A Single Forward-Velocity Control Signal for Stochastic Source Seeking With Multiple Nonholonomic Vehicles

With a single stochastic extremum seeking control signal, we steer multiple autonomous vehicles, modeled as nonholonomic unicycles, toward the maximum of an unknown, spatially distributed signal field. The vehicles, whose angular velocities are constant and distinct, travel at the same forward velocity, which is controlled by the stochastic extremum seeking controller. To determine the vehicles’ velocity, the controller uses sinusoidal perturbations to estimate the gradient while stochastic excitation based on filtered white noise. The positions of the vehicles are not measured. We prove local exponential convergence, both almost surely and in probability, to a small neighborhood near the source and provide a numerical example to illustrate the effectiveness of the algorithm. [DOI: 10.1115/1.4027577]

1 Introduction

For multi-agent systems operating in difficult environments, such as under water, under ice, in caves, or in urban areas, global positioning system (GPS) may be unavailable for navigation. Additionally, design constraints may deem inertial navigation system (INS) systems too expensive or infeasible as is the case for nano/microscale motors being developed for biomedical applications [1–3]. This lack of reliable navigation information has inspired research focused on coordinated control of autonomous agents without position information. We consider the signal source tracking problem with N nonholonomic vehicles that lack position information and are modeled with unicycle dynamics. To achieve convergence to the maximum of the unknown field, we develop a single, scalar stochastic extremum seeking controller to tune simultaneously the forward velocity of each vehicle while the vehicles’ angular velocities are held constant and nonzero.

1.1 Literature Review. Extremum seeking is a nonmodel based real time optimization method that, on the average, estimates the gradient of an unknown function and drives the gradient to zero [4]. Standard deterministic extremum seeking utilizes sinusoidal perturbations to estimate the gradient while stochastic extremum seeking, drawing inspiration from biological systems such as bacterial chemotaxis, employs stochastic perturbations [5]. Recently, extremum seeking has been the focus of much research in theory [6–14] and in applications [15–22], including adaptive fluid flow control in Ref. [23], control of a tunable thermoacoustic cooler in Ref. [24], and control of plasmas in fusion reactors [25,26].

To locate the source of a signal field, vehicles use position information in Refs. [27] and [28] while in Refs. [29] and [30], deterministic extremum seeking controllers steer a single nonholonomic vehicle without any position information toward the source. In Ref. [29], the forward velocity of the vehicle center is controlled and the angular velocity is constant. In Ref. [30], the angular velocity is controlled and forward velocity is constant. Stochastic extremum seeking controllers are used to deploy fully-actuated agents about a source in the plane in Ref. [31] and to steer a single nonholonomic vehicle to a source by tuning its angular velocity in Ref. [32] and forward velocity in Ref. [33].

Many other works have studied multi-agent formation and coverage control with nonholonomic vehicles and/or optimization techniques. Stabilization of any geometric pattern using Laplacian techniques. Stabilization of any geometric pattern using Laplacian

1.2 Contributions. We investigate a stochastic source seeking controller to steer N unicycles to the maximum of an unknown, spatially distributed signal field by tuning the forward velocity of all the vehicles with a single, scalar control signal, extending the results in Ref. [33]. The angular velocity of each vehicle is nonzero, constant, and distinct. The forward velocity command is generated by an extremum seeking controller that is driven by stochastic excitation and the mean of the signal field measurements obtained at the respective vehicle positions. The vehicles’ positions are not measured. Such a controller results in a system with distributed sensing and centralized control achieved by a scalar signal that can be broadcasted to the agents, which is in contrast to the N-dimensional signal one would expect to be necessary.

We prove local exponential convergence, both almost surely and in probability, to a small neighborhood near the source, which requires the stability analysis of a time-varying linear system. The convergence result also holds for the single vehicle case, which is the stochastic counterpart to [29] in the same way that [32] is a stochastic counterpart to [30]. Only local results for stochastic averaging exist to date [5], and while an extension to the
1.3 Organization. In the following section, we provide the vehicle model, signal field map, and stochastic forward velocity controller. Then, we prove local exponential convergence to a small neighborhood of signal field’s maximum, present simulation results, and offer some concluding remarks.

