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Robustness of the Tuning Functions Adaptive Backstepping Design for Linear Systems

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***Abstract*—In this paper we study robustness of the recently developed adaptive backstepping design with *tuning functions* for linear systems. Under assumptions on unmodeled dynamics and disturbances equal to those for certainty equivalence schemes, we address—for the first time—an adaptive scheme *not* based on the certainty equivalence principle. In the process of redesign for robustness we employ only leakage in the estimator—we do not employ normalization, neither static nor dynamic. A fundamental difference between the tuning functions design and the certainty equivalence designs is that the controller in the former is inherently nonlinear, while in the latter it is nonlinear only in the parameter estimate. As a result, achievable robustness results for the tuning functions scheme are not global but regional, with a region of attraction inversely proportional to the "size" of the unmodeled dynamics. The tracking error is proportional to the size of the uncertainties.**

***Index Terms*—Adaptive backstepping, leakage, robustness, tuning functions, unmodeled dynamics.**

I. Introduction and Problem Statement

Standard results on robust adaptive control apply to certainty-equivalence schemes [2]. Lyapunov-type designs—designs which incorporate the complete state of the plant, filters, and estimators into a Lyapunov function—have been in existence since Feuer and Morse [1] but have only recently become popular in the context of integrator backstepping [3], [4] (the only other Lyapunov scheme that has attracted some attention is Morse's scheme with high-order tuners [6]). In this paper we study robustness of the adaptive backstepping design with *tuning functions* for linear systems. This is the first result available for a Lyapunov-based scheme under general assumptions used in certainty-equivalence robust adaptive control [2].

Adaptive backstepping has so far spawned two classes of methods. The better known tuning functions design [4, Secs. 10.2–10.4] is a

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Lyapunov-based method and is of interest because in the absence of modeling errors, it exhibits the strongest transient performance properties available in the literature (both $\mathcal{L}_2$ and $\mathcal{L}_\infty$). The lesser known modular design [4, Secs. 10.6] is of certainty equivalence type, and, while it inherits some of the advantages of the backstepping method in the nonadaptive context, its adaptive transient performance properties are not as strong. Because of the certainty-equivalence structure, the study of robustness of the modular design follows the route standard in robust adaptive control [2]. In contrast, the robustness study for the tuning functions design, undertaken in this paper, requires a new approach suited for the Lyapunov framework and applicable to controllers which are truly nonlinear.

In the process of redesign for robustness, we make only one modification to the original tuning functions design [3]—we add a switching σ-modification to the tuning functions. This modification affects both the parameter update law and the actual control law because the tuning functions controller incorporates the tuning functions. An important difference from standard robust adaptive control is that we do not employ update law normalization—neither dynamic [2] nor static [7]. Normalization is incompatible with (and even detrimental to) Lyapunov designs because their stability depends on fast adaptation, even in the ideal case.

The result of our paper is that, for sufficiently small μ, the state of the closed-loop system is uniformly ultimately bounded when Δ is improper or has relative degree zero. The region of attraction is proportional to $1/\mu$. When Δ is strictly proper and for sufficiently small μ, the closed-loop state is *globally* uniformly ultimately bounded. In both cases, the mean square of the tracking error is proportional to the size of the uncertainties. The reason for the loss of globality is the nonlinear character of the tuning functions controller. The loss of globality is the price paid for achieving improved transient performance properties in the absence of perturbations [4, Sec. 10.4].

Since the time of the original submission of this paper, the robustness of adaptive backstepping designs has become a topic of active study. Li *et al.* [5] presented a simulation study which indicates that in the absence of robustification tools, the tuning functions design possesses a much higher degree of robustness than certainty-equivalence type designs. Zhang and Ioannou [8]–[10] obtained several results.

- In [8] they provided a robustification to the tuning functions design restricted to the *relative degree two* case. In [9] they studied plants of general relative degree but the unmodeled dynamics were assumed to be strictly proper. Our results are more general and apply to plants with arbitrary relative degree and improper unmodeled dynamics.
- In [10] they develop a certainty equivalence design based on backstepping, similar to our modular design [4, Sec. 10.6]. This design is compatible with standard robustification tools (projection, dynamic normalization, etc.); however, as a certainty-equivalence design, it does not possess the transient performance properties of the tuning functions design.

This paper is organized as follows. In Section II we present the design procedure. Section III deals with the stability and asymptotic performance analysis of the closed-loop system when the transfer function Δ is improper. In Sections IV and V we address the case where Δ is respectively proper and strictly proper.

A. Problem Statement

The control objective is to asymptotically track a reference signal $y_r(t)$ with the output y of the plant

$$y(t) = \frac{B(s)}{A(s)}(1 + \mu\Delta(s))u(t) + d(t) \tag{1}$$

where the polynomials $A(s)$ and $B(s)$ are defined as follows:

$$A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{2}$$

$$B(s) = b_m s^m + \cdots + b_1 s + b_0. \tag{3}$$

The parameters a_i and b_i are unknown. Without loss of generality, we assume that $\mu \geq 0$.

Assumption 1.1: The plant is minimum phase, i.e., the polynomial $B(s)$ is Hurwitz. The plant order (n), relative degree $(\rho = n - m)$, and sign of the high-frequency gain $(\mathrm{sgn}(b_m))$ are known.

Assumption 1.2: The reference signal $y_r(t)$ and its first ρ derivatives are known and bounded and, in addition, $y_r^{(\rho)}$ is piecewise continuous.

Assumption 1.3: The transfer function Δ is stable and its relative degree is no lower than $-\rho + 1$. The output disturbance $d(t)$ and its first derivative are uniformly bounded.

Assumption 1.4: Upper bounds M_θ and M_ϱ of $\|\theta\|$ and $|\varrho| = |1/b_m|$ are known, where $\theta = (b_m, \cdots, b_0, a_{n-1}, \cdots, a_0)^T$ is the unknown parameter vector.

