_{1} \leq -\frac{\varepsilon_{R}}{8R^{2}} \|\tilde{w}\|_{L^{2}}^{2} - \left(\frac{\varepsilon_{R}}{8R^{2}} - \frac{1}{2\gamma_{1}}\right) \|\hat{w}\|_{L^{2}}^{2} - \frac{\varepsilon_{R}}{2} \|D\tilde{w}\|_{L^{2}}^{2} - \frac{\varepsilon_{R}}{2} \|D\tilde{w}\|_{L^{2}}^{2} - (A_{1}\varepsilon_{R} - \gamma_{1}C_{5}R) \|\Delta\tilde{w}\|_{L^{2}}^{2} - \gamma_{1}C_{5}R \|D^{2}\tilde{w}\|_{L^{2}}^{2} - \left(\varepsilon_{R} - \frac{1}{2\gamma_{1}}\right) \|\Delta\hat{w}\|_{L^{2}}^{2} \leq -\alpha_{3}V_{1}.$$
(70)

Take now  $V_2$ , equivalent to the  $H^2$  norm, as

$$V_2 = V_1 + \frac{A_2}{2} \| \triangle \tilde{w} \|_{L^2}^2 + \frac{1}{2} \| \triangle \hat{w} \|_{L^2}^2.$$
(71)

By Cauchy-Schwarz's, Young's, and Poincare's inequalities, we find

$$\begin{split} \dot{V}_{2} &\leq \dot{V}_{1} - A_{2} \varepsilon_{R} \|D(\bigtriangleup \tilde{w})\|_{L^{2}}^{2} - \varepsilon_{R} \|D(\bigtriangleup \hat{w})\|_{L^{2}}^{2} \\ &+ \frac{1}{2\gamma_{2}} \int_{0}^{R} \int_{-\pi}^{\pi} |(\bigtriangleup \hat{w})_{r}|^{2} r \, d\theta \, dr \\ &+ \frac{\gamma_{2}}{2} \int_{0}^{R} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \bar{F}_{r}(R, r, \theta, \phi) \tilde{w}_{r}(R, \phi) d\phi \right|^{2} d\theta \, dr \\ &+ \frac{1}{2\gamma_{2}} \int_{0}^{R} \int_{-\pi}^{\pi} |\bigtriangleup \hat{w}_{\theta}|^{2} \frac{1}{r} \, d\theta \, dr \\ &+ \frac{\gamma_{2}}{2} \int_{0}^{R} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \bar{F}_{\theta}(R, r, \theta, \phi) \tilde{w}_{r}(R, \phi) d\phi \right|^{2} d\theta \, dr. \end{split}$$
(72)

By using Lemma 8, we obtain

$$\dot{V}_{2} \leq -\frac{\varepsilon_{R}}{8R^{2}} \|\tilde{w}\|_{L^{2}}^{2} - \left(\frac{\varepsilon_{R}}{8R^{2}} - \frac{1}{2\gamma_{1}}\right) \|\hat{w}\|_{L^{2}}^{2} - \frac{\varepsilon_{R}}{2} \|D\tilde{w}\|_{L^{2}}^{2} 
- \left(\varepsilon_{R} - \frac{1}{2\gamma_{2}}\right) \|D(\Delta\hat{w})\|_{L^{2}}^{2} - \frac{\varepsilon_{R}}{2} \|D\hat{w}\|_{L^{2}}^{2} 
- \left(A_{1}\varepsilon_{R} - \left(\gamma_{1}C_{5} + \gamma_{2}\frac{C_{6}}{2}\right)R\right) \|\Delta\tilde{w}\|_{L^{2}}^{2} 
- A_{2}\varepsilon_{R} \|(\Delta\tilde{w})_{r}\|_{L^{2}}^{2} - \left(\varepsilon_{R} - \frac{1}{2\gamma_{1}}\right) \|\Delta\hat{w}\|_{L^{2}}^{2} 
- \left(A_{2}\varepsilon_{R} - \frac{\gamma_{2}}{2}(C_{7} + C_{8})R^{3}\right) \left\|\frac{\Delta\tilde{w}_{\theta}}{r^{2}}\right\|_{L^{2}}^{2} 
\leq -\alpha_{4}V_{2}$$
(73)

where we now take

$$\gamma_1 = \frac{1+4R^2}{\varepsilon_R}, \quad A_1 = \frac{2}{\varepsilon_R^2} \left( (1+4R^2)C_5 + C_6 \right),$$
  
$$\gamma_2 = \frac{1}{\varepsilon_R}, \qquad A_2 = \frac{R^3}{\varepsilon_R^2} (C_7 + C_8).$$

Here,  $C_5$ ,  $C_6$ ,  $C_7$ , and  $C_8$  follow the definition of Lemma 8. Thus, the proposition is proved. Since the transformations (59) and (60) are invertible, we reach the final result by combining Propositions 1 and 3.

Theorem 2: Consider the system (1) and (3), the observer (57) and boundary controller (58) with initial conditions  $u_0(r,\theta)$ ,  $\hat{u}_0(r,\theta)$ . Then there exist  $D, \alpha > 0$  such that if  $u_0, \hat{u}_0 \in H^2$  and satisfy the compatibility conditions, i.e., (44) is satisfied and  $\hat{u}_0(R,\theta) = u_0(R,\theta)$ , then  $u, \hat{u} \in \mathcal{C}([0, \infty), H^2)$  and  $||u(t, \cdot)||_{H^2} + ||\hat{u}(t, \cdot)||_{H^2} \leq De^{-\alpha t}(||u_0||_{H^2} + ||\hat{u}_0||_{H^2})$ .

Similar results hold for the real-valued system  $z(t, r, \theta)$ .

### VI. NUMERICAL SIMULATIONS FOR DISCRETIZED AGENT CONTROL LAWS

#### A. Discretized Agent Control Laws

To apply the feedback laws to a finite number of agents, we discretize the PDE model (1) and (2) in space by using the discretized grid defined in [48], namely

$$r_i = (i - \frac{1}{2})h_r, \quad \theta_j = (j - 1)h_\theta$$
 (74)

where  $h_r = R/(M - (1/2))$ ,  $h_\theta = 2\pi/(N - 1)$ , and  $i = 1, \dots, M$ ,  $j = 1, \dots, N$ . The grid points are half-integer in the radial direction to avoid the singularity at the disk center, while the boundary is defined on the grid points. Using a three-point central difference approximation,<sup>2</sup> we obtain, for  $i = 2, 3 \dots, M - 1, j = 1, 2, \dots, N - 1$ 

$$\dot{u}_{i,j} = \varepsilon \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_r^2} + \frac{\varepsilon}{r_i} \frac{u_{i+1,j} - u_{i-1,j}}{2h_r} + \frac{\varepsilon}{r_i^2} \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_\theta^2} + \lambda u_{i,j},$$
(75)  
$$\dot{z}_{i,j} = \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{h_\theta^2} + \frac{1}{2} \frac{z_{i+1,j} - z_{i-1,j}}{2h_r}$$

$$h_{i,j} = \frac{h_r^2}{h_r^2} + \frac{1}{r_i} \frac{2h_r}{2h_r} + \frac{1}{r_i^2} \frac{z_{i,j+1} - 2z_{i,j} + z_{i,j-1}}{h_{\theta}^2} + \mu z_{i,j}.$$
(76)