2 Problem Statement

Consider N nonholonomic vehicles in an unknown, distributed signal field. Our goal is to design a single, scalar forward velocity controller that is driven by all the vehicles’ signal field measurements to achieve convergence to the maximum of the signal field. A distributed sensor network with a central monitoring station or leader agent that issues commands to a group of agents fits within this framework.

2.1 Autonomous Vehicle Model. For each of the N nonholonomic vehicles, we consider a unicycle model with a sensor located at the center of the vehicle. The equations of motion for the vehicle center are

\[
\begin{align*}
\dot{x}_i &= v_i \cos(\theta_i) \\
\dot{y}_i &= v_i \sin(\theta_i) \\
\dot{\theta}_i &= \omega_i
\end{align*}
\]

where \((x_i, y_i) = z_i\) is the center of vehicle \(i\), \(\theta_i\) is the orientation, \(v_i\), \(\omega_i\) are the forward and angular velocity inputs, and \(i \in \{1, \ldots, N\}\). A diagram depicting the position, heading, angular and forward dynamics of the vehicles can be written in vector form as

\[
\begin{align*}
\mathbf{x}_i &= \begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix}, \\
G(\Theta) &= \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ \vdots \\ \cos(\theta_N) \\ \sin(\theta_N) \end{bmatrix}
\end{align*}
\]

For our analysis, we assume that the nonlinear map is quadratic

\[
Q(x, y) = Q^* + \frac{1}{2} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}^T H \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}
\]

where \(H = H^T < 0\). For nonquadratic maps, Eq. (7) serves as a local approximation.

2.2 Signal Field. The vehicles wish to track the source of an unknown signal field that is distributed according to the nonlinear map \(Q(x, y)\), which has an isolated maximum \(Q^* = Q(x^*, y^*)\) at \((x^*, y^*)\). Our goal is to tune the forward velocity of the N vehicles to achieve local convergence to the maximizer \((x^*, y^*)\) using only measurements of the signal field at each vehicle position \((x_i, y_i)\). The vehicle positions \((x_i, y_i)\) are not measured and the shape of \(Q(x, y)\) is unknown.

For each of the \(N\) vehicles, we consider a vehicle model with a sensor located at the center of the vehicle, \(z_i\) is the forward velocity, and \(\omega_i\) is the angular velocity.

2.3 Stochastic Forward Velocity Controller. To achieve convergence to the signal field source, we employ the following centralized controller that tunes the forward velocity of each vehicle:

\[
\begin{align*}
\dot{\mathbf{z}} &= G(\Theta) \mathbf{v} \\
\dot{\Theta} &= \Omega
\end{align*}
\]

where \(z = [x_1, y_1, \ldots, x_N, y_N]^T, \Theta = [\theta_1, \ldots, \theta_N]^T, \Omega = [\omega_1, \ldots, \omega_N]^T, \) and

\[
G(\Theta) = \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ \vdots \\ \cos(\theta_N) \\ \sin(\theta_N) \end{bmatrix}
\]

In Ref. [32], the sensor is noncollocated with the vehicle’s center and the forward velocity is held constant while the angular velocity is tuned. When tuning the forward velocity while the angular velocity is constant, there is no essential difference between the collocated and noncollocated sensor cases, but the calculations are more complex in the latter scenario.

Selecting the parameter values of \(c, a, g\) is subject to several tradeoffs related to transient performance (convergence process) and asymptotic performance (radius of residual set within which the vehicles move after converging to the source). Larger values speed up the convergence but require more control effort and may increase the overshoot when arriving near the source. Larger values also speed up the convergence, whose rate is dictated by \(a(1 - e^{-x}) \approx ag^2/2\), but increase the set of residual motion and the range of exploration (wandering during convergence) for each vehicle.