B. Notation

- c Generic positive constant independent of μ, d, $\dot{d}$ and the initial conditions.
- g Generic positive constant independent of μ, d, $\dot{d}$ and possibly depending on the initial conditions.
- h Generic constant scalar, vector or matrix independent of d, $\dot{d}$ and the initial conditions, uniformly bounded with respect to μ.
- κ Generic bounded function of time independent of $\dot{d}$ and the initial conditions, possibly depending on d and μ, and uniformly bounded with respect to μ.

II. Design Procedure

The design procedure follows the steps in [4, Sec. 10.2.1]. The only difference here is that we employ σ-modification in the update law and accordingly modify the control law. We first represent the plant (1) in the observer canonical form

$$\begin{aligned} \dot{x} &= A_0 x + (k - a)x_1 + bu \\ y &= (1 + \mu\Delta)x_1 + d \end{aligned} \tag{4}$$

where

$$A_0 = \begin{pmatrix} & I_{n-1} \\ -k & \\ & 0 \cdots 0 \end{pmatrix}, \qquad k = (k_1, \cdots, k_n)^T$$

$$a = (a_{n-1}, \cdots, a_0)^T, \qquad b = (0_{(\rho-1)\times 1}, b_m, \cdots, b_0)^T. \tag{5}$$

By filtering u and y with two n-dimensional filters

$$\begin{aligned} \dot{\eta} &= A_0\eta + e_n y \\ \dot{\lambda} &= A_0\lambda + e_n u \end{aligned} \tag{6}$$

(where $e_n = [0 \ \cdots \ 0 \ 1]^T$), the state estimate is formed as

$$\hat{x} = B(A_0)\lambda - A(A_0)\eta \tag{7}$$

where $A(\cdot)$ and $B(\cdot)$ are polynomial matrices with argument A_0. Then the estimation error satisfies

$$\begin{aligned} \varepsilon &= x - (B(A_0)\lambda - A(A_0)\eta) \\ \dot{\varepsilon} &= A_0\varepsilon + (a - k)(\mu\Delta x_1 + d). \end{aligned} \tag{8}$$

The adaptive control law is given in Table I. The only differences from the controller in [4, p. 432] are the underbraced terms in (20) and (23). (Note that these terms propagate through the stabilizing functions α_i.) The switching σ-modification is defined as

$$\sigma_\theta = \begin{cases} 0, & \text{if } \|\hat{\theta}\| \le M_\theta \\ \sigma_{s\theta}, & \text{if } \|\hat{\theta}\| \ge 2M_\theta \\ \text{smooth connecting function,} & \text{otherwise} \end{cases}$$
$$\sigma_\varrho = \begin{cases} 0, & \text{if } \|\hat{\varrho}\| \le M_\varrho \\ \sigma_{s\varrho}, & \text{if } \|\hat{\varrho}\| \ge 2M_\varrho \\ \text{smooth connecting function,} & \text{otherwise} \end{cases} \tag{9}$$

for some design constants $\sigma_{s\theta}$ and $\sigma_{s\varrho}$. For an example of an arbitrarily many times differentiable connecting function, please see [9, eq. (27)] (we only need $C^{\rho-1}$). For a discussion of the effect of $\sigma_{s\theta}, \sigma_{s\varrho} > 0$ on performance, the reader is referred to [2].

Consider the Lyapunov function candidate

$$V_\rho = \sum_{j=1}^{\rho}\left(\frac{1}{2}z_j^2 + \frac{1}{d_j}\varepsilon^T P_0 \varepsilon\right) + \frac{|b_m|}{2\gamma}(\varrho - \hat{\varrho})^2 + \frac{1}{2}(\theta - \hat{\theta})^T \Gamma^{-1}(\theta - \hat{\theta}). \tag{10}$$

Noting that the derivative of the tracking error is

$$\dot{z}_1 = x_2 - a_{n-1}y - \dot{y}_r + \mu\Delta(s + a_{n-1})x_1 + \dot{d} + a_{n-1}d \tag{11}$$

and using (23) and (10), by following a derivation similar to [4] it is readily shown that

$$\begin{aligned} \dot{V}_\rho \le & -c_1 z_1^2 + z_1 \underbrace{(\mu\Delta(s + a_{n-1})x_1 + \dot{d} + a_{n-1}d)} \\ & - \sum_{j=2}^{\rho} c_j z_j^2 - \frac{1}{2}\sum_{j=1}^{\rho}\frac{1}{d_j}\|\varepsilon\|^2 \\ & + \Psi^T \varepsilon \underbrace{(\mu\Delta x_1 + d)} - \sum_{j=2}^{\rho}\frac{d_j}{2}\left(\frac{\partial\alpha_{j-1}}{\partial y}\right)^2 z_j^2 \\ & - \sum_{j=2}^{\rho} z_j \frac{\partial\alpha_{j-1}}{\partial y}\underbrace{(\mu\Delta(s + a_{n-1})x_1 + \dot{d} + a_{n-1}d)} \\ & - \sigma_\theta \tilde{\theta}^T(\tilde{\theta} - \theta) - \sigma_\varrho \tilde{\varrho}(\tilde{\varrho} - \varrho) \end{aligned} \tag{12}$$

where we denote

$$\Psi^T = 2\sum_{j=1}^{\rho}\frac{1}{d_j}(a - k)^T P_0. \tag{24}$$

The terms with uncertainties (underbraced) reduce negativity of the Lyapunov inequality (12). Our task in the next section is to quantify the effect of these terms.