All variables are  $2\pi$  periodic in  $\theta$ , which gives  $u_{i,1} = u_{i,N}$  and  $z_{i,1} = z_{i,N}$ . We set a virtual grid point at the center with index i = 0. Since  $r_1 = h_r/2$ , the coefficient of  $u_{0,j}$  and  $z_{0,j}$  can be eliminated, thus obtaining at i = 1

$$\dot{u}_{1,j} = 2\varepsilon \frac{u_{2,j} - u_{1,j}}{h_r^2} + \frac{\varepsilon}{r_1^2} \frac{u_{1,j+1} - 2u_{1,j} + u_{1,j-1}}{h_\theta^2} + \lambda u_{1,j}.$$
 (77)

Similar formulas can be obtained for z.

For the leader agents, whose controllers contain the desired deployment information and state feedback

$$u_{M,j} = \bar{u}_{M,j} + \sum_{m=1}^{M-1} \sum_{l=1}^{N} h_r h_\theta a_{m,l} K_{j,m,l} (u_{m,l} - \bar{u}_{m,l}) + F_M (u_{M,j} - \bar{u}_{M,j}),$$
(78)  
$$z_{M,j} = \bar{z}_{M,j} + \sum_{m=1}^{M-1} \sum_{l=1}^{N} h_r h_\theta a_{m,l} K_{j,m,l}^z (z_{m,l} - \bar{z}_{m,l}) + F_M^z (z_{M,j} - \bar{z}_{M,j})$$
(79)

<sup>2</sup>A more flexible approach based on multi-indexes can be found in [18], which increases the possible underlying graph topologies. In this work, we limit ourselves to three-point central differences for the sake of simplicity, to obtain a discrete model with a minimal number of neighbors.

where  $K_{j,m,l} = F_m P_{j,m,l}$  and  $K_{j,m,l}^z = F_m^z P_{j,m,l}$  are the discretized control kernel for u and z, respectively, with  $F_m = -(\lambda/\varepsilon)r_m(\mathbf{I}_1[\sqrt{(\lambda/\varepsilon)(R^2 - r_m^2)}]/\sqrt{(\lambda/\varepsilon)(R^2 - r_m^2)}), F_m^z = -\mu r_m(\mathbf{I}_1[\sqrt{\mu(R^2 - r_m^2)}]/\sqrt{\mu(R^2 - r_m^2)})$ , and  $P_{j,m,l} = 1/(2\pi/(R^2 - r_m^2R^2 + r_m^2 - 2Rr_m\cos(\theta_j - \theta_l)))$ . Here we use the property of Poisson Kernel to get the last terms of (78) and (79). The coefficients  $a_{m,l}$  are determined by the Simpson's rules of numerical integration

$$\begin{split} a_{m,l} &= \frac{2}{9}, \{m = 1\} \cap \{l = 1, 3, \cdots, N - 2\}; \\ a_{m,l} &= \frac{8}{9}, \{m = 2, 4, \cdots, M - 1\} \cap \{l = 1, 3, \cdots, N - 2\} \\ &\cup \{m = 3, 5, \cdots, M - 2\} \cap \{l = 2, 4, \cdots, N - 1\}; \\ a_{m,l} &= \frac{4}{9}, \{m = 1\} \cap \{l = 2, 4, \cdots, N - 1\} \\ &\cup \{m = 3, 5, \cdots, M - 2\} \cap \{l = 1, 3, \cdots, N - 2\}; \\ a_{m,l} &= \frac{16}{9}, \{m = 2, 4, \cdots, M - 1\} \cap \{l = 2, 4, \cdots, N - 1\} \end{split}$$

where M and N must be chosen as odd numbers according to Simpson's rules.

The observer can be discretized in a similar manner if output feedback is employed. In that case, u in (78) and z in (79) must be replaced by their respective state estimates. The approximating error due to the discretization is  $O(h_r^2 + h_{\theta}^2)$  which depends on the spatial discretization method [49]. The PDEbased method is most suitable to analyze large scale systems, as larger numbers of agents greatly decrease the error. Higher order (such as five-point central difference) approximations will reduce the error at the cost of increasing the complexity of both the communication network and the controller. To avoid this complication, in our simulations we employ a method based on Fourier series expansions which greatly reduces the error. In this way, we calculate the harmonics  $u_n$  which only need discretization in the radial direction, and then sum a finite number S of harmonics to recover u

$$u_{i,j} = \sum_{n=1}^{S} u_{ni} \mathrm{e}^{\mathrm{j}n\theta_j} \tag{80}$$

where S > 0 is a large integer. The error caused by using a finite number of harmonics is much smaller than the azimuthal discretization error.

#### B. Numerical Simulations

A variety of deployment manifolds are simulated to demonstrate the effectiveness of the feedback controller. We also show how the agents transition smoothly between different deployment references. The smoothness is guaranteed by the  $H^2$  closed-loop stability of the system.

The simulation consists on a grid of  $79 \times 81$  agents in 3-D space with parameters  $\varepsilon = 1$ ,  $\lambda = 100$ ,  $\mu = 15$ , R = 1, and C = 1, transitioning along a sequence of deployment patterns. A video of the simulation can be downloaded from the Web [50]. For the readers' convenience, we show several snapshots of the simulation in Figs. 4 and 5.

The deployment profiles are ordered following the sequence shown in Fig. 2, i.e., from circle, to asteroid, to quadrifolium, to deltoid, to polar rose(3-petal), to epicycloid(3-petal), and



Fig. 4. Agents' deployment snapshots. The beginning (a) and intermediate (b) stages of the transition from asteroid pattern to quadrifolium. (c) The formation settles to the quadrifolium pattern. The beginning (d) and intermediate (e) stages of the transition from quadrifolium pattern to deltoid. (f) The formation settles to deltoid pattern. For each figure, the upper-left corner displays the reference boundary shape and the upper-right corner shows the leaders actuation.