A block diagram of the stochastic extremum seeking scheme with a washout filter is shown in Fig. 2. The washout filter can be realized as

\[
\dot{z} = -h_\omega + hJ
\]
controller derived for a single vehicle in Ref. [29], which is shown below for reference:

\[
\xi = J - \xi
\]

(12)

with \( \xi \) replacing the occurrence of \( J \) in Eq. (8). Since the washout filter is not needed to establish our convergence result, we omit it in our analysis to avoid increasing the state dimension and obfuscating the convergence of the vehicle positions to \((x^*, y^*)\). While not necessary, the washout filter, when implemented removes the DC component of \( J \), which typically improves performance.

The controller (8) is analogous to the deterministic velocity controller derived for a single vehicle in Ref. [29], which is shown below for reference:

\[
v = cJ \sin(\omega t) + a - \frac{\sin(\omega t)}{s}
\]

(13)

where \( \omega \) is the frequency of the deterministic perturbation signal. Both Eqs. (8) and (13) have two terms: (1) the signal field measurement multiplied by the perturbation signal, which follows from the standard extremum seeking design [4], and (2) the derivative of the perturbation signal, which when integrated by the unicycle dynamics results in the additive modulation signal found in the standard designs. The computation of the derivative term in Eq. (8) follows from Ito’s formula. As noted in Ref. [32], \( \sin(\eta) \) is used in the stochastic case to develop a gradient estimate just as \( \sin(\omega t) \) is used in the deterministic design. Choosing the sinusoidal nonlinearity for the stochastic design, however, is not required; this choice is made mainly to facilitate the stability analysis.

3 Convergence Result

Before stating the convergence result, we develop the error system dynamics for the vehicles’ position error relative to the maximizer \((x^*, y^*)\). For \( i \in \{1, \ldots, N\} \), define

\[
\begin{align*}
\tilde{x}_i &= x_i - x^* - a \cos(\theta_i) \sin(\eta) \\
\tilde{y}_i &= y_i - y^* - a \sin(\theta_i) \sin(\eta)
\end{align*}
\]

(14)

(15)

which leads to the vector error variable

\[
\tilde{z}^e = z - \tilde{z}^e - aG(\Theta) \sin(\eta)
\]

(16)

where \( \tilde{z}^e = [x^e, y^e, \ldots, x^e, y^e]^T \) is a \( 2N \times 1 \) vector and the superscript \( e \) is used to make the dependence on the small parameter \( \varepsilon \) more clear. The cost value (10) can then be expressed as

\[
J = Q^* + \frac{1}{2N} [\tilde{z}^e + aG(\Theta) \sin(\eta)]^T H [\tilde{z}^e + aG(\Theta) \sin(\eta)]
\]

(17)

where \( H = \text{diag}[H, \ldots, H] \) with \( N \) entries and \( \text{diag}[,] \) denotes a block diagonal matrix. Hence, \( H = \varepsilon I < 0 \) is a \( 2N \times 2N \) matrix. From Eqs. (4), (8), (16), and Ito’s formula, we obtain the following error dynamics:

\[
\frac{d\tilde{z}^e}{dt} = (-aF(\Theta, \Omega) + aG(\Theta)) \sin(\eta)
\]

\[
= -aF(\Theta, \Omega) \sin(\eta) + c \sin(\eta)G(\Theta)Q^*
\]

\[
+ \frac{e^2}{2N} \sin(\eta) G(\Theta)^T \tilde{z}^e H \tilde{z}^e + 2aG^T(\Theta)H \tilde{z}^e \sin(\eta)
\]

\[
+ a^2G^T(\Theta)H G(\Theta) \sin^2(\eta)
\]

(18)

\[
\frac{d\theta}{dt} = \Omega
\]