III. Robustness Properties with Δ Improper

In this section, we treat the most general case where Δ is improper, with a relative degree no smaller than $-\rho + 1$. The stability analysis is carried out by using a similarity transformation to represent (4) as

$$\begin{aligned} \dot{x}_1 &= x_2 - a_{n-1}x_1 \\ &\vdots \\ \dot{x}_\rho &= c_b^T \bar{x} - a_m x_1 + b_m u \\ \dot{\zeta} &= A_b \zeta + b_b x_1 \\ y &= (1 + \mu\Delta)x_1 + d \end{aligned} \tag{25}$$

TABLE I
Tuning Functions Design with σ-Modification

$$z_1 = y - y_{\mathrm{r}} \tag{13}$$
$$z_i = (A_0^m \lambda)_{i+1} - \hat{\varrho} y_{\mathrm{r}}^{(i-1)} - \alpha_{i-1} \qquad i = 2, \cdots, \rho \tag{14}$$

$$\alpha_1 = \hat{\varrho}\bar{\alpha}_1 \tag{15}$$
$$\bar{\alpha}_1 = -(c_1 + d_1)z_1 - \xi_2 - \bar{\omega}^{\mathrm{T}}\hat{\theta} \tag{16}$$
$$\alpha_2 = -\hat{b}_m z_1 - \left[c_2 + d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2\right]z_2 + \beta_2 + \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\tau_2 \tag{17}$$
$$\alpha_i = -z_{i-1} - \left[c_i + d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\right]z_i + \beta_i + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma\tau_i - \sum_{j=2}^{i-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma\frac{\partial\alpha_{i-1}}{\partial y}z_j \qquad i = 3, \cdots, \rho \tag{18}$$
$$\begin{aligned} \beta_i = & \frac{\partial\alpha_{i-1}}{\partial y}(\xi_2 + \omega^{\mathrm{T}}\hat{\theta}) + \frac{\partial\alpha_{i-1}}{\partial\eta}(A_0\eta + e_n y) \\ & + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_{\mathrm{r}}^{(j-1)}}y_{\mathrm{r}}^{(j)} + k_i\left(A_0^m\lambda\right)_1 + \sum_{j=1}^{m+i-1}\frac{\partial\alpha_{i-1}}{\partial\lambda_j} \\ & \times(-k_j\lambda_1 + \lambda_{j+1}) + \left(y_{\mathrm{r}}^{(i-1)} + \frac{\partial\alpha_{i-1}}{\partial\hat{\varrho}}\right)\dot{\hat{\varrho}} \end{aligned} \tag{19}$$

$$\tau_1 = (\omega - \hat{\varrho}(\dot{y}_{\mathrm{r}} + \bar{\alpha}_1)e_1)z_1 - \underbrace{\sigma_\theta\hat{\theta}} \tag{20}$$
$$\tau_i = \tau_{i-1} - \frac{\partial\alpha_{i-1}}{\partial y}\omega z_i \quad i = 2, \cdots, \rho \tag{21}$$

Adaptive control law:

$$u = \alpha_\rho - \left(A_0^m\lambda\right)_{\rho+1} + \hat{\varrho}y_{\mathrm{r}}^{(\rho)} \tag{22}$$

Parameter update laws:

$$\begin{aligned} \dot{\hat{\theta}} &= \Gamma\tau_\rho \\ \dot{\hat{\varrho}} &= -\gamma\,\mathrm{sgn}(b_m)(\dot{y}_{\mathrm{r}} + \bar{\alpha}_1)z_1 - \underbrace{\gamma\sigma_\varrho\hat{\varrho}} \end{aligned} \tag{23}$$

where $\bar{x} = (x_1, \cdots, x_\rho, \zeta^T)^T$. The vectors b_b and c_b are defined in [4, eqs. (10.129)–(10.133)] and their exact form is not needed in our analysis. The matrix A_b is a companion matrix associated with the polynomial $B(s)$, which means that it is Hurwitz. For stability analysis, we are interested in the deviation $\tilde{\zeta} = \zeta - \zeta_r$ which is governed by

$$\dot{\tilde{\zeta}} = A_b\tilde{\zeta} + b_b\tilde{x}_1, \quad \tilde{\zeta}(0) = 0 \tag{26}$$

where ζ_r is defined as

$$\dot{\zeta}_r = A_b\zeta_r + b_b y_r, \qquad \zeta_r(0) = \zeta(0) \tag{27}$$

and $\tilde{x}_1$ is defined as

$$\tilde{x}_1 = x_1 - y_r. \tag{28}$$

For the η variables we define analogously $(\tilde{\eta} = \eta - \eta_r)$

$$\begin{aligned} \dot{\tilde{\eta}} &= A_0\tilde{\eta} + e_n z_1, \qquad \tilde{\eta}(0) = 0 \\ \dot{\eta}_r &= A_0\eta_r + e_n y_r, \qquad \eta_r(0) = \eta(0). \end{aligned} \tag{29}$$

Define the strictly proper and stable transfer functions Δ_1 and Δ_2 and the states ν_1 and ν_2 as

$$\begin{aligned}\Delta(s+a_{n-1}) &= \sum_{j=0}^{\rho} \varsigma_{1j} s^j + \Delta_1 \\ \Delta &= \sum_{j=0}^{\rho-1} \varsigma_{2j} s^j + \Delta_2 \\ \dot{\nu}_1 &= A_1\nu_1 + b_{1\nu}\tilde{x}_1 \\ \Delta_1\tilde{x}_1 &= (1,0,\cdots,0)\nu_1 = \nu_{11} \\ \dot{\nu}_2 &= A_2\nu_2 + b_{2\nu}\tilde{x}_1 \\ \Delta_2\tilde{x}_1 &= (1,0,\cdots,0)\nu_2 = \nu_{21}.\end{aligned} \tag{30}$$

The matrices A_1 and A_2 are Hurwitz since Δ is stable. We are now ready to introduce the augmented Lyapunov function V

$$V = V_\rho + \frac{1}{k_\eta}\tilde{\eta}^T P_0 \tilde{\eta} + \frac{1}{k_\zeta}\tilde{\zeta}^T P_b \tilde{\zeta} + q_1 \nu_1^T P_1 \nu_1 + q_2 \nu_2^T P_2 \nu_2. \tag{31}$$