Fig. 5. Agents deployment snapshots, continued from Fig. 4. The beginning (g) and intermediate (h) stages of the transition from deltoid pattern to 3-petal polar rose. (i) The formation settles to 3-petal polar rose pattern. The beginning (j) and intermediate (k) stages of the transition from 3-petal polar rose pattern to 3-petal epicycloid. (l) The formation settles to 3-petal epicycloid pattern.

finally to epicycloid(4-petal). To avoid large transients we use continuous, slowly-varying reference trajectories connecting consecutive deployment pattern; the transitions are also illustrated in the figures.

Figs. 4 and 5 depict four groups of transitions between two different deployment patterns, where (a–c) correspond to the transition from asteroid to quadrifolium, (d–f) to the transition from quadrifolium to deltoid, (g–i) to the transition from deltoid to 3-petal polar rose, and (j–l) to the transition from 3-petal polar rose to 3-petal epicycloid. The insets of Figs. 4 and 5 display the reference boundary curves  $\bar{U}(\theta)$  in (30) that the leaders should track (on the upper-left corner) and the actual shapes that the leaders form (on the upper-right corner) to con-

trol the followers, respectively. Since the reference boundary is also dynamic in time, the upper-left inset in each figure shows different curves representing different evolution stages.

In general, it can be seen that, at the beginning of each transition, the reference changes a little, but it leads to a dramatic change in the boundary formation (actuation). This is due to the discrepancy between the desired formation and the actual deployment, with the last two terms of (30) being nonzero. Since the system behaves as a diffusion process, the agents closest in the communication layer to the leaders are the ones to respond first, followed by the next layer, and so on. Thus, the reference signals propagate from boundary to center, eventually enabling all agents to reach the desired deployment.



Fig. 6. Agents tracking error and control effort during the simulation. (a) Tracking error between the actual formation and the reference manifold. (b) Control effort.



Fig. 7. Agents rendezvous at the origin using the output-feedback control law. (a) Tracking error. (b) Control effort (c) Observer error. A semi-log plot is used for clarity; note the different time scale in (c).

In the intermediate stage, with the reference still varying gradually, the leaders' tracking error is still increasing until the reference stops changing. Then, as the reference is finally kept constant, the tracking error decreases and at last the agents (and in particular the leaders) converge to the reference shape (see (c), (f), (i), and (l) in Figs. 4 and 5).

We plot the time evolution of the  $L^2$  norm of the tracking error between the reference deployment manifold and the actual formation in Fig. 6(a), which shows sudden increases in error at transitions and fast convergence of the error to zero when the reference stops changing. The agents' tracking error at different layers, namely r = 0.005, r = 0.5, and r = 0.88, is shown, as well as the tracking error of the leaders and the average error for all the agents. The error at the innermost layer is smallest because changes in reference position are minimal at r = 0.005. On the other hand, the average error for all agents is smaller than the leaders' due to their control role. Fig. 6(b) gives the control effort exerted by the agents. The average control effort from all the agents (excluding the leaders) is computed as  $\|(\dot{x}, \dot{y}, \dot{z})\|_{L^2}$ , while for a layer  $(i = 1, \dots, M - 1)$  at r =(2i-1)R/(2M-1) is given by  $\|(\dot{x}_i, \dot{y}_i, \dot{z}_i)\|_{L^2}$ . The leaders' effort  $||(x_M, y_M, z_M)||_{L^2}$  is different from the followers' in the sense that their position is directly controlled. Notice how the error and control effort vary with the reference trajectory; in particular, the last reference transitions result in rather smooth and mild transients, and the corresponding tracking error and control effort are comparatively small.

To test the output-feedback control, we provide an additional simulation of a rendezvous scenario at the origin with the same PDE parameters. The agents start at random, normally distributed positions, that follow a Gaussian distribution with zero mean and  $\sigma^2 = 0.3$ . The observer's initial condition is set as the agents' actual position plus a Gaussian distributed error, with zero mean and  $\sigma^2 = 0.2$ . Fig. 7(a) shows the  $L^2$  norm of the tracking error, while Fig. 7(b) and (c) depicts, respectively, the control effort and the observer error. These quantities are shown for all agents, for the leaders, and for the agents of three selected layers. It is clear that the observer converges first, taking about one second, and during this period the trajectory is oscillatory. After that, the behavior is analogous to the full-state case and the agents quickly converge to the origin.

# VII. CONCLUSION

In this paper, we have introduced a distributed cooperative deployment framework for multi-agent deployment control in 3-D space on a 2-D lattice neighborhood topology, by using a PDE-based method.

The agents' communication graph is directly determined from the discretization of the spatial derivatives of the PDEs, resulting in a distributed scheme in which each agent requires only local neighbor-to-neighbor information exchange. Given practical limitations in communication range and bandwidth, this framework is very suitable for large-scale agent deployment.

The merit of our framework is that it allows the exploitation of powerful tools from the field of PDE control. We begin by finding a family of explicit deployment profiles, found from the PDEs' equilibrium. Since these are potentially unstable, we introduce a new class of (also explicit) backstepping boundary control laws that solve the stabilization problem on a disk topology (a previously open and challenging problem of PDE stabilization). In addition, an observer (whose design is dual to that of the controller) is formulated, allowing to obtain an all-explicit output feedback law requiring as sole measurement the leaders' neighbor positions. Our paradigm also allows for smooth transitions between different deployment manifolds by only adjusting a few terms in the leaders' control law.

A simulation study shows how the discretized dynamics inherit the properties of the continuous PDE model. In particular, we observe that the agents' collective dynamics behave as a diffusion process, with leaders' motion propagating among the followers by proximity. Hence, the agents gradually form the desired deployment manifold from the boundary to the topological center.

Further research includes the extension of this paradigm to a movable formation driven by an anchor (fixed agent) in the topological center. Another possible extension is considering a communication topology in 3D (a *blob* of agents instead of a lattice); in particular, if the agents communicate according to a *sphere* topology, the methodology of the paper can be applied (using Spherical Harmonics instead of Fourier Series), obtaining similar deployment laws. One intriguing line of research would be to consider Neumann or Robin boundary conditions in the PDE model to increase the range of feasible 3-D deployment profiles; this would amount to treating the leaders as virtual agents. In addition, some practical issues need to be addressed, for instance how to deal with problems caused by agents' faults, and obstacle and collision avoidance.

# APPENDIX TECHNICAL LEMMAS

To prove Proposition 1, we state and prove Lemmas 1–4. Lemma 1: For  $n_1, n_2 > 0$ , it holds that

$$\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} P(r,\rho,\theta-\psi) f(\rho,\psi) d\psi \right|^2 d\theta \le \int_{-\pi}^{\pi} |f(\rho,\theta)|^2 d\theta, \quad (81)$$

$$\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} P(r,\rho,\theta-\psi) \cos^{n_1}(\theta-\psi) \sin^{n_2}(\theta-\psi) \right|^2 d\theta, \quad (82)$$

$$f(\rho,\psi) d\psi|^2 d\theta \le \int_{-\pi}^{\pi} |f(\rho,\theta)|^2 d\theta.$$

Proof: Using Fourier series, and remembering that

$$P(r,\rho,\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=\infty} \left(\frac{\rho}{r}\right)^{|n|} e^{jn\theta}, \tag{83}$$

$$f(\rho,\theta) = \sum_{n=-\infty} f_n(\rho) e^{jn\theta}$$
(84)

we get

$$\int_{-\pi}^{\pi} P(r,\rho,\theta-\psi)f(\rho,\psi)d\psi$$

$$=\frac{1}{2\pi}\sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty}\left(\frac{\rho}{r}\right)^{|n|}f_{m}(\rho)\int_{-\pi}^{\pi}e^{jn(\theta-\psi)}e^{jm\psi}d\psi$$

$$=\sum_{n=-\infty}^{n=\infty}\left(\frac{\rho}{r}\right)^{|n|}f_{n}(\rho)e^{jn\theta}.$$
(85)

Here, we have used the orthogonality property of Fourier series (the same conclusion is reached using the convolution theorem). Now, by Parseval's theorem

$$\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} P(r,\rho,\theta-\psi) f(\rho,\psi) d\psi \right|^2 d\theta$$
$$= 2\pi \sum_{n=-\infty}^{n=\infty} \left| \left(\frac{\rho}{r}\right)^{|n|} f_n(\rho) \right|^2$$
$$\leq 2\pi \sum_{n=-\infty}^{n=\infty} |f_n(\rho)|^2 = \int_{-\pi}^{\pi} |f(\rho,\theta)|^2 d\theta.$$
(86)

Similarly, (82) can be proved by showing that the Fourier coefficients of  $P(r, \rho, \theta) \cos^{n_1}(\theta) \sin^{n_2}(\theta)$  are less than or equal to one, which we omit due to limited space.

*Lemma* 2: Call  $\hat{P} = P(r, \rho, \theta - \psi)(\rho/r)^{n_1} \cos(\theta - \psi)^{n_2} \sin(\theta - \psi)^{n_3}$  for any integer  $n_1, n_2, n_3 \ge 0$ . Then, if

$$g(r,\theta) = \int_{0}^{r} \int_{-\pi}^{\pi} \hat{F}(r,\rho,\theta) \hat{P}(r,\rho,\theta-\psi) f(\rho,\psi) \, d\psi \, d\rho \quad (87)$$

where  $\hat{F}(r,\rho,\theta) \in \mathcal{C}^1(\mathcal{T}) \times \mathcal{C}^1([-\pi,\pi])$  and  $\hat{F}(r,0,\theta) = 0$ , then

$$g_x = \int_{0}^{r} \int_{-\pi}^{\pi} \left( (\hat{F}_r + \hat{F}_\rho \frac{\rho}{r} + \frac{\hat{F}}{r}) \cos \theta - \frac{\sin \theta}{r} \hat{F}_\theta \right) \\ \times \hat{P}(r, \rho, \theta - \psi) f(\rho, \psi) \, d\psi \, d\rho \\ + \int_{0}^{r} \int_{-\pi}^{\pi} \frac{\rho}{r} \hat{F} \hat{P}(r, \rho, \theta - \psi) \\ \times \left[ \cos(\theta - \psi) f_x(\rho, \psi) - \sin(\theta - \psi) f_y(\rho, \psi) \right] d\psi \, d\rho.$$
(88)

Proof: In polar coordinates

$$g_{x} = \cos\theta \frac{\partial g}{\partial r} - \frac{\sin\theta}{r} \frac{\partial g}{\partial \theta}$$
  
=  $\cos\theta \hat{F}(r, r, \theta) f(r, \theta) \chi(n_{3})$   
+  $\int_{0}^{r} \int_{-\pi}^{\pi} \left(\hat{F}_{r} \cos\theta - \frac{\sin\theta}{r} \hat{F}_{\theta}\right) \hat{P}(r, \rho, \theta - \psi) f(\rho, \psi) d\psi d\rho$   
+  $\int_{0}^{r} \int_{-\pi}^{\pi} \hat{F}(r, \rho, \theta) \cos\theta \hat{P}_{r}(r, \rho, \theta - \psi) f(\rho, \psi) d\psi d\rho$   
-  $\int_{0}^{r} \int_{-\pi}^{\pi} \hat{F}(r, \rho, \theta) \sin\theta r \hat{P}_{\theta}(r, \rho, \theta - \psi) f(\rho, \psi) d\psi d\rho$  (89)

where  $\chi(n_3) = 1$  if  $n_3 = 0$ , and zero otherwise. Noticing that

$$\frac{\partial}{\partial r}\hat{P}(r,\rho,\theta-\psi) = -\frac{\rho}{r}\frac{\partial}{\partial \rho}\hat{P}(r,\rho,\theta-\psi),\qquad(90)$$

$$\frac{\partial}{\partial\theta}\hat{P}(r,\rho,\theta-\psi) = -\frac{\partial}{\partial\psi}\hat{P}(r,\rho,\theta-\psi)$$
(91)

and, integrating by parts the last two lines of (89) and expanding  $\cos \theta = \cos(\theta - \psi + \psi)$  and  $\sin \theta = \sin(\theta - \psi + \psi)$  as sums,

we rewrite the last two lines of (89) as

$$\int_{0}^{r} \int_{-\pi}^{\pi} \hat{F}(r,\rho,\theta) \frac{\rho}{r} \cos(\theta-\psi) \hat{P}(r,\rho,\theta-\psi) \\ \left(\cos(\psi) f_r(\rho,\psi) - \sin\psi \frac{f_{\theta}(\rho,\psi)}{\rho}\right) d\psi d\rho \\ - \int_{0}^{r} \int_{-\pi}^{\pi} \hat{F}(r,\rho,\theta) \frac{\rho}{r} \sin(\theta-\psi) \hat{P}(r,\rho,\theta-\psi) \\ \left(\sin\psi f_r(\rho,\psi) + \cos(\psi) \frac{f_{\theta}(\rho,\psi)}{\rho}\right) d\psi d\rho \\ = \int_{0}^{r} \int_{-\pi}^{\pi} \hat{F}(r,\rho,\theta) \frac{\rho}{r} \hat{P}(r,\rho,\theta-\psi) \\ \left[\cos(\theta-\psi) f_x(\rho,\psi) - \sin(\theta-\psi) f_y(\rho,\psi)\right] d\psi d\rho.$$
(92)

Lemma 3: If

$$g(r,\theta) = f(r,\theta) + \int_{0}^{r} \int_{-\pi}^{\pi} \rho F(r,\rho) P(r,\rho,\theta-\psi) f(\rho,\psi) \, d\psi \, d\rho$$
(93)

where  $F(r, \rho) \in \mathcal{C}^2(\mathcal{T})$ , then

where

$$H_1(r,\rho) = 2F(r,\rho)\frac{\rho}{r} + \rho F_\rho(r,\rho)\frac{\rho}{r} + \rho F_r(r,\rho),$$
  

$$H_2(r,\rho) = \left(\partial_r + \rho r \partial_\rho + \frac{1}{r}\right)H_1(r,\rho),$$
  

$$H_3(r,\theta,\psi) = \cos(\theta - \psi)f_x(r,\psi) - \sin(\theta - \psi)f_y(r,\psi)$$

*Proof:* The proof is carried out by applying Lemma 2. Let  $H_4(r, \theta, \psi) = \sin(\theta - \psi) f_x(r, \psi) + \cos(\theta - \psi) f_y(r, \psi)$ . Then we have

$$g_{xx} = f_{xx} + \int_{-\pi}^{\pi} \cos^2 \theta H_1(r,r) f(r,\psi) P(r,r,\theta-\psi) d\psi$$
$$+ \int_{-\pi}^{\pi} \cos \theta F(r,r) H_3(r,\theta,\psi) P(r,r,\theta-\psi) r d\psi$$
$$+ \cos^2 \theta \int_{0}^{r} \int_{-\pi}^{\pi} (H_1(r,\rho))_r P f(\rho,\psi) d\psi d\rho$$

$$+\frac{\sin^{2}\theta}{r}\int_{0}^{r}\int_{-\pi}^{\pi}H_{1}Pf(\rho,\psi)\,d\psi\,d\rho$$

$$+\cos\theta\int_{0}^{r}\int_{-\pi}^{\pi}(\frac{\rho^{2}F}{r})_{r}PH_{3}\,d\psi\,d\rho$$

$$+\frac{\sin\theta}{r}\int_{0}^{r}\int_{-\pi}^{\pi}\frac{\rho^{2}F(r,\rho)}{r}PH_{4}\,d\psi\,d\rho$$

$$+\int_{0}^{r}\int_{-\pi}^{\pi}\left(\cos^{2}\theta H_{1}f(\rho,\psi)+\frac{\cos\theta\rho^{2}}{r}F(r,\rho)H_{3}\right)P_{r}\,d\psi\,d\rho$$
(95)
$$-\int_{0}^{r}\int_{-\pi}^{\pi}\left(\frac{\sin\theta\cos\theta H_{1}f(\rho,\psi)}{r}+\frac{\rho^{2}\sin\theta F(r,\rho)H_{3}}{r^{2}}\right)$$

$$\times P_{\theta}\,d\psi\,d\rho.$$
(96)

Analogous to the derivation process of Lemma 2, we rewrite (95) as follows integrating by parts:

$$-\cos^{2}\theta \int_{-\pi}^{\pi} H_{1}(r,r)P(r,r,\theta-\psi)f(r,\psi) d\psi$$

$$-\cos\theta \int_{-\pi}^{\pi} rF(r,r)P(r,r,\theta-\psi)H_{3}(r,\cdot) d\psi$$

$$+\cos^{2}\theta \int_{0}^{r} \int_{-\pi}^{\pi} \left(\frac{\rho}{r}H_{1}(r,\rho)\right)_{\rho}Pf(\rho,\psi) d\psi d\rho$$

$$+\cos^{2}\theta \int_{0}^{r} \int_{-\pi}^{\pi} \frac{\rho}{r}H_{1}(r,\rho)Pf_{\rho}(\rho,\psi) d\psi d\rho$$

$$+\cos\theta \int_{0}^{r} \int_{-\pi}^{\pi} \left(\frac{\rho^{3}}{r^{2}}F\right)_{\rho}PH_{3}(\rho,\cdot) d\psi d\rho$$

$$+\cos\theta \int_{0}^{r} \int_{-\pi}^{\pi} \frac{\rho^{3}}{r^{2}}FPH_{3\rho}(\rho,\cdot) d\psi d\rho \qquad (97)$$

Also, (96) can be rewritten by use of (91) and integration by parts as

$$-\sin\theta\cos\theta r \int_{0}^{r} \int_{-\pi}^{\pi} H_1 P f_{\psi}(\rho,\psi) d\psi d\rho$$
$$-\frac{\sin\theta}{r} \int_{0}^{r} \int_{-\pi}^{\pi} \frac{\rho^2}{r} F(r,\rho) P H_{3\psi}(\rho,\cdot) d\psi d\rho.$$
(98)

Expanding  $\cos \theta$  and  $\sin \theta$  as in the proof of Lemma 2 and substitute them in  $H_{3\rho}$  and  $H_{3\psi}$ 

$$\cos\theta \int_{0}^{r} \int_{-\pi}^{\pi} \frac{\rho^{3}}{r^{2}} FP \cos(\theta - \psi) f_{x\rho}(\rho, \psi) \, d\psi \, d\rho$$
$$-\cos\theta \int_{0}^{r} \int_{-\pi}^{\pi} \frac{\rho^{3}}{r^{2}} FP \sin(\theta - \psi) f_{y\rho}(\rho, \psi) \, d\psi \, d\rho$$

$$-\frac{\sin\theta}{r}\int_{0}^{r}\int_{-\pi}^{\pi}\frac{\rho^{2}}{r}FP\cos(\theta-\psi)f_{x\psi}(\rho,\psi)\,d\psi\,d\rho$$
$$-\frac{\sin\theta}{r}\int_{0}^{r}\int_{-\pi}^{\pi}\frac{\rho^{2}}{r}FP\sin(\theta-\psi)f_{y\psi}(\rho,\psi)\,d\psi\,d\rho$$
$$=\int_{0}^{r}\int_{-\pi}^{\pi}\frac{\rho^{2}}{r^{2}}FP\left[\cos^{2}(\theta-\psi)f_{xx}\right]$$
$$-2\sin(\theta-\psi)\cos(\theta-\psi)f_{xy}+\sin^{2}(\theta-\psi)f_{yy}d\psi\,d\rho. \tag{99}$$

Similarly,

$$\cos^{2}\theta \int_{0}^{r} \int_{-\pi}^{\pi} \frac{\rho}{r} H_{1}Pf_{\rho}(\rho,\psi) d\psi d\rho$$
  
$$- \frac{\sin\theta\cos\theta}{r} \int_{0}^{r} \int_{-\pi}^{\pi} H_{1}Pf_{\psi}(\rho,\psi) d\psi d\rho$$
  
$$= \cos\theta \int_{0}^{r} \int_{-\pi}^{\pi} \rho r H_{1}P \left[\cos(\theta-\psi)f_{x}(\rho,\psi)\right]$$
  
$$- \sin(\theta-\psi)f_{y}(\rho,\psi) d\psi d\rho.$$
(100)

Substituting in (96) we find the expression of  $g_{xx}$ .

Based on the expression of the derivatives of the transformation, we now can derive a bound on these derivatives.

Lemma 4: The function

$$g(r,\theta) = f(r,\theta) + \int_{0}^{r} \int_{-\pi}^{\pi} \rho F(r,\rho) P(r,\rho,\theta-\psi) f(\rho,\psi) \, d\psi \, d\rho$$
(101)

where  $F(r, \rho) \in \mathcal{C}^2(\mathcal{T})$ , satisfies

$$||g||_{L^{2}} \le C_{0}||f||_{L^{2}}, \tag{102}$$

$$||g_{-}||_{L^{2}} \le C_{1}||f||_{L^{2}} + C_{2}(||f_{-}||_{L^{2}} + ||f_{-}||_{L^{2}}) \tag{103}$$

$$\begin{aligned} \|g_x\|_{L^2} &\leq C_1 \|f\|_{L^2} + C_2 (\|f_x\|_{L^2} + \|f_y\|_{L^2}), \\ \|g_{xx}\|_{L^2} &\leq C_3 \|f\|_{H_1} + C_4 \|f_{xx}\|_{L^2} \\ &+ C_4 \|f_{xy}\|_{L^2} + C_4 \|f_{yy}\|_{L^2}. \end{aligned}$$
(104)

*Proof:* First, since  $F(r, \rho) \in C^2(\mathcal{T})$ , we can bound F:

$$\begin{aligned} \forall (r,\rho) \in \mathcal{T} \quad |F(r,\rho)| \leq M, \quad |F_r(r,\rho)| \leq M_r, \\ |F_\rho(r,\rho)| \leq M_\rho, \quad |F_{rr}(r,\rho)| \leq M_{rr}, \\ |F_{r\rho}(r,\rho)| \leq M_{r\rho}, \quad |F_{\rho\rho}(r,\rho)| \leq M_{\rho\rho}. \end{aligned}$$

Then

$$\begin{split} \|g\|_{L^{2}}^{2} &= \int_{0}^{R} \int_{-\pi}^{\pi} |f(r,\theta)| \\ &+ \int_{0}^{r} \int_{-\pi}^{\pi} F(r,\rho) P(r,\rho,\theta-\psi) f(\rho,\psi) \, d\psi\rho \, d\rho \bigg|^{2} r \, d\theta \, dr \\ &\leq 2 \|f\|_{L^{2}}^{2} + 2M^{2} \int_{0}^{R} \frac{r^{3}}{2} \\ &\int_{0}^{r} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} P(r,\rho,\theta-\psi) f(\rho,\psi) d\psi \right|^{2} d\theta\rho \, d\rho \, dr \end{split}$$

$$\leq 2\|f\|_{L^{2}}^{2} + 2M^{2} \int_{0}^{R} \frac{r^{3}}{2} \int_{0}^{r} \int_{-\pi}^{\pi} |f(\rho,\theta)|^{2} \rho \, d\theta \, d\rho \, dr$$
  
$$\leq 2\left(1 + \frac{M^{2}R^{4}}{8}\right) \|f\|_{L^{2}}^{2}.$$
(105)

Hence, by setting  $C_0 = 2(1 + (M^2R^4/8))$  we have proven (102). We can bound the first and second derivatives of g by using Lemma 1–3, Cauchy–Schwarz inequality, and Hölder's inequality in the same fashion.

We have shown expressions and bounds for  $g_x$  and  $g_{xx}$ . Analogous results follow for  $g_y$ ,  $g_{xy}$ , and  $g_{yy}$ . Using these bounds we reach Proposition 1.

To prove Proposition 3, we state and prove Lemmas 5–8. For simplicity, let

$$\bar{P}(R,\rho,\psi) = \int_{-\pi}^{\pi} P(R,\rho,\psi-\phi)f(\rho,\phi)d\phi.$$
(106)

Lemma 5: The following inequalities hold

$$\int_{-\pi}^{\pi} \left| \int_{0}^{r} \int_{-\pi}^{\pi} P(r,\rho,\theta-\psi) \bar{P}(R,\rho,\psi) \, d\psi \, d\rho \right|^{2} d\theta$$

$$\leq \int_{-\pi}^{\pi} \int_{0}^{\pi} |f(\rho,\theta)|^{2} \, d\rho \, d\theta, \qquad (107)$$

$$\int_{-\pi}^{\pi} \left| \lim_{\rho \to r} \left[ \int_{-\pi}^{\pi} P(r,\rho,\theta-\psi) \bar{P}(R,\rho,\psi) d\psi \right] \right|^{2} \, d\theta$$

$$\leq \int_{-\pi}^{\pi} |f(r,\theta)|^{2} \, d\theta. \qquad (108)$$

We skip the proof, which mimics that of Lemma 1 by using the Convolution Theorem and Parseval's theorem.

*Lemma 6:* The following inequalities hold

$$\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} P_{\theta}(R, r, \theta - \psi) f(\rho, \psi) d\psi \right|^{2} d\theta \leq \int_{-\pi}^{\pi} |f_{\theta}(\rho, \theta)|^{2} d\theta,$$
(109)
$$\int_{-\pi}^{\pi} \left| \int_{0}^{r} \int_{-\pi}^{\pi} P_{\theta}(r, \rho, \theta - \psi) \bar{P}(R, \rho, \psi) d\psi d\rho \right|^{2} d\theta \\
\leq \int_{-\pi}^{\pi} \int_{0}^{r} |f_{\theta}(\rho, \theta)|^{2} d\rho d\theta.$$
(110)

*Proof:* We have that

$$\int_{-\pi}^{\pi} \left( P_{\theta}(R, r, \theta - \psi) \right) f(\rho, \psi) d\psi$$
$$= -\int_{-\pi}^{\pi} \left( P_{\psi}(R, r, \theta - \psi) \right) f(\rho, \psi) d\psi$$
$$= \int_{-\pi}^{\pi} P(R, r, \theta - \psi) f_{\theta}(\rho, \psi) d\psi. \tag{111}$$

Using Lemma 1, we get (109). Integrating by parts twice yields

$$\int_{-\pi}^{\pi} P_{\theta}(r,\rho,\theta-\psi) \int_{-\pi}^{\pi} P(R,\rho,\psi-\phi) f(\rho,\phi) \, d\phi \, d\psi$$
$$= \int_{-\pi}^{\pi} P(r,\rho,\theta-\psi) \int_{-\pi}^{\pi} P(R,\rho,\psi-\phi) f_{\theta}(\rho,\phi) \, d\phi \, d\psi.$$
(112)

Using Lemma 5 the result follows.

*Lemma 7*: The following inequalities hold:

$$\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} P_r(R, r, \theta - \phi) f(\rho, \phi) d\phi \right|^2 d\theta$$

$$\leq \int_{-\pi}^{\pi} \left| \frac{1}{R} f_{\theta}(\rho, \theta) \right|^2 d\theta, \qquad (113)$$

$$\int_{-\pi}^{\pi} \left| \int_{0}^{r} \int_{-\pi}^{\pi} P_r(r, \rho, \theta - \psi) \bar{P}(R, \rho, \psi) d\psi d\rho \right|^2 d\theta$$

$$\leq \frac{1}{r} \int_{-\pi}^{\pi} \int_{0}^{r} |f_{\theta}(\rho, \theta)|^2 d\rho d\theta. \qquad (114)$$

 $-\pi$  0

Lemma 7 is proved by combining Parseval's Theorem and integration by parts with the Fourier series

$$f_{\theta}(\rho,\theta) = \sum_{n=-\infty}^{n=\infty} jn f_n(\rho) e^{jn\theta}.$$
 (115)

For simplicity rewrite (64) as

$$F(R, r, \theta, \phi) = f_1(r)P(R, r, \theta - \phi) - \int_0^r \int_{-\pi}^{\pi} f_2(r, \rho)P(r, \rho, \theta - \psi)f_1(\rho)P(R, \rho, \psi - \phi)d\rho \,d\psi \quad (116)$$

where  $f_1(r) \in \mathcal{C}^2([0, R])$  and  $f_2(r, \rho) \in \mathcal{C}^2(\mathcal{T})$ . Based on Lemmas 5–7, we obtain the following lemma.

*Lemma 8:* The following inequalities hold:

$$\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \bar{F}(R, r, \theta, \phi) \tilde{w}_r(R, \phi) d\phi \right|^2 d\theta$$

$$\leq C_5 \int_{-\pi}^{\pi} |\tilde{w}_r(R, \theta)|^2 d\theta, \qquad (117)$$

$$\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \bar{F}_r(R, r, \theta, \phi) \tilde{w}_r(R, \phi) d\phi \right|^2 d\theta$$
$$\leq C_6 \int_{-\pi}^{\pi} |\tilde{w}_r(R, \theta)|^2 d\theta + C_7 \int_{-\pi}^{\pi} |\tilde{w}_{r\theta}(R, \theta)|^2 d\theta, \quad (118)$$

$$\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \bar{F}_{\theta}(R, r, \theta, \phi) \tilde{w}_{r}(R, \phi) d\phi \right|^{2} d\theta$$

$$\leq C_{8} \int_{-\pi}^{\pi} |\tilde{w}_{r\theta}(R, \theta)|^{2} d\theta \qquad (119)$$

where the constants  $C_i$  depend only on R,  $f_1$ , and  $f_2$ .

*Proof:* We can bound  $f_1$ ,  $f_2$  and their derivatives. Then, from Lemma 5, we find (117). Combining Lemmas 5 and 7, we get (118) and Lemmas 5 and 6 give (119).

#### REFERENCES

- R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.
- [2] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," in *Proc. Amer. Control Conf.*, 2005, pp. 1859–1864, IEEE.
- [3] R. Carli and F. Bullo, "Quantized coordination algorithms for rendezvous and deployment," *SIAM J. Control Optimiz.*, vol. 48, no. 3, pp. 1251– 1274, 2009.
- [4] F. Zhang and N. Leonard, "Cooperative filters and control for cooperative exploration," *IEEE Trans. Autom. Control*, vol. 55, no. 3, pp. 650–663, Mar. 2010.
- [5] S. Kalantar and U. R. Zimmer, "Distributed shape control of homogeneous swarms of autonomous underwater vehicles," *Auton. Robots*, vol. 22, no. 1, pp. 37–53, 2007.
- [6] E. Justh and P. Krishnaprasad, "Equilibria and steering laws for planar formations," Syst. Control Lett., vol. 52, no. 1, pp. 25–38, 2004.
- [7] D. van der Walle, B. Fidan, A. Sutton, C. Yu, and B. D. O. Anderson, "Non-hierarchical UAV formation control for surveillance tasks," in *Proc. Amer. Control Conf.*, 2008, pp. 777–782.
- [8] M. Turpin, N. Michael, and V. Kumar, "Decentralized formation control with variable shapes for aerial robots," in *Proc. IEEE Int. Conf. Robotics Autom. (ICRA)*, 2012, pp. 23–30.
- [9] M. Dunbabin and L. Marques, "Robots for environmental monitoring: Significant advancements and applications," *IEEE Robotics Autom. Mag.*, vol. 19, no. 1, pp. 24–39, 2012.
- [10] J. C. F. Bullo and S. Martinez, Distributed Control of Robotic Networks: A Mathematical Approach to Motion Coordination Algorithms. Princeton, NJ, USA: Princeton Univ. Press, 2009.
- [11] P. Frihauf and M. Krstic, "Leader-enabled deployment onto planar curves: A PDE-based approach," *IEEE Trans. Autom. Control*, vol. 56, no. 8, pp. 1791–1806, Aug. 2011.
- [12] S. Martinez, J. Cortes, and F. Bullo, "Motion coordination with distributed information," *IEEE Control Syst.*, vol. 27, no. 4, pp. 75–88, Aug. 2007.
- [13] M. Zavlanos, M. Egerstedt, and G. Pappas, "Graph-theoretic connectivity control of mobile robot networks," *Proc. IEEE*, vol. 99, no. 9, pp. 1525– 1540, Sep. 2011.
- [14] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.
- [15] Y. Q. Chen and Z. Wang, "Formation control: A review and a new consideration," in *Proc. IEEE/RSJ Int. Conf. Intell. Robots Syst. (IROS '05)*, 2005, pp. 3181–3186.
- [16] M. Mesbahi and M. Egerstedt, Graph Theoretic Methods in Multiagent Networks. Princeton, NJ, USA: Princeton Univ. Press, 2010.
- [17] P. Yang, R. A. Freeman, and K. M. Lynch, "Multi-agent coordination by decentralized estimation and control," *IEEE Trans. Autom. Control*, vol. 53, no. 11, pp. 2480–2496, Nov. 2008.
- [18] T. Meurer, Control of Higher-Dimensional PDEs. Berlin/Heidelberg, Germany: Springer, 2013.
- [19] P. Frihauf and M. Krstic, "Rapidly convergent leader-enabled multi-agent deployment into planar curves," in *Proc. Amer. Control Conf.*, 2009, pp. 1994–1999, IEEE.
- [20] D. Helbing, "Traffic and related self-driven many-particle systems," *Rev. Modern Phys.*, vol. 73, no. 4, p. 1067, 2001.
- [21] S. Kalantar and U. R. Zimmer, "Distributed shape control of homogeneous swarms of autonomous underwater vehicles," *Auton. Robots*, vol. 22, no. 1, pp. 37–53, 2007.

- [22] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis, "Continuous-time average-preserving opinion dynamics with opinion-dependent communications," *SIAM J. Control Optimiz.*, vol. 48, no. 8, pp. 5214–5240, 2010.
- [23] C. Canuto, F. Fagnani, and P. Tilli, "An Eulerian approach to the analysis of rendez-vous algorithms," in *Proc. IFAC World Congr.*, 2008, pp. 9039–9044.
- [24] G. Ferrari-Trecate, A. Buffa, and M. Gati, "Analysis of coordination in multi-agent systems through partial difference equations," *IEEE Trans. Autom. Control*, vol. 51, no. 6, pp. 1058–1063, Jun. 2006.
- [25] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.
- [26] A. Sarlette and R. Sepulchre, "A PDE viewpoint on basic properties of coordination algorithms with symmetries," in *Proc. 48th IEEE Conf. Decision Control and 28th Chinese Control Conf.*, 2009, pp. 5139–5144.
- [27] P. Barooah, P. G. Mehta, and J. P. Hespanha, "Mistuning-based control design to improve closed-loop stability margin of vehicular platoons," *IEEE Trans. Autom. Control*, vol. 54, no. 9, pp. 2100–2113, Sep. 2009.
- [28] W. Ren and N. Sorensen, "Distributed coordination architecture for multirobot formation control," *Robotics Auton. Syst.*, vol. 56, no. 4, pp. 324– 333, 2008.
- [29] W. Ren, K. L. Moore, and Y. Chen, "High-order and model reference consensus algorithms in cooperative control of multivehicle systems," *J. Dynam. Syst., Meas., Control*, vol. 129, no. 5, pp. 678–688, 2007.
- [30] G. Antonelli, F. Arrichiello, F. Caccavale, and A. Marino, "Decentralized centroid and formation control for multi-robot systems," in *Proc. IEEE Int. Conf. Robotics Autom. (ICRA)*, 2013, pp. 3511–3516, IEEE.
- [31] J. A. Gouvea, F. Lizarralde, and L. Hsu, "Formation control of dynamic nonholonomic mobile robots with curvature constraints via potential functions," in *Proc. Amer. Control Conf. (ACC)*, 2013, pp. 3039–3044.
- [32] M. A. Hsieh, V. Kumar, and L. Chaimowicz, "Decentralized controllers for shape generation with robotic swarms," *Robotica*, vol. 26, no. 5, pp. 691–701, 2008.
- [33] D. V. Dimarogonas and K. H. Johansson, "Stability analysis for multiagent systems using the incidence matrix: Quantized communication and formation control," *Automatica*, vol. 46, no. 4, pp. 695–700, 2010.
- [34] F. Lin, M. Fardad, and M. R. Jovanovic, "Optimal control of vehicular formations with nearest neighbor interactions," *IEEE Trans. Autom. Control*, vol. 57, no. 9, pp. 2203–2218, Sep. 2012.
- [35] X. Yan, J. Chen, and D. Sun, "Multilevel-based topology design and shape control of robot swarms," *Automatica*, vol. 48, no. 12, pp. 3122–3127, 2012.
- [36] J. Kim, K.-D. Kim, V. Natarajan, S. D. Kelly, and J. Bentsman, "PdEbased model reference adaptive control of uncertain heterogeneous multi-agent networks," *Nonlinear Anal.: Hybrid Syst.*, vol. 2, no. 4, pp. 1152–1167, 2008.
- [37] T. Meurer and M. Krstic, "Finite-time multi-agent deployment: A nonlinear PDE motion planning approach," *Automatica*, vol. 47, no. 11, pp. 2534–2542, 2011.
- [38] N. Ghods and M. Krstic, "Multiagent deployment over a source," *IEEE Trans. Control Syst. Technol.*, vol. 20, no. 1, pp. 277–285, Jan. 2012.
- [39] H. Hao, P. Barooah, and P. G. Mehta, "Stability margin scaling laws for distributed formation control as a function of network structure," *IEEE Trans. Autom. Control*, vol. 56, no. 4, pp. 923–929, Apr. 2011.
- [40] A. Mauroy and R. Sepulchre, "Global analysis of a continuum model for monotone pulse-coupled oscillators," *IEEE Trans. Autom. Control*, vol. 58, no. 5, pp. 1154–1166, May 2013.
- [41] K. L. C. Pimenta, N. Michael, R. C. Mesquita, G. A. Pereira, and V. Kumar, "Control of swarms based on hydrodynamic models," in *Proc. IEEE Int. Conf. Robotics Autom.*, 2008, pp. 1948–1953.
- [42] J. Kim, V. Natarajan, S. D. Kelley, and J. Bentsman, "Disturbance rejection in robust pde-based mrac laws for uncertain heterogeneous multiagent networks under boundary reference," *Nonlinear Anal.: Hybrid Syst. Applicat.*, vol. 4, no. 3, pp. 484–495, 2010.
- [43] A. Smyshlyaev and M. Krstic, "Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations," *IEEE Trans. Autom. Control*, vol. 49, no. 12, pp. 2185–2202, Dec. 2004.
- [44] M. Krstic and A. Smyshlyaev, Boundary Control of PDEs: A Course on Backstepping Designs, vol. 16. Philadelphia, PA, USA: SIAM, 2008.
- [45] O. M. Aamo, A. Smyshlyaev, and M. Krstic, "Boundary control of the linearized Ginzburg-Landau model of vortex shedding," *SIAM J. Control Optimiz.*, vol. 43, no. 6, pp. 1953–1971, 2005.

- [46] R. Vazquez and M. Krstic, "Boundary observer for output-feedback stabilization of thermal-fluid convection loop," *IEEE Trans. Control Syst. Technol.*, vol. 18, no. 4, pp. 789–797, Jul. 2010.
- [47] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations. New York, NY, USA: Springer, 2010.
- [48] M.-C. Lai, "A note on finite difference discretizations for Poisson equation on a disk," in *Numerical Methods for Partial Differential Equations*, vol. 17. New York, NY, USA: Wiley, 2001, pp. 199–203.
- [49] R. J. LeVeque, Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems. Philadelphia, PA, USA: SIAM, 2007.
- [50] J. Qi, R. Vazquez, and M. Krstic, Simulation Movie of an Example 3-D Deployment, 2014. [Online]. Available: http://flyingv.ucsd.edu/krstic/ MASDeployment.avi



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