(19)

where

\[
F(\Theta, \Omega) = \frac{d}{dt} G(\Theta) = \\
\begin{bmatrix}
-\omega_1 \sin(\theta_1) \\
\omega_1 \cos(\theta_1) \\
\vdots \\
-\omega_N \sin(\theta_N) \\
\omega_N \cos(\theta_N)
\end{bmatrix}
\]

(20)

By the definition of the Ito stochastic differential equation, we have

\[
\eta(t) = \eta(0) - \int_0^t \frac{1}{\sqrt{\varepsilon}} \eta(s) ds + \int_0^t \frac{\varepsilon}{\sqrt{\varepsilon}} dW(s)
\]

(21)

Thus, it holds that

\[
\eta(xt) = \eta(0) - \int_0^t \frac{1}{\sqrt{\varepsilon}} \eta(xu) du + \int_0^t \frac{\varepsilon}{\sqrt{\varepsilon}} dW(xu)
\]

(22)

Now define

\[
B(t) = \frac{1}{\sqrt{\varepsilon}} W(xt)
\]

(23)

\[
\chi(t) = \eta(xt)
\]

(24)

Then, the error dynamics (18) and (19) become

\[
\frac{d\tilde{z}^e}{dt} = -aF(\Theta, \Omega) \sin(\chi(xt)) + c \sin(\chi(xt))G(\Theta)Q^*
\]

\[
+ \frac{e^2}{2N} \sin(\chi(xt)) G(\Theta)^T \tilde{z}^e H \tilde{z}^e + 2aG^T(\Theta)H \tilde{z}^e \sin(\chi(xt))
\]

\[
+ a^2G^T(\Theta)H G(\Theta) \sin^2(\chi(xt))
\]

(25)

\[
\frac{d\theta}{dt} = \Omega
\]

(26)

where

\[
\text{d}\chi(t) = -\chi(t)dt + \text{d}B(t)
\]

(27)

\( B(t) \) is a standard Brownian motion, and the process \( \chi(t) \) is an Ornstein–Uhlenbeck (OU) process, which is ergodic with invariant distribution

\[
\mu(dy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\]

(28)

We are now able to state the convergence result:

**Theorem 1.** Consider the vehicle (1)–(3) with the forward velocity controller (8). If the error dynamics (25) and (26) have a
unique continuous solution on \([0, \infty)\), then there exist constants \(r, M, \gamma > 0\) and a function \(T(e) : (0, \varepsilon_0) \to \mathbb{N}\) such that for any initial condition \(z(0) - z^* < r\) and any \(\delta > 0\)

\[
\lim_{t \to 0} \left\{ t \geq 0 : \left[ \begin{array}{c} z^*(t) - z^* \\ \Theta(t) - \Omega - \Theta_0 \end{array} \right] \geq M|z^*(0) - z^*| + \delta + O(a) \right\} = +\infty, \text{ a.s.}
\]

(29)

and

\[
\lim_{t \to 0} P \left\{ t \geq 0 : \left[ \begin{array}{c} z^*(t) - z^* \\ \Theta(t) - \Omega - \Theta_0 \end{array} \right] \leq M|z^*(0) - z^*| + \delta + O(a) \right\} = 1
\]

(30)

with \(\lim_{t \to 0} T(e) = +\infty\).

**Proof.** To analyze the stability properties of Eq. (25), we apply the stochastic averaging theory [5] and compute Eq. (A3), where \(a(x, y)\) is the right-hand side of Eqs. (25) and (26), to obtain the average system

\[
\frac{d\mathbf{z}^{\text{ave}}}{dt} = \frac{ca}{2N} \left(1 - e^{-\frac{x^2}{\sigma^2}}\right) G \left(\mathbf{z}^{\text{ave}}\right) G^T \left(\mathbf{z}^{\text{ave}}\right) H \mathbf{z}^{\text{ave}}
\]

(31)

\[
\frac{d\Theta^{\text{ave}}}{dt} = \Omega
\]