Note that V is a quadratic function $V = \chi^T P_\chi \chi$ of the vector

$$\chi = \left(z^T, \varepsilon^T, \tilde{\eta}^T, \tilde{\zeta}^T, \nu_1^T, \nu_2^T, \tilde{\theta}^T, \tilde{\varrho}\right)^T. \tag{32}$$

On the other hand, using (4) and (28) we obtain

$$\tilde{x}_1 = z_1 - \mu\Delta\tilde{x}_1 - \mu\Delta y_r - d. \tag{33}$$

With (12), (31), and (33) we get

$$\begin{aligned}\dot{V} \le& -\frac{c_1}{4}z_1^2 + z_1(\mu\Delta(s+a_{n-1})y_r + \dot{d} + a_{n-1}d) \\ &- \frac{c_1}{4}z_1^2 + \mu z_1\Delta(s+a_{n-1})\tilde{x}_1 - \frac{c_1}{8}z_1^2 - \frac{1}{2k_\eta}\|\tilde{\eta}\|^2 \\ &+ \frac{2}{k_\eta}\tilde{\eta}P_0 e_n z_1 - \frac{c_1}{8}z_1^2 - \frac{1}{4k_\zeta}\|\tilde{\zeta}\|^2 + \frac{2}{k_\zeta}\tilde{\zeta}P_b b_b z_1 \\ &- \frac{c_1}{16}z_1^2 + 2q_1 b_{1\nu}^T P_1 \nu_1 z_1 - \frac{q_1}{8}\|\nu_1\|^2 - \frac{c_1}{16}z_1^2 \\ &+ 2q_2 b_{2\nu}^T P_2 \nu_2 z_1 - \frac{q_2}{8}\|\nu_2\|^2 - \frac{1}{8}\sum_{j=1}^{\rho}\frac{1}{d_j}\|\varepsilon\|^2 \\ &+ \Psi^T\varepsilon(\mu\Delta y_r + d) - \frac{1}{8}\sum_{j=1}^{\rho}\frac{1}{d_j}\|\varepsilon\|^2 + \mu\Psi^T\varepsilon\Delta\tilde{x}_1 \\ &- \frac{1}{8k_\zeta}\|\tilde{\zeta}\|^2 - \frac{2}{k_\zeta}\tilde{\zeta}P_b b_b(\mu\Delta y_r + d) - \frac{1}{8k_\zeta}\|\tilde{\zeta}\|^2 \\ &- \mu\frac{2}{k_\zeta}\tilde{\zeta}P_b b_b\Delta\tilde{x}_1 - \frac{q_1}{2}\|\nu_1\|^2 - 2q_1 b_{1\nu}^T P_1\nu_1(\mu\Delta y_r + d) \\ &- \frac{q_2}{2}\|\nu_2\|^2 - 2q_2 b_{2\nu}^T P_2\nu_2(\mu\Delta y_r + d) - \frac{q_1}{4}\|\nu_1\|^2 \\ &- 2\mu q_1 b_{1\nu}^T P_1\nu_1\Delta\tilde{x}_1 - \frac{q_2}{4}\|\nu_2\|^2 - 2\mu q_2 b_{2\nu}^T P_2\nu_2\Delta\tilde{x}_1 \\ &- \sigma_\theta\|\tilde{\theta}\|(\|\tilde{\theta}\| - \|\theta\|) + \sigma_{s\theta}\|\tilde{\theta}\|^2 - \sigma_\varrho\tilde{\varrho}(\tilde{\varrho} - \varrho) + \sigma_{s\varrho}\tilde{\varrho}^2 \\ &+ \sum_{j=2}^{\rho}\left(-\frac{d_j}{4}\left(\frac{\partial\alpha_{j-1}}{\partial y}z_j\right)^2 \right. \\ &\qquad \left. - \frac{\partial\alpha_{j-1}}{\partial y}z_j(\mu\Delta(s+a_{n-1})y_r + \dot{d} + a_{n-1}d)\right) \\ &+ \sum_{j=2}^{\rho}\left(-\frac{d_j}{4}\left(\frac{\partial\alpha_{j-1}}{\partial y}z_j\right)^2 - \mu z_j\frac{\partial\alpha_{j-1}}{\partial y}\Delta(s+a_{n-1})\tilde{x}_1\right) \\ &- \frac{c_1}{8}z_1^2 - \sum_{j=2}^{\rho}c_j z_j^2 - \frac{1}{4}\sum_{j=1}^{\rho}\frac{1}{d_j}\|\varepsilon\|^2 - \frac{1}{2k_\eta}\|\tilde{\eta}\|^2 \\ &- \frac{1}{2k_\zeta}\|\tilde{\zeta}\|^2 - \frac{q_1}{8}\|\nu_1\|^2 - \frac{q_2}{8}\|\nu_2\|^2 - \sigma_{s\theta}\|\tilde{\theta}\|^2 - \sigma_{s\varrho}\tilde{\varrho}^2.\end{aligned} \tag{34}$$

Thus, if we choose k_η, k_ζ, q_1, and q_2 as

$$\begin{aligned}k_\eta &\ge \frac{16}{c_1}\|P_0 e_n\|^2 k_\zeta \ge \frac{32}{c_1}\|P_b b_b\|^2 \\ q_1 &\le \frac{c_1}{2^7\left\|b_{1\nu}^T P_1\right\|^2} q_2 \le \frac{c_1}{2^7\left\|b_{2\nu}^T P_2\right\|^2}\end{aligned} \tag{35}$$

we obtain

$$\dot{V} \le -\alpha V + \beta + \mu^2 c\left((\Delta(s+a_{n-1})\tilde{x}_1)^2 + (\Delta\tilde{x}_1)^2\right) \tag{36}$$

where

$$\begin{aligned}\alpha &= \min\left\{\frac{c_1}{4}, 2c_2, \ldots, 2c_\rho, \frac{\lambda_{\min}^{-1}(P_0)}{4}, \frac{\lambda_{\min}^{-1}(P_b)}{2}, \right. \\ &\qquad \left. \frac{\lambda_{\min}^{-1}(P_1)}{8}, \frac{\lambda_{\min}^{-1}(P_2)}{8}, \frac{2\gamma\sigma_{s\varrho}}{|b_m|}, \frac{2\sigma_{s\theta}}{\lambda_{\min}(\Gamma^{-1})}\right\} \\ \beta &= c(\mu^2 + \dot{d}^2 + d^2 + \sigma_{s\varrho} + \sigma_{s\theta}).\end{aligned} \tag{37}$$

From (4) and (30) we obtain

$$\begin{aligned}\Delta(s+a_{n-1})\tilde{x}_1 &= hx + h\zeta + hu + \nu_{11} + \kappa \\ \Delta\tilde{x}_1 &= hx + \nu_{21} + \kappa.\end{aligned} \tag{38}$$

Equations (38) show that the residual terms $\Delta(s+a_{n-1})\tilde{x}_1$ and $\Delta\tilde{x}_1$ depend on the coordinates x and λ, while the term $-\alpha V$ in (36) does not contain these coordinates. Thus, we need to express x and λ in terms of χ. Introducing

$$\bar{\lambda}_m = (\lambda_1, \cdots, \lambda_m)^T \tag{39}$$

we obtain from (8)

$$x_j = b_m\lambda_{m+j} + h\bar{\lambda}_{m+j-1} + h\tilde{\eta} + \varepsilon_j + \kappa. \tag{40}$$

It can be shown [4, p. 345] that whenever the polynomials $B(s)$ and $K(s) = s^n + k_1 s^{n-1} + \cdots + k_n$ are coprime, the vector $\bar{\lambda}_m$ can be written as

$$\bar{\lambda}_m = h\varepsilon + h\tilde{\eta} + h\tilde{\zeta} + \kappa. \tag{41}$$

Due to Assumption 1.4, a lower bound on the leading coefficient of $B(s)$ and an upper bound on its other coefficients are known so that all the roots of $B(s)$ lie in a known compact set $\mathcal{K}$ in the complex plane. By choosing the roots of $K(s)$ outside $\mathcal{K}$, we avoid cancellations with the polynomial $B(s)$. With (33) and (38) we get

$$\tilde{x}_1 = z_1 - \mu\left(h\tilde{x}_1 + \sum_{j=2}^{\rho}hx_j + \nu_{21}\right) + \kappa. \tag{42}$$

Combining (40)–(42), we obtain

$$\tilde{x}_1 = \mu h\lambda + h\chi + \kappa \tag{43}$$

for $\mu \le 1/2|h|$. From (43) and (40) for $j = 1$ it follows that

$$\lambda_{m+1} = \mu h\lambda_{m+2,n} + h\chi + \kappa \tag{44}$$

for sufficiently small μ. The vector $\lambda_{m+2,n}$ is defined as

$$\lambda_{m+2,n} = (\lambda_{m+2}, \cdots, \lambda_n)^T. \tag{45}$$

Using (44) and (22) it follows that

$$\lambda_{m+2} = \mu\lambda_{m+2,n}^T P(\chi, d) + Q(\chi, d) + \kappa. \tag{46}$$

The quantities P and Q are vectors whose components are polynomials in χ_j and d. From (44), (46), and (22) it follows recursively that

$$\lambda_{m+2,n} = F\big(\mu\lambda_{m+2,n}, \chi, y_r, \dot{y}_r, \cdots, y_r^{(\rho)}, d\big) \tag{47}$$

where F is a C^1 function with respect to its arguments. Consider the function G and the vector Y defined as

$$\begin{aligned} G(\lambda_{m+2,n}, Y) &= \lambda_{m+2,n} - F(\mu\lambda_{m+2,n}, Y) \\ Y &= \big(\chi^T, y_r, \dot{y}_r, \cdots, y_r^{(\rho)}, d\big)^T. \end{aligned} \tag{48}$$

Note that G vanishes along the trajectories of the closed-loop system. In particular we have

$$G(0,0) = 0. \tag{49}$$

We define the constant matrices

$$\begin{aligned} G_\lambda &= \frac{\partial G}{\partial \lambda_{m+2,n}}(0,0) \\ F_\lambda &= \frac{\partial F}{\partial \mu\lambda_{m+2,n}}(0,0). \end{aligned} \tag{50}$$

The matrix F_λ (respectively, G_λ) is computed by first differentiating the function F (respectively, G) with respect to the vector $\lambda_{m+2,n}$, then putting $\lambda_{m+2,n} = 0$ and $Y = 0$ in the resulting derivative. Thus F_λ and G_λ do not depend on the initial conditions. Using (48) and (50) we obtain

$$G_\lambda = I_{\rho-1} - \mu F_\lambda. \tag{51}$$

Since all vectors h in (44) are uniformly bounded with respect to μ, the matrix F_λ is uniformly bounded with respect to μ. Thus, using the Implicit Function theorem we conclude from (49) and (51) that for $\mu \le 1/2\|F_\lambda\|$ the vector $\lambda_{m+2,n}$ can be written as a C^1 function of Y in a neighborhood $N_{1/\mu}$ of $Y = 0$. In a similar fashion, we can argue that $\lambda_{m+2,n}$ is a C^1 function of μ. We now estimate the size of $N_{1/\mu}$. From (48) we obtain

$$\begin{aligned} &\frac{\partial G}{\partial \lambda_{m+2,n}}(\lambda_{m+2,n}, Y) \\ &\quad = I_{\rho-1} - \mu F_\lambda + \mu F_\lambda - \mu \frac{\partial F}{\partial \mu\lambda_{m+2,n}}(\lambda_{m+2,n}, Y). \end{aligned} \tag{52}$$

Using (48) and (22) it can be shown that

$$\left\| F_\lambda - \frac{\partial F}{\partial \mu\lambda_{m+2,n}}(\lambda_{m+2,n}, Y)\right\| \le c\|\lambda_{m+2,n}, Y\|^{m_1 2^\rho} \tag{53}$$

for some positive integer m_1 independent of ρ. Combining (53) and (52) we obtain

$$\left\|\frac{\partial G}{\partial \lambda_{m+2,n}}(\lambda_{m+2,n}, Y)\right\| \ge \|I_{\rho-1} - \mu F_\lambda\| - \mu c\|\lambda_{m+2,n}, Y\|^{m_1 2^\rho} \tag{54}$$

which is nonzero whenever

$$\|\lambda_{m+2,n}, Y\| < \left(\frac{\|I_\rho - \mu F_\lambda\|}{c\mu}\right)^{\frac{1}{m_1 2^\rho}}. \tag{55}$$

Since $\mu \le 1/2\|F_\lambda\|$ we have from (55)

$$\|Y\| \le c/\mu^{1/m_1 2^\rho}. \tag{56}$$

Thus, we have proved that $\frac{\partial G}{\partial \lambda_{m+2,n}}(\lambda_{m+2,n}, Y)$ is nonsingular in every point of the ball of radius $c/\mu^{1/m_1 2^\rho}$, which implies that $N_{1/\mu}$ contains a ball of radius $c/\mu^{1/m_1 2^\rho}$. It follows also that $\lambda_{m+2,n}$ can be written as a C^1 function of Y and μ inside $N_{1/\mu}$ such that

$$\lambda_{m+2,n} = \Lambda(Y, \mu). \tag{57}$$

Combining (38), (40), (41), (44), (57), (22), and the fact that $\Lambda(Y,\mu)$ is continuous in μ, it follows that for sufficiently small μ, and in $N_{1/\mu}$, we have

$$\begin{aligned} |u| &\le c\|Y\| \\ |\Delta(s + a_{n-1})\tilde{x}_1| &\le c\|Y\| \\ |\Delta\tilde{x}_1| &\le c\|Y\|. \end{aligned} \tag{58}$$

From (36) and (58) it follows that

$$\dot{V} \le -\frac{\alpha}{2}V + 2\beta \tag{59}$$

in $N_{1/\mu}$ and for sufficiently small μ. To conclude that V is uniformly bounded, we need to have

$$\sqrt{\frac{4\beta}{\alpha\lambda_{\max}(P_\chi)}} < c/\mu^{1/m_1 2^\rho} \tag{60}$$

which can be guaranteed for

$$\|d\|_\infty \le \frac{c}{\mu^{1/m_1 2^{\rho+1}}} \quad \text{and} \quad \mu \le \wp(\|\dot{d}\|_\infty) \tag{61}$$

where $\wp$ is a continuous scalar function verifying

$$\wp(x) > 0 \quad \text{and} \quad \lim_{x\to\infty} \wp(x) = 0. \tag{62}$$

The boundedness of the vector $\lambda_{m+2,n}$ follows from (57), the boundedness of V, and the continuity of Λ, whenever $\lambda_{m+2,n}(0) \le c/\mu^{1/m_1 2^\rho}$. The boundedness of the vector λ is derived from (41) and (44). The boundedness of the control u and the state x follow, respectively, from (22) and (40). We now focus on the asymptotic performance of the closed-loop system. Using (34), (58), and the fact that $\sigma_\theta\tilde{\theta}^{\mathrm{T}}(\hat{\theta} - \theta) \ge 0$ and $\sigma_\varrho\tilde{\varrho}(\hat{\varrho} - \varrho) \ge 0$, we obtain

$$\begin{aligned} \dot{V} \le &-\frac{\alpha}{2}\left(\sum_{j=1}^{\rho} z_j^2 + \|\varepsilon\|^2 + \|\tilde{\eta}\|^2 + \|\tilde{\zeta}\|^2 + \|\nu_1\|^2 + \|\nu_2\|^2\right) \\ &+ c(\mu^2 + \dot{d}^2 + d^2). \end{aligned} \tag{63}$$

Integrating both sides of (63) and noting that V is uniformly bounded, the asymptotic performance of our robust adaptive scheme is

$$\begin{aligned} &\int_t^{t+T}\left(\sum_{j=1}^{\rho} z_j^2 + \|\varepsilon\|^2 + \|\tilde{\eta}\|^2 + \|\tilde{\zeta}\|^2 + \|\nu_1\|^2 + \|\nu_2\|^2\right) dt \\ &\quad \le g + c\int_t^{t+T}(\mu^2 + \dot{d}^2 + d^2)\, dt \quad \forall t, T \ge 0. \end{aligned} \tag{64}$$

We now state the main result of this section.

Theorem 3.1: Consider the plant (1) subject to Assumptions 1.1–1.4 and the adaptive controller composed of the control law (22) and the parameter update law (23). There exist positive constants $\mu^\star$, g, and c independent of μ, $\dot{d}$, and d, and a positive integer m_1 independent of ρ such that for $\|\lambda(0), \chi(0)\| \le c/\mu^{1/m_1 2^\rho}$, for $\|y_r\|_\infty + \|\dot{y}_r\|_\infty + \cdots + \|y_r^{(\rho)}\|_\infty \le c/\mu^{1/m_1 2^\rho}$, for $\|d\|_\infty \le$

$c/\mu^{1/m_1 2^{\rho+1}}$ and for any $0 \le \mu < \mu^\star$ we have

1) All the signals of the closed loop are bounded.
2) The tracking error is proportional to the size of perturbations

$$\int_t^{t+T} (y(t) - y_r(t))^2\, dt \le g + c\int_t^{t+T} (\mu^2 + \dot{d}(t)^2 + d(t)^2)\, dt. \tag{65}$$

We point out that c and g are positive constants independent of μ, d, and $\dot{d}$; only g depends on initial conditions.

IV. Robustness Properties with Δ Proper

In this section we suppose that Δ is proper, that is, its relative degree is zero. The stability result is qualitatively the same as for the case Δ improper. Our aim in this section is to give a better estimate of the region of attraction, the allowable disturbances, and reference signals. Similarly to Section III we introduce an augmented Lyapunov function as in (31). Define the state ν_3 as

$$\begin{aligned} \dot{\nu_3} &= A_3\nu_3 + b_3 z_1 \\ \left(\frac{1}{1+\mu\Delta} - \frac{1}{1+\mu\varsigma_{20}}\right) z_1 &= (1, 0, \cdots, 0)\nu_3 = \nu_{31}. \end{aligned} \tag{66}$$

Note that the proper transfer function $1/(1+\mu\Delta)$ is stable and the term $1/(1+\mu\varsigma_{20})$ is well defined for sufficiently small μ. We introduce the final Lyapunov function for our closed-loop system as

$$\begin{aligned} V = V_\rho &+ \frac{1}{k_\eta}\tilde{\eta} P_0 \tilde{\eta} + \frac{1}{k_\zeta}\tilde{\zeta} P_b \tilde{\zeta} + q_1 \nu_1^T P_1 \nu_1 \\ &+ q_2 \nu_2^T P_2 \nu_2 + q_3 \nu_3^T P_3 \nu_3. \end{aligned} \tag{67}$$

Observe that V is a quadratic function $V = \chi^T P_\chi \chi$ of the vector

$$\chi = \left(z^T, \varepsilon^T, \tilde{\eta}^T, \tilde{\zeta}^T, \nu_1^T, \nu_2^T, \nu_3^T, \tilde{\varrho}, \tilde{\theta}^T\right)^T. \tag{68}$$

Similarly to Section III we obtain

$$\dot{V} \le -\alpha V + \beta + \mu^2 c((\Delta(s + a_{n-1})\tilde{x}_1)^2 + (\Delta\tilde{x}_1)^2). \tag{69}$$

From (30), (33), and noting that $\varsigma_{1j} = 0$ for $j \ge 2$ and $\varsigma_{2j} = 0$ for $j \ge 1$ it follows that

$$\begin{aligned} \Delta(s + a_{n-1})\tilde{x}_1 &= \nu_{11} + \varsigma_{10}(z_1 - \mu\nu_{21} - \mu\Delta y_r - d) + \varsigma_{11}\dot{x}_{1r} \\ \Delta\tilde{x}_1 &= \nu_{21} + \varsigma_{20}(z_1 - \mu\nu_{21} - \mu\Delta y_r - d). \end{aligned} \tag{70}$$

Using (4), (5), (30), and (40) for $j = 1$ we obtain

$$\dot{x}_{1r} = h\varepsilon + \mu h\nu_{11} + h\bar{\lambda}_{m+2} + h\eta + hz_1 + \kappa. \tag{71}$$

From (71) it can be seen that the term $\Delta(s+a_{n-1})\tilde{x}_1$ in (70) contains the state $\bar{\lambda}_{m+2}$. Thus, this term cannot be directly cancelled by $-\alpha V$ in (69). The aim of the subsequent analysis is to express this state in terms of the vector χ. From (4) and (66) we have

$$\tilde{x}_1 = \frac{1}{1+\mu\varsigma_{20}} z_1 + \nu_{31} + \kappa. \tag{72}$$

With (72) and (40) for $j = 1$ we get

$$\lambda_{m+1} = h\tilde{\eta} + h\varepsilon + hz_1 + h\tilde{\zeta} + h\nu_{31} + \kappa. \tag{73}$$

Using (22) we obtain

$$\lambda_{m+2} = z_2 + \alpha_1 + h\bar{\lambda}_m. \tag{74}$$

Combining (74), (73), and (41) it follows that

$$\begin{aligned} |\lambda_{m+1}| &\le c(\|\chi\| + \mathcal{Y}_r + \|d\|_\infty) \\ |\lambda_{m+2}| &\le c(\|\chi\| + \mathcal{Y}_r + \|d\|_\infty)^3 \end{aligned} \tag{75}$$

where

$$\mathcal{Y}_r = \|y_r\|_\infty + \|\dot{y}_r\|_\infty. \tag{76}$$

From (71) and (75) we obtain

$$|\dot{x}_{1r}| \le c(\|\chi\| + \mathcal{Y}_r + \|d\|_\infty)^3. \tag{77}$$

With (77), (69), and (70) we get

$$\dot{V} \le -\frac{\alpha}{2} V + 2\beta \tag{78}$$

for

$$\|\chi\| \le \frac{c}{\sqrt{\mu}}, \qquad \mathcal{Y}_r \le \frac{c}{\sqrt[4]{\mu}}, \qquad \|d\|_\infty \le \frac{c}{\sqrt[4]{\mu}}. \tag{79}$$

To conclude that V is uniformly bounded, we need to have

$$\sqrt{\frac{4\beta}{\alpha\lambda_{\max}(P_\chi)}} < \frac{c}{\sqrt{\mu}} \tag{80}$$

which can be guaranteed for

$$\mu \le \wp(\|\dot{d}\|_\infty) \tag{81}$$

where $\wp$ is a continuous scalar function verifying

$$\wp(x) > 0 \quad \text{and} \quad \lim_{x\to\infty} \wp(x) = 0. \tag{82}$$

The boundedness of the control u and the vectors λ and x is shown as in [4, Sec. 10.2.2]. The asymptotic performance is as in (64). We now state the main result of this section

Theorem 4.1: Consider the plant (1) subject to Assumptions 1.1–1.4 and the adaptive controller composed of the control law (22) and the parameter update law (23). If Δ is proper, then there exist positive constants $<\mu^\star$, c, and g independent of μ, d, and $\dot{d}$ such that for $\|\chi(0)\| \le c/\sqrt{\mu}$, for $\|d\|_\infty \le c/\sqrt[4]{\mu}$, for $\|y_r\|_\infty + \|\dot{y}_r\|_\infty \le c/\sqrt[4]{\mu}$ and for any $0 \le \mu < \mu^\star$, we have the following.

1) All the signals of the closed loop are bounded.
2) The tracking error is proportional to the size of perturbations

$$\int_t^{t+T} (y(t) - y_r(t))^2\, dt \le g + c\int_t^{t+T} (\mu^2 + \dot{d}(t)^2 + d(t)^2)\, dt. \tag{83}$$

V. Robustness Properties with Δ Strictly Proper

In this section we suppose that the transfer function Δ is strictly proper. The Lyapunov function V for the closed-loop system is defined as in (31). The derivative $\dot{V}$ is computed as in (25). Using (30) and noting that $\varsigma_{1j} = 0$ for $j \ge 1$ and $\varsigma_{2j} = 0$ for $j \ge 0$, we obtain

$$\begin{aligned} \Delta(s + a_{n-1})\tilde{x}_1 &= \varsigma_{10}(z_1 - \mu\nu_{21} - \mu\Delta y_r - d) + \nu_{11} \\ \Delta\tilde{x}_1 &= \nu_{21}. \end{aligned} \tag{84}$$

From (84) and (36) it follows that

$$\dot{V} \le -\frac{\alpha}{2} V + 2\beta \tag{85}$$

for sufficiently small μ. From (85) we conclude that V is *globally* uniformly bounded. The boundedness of the vectors λ and x and the control u can be shown as in [4, Sec. 10.2.2]. The asymptotic performance is as in (64). We now state the main result of this section.

Theorem 5.1: Consider the plant (1) subject to Assumptions 1.1–1.4 and the adaptive controller composed of the control law (22) and the parameter update law (23). If Δ is strictly proper, then there exist positive constants $\mu^\star$, c, and g independent of μ, d, and $\dot{d}$ such that for every $0 \le \mu < \mu^\star$ we have the following.

1) All the signals of the closed loop are *globally* bounded.
2) The tracking error is proportional to the size of perturbations

$$\int_t^{t+T} (y(t) - y_r(t))^2\, dt \le g + c \int_t^{t+T} (\mu^2 + \dot{d}(t)^2 + d(t)^2)\, dt. \tag{86}$$

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Correction to "Local l_p-Stability and Local Small Gain Theorem for Discrete-Time Systems"

Henri Bourlès

I. Introduction

Section VI of the above-mentioned paper[1] contains an error. The following system Σ_d is considered:

$$y(t+1) = f(y(t-h), \cdots, y(t-1), y(t)) + u(t) \tag{1}$$

where $y(t) \in R^q$ and $u(t) \in R^q$. The state $x(t)$ is defined by $x(t) = (y(t-h), \cdots, y(t-1), y(t)) \in R^n$ with $n = q(h+1)$. The function f is assumed to satisfy $f(0) = 0$ so that zero is an equilibrium point for the unforced system associated with Σ_d (and obtained for $u = 0$). Moreover, it is assumed that f is continuous in a neighborhood U of the zero in R^n and satisfies[2]

$$\sup_{x \in U - \{0\}} \frac{|f(x)|_p}{|x|_p} < 1. \tag{2}$$

The following are proven.[1]

1) Σ_d is locally l_p-reachable (and globally l_p-reachable if $U = R^n$; see Proposition 6 in the above-mentioned paper[1]).
2) The input–output operator associated with Σ_d is locally l_p-stable (and globally l_p-stable if $U = R^n$; see Proposition 7[1]).
3) Consider more generally a nonlinear time-invariant system Σ such that zero is an equilibrium point for the associated unforced system. If Σ is locally (respectively, globally) l_p-reachable, locally (respectively, globally) l_p-observable, and if the associated input–output operator is locally (respectively, globally) l_p-observable, then zero is a locally (respectively, globally) asymptotically stable equilibrium point for the unforced system (see Proposition 4[1]).
4) Σ_d is globally l_p-observable (see Proposition 6[1]).
5) zero is a locally (globally if $U = R^n$) asymptotically stable equilibrium point for the unforced system associated with Σ_d (see Proposition 8[1]).

Remark 1: From the above list, 3) has been established independently in [2] in the continuous-time case; see, also, [1, Th. 2]. In the global continuous-time case, this result was established in [3].

The point is that 4) is erroneous, as shown by the following linear example:

$$y(t+1) = Ay(t) + By(t-1) + u(t). \tag{3}$$

In the linear case, indeed, l_p-observability is equivalent to the usual observability (see Proposition 3 in the above-mentioned paper[1]). It is easy to verify that (3) is not necessarily observable if the matrix A is singular.

In what follows, it is proved that

i) 3) can be improved, replacing l_p-observability by a novel notion called l_p-constructibility. In the linear case, this notion is equivalent to the usual constructibility;
ii) 4) must be replaced by the following proposition: Σ_d is globally l_p-constructible;
iii) and therefore, 5) is true.

II. l_p-Constructibility and Its Use

Consider the time-invariant nonlinear system Σ defined by

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)), \qquad x(t) \in R^n,\ u(t) \in R^m,\ y(t) \in R^q \end{aligned} \tag{4}$$

where f and g are continuous and satisfy $f(0,0) = 0, g(0,0) = 0$. Let us denote as $\phi(t, t_0, x_0, u)$ the state $x(t)$ satisfying (4) with $x(t_0) = x_0$.

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[2] In everything that follows, the notation and the definitions are those of the above-mentioned paper.[1]