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Brief paper

Adaptive output-feedback stabilization of non-local hyperbolic PDEs[☆]

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ABSTRACT

We address the problem of adaptive output-feedback stabilization of general first-order hyperbolic partial integro-differential equations (PIDE). Such systems are also referred to as PDEs with non-local (in space) terms. We apply control at one boundary, take measurements on the other boundary, and allow the system's functional coefficients to be unknown. To deal with the absence of both full-state measurement and parameter knowledge, we introduce a pre-transformation (which happens to be based on backstepping) of the system into an observer canonical form. In that form, the problem of adaptive observer design becomes tractable. Both the parameter estimator and the control law employ only the input and output signals (and their histories over one unit of time). Prior to presenting the adaptive design, we present the non-adaptive/baseline controller, which is novel in its own right and facilitates the understanding of the more complex, adaptive system. The parameter estimator is of the gradient type, based on a parametric model in the form of an integral equation relating delayed values of the input and output. For the closed-loop system we establish boundedness of all signals, pointwise in space and time, and convergence of the PDE state to zero pointwise in space. We illustrate our result with a simulation.

1. Introduction

After a spurt of activity in boundary control of various parabolic PDEs in the early 2000s (Balogh, Liu, & Krstic, 2001; Liu & Krstic, 2000), much attention has been dedicated in recent years to hyperbolic PDEs and to their stabilization (Bastin & Coron, 2011; Coron, Bastin, & d'Andrea Novel, 2008; Coron, d'Andrea Novel, & Bastin, 2007). In this paper, we focus on the stabilization of a general first-order hyperbolic PIDE, where the state is controlled at one boundary (input), and measured at the other (output). Our work's novelty is in how little knowledge we require to stabilize the system: the state is measured at only one boundary, and we allow the system's functional coefficients to be unknown. The key to our result is our introduction of an "observer canonical form" for this class of systems, which enables the design of an adaptive observer for stabilization of the system.

Despite a growing number of publications on the topic of boundary control of hyperbolic PDEs, stabilization by adaptive output feedback has been pursued only for matched uncertainties (He & Ge, 2012; He, Ge, & Zhang, 2011). For unmatched parametric uncertainties, the backstepping method, introduced in Smyshlyaev and Krstic (2004) for parabolic systems, has seen use in increasingly complex systems of coupled hyperbolic PDEs (Coron, Vazquez, Krstic, & Bastin, 2013; Di Meglio, Krstic, Vazquez, & Petit, 2012; Di Meglio, Vazquez, & Krstic, 2013; Vazquez, Krstic, Coron, & Bastin, 2012), as well as in Krstic (2009) and Krstic and Smyshlyaev (2008) for the hyperbolic PIDE that we tackle here.

We provide in this paper two novel contributions: a new output-feedback controller to face the absence of full-state measurement and, more importantly, the output-feedback controller's adaptive version for the case of unknown parameters.

The key new ingredient in our approach lies in the use of backstepping to transform the system into an observer canonical form, analogous to the transform used in Smyshlyaev and Krstic (2010) to transform parabolic PIDEs into a parabolic observer canonical form. Unlike the original plant in which a product of unknown coefficients and unmeasured state appears, our pre-transformation leads to a system structure in which only one infinite-dimension parameter is unknown but is multiplied by the measured output, making simultaneous state and parameter estimation feasible.

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For parameter estimation we use a gradient-based update law similar to those employed in Smyshlyaev and Krstic (2007), which differ from Lyapunov-based update laws developed in Krstic and Smyshlyaev (2005), and then in Bresch-Pietri, Chauvin, and Petit (2012), Bresch-Pietri and Krstic (2009), Bresch-Pietri and Krstic (2010), and Krstic and Bresch-Pietri (2009) to estimate delays or unknown parameters. This gradient update law is obtained via a parametric model in the form of an integral equation relating delayed values of the input and output. The use of projection enables to keep the estimated parameter within an a-priori bound, which we assume known.

As for the problem of state estimation, it was already addressed in Vazquez, Krstic, and Coron (2011) for a 2×2 hyperbolic linear system, through the design of a collocated boundary observer. In our paper, however, we present an explicit state observer employing the delayed values of both the input and the output over one unit of time, which enables us to design an output-feedback controller. Associated with the parameter estimation, and using the certainty equivalence principle, we get an adaptive output-feedback controller which achieves pointwise-in-space convergence of the PDE state to zero. All signals are established to be bounded pointwise in space and time.

Integral equations play prominent roles in our development. One of the plant's representations, the parametric model for the parameter estimator design, the control law, and the control gain kernel are all governed by integral equations. The relation between hyperbolic PDE systems and integral delay equations was recently thoroughly studied in Karafyllis and Krstic (2014).

Outline. After introducing our system in Section 2, we transform it to the observer canonical form in Section 3. Once this step is accomplished, we start by presenting the non-adaptive controller in Section 4 in order to facilitate the understanding of the more complex, adaptive design, which follows in Section 5 with the statement of the main stability theorem. Section 6 then consists of its proof. We finally end our paper with an illustration of our result through a simulation in Section 7.

Notation. For any functions f and g defined on $[0, 1]$, we use the convolution notation

$$f * g(x) = \int_0^x f(x-y)g(y)dy = \int_0^x f(y)g(x-y)dy$$

where $x \in [0, 1]$ and for any function f defined on $[0, 1] \times [0, \infty)$, we denote the L^2-norm as

$$\|f\|(t) = \sqrt{\int_0^1 f(x,t)^2 dx}.$$

2. General first-order hyperbolic PIDE

We consider the following class of first-order hyperbolic PIDE:

$$\bar{u}_t(x,t) = \bar{u}_x(x,t) + \lambda(x)\bar{u}(x,t) + \bar{g}(x)\bar{u}(0,t) + \int_0^x \bar{f}(x,y)\bar{u}(y,t)dy \tag{1}$$

$$\bar{u}(1,t) = U(t) \tag{2}$$

$$Y(t) = \bar{u}(0,t), \tag{3}$$

where $\lambda, \bar{g}$ and $\bar{f}$ are unknown, continuous functions. The goal is to regulate $\bar{u}(x,t)$ to zero for all $x \in [0, 1]$ using the measurement of only $Y(t) = \bar{u}(0,t)$ and using boundary control $U(t)$.

PIDEs in the form (1) would be obtained from models of various coupled PDE dynamics that incorporate at least one transport process. Such dynamics would arise in certain chemical processes and in slugging flows in oil risers—both of which involve two or more PDEs. An equation in the form (1) would be obtained after various changes of variables, re-scaling, linearization around an equilibrium profile, and, most importantly, after applying a singular perturbation reduction relative to all the PDEs except for the slowest one. Such systems are often actuated by boundary control and the control objective is stabilization to a (typically non-zero) equilibrium profile.

We first remove the reaction term in $\lambda\bar{u}$ by introducing the scaled state

$$u(x,t) = \exp\left(\int_0^x \lambda(\xi)d\xi\right)\bar{u}(x,t) \tag{4}$$

which is governed by

$$u_t(x,t) = u_x(x,t) + g(x)u(0,t) + \int_0^x f(x,y)u(y,t)dy \tag{5}$$

$$u(1,t) = \rho\, U(t) \tag{6}$$

$$Y(t) = u(0,t), \tag{7}$$

where

$$g(x) = \exp\left(\int_0^x \lambda(\xi)d\xi\right)\bar{g}(x) \tag{8}$$

$$f(x,y) = \exp\left(\int_y^x \lambda(\xi)d\xi\right)\bar{f}(x,y) \tag{9}$$

$$\rho = \exp\left(\int_0^1 \lambda(\xi)d\xi\right). \tag{10}$$

We impose the following assumptions.

Hypothesis 1. ρ is known and, without loss of generality, we set it to $\rho = 1$ (by absorbing any non-unity ρ into U).

Although an adaptive controller can be designed for λ, g and f completely unknown, with a separate estimator of the "high-frequency gain" ρ, Hypothesis 1 allows us to maintain clarity of exposition and stay within the page limit.

Hypothesis 2. Constants M_g and M_f are known such that, for all $0 \le y \le x \le 1$, $|g(x)| \le M_g$ and $|f(x,y)| \le M_f$.

This assumption does not constitute a limitation. Indeed, bounds – which may be as large as necessary – can always be found based on the physical knowledge of the system, and they do not affect the control, however overestimated they may be: f and g remain unknown.

3. Observer canonical form

The key challenge for feedback design for the plant (5)–(6) is that the term $\int_0^x f(x,y)u(y,t)dy$ is a product of the unmeasured state $u(x,t)$ and of the unknown parameter $f(x,y)$. We overcome this challenge by transforming the system into a form in which an unknown parameter multiplies only the measured output $Y(t) = u(0,t)$.

We introduce the backstepping pre-transformation

$$v(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)dy \tag{11}$$

where q is the solution to the PDE

$$q_y(x,y) + q_x(x,y) = \int_y^x q(x,s)f(s,y)ds - f(x,y) \tag{12}$$

$$q(1,y) = 0 \tag{13}$$

and which maps the system (5)–(6) into

$$v_t(x,t) = v_x(x,t) + \theta(x)v(0,t) \tag{14}$$
$$v(1,t) = U(t), \tag{15}$$

where

$$Y(t) = v(0,t) = u(0,t) = \bar{u}(0,t) \tag{16}$$

is measured and

$$\theta(x) = q(x,0) + g(x) - \int_0^x q(x,y)g(y)dy. \tag{17}$$

We refer to the form (14), (15), (16) as the *observer canonical form* due to its analogy with the eponymous form for finite-dimensional systems. The transformation (11) is not a part of design but of analysis only. The kernel $q(x,y)$ is unknown and so is the new system parameter $\theta(x)$. Unlike the term $\int_0^x f(x,y)u(y,t)dy$ in (5)–(6), which is a product of two unknown quantities, the term $\theta(x)v(0,t) = \theta(x)Y(t)$ in (14) has only $\theta(x)$ as an unknown. This is the key feature with which the observer canonical form (14), (15), (16) enables us to perform adaptive output-feedback design.

In the following theorem, proved in Section 6.1, we show that the PDE (12)–(13) is well posed.

Theorem 3. *The PDE (12)–(13) has a unique $C^1([0,1]\times[0,1])$ solution with the bound*

$$|q(x,y)| \leq M_f(1-x)\, e^{M_f(x-y)(1-x)}, \tag{18}$$

where M_f is a bound for the function f on $[0,1]\times[0,1]$.

The inverse of the pre-transformation (11) is found in exactly the same way as the direct transformation, but by transforming the observer canonical form (v) into the original plant model (u) and by proving the well posedness of the underlying kernel PDE as in Theorem 3.

4. Non-adaptive output-feedback control design

Our non-adaptive controller is given by

$$\begin{aligned} U(t) &= \int_{t-1}^t \kappa(t-\tau)U(\tau)d\tau \\ &\quad + \int_{t-1}^t \left(\int_{t-\tau}^1 \kappa(\mu)\theta(1-\mu+t-\tau)d\mu\right) Y(\tau)d\tau, \end{aligned} \tag{19}$$

where κ is solution of the Volterra equation

$$\kappa(x) = -\theta(x) + \int_0^x \kappa(y)\theta(x-y)dy, \tag{20}$$

and where arbitrary functions $U(\tau), Y(\tau)$ can be employed in the controller (19) for $\tau \in [-1,0)$. We shed some light on the construction of this controller after the statement of the following theorem.

Theorem 4. *For the system consisting of the plant (5)–(6) and the controller (19)–(20), there exist $M_o \geq 1$ and $\delta > 0$ such that the following holds:*

$$\Omega(t) \leq M_o e^{-\delta t}\Omega(0), \quad \forall t \geq 0, \tag{21}$$

$$\Omega(t) \triangleq \int_0^1 u^2(x,t)dx + \int_{t-1}^t \left(U^2(\tau) + Y^2(\tau)\right) d\tau. \tag{22}$$

To make the adaptive design and analysis in the subsequent sections easier to follow, we sketch the proof of the stability result for the non-adaptive design.

We represent the delayed input and output signals with the transport PDEs

$$\phi_t(x,t) = \phi_x(x,t), \qquad \phi(x,0) = \phi_0(x), \quad x \in [0,1] \tag{23}$$
$$\phi(1,t) = Y(t) \tag{24}$$

and

$$\psi_t(x,t) = \psi_x(x,t), \qquad \psi(x,0) = \psi_0(x), \quad x \in [0,1] \tag{25}$$
$$\psi(1,t) = U(t). \tag{26}$$

where ϕ_0, ψ_0 are arbitrary initial conditions verifying

$$\phi_0(1) = Y(0) \tag{27}$$
$$\psi_0(1) = U(0). \tag{28}$$

We can define for $x \in [0,1]$,

$$Y(x-1) = \phi_0(x) \tag{29}$$
$$U(x-1) = \psi_0(x) \tag{30}$$

and the explicit solutions to the PDE filters, for $x \in [0,1]$, $t \geq 0$, are given by

$$\phi(x,t) = Y(t+x-1) \tag{31}$$
$$\psi(x,t) = U(t+x-1). \tag{32}$$

The transformed plant state v, represented, based on (11), compactly as

$$v = (\mathrm{Id} - \mathscr{Q})u \tag{33}$$

$$\mathscr{Q}[u](x,t) \triangleq \int_0^x q(x,y)u(y,t)dy, \tag{34}$$

is estimated using the filter-based observer

$$\hat{v} = \psi + \mathscr{G}[\phi] \tag{35}$$

$$\mathscr{G}[\phi](x,t) \triangleq \int_x^1 \theta(\xi)\phi(1-(\xi-x),t)d\xi \tag{36}$$

and the observer error is denoted as

$$e = v - \hat{v}. \tag{37}$$

The control design is based on a backstepping transformation of the state observer signal $\hat{v}$, with the direct and inverse form of the transformation given by

$$w = \hat{v} - \kappa * \hat{v} \triangleq (\mathrm{Id} - \mathscr{K})[\hat{v}] \tag{38}$$
$$\hat{v} = w - \theta * w. \tag{39}$$

The plant's state is u, whereas ϕ and ψ are the states of the dynamic controller. Our interest is in proving properties of (ϕ, ψ, u). However, it is easier to study the system in the equivalent variables (ϕ, w, e). With a lengthy but straightforward algebraic calculation the reader can verify that the transformed variables (ϕ, w, e) are governed by the PDE systems

$$e_t(x,t) = e_x(x,t) \tag{40}$$
$$e(1,t) = 0 \tag{41}$$
$$w_t = w_x - \kappa(x)e(0) \tag{42}$$
$$w(1) = 0 \tag{43}$$
$$\phi_t = \phi_x \tag{44}$$
$$\phi(1) = w(0) + e(0). \tag{45}$$

Controller (19) and the Volterra equation (20) for κ are selected to ensure (43). The other key idea in our design is the construction of the integral operator $\mathscr{G}$ in (36), which yields a stable autonomous observer error system (40), (41).

For the reader's benefit we summarize the direct transformations from original states (ϕ, ψ, u) to transformed states (ϕ, w, e) as

$$\phi = \phi \tag{46}$$
$$w = (\mathrm{Id} - \mathscr{K})[\psi + \mathscr{G}[\phi]] \tag{47}$$
$$e = (\mathrm{Id} - \mathscr{Q})u - \psi - \mathscr{G}[\phi] \tag{48}$$

and stress that we obtain the original states from the transformed ones as

$$\phi = \phi \tag{49}$$
$$\psi = w - \theta * w - \mathscr{G}[\phi] \tag{50}$$
$$u = (\mathrm{Id} - \mathscr{Q})^{-1}[w - \theta * w + e]. \tag{51}$$

So we study the stability of the system (ϕ, ψ, u), using the system (40)–(45). The initial condition of the (ϕ, w, e) system is defined with (46), (47), (48), whereas the solution (ϕ, ψ, u) is defined with (49), (50), (51).

The structure of the system (40)–(45) is such that (1) the e-system is autonomous and exponentially stable, (2) the w-system is exponentially stable but driven by the e-system, and (3) the ϕ-system is exponentially stable and driven by both e and w systems. This observation will influence how we put together a Lyapunov function for the overall (ϕ, w, e)-system.

We consider the following component Lyapunov functions:

$$V_1 = \frac{1}{2}\int_0^1 (1+x)\phi^2(x)dx \tag{52}$$
$$V_2 = \frac{1}{2}\int_0^1 (1+x)w^2(x)dx \tag{53}$$
$$V_3 = \frac{1}{2}\int_0^1 (1+x)e^2(x)dx. \tag{54}$$

Differentiating V_3, we get

$$\dot{V}_3 = -\frac{1}{2}e^2(0) - \frac{1}{2}\|e\|^2. \tag{55}$$

Using the PDEs (44)–(45), (42)–(43), and Young's inequality, we get the following majorizations:

$$\dot{V}_1 \le \frac{3}{2}w^2(0) + \frac{3}{2}e^2(0) - \frac{1}{2}\phi^2(0) - \frac{1}{2}\|\phi\|^2 \tag{56}$$
$$\dot{V}_2 \le -\frac{1}{2}w^2(0) - \left(\frac{1}{2} - c\right)\|w\|^2 + \frac{K}{c}e^2(0), \tag{57}$$

where K is an upper bound on $\kappa(x)$ and c is an arbitrary positive constant. Taking $c = \frac{1}{4}$, we get

$$\dot{V}_2 \le -\frac{1}{2}w^2(0) - \frac{1}{4}\|w\|^2 + 4Ke^2(0). \tag{58}$$

Taking

$$V = V_1 + 3V_2 + (3 + 24K)\,V_3 \tag{59}$$

we get

$$\dot{V} \le -\frac{1}{2}\|\phi\|^2 - \frac{3}{4}\|w\|^2 - \left(\frac{3}{2} + 12K\right)\|e\|^2 \le -\frac{1}{4}V. \tag{60}$$

With routine calculations we obtain an L_2-stability estimate in terms of the norm of (ϕ, w, e). With some additional calculations, relying on (46), (47), (48) and (49), (50), (51) we get an L_2-stability estimate in terms of the norm of (ϕ, ψ, u). With (31), (32), we get the estimate (21) and complete the proof of Theorem 4.

5. Adaptive design

We apply the certainty equivalence principle and use an adaptive version of controller (19), namely, we replace θ and κ by their estimate $\hat{\theta}$ and $\hat{\kappa}$, obtaining the control law

$$U(t) = \int_{t-1}^{t} \hat{\kappa}(t-\tau, t)U(\tau)d\tau + \int_{t-1}^{t}\left(\int_{t-\tau}^{1} \hat{\kappa}(\mu, t)\hat{\theta}(1-\mu+t-\tau, t)d\mu\right)Y(\tau)d\tau, \tag{61}$$

where $\hat{\theta}$ is generated by an estimator (to be designed) and $\hat{\kappa}$ is obtained from $\hat{\theta}$ by real-time solution of the Volterra equation

$$\hat{\kappa}(x, t) = -\hat{\theta}(x, t) + \int_0^x \hat{\kappa}(y, t)\hat{\theta}(x-y, t)dy. \tag{62}$$

For the design of a parameter estimator for $\theta(x)$, we need a parametric model. The observer canonical form (14), (15), (16) serves as our parametric model, however, we use the following alternative representation of the observer canonical form to motivate our choice of the estimator:

$$Y(t) = U(t-1) + \int_{t-1}^{t} \theta(t-\tau)Y(\tau)d\tau + \varepsilon(t), \tag{63}$$

where the function $\varepsilon(t) = Y(t) - \psi_0(t) + \int_t^{t+1}\theta(t+1-\tau)\phi_0(\tau)d\tau$ is arbitrary for $t \in [0, 1]$ and $\varepsilon(t) = 0$ for $t > 1$.

Our parameter update law will need to employ projection to keep the estimate $\hat{\theta}(x, t)$ within an a priori known bounded interval for each $x \in [0, 1]$. We make an assumption in Hypothesis 2, which enables us to determine an a priori bound on the true $\theta(x)$.

Reminding the reader that we have assumed (without loss of generality) that $\rho = 1$, from the expression (17) for θ, we get that, for all $x \in [0, 1]$,

$$\begin{aligned} |\theta(x)| &\le M_f(1-x)e^{M_f x(1-x)}(1+M_g) + M_g \\ &\le M_f\, e^{M_f}(1+M_g) + M_g \triangleq M, \end{aligned} \tag{64}$$

which is a bound that we shall employ to limit the estimate $\hat{\theta}(x)$ using projection.

Now, guided by the parametric model (63), we introduce the update law

$$\hat{\theta}_t(x, t) = \frac{\gamma(x)}{1 + \int_{t-1}^{t} Y^2(\tau)d\tau}\mathrm{Proj}\left(Y(t-x)\hat{e}(0, t), \hat{\theta}(x, t)\right), \tag{65}$$

where γ is a positive-valued adaptation gain function, $Y(t-x)$ is the "regressor",

$$\hat{e}(0, t) = Y(t) - U(t-1) - \int_{t-1}^{t} \hat{\theta}(t-\tau, t)Y(\tau)d\tau \tag{66}$$

is the "estimation error", and the projection is given by

$$\mathrm{Proj}(a, b) = \begin{cases} 0, & \text{if } |b| = M \text{ and } ab > 0 \\ a, & \text{otherwise.} \end{cases} \tag{67}$$

The gain γ is chosen for the desired convergence speed of θ.

We employ projection in order to guarantee pointwise (rather than merely L_2) boundedness of $\hat{\theta}(x)$ and $\hat{\kappa}(x)$, which appear in the adaptive versions of the backstepping (inverse and direct) transformations in the proof of pointwise boundedness and regulation.

Our main theorem is stated next.

Theorem 5. *Consider the plant (5)–(6) under Hypotheses 1 and 2 with the controller (61)–(62) and the update law (65)–(66). Then, for any initial conditions* $\hat{\theta}(\cdot, 0) \in C^1(0, 1)$, *and* ϕ_0, ψ_0 *verifying (27)–(28), the solution* $(u, \hat{\theta})$ *and the control U are bounded for all* $x \in [0, 1]$, $t \geq 0$ *and*

$$\lim_{t\to\infty} u(x,t) = 0, \quad \forall x \in [0,1] \tag{68}$$

$$\lim_{t\to\infty} U(t) = 0. \tag{69}$$

6. Proof of Theorem 5

The closed-loop system is infinite dimensional, nonlinear, and involves discontinuity in the projection operator. We do not prove existence and uniqueness of the closed-loop solutions in the Filippov sense for the nonlinear PDE system but assume them in our proofs of stability.

The stability proof consists of several steps which are presented in the following subsections.

6.1. Well-posedness of the transformation into the observer canonical form

The PDE (12)–(13) is defined on the triangular domain: $\tau = \{(x, y), 0 \leq y \leq x \leq 1\}$. The change of variables $\tilde{x} = 1 - y$, $\tilde{y} = 1 - x$, $\tilde{f}(\tilde{x}, \tilde{y}) = f(x, y)$, $\tilde{q}(\tilde{x}, \tilde{y}) = q(x, y)$ leads us to a new PDE, defined on τ:

$$\tilde{q}_{\tilde{y}}(\tilde{x}, \tilde{y}) + \tilde{q}_{\tilde{x}}(\tilde{x}, \tilde{y}) = -\int_{\tilde{y}}^{\tilde{x}} \tilde{q}(\tilde{x}, s)\tilde{f}(s, \tilde{y})ds + \tilde{f}(\tilde{x}, \tilde{y}) \tag{70}$$

$$\tilde{q}(\tilde{x}, 0) = 0. \tag{71}$$

The function $\tilde{q}(\tilde{x}, \tilde{y})$ satisfies the integral equation

$$\tilde{q}(\tilde{x}, \tilde{y}) = F_0(\tilde{x}, \tilde{y}) + F[\tilde{q}](\tilde{x}, \tilde{y}), \tag{72}$$

where

$$F_0(\tilde{x}, \tilde{y}) = \int_0^{\tilde{y}} \tilde{f}(\tilde{x} - \tilde{y} + \xi, \xi)d\xi \tag{73}$$

$$F[\tilde{q}](\tilde{x}, \tilde{y}) = -\int_0^{\tilde{y}} \int_0^{\tilde{x}-\tilde{y}} \tilde{q}(\tilde{x} - \tilde{y} + \eta, \xi + \eta) \times \tilde{f}(\xi + \eta, \eta)d\xi d\eta. \tag{74}$$

We solve this equation by the method of successive approximations. We define the sequence

$$\tilde{q}^0(\tilde{x}, \tilde{y}) = F_0(\tilde{x}, \tilde{y}) \tag{75}$$

$$\tilde{q}^{n+1}(\tilde{x}, \tilde{y}) = F_0(\tilde{x}, \tilde{y}) + F[\tilde{q}^n](\tilde{x}, \tilde{y}) \tag{76}$$

and the differences

$$\Delta\tilde{q}^n = \tilde{q}^{n+1} - \tilde{q}^n. \tag{77}$$

Then, we get

$$\Delta\tilde{q}^{n+1}(\tilde{x}, \tilde{y}) = F[\Delta\tilde{q}^n](\tilde{x}, \tilde{y}). \tag{78}$$

By induction, we prove that, for all integers n,

$$|\Delta\tilde{q}^n(\tilde{x}, \tilde{y})| \leq \frac{M_f^{n+1}(\tilde{x} - \tilde{y})^n}{n!}\tilde{y}^{n+1}. \tag{79}$$

Therefore, the series

$$\tilde{q}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \tilde{q}^n(\tilde{x}, \tilde{y}) = F_0(\tilde{x}, \tilde{y}) + \sum_{n=0}^{\infty} \Delta\tilde{q}^n(\tilde{x}, \tilde{y}) \tag{80}$$

uniformly converges in τ to solution of (72) with the bound $|\tilde{q}(\tilde{x}, \tilde{y})| \leq M_f \tilde{y}\, e^{M_f(\tilde{x}-\tilde{y})\tilde{y}}$. Thus, we also have that $\tilde{q} \in C^1(\tau)$ since $\tilde{q}^n \in C^1(\tau)$ according to (73) and (74). The bound (18) on q is easily deduced.

If we suppose $\tilde{q}_1$ and $\tilde{q}_2$ are two solutions and we consider their difference $\delta\tilde{q} = \tilde{q}_1 - \tilde{q}_2$, then we get

$$\delta\tilde{q}(\tilde{x}, \tilde{y}) = F[\delta\tilde{q}](\tilde{x}, \tilde{y}) \tag{81}$$

and for all integer n,

$$|\delta\tilde{q}(\tilde{x}, \tilde{y})| \leq \frac{M_f^{n+1}(\tilde{x} - \tilde{y})^n}{n!}\tilde{y}^{n+1}. \tag{82}$$

Thus, $\delta\tilde{q} = 0$ and $\tilde{q}_1 = \tilde{q}_2$. Hence, we establish uniqueness of the solution.

6.2. Nonadaptive observer

We use the filters introduced in (23)–(32). The non-adaptive observer error

$$e(x, t) = v(x, t) - \psi(x, t) - \int_x^1 \theta(\xi)\phi(1 - (\xi - x), t)d\xi \tag{83}$$

satisfies the autonomous PDE

$$e_t(x, t) = e_x(x, t) \tag{84}$$

$$e(1, t) = 0. \tag{85}$$

Lemma 6. *With* $\|\cdot\|$ *denoting the* L_2 *norm in* $x \in [0, 1]$, *and with* $\mathscr{L}_2$ *and* $\mathscr{L}_\infty$ *denoting the usual function spaces in* $t \in [0, \infty)$, *the solutions of (84)–(85) satisfy the following properties as functions of time:*

$$\|e\| \in \mathscr{L}_2 \cap \mathscr{L}_\infty \quad \text{and} \quad \|e(\cdot, t)\| \to 0 \text{ as } t \to \infty \tag{86}$$

$$e(x) \in \mathscr{L}_2 \cap \mathscr{L}_\infty \quad \text{and} \quad e(x, t) \to 0 \text{ as } t \to \infty, \text{ for all } x \in [0, 1]. \tag{87}$$

Proof. Taking $V = \int_0^1 (1 + x)e^2(x)dx$, we get $\dot{V} = -\frac{1}{2}e^2(0) - \frac{1}{2}\|e\|^2$, which guarantees that $\|e\| \in \mathscr{L}_2 \cap \mathscr{L}_\infty$. It also follows in a straightforward manner that $V \to 0$ and $\|e\| \to 0$.

Differentiating the e-system in x, we get that e_x obeys the same PDE with the same boundary condition. Therefore, $\|e_x\| \in \mathscr{L}_2 \cap \mathscr{L}_\infty$ and, by Agmon's inequality (with $e(1) = 0$), $e(x) \in \mathscr{L}_2 \cap \mathscr{L}_\infty$ and $e(x) \to 0$ for all $x \in [0, 1]$.

6.3. Properties of the update law

Our update law for the estimate $\hat{\theta}(x, t)$ is based on the parametric model

$$e(0, t) = v(0, t) - \psi(0, t) - \int_0^1 \theta(\xi)\phi(1 - \xi, t)d\xi. \tag{88}$$

The estimation error (66) is alternatively written as

$$\hat{e}(0, t) = v(0, t) - \psi(0, t) - \int_0^1 \hat{\theta}(\xi, t)\phi(1 - \xi, t)d\xi \tag{89}$$

and the parameter estimation error $\tilde{\theta}(x, t) = \theta(x) - \hat{\theta}(x, t)$ satisfies

$$\hat{e}(0, t) = e(0, t) + \int_0^1 \tilde{\theta}(\xi, t)\phi(1 - \xi, t)d\xi. \tag{90}$$

With the filters, we rewrite the update law as

$$\hat{\theta}_t(x) = \frac{\gamma(x)}{1 + \|\phi\|^2}\text{Proj}(\hat{e}(0)\phi(1 - x), \hat{\theta}(x)) \tag{91}$$

(we remove the time dependence for clarity).

Lemma 7. *Following the same norm and space nomenclature as in Lemma 6, the adaptive law (91) guarantees that*

$$|\hat{\theta}(x)| \leq M, \quad \text{for all } (x,t) \in [0,1] \times [0,\infty) \tag{92}$$

$$\|\tilde{\theta}\| \in \mathscr{L}_\infty \tag{93}$$

$$\|\hat{\theta}_t\| \in \mathscr{L}_2 \cap \mathscr{L}_\infty \tag{94}$$

$$\frac{\hat{e}(0)}{\sqrt{1+\|\phi\|^2}} \in \mathscr{L}_2 \cap \mathscr{L}_\infty. \tag{95}$$

Proof. First, we define

$$\tau(x) = \gamma(x)\frac{\hat{e}(0)\phi(1-x)}{1+\|\phi\|^2} \tag{96}$$

so that

$$\hat{\theta}_t(x) = \mathrm{Proj}(\tau(x), \hat{\theta}(x)). \tag{97}$$

This scalar projection is a particular case of the vector projection described in the Appendix E of Krstic (2009). Therefore, we have the following properties:

$$\hat{\theta}_t(x)^2 \leq \tau^2(x) \tag{98}$$

$$-\tilde{\theta}(x)\hat{\theta}_t(x) \leq -\tilde{\theta}(x)\tau(x) \tag{99}$$

and θ remains in the target domain.

Let us now consider the Lyapunov function:

$$V = \frac{1}{2}\|e\|^2 + \int_0^1 \frac{\tilde{\theta}^2(x)}{2\gamma(x)}dx. \tag{100}$$

Then,

$$\begin{aligned}
\dot{V} &= \int_0^1 e(x,t)e_x(x,t)dx - \int_0^1 \frac{\tilde{\theta}(x)\hat{\theta}_t(x)}{\gamma(x)}dx \\
&\leq \int_0^1 e(x,t)e_x(x,t)dx - \frac{\int_0^1 \tilde{\theta}(x)\phi(1-x)dx}{1+\|\phi\|^2}\hat{e}(0) \\
&\leq -\frac{1}{2}e^2(0) + \frac{\hat{e}(0)e(0) - \hat{e}^2(0)}{1+\|\phi\|^2} \\
&\leq -\frac{1}{2}e^2(0) + \frac{1}{2}\frac{e^2(0)}{1+\|\phi\|^2} - \frac{1}{2}\frac{\hat{e}^2(0)}{1+\|\phi\|^2} \\
&\leq -\frac{1}{2}\frac{\hat{e}^2(0)}{1+\|\phi\|^2}.
\end{aligned}$$

Hence, V is bounded, $\frac{\hat{e}(0)}{\sqrt{1+\|\phi\|^2}} \in \mathscr{L}_2$, and $\|\tilde{\theta}\| \in \mathscr{L}_\infty$. The other properties come from the update law and relations (98) and (90).

The properties in Lemma 7 are established with projection (67). Under such discontinuous projection, the solutions of the infinite-dimensional system should be understood in Filippov's sense. Alternatively, to ensure continuity of signals, we can employ a continuous projection operator with a boundary layer $\varepsilon > 0$, which uses a linear transition between a and 0:

$$\mathrm{Proj}(a,b) = a\begin{cases} \dfrac{M+\varepsilon-b}{\varepsilon}, & \text{if } M \leq |b| \leq M+\varepsilon \\ & \text{and } ab > 0 \\ 1, & \text{otherwise.}\end{cases} \tag{101}$$

While the properties in Lemma 7 can be established for the continuous projection, we use the basic discontinuous projection because the implementation and proofs are simpler.

6.4. Backstepping transformation

Based on (83), we introduce the adaptive state estimate

$$\hat{v}(x) = \psi(x) + \int_x^1 \hat{\theta}(\xi)\phi(1-(\xi-x))d\xi \tag{102}$$

and apply the following backstepping transformation:

$$w(x) = \hat{v}(x) - \hat{\kappa} * \hat{v}(x) \triangleq T[\hat{v}](x), \tag{103}$$

where κ is the solution to the Volterra equation

$$\hat{\kappa}(x) = -\hat{\theta}(x) + \hat{\kappa} * \hat{\theta}(x) = -T[\hat{\theta}](x). \tag{104}$$

Transformation (103) is invertible,

$$\hat{v}(x) = w(x) - \hat{\theta} * w(x), \tag{105}$$

and leads to the target system

$$\begin{aligned} w_t &= w_x - \hat{\kappa}(x)\hat{e}(0) + w * T[\hat{\theta}_t](x) \\ &\quad + T\left[\int_x^1 \hat{\theta}_t(\xi)\phi(1-(\xi-x))d\xi\right] \end{aligned} \tag{106}$$

$$w(1) = 0. \tag{107}$$

This leads to the controller

$$U(t) = \hat{v}(1) = \int_0^1 \hat{\kappa}(1-y)\hat{v}(y,t)dy, \tag{108}$$

i.e.,

$$\begin{aligned} U(t) = \int_0^1 \hat{\kappa}(1-y)\Big[&\psi(y,t) \\ &+ \int_y^1 \hat{\theta}(\xi)\phi(1-(\xi-y),t)d\xi\Big]dy, \end{aligned} \tag{109}$$

which corresponds to the controller (61) presented in Theorem 5.

The ϕ system can be rewritten as

$$\phi_t = \phi_x \tag{110}$$

$$\phi(1) = w(0) + \hat{e}(0). \tag{111}$$

We now have two interconnected systems, ϕ and w, given by (110)–(111) and (106)–(107).

6.5. $\mathscr{L}_2$ boundedness

From (104) and the Gronwall inequality, we get the following bound:

$$|\hat{\kappa}(x)| \leq Me^M \triangleq K. \tag{112}$$

We also know from the previous section, that $\|\hat{\theta}_t\|$ is bounded.

Let us now consider the Lyapunov functions:

$$V_1 = \frac{1}{2}\int_0^1 (1+x)\phi^2(x)dx \tag{113}$$

$$V_2 = \frac{1}{2}\int_0^1 (1+x)w^2(x)dx. \tag{114}$$

Using the PDEs (110)–(111), (106)–(107), and the Young inequality, we get the following majorizations:

$$\dot{V}_1 \leq \frac{3}{2}w^2(0) + \frac{3}{2}\hat{e}^2(0) - \frac{1}{2}\phi^2(0) - \frac{1}{2}\|\phi\|^2 \tag{115}$$

$$\begin{aligned} \dot{V}_2 \;\leq\; & -\frac{1}{2}w^2(0) - \left(\frac{1}{2} - c_1 - c_2 - c_3 - c_4\right)\|w\|^2 \\ & + \frac{K}{c_1}\hat{e}^2(0) + l_1\|w\|^2 + l_2\|\phi\|^2 \end{aligned} \tag{116}$$

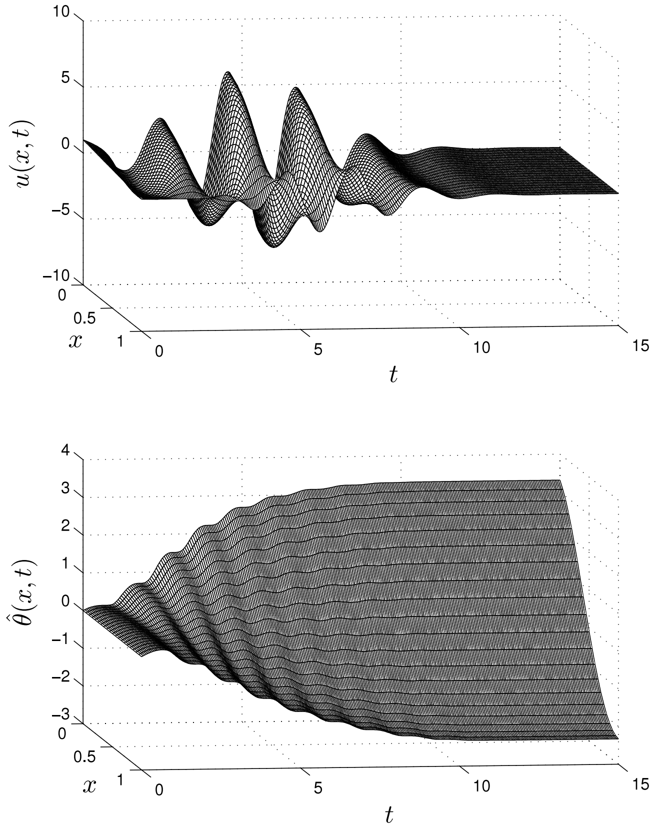

Fig. 1. Response of system (121) to the adaptive control (61)–(62): evolution of state u (top), and of the estimate $\hat{\theta}$ of the unknown infinite-dimension parameter θ (bottom).

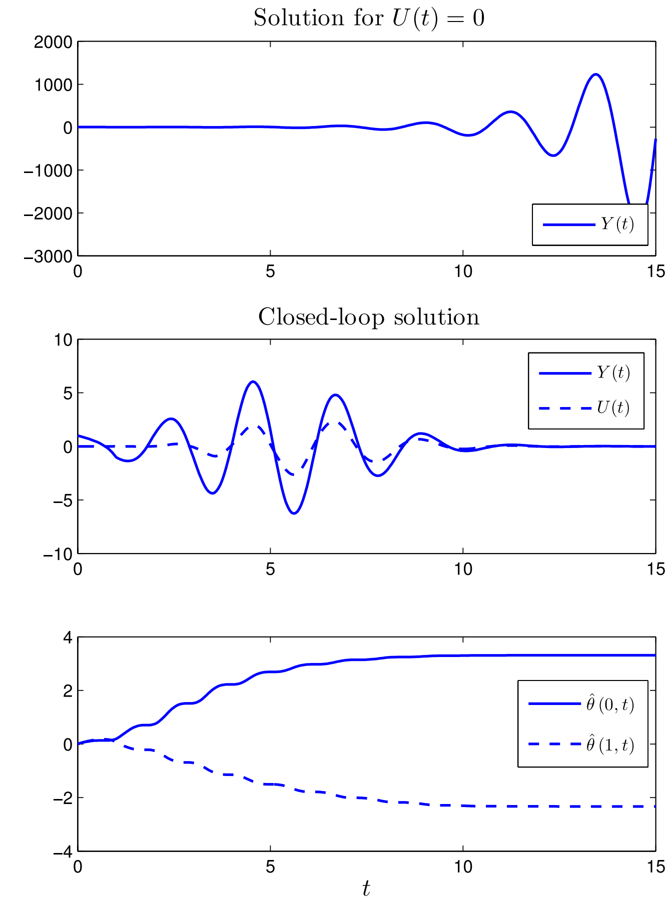

Fig. 2. Comparison of the open and closed loops. For the closed loop, we represent the input U and the output Y (middle), and the boundary values of the estimation $\hat{\theta}$ (bottom).

where the c_i are arbitrary positive constants and l_i are integrable bounded nonnegative functions.

We consider next the Lyapunov function

$$V = V_1 + 4V_2. \tag{117}$$

Taking $c_1 = c_2 = c_3 = c_4 = \frac{1}{16}$, we get, from (115)–(117), the inequality

$$\begin{aligned}\dot{V} &\leq -\frac{1}{4}V + lV + l_4 - \frac{1}{2}\phi^2(0) - \frac{1}{2}w^2(0)\\ &\leq -\frac{1}{4}V + lV + l_4\end{aligned} \tag{118}$$

since $\frac{1}{2}\|\phi\|^2 \leq V_1 \leq \|\phi\|^2$, $\frac{1}{2}\|w\|^2 \leq V_2 \leq \|w\|^2$.

Therefore, V is bounded and integrable (Lemma D.3 in Smyshlyaev & Krstic, 2010), and $\|\phi\|, \|w\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. The transformation (105) gives $\hat{v} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and with (102), $\|\psi\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

Then, from (86) and (83), we get that $\|v\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and from (11) that $\|u\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

6.6. Pointwise boundedness

In this section the notation $z(x, \cdot)$ refers to a function of the second argument (time) for a fixed value of x. Hence, when we say $z(x, \cdot) \in \mathcal{L}_p$ we refer to the $\mathcal{L}_p$-boundedness in time for a given x.

The $\mathcal{L}_2$ boundedness of $\|\phi\|$ and $\|\psi\|$ gives $U \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ (see (109)). Therefore, (32) ensures that for all x, $\psi(x, \cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

The following equalities hold:

$$\hat{e}(x) = e(x) - \int_x^1 \tilde{\theta}(\xi)\phi(1 - (\xi - x))d\xi \tag{119}$$

$$\hat{e}(x) = v(x) - \psi(x) - \int_x^1 \hat{\theta}(\xi)\phi(1 - (\xi - x))d\xi. \tag{120}$$

Therefore, with (87) and (119), $\hat{e}(x, \cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and then with (120), $v(x, \cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Then, (11) gives $u(x, \cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ for all $x \in [0, 1]$. With (31), we finally get $\phi(x, \cdot) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

In summary, the solution $(u, \phi, \psi, \hat{\theta})$ is pointwise bounded.

6.7. Convergence

With (118), $\dot{V}$ is bounded from above. As V is also positive and integrable, we obtain that $V \to 0$, that is, $\|w\| \to 0$ and $\|\phi\| \to 0$. From (105), we get $\|\hat{v}\| \to 0$, and from (102), $\|\psi\| \to 0$ follows. (83) and then (11), lead to $\|v\| \to 0$ and $\|u\| \to 0$.

Moreover, with (109), we get $U(t) \to 0$. Therefore, $\psi(x, \cdot)$ tends to 0 (from (32)) and, with (83), we get $v(x, \cdot) \to 0$. Finally, with (11) we get the convergence of $u(x, \cdot)$ to zero.

This completes the proof of Theorem 5.

7. Simulations

We take the example of the Korteweg–de Vries-like equations used in Krstic and Smyshlyaev (2008). The system is determined by three coefficients, a, δ, and ε and a transformation leads to the following PIDE ($b = \sqrt{\frac{a}{\varepsilon}}$):

$$\begin{aligned}u_t(x,t) &= \varepsilon u_x(x,t) - \delta b \sinh(bx)\, u(0,t)\\ &\quad + \delta b^2 \int_0^x \cosh\left(b(x-y)\right)\, u(y,t)dy.\end{aligned} \tag{121}$$

Taking $\varepsilon = 1$ and assuming we want to control PIDE (121) without knowing a and δ, we apply the adaptive output-feedback presented in Section 5. The results of the simulation for $a = 1$, $\delta = 4$, and a constant gain function $\gamma(x) = 1$ in the update law, are given in Fig. 1.

We see on the first graph of Fig. 2 that the open-loop is unstable and oscillatory; the two other graphs in Fig. 2 describe how the adaptive control works. $\hat{\theta}$ is initialized at zero, which makes the start of control slow (very small for at least 2 time units); this slow start of control allows u to grow, which excites the update law, enabling $\hat{\theta}$ to converge towards (but not exactly to) $\theta(t)$. Control then catches up and ensures the convergence of u to zero for all $x \in [0, 1]$. A higher gain function would induce a faster convergence: for instance, $\gamma(x) = 10$ doubles the convergence speed.

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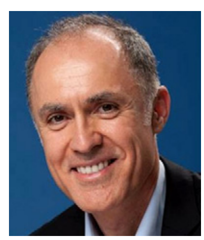

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