(32)

which is found by solving integrals of the following forms:

\[
\int_{-\infty}^{\infty} \sin^{2k+1}(y) \text{d}y = \begin{cases} 0, & k \in \{0, 1\}, \\ 1, & k \geq 2. \end{cases}
\]

(33)

The average system (31) and (32) is not exponentially stable due to the constant forcing in Eq. (32). However, Eq. (32) can be solved directly, yielding

\[
\Theta^{\text{ave}}(t) = \Omega + \Theta_0, \quad \forall t \geq 0
\]

(34)

where \(\Theta_0 = [\theta_1, \ldots, \theta_N]^T\), which allows Eq. (31) to be rewritten as the linear time-varying system

\[
\Psi(t) = \begin{bmatrix} \cos(\omega_1 t + \theta_{10}) & \cos(\omega_1 t + \theta_{10}) \\ \sin(\omega_1 t + \theta_{10}) & \sin(\omega_1 t + \theta_{10}) \\ \vdots & \vdots \\ \cos(\omega_N t + \theta_{N0}) & \cos(\omega_N t + \theta_{N0}) \\ \sin(\omega_N t + \theta_{N0}) & \sin(\omega_N t + \theta_{N0}) \end{bmatrix}
\]

\[
G(\Omega + \Theta_0) G^T (\Omega + \Theta_0)
\]

(35)

By the persistency of excitation property of \(G(\Omega + \Theta_0)\), Eq. (35) can be shown to be exponentially stable for any \(\omega_i \neq \omega_j, i \neq j\), and for any \(\theta_{ij}\), where \(i, j \in \{1, \ldots, N\}\) (see Ref. [46], Lemma 3.4). Thus, there exist constants \(M, \gamma > 0\) such that the bound

\[
\liminf_{t \to 0} \left\{ t \geq 0 : \left[ \begin{array}{c} x^*(t) - x^* \\ \Psi(t) - \Omega - \Theta_0 \end{array} \right] \geq M \left[ x^*(0) - x^* \right] e^{-\gamma t} + \delta + O(a) \right\} = +\infty, \text{ a.s.}
\]

(42)

and

\[
\lim_{t \to 0} P \left\{ t \geq 0 : \left[ \begin{array}{c} x^*(t) - x^* \\ \Psi(t) - \Omega - \Theta_0 \end{array} \right] \leq M \left[ x^*(0) - x^* \right] e^{-\gamma t} + \delta + O(a) \right\} = 1
\]

(43)

with \(\lim_{t \to 0} T(e) = +\infty\).
4 Simulation Results

To demonstrate the forward velocity controller (8) shown in Fig. 2, we now present numerical examples with \( N = \{5, 20, 40\} \) vehicles and initial positions in either a circle formation or about a single point. We also explore the use of a weighted cost function where the signal measurement weights may indicate the quality of a vehicle’s measurement.

For the simulations, the unknown signal field is

\[
Q(x, y) = 1 - \frac{1}{2}(x - x^*)^2 - \frac{7}{4}(y - y^*)^2
\]

where \((x^*, y^*) = (0, 0)\), and unless otherwise stated, the controller parameters are \( c = 150, a = 0.05, e = 0.03, \) and \( g = 1.0 \). The washout filter parameter value is \( h = 1 \). The system is simulated in continuous time with the forward velocity commands sampled at 100 Hz and held constant between samples.

We denote the unicycles as \( U_i \) where \( i \in \{1, \ldots, N\} \) and initialize them either on a circle about the source or randomly clustered about a single point that is offset from the source to show that the convergence is not dependent on the source being encircled by the vehicles. The initial orientation for each vehicle is randomly drawn from the range \([0, 2\pi) \) rad, and the angular velocity, from the range \([1, 30] \) rad/s.

First, we consider simulations with the unicycles initialized on a circle with radius 2.5 about the source. The planar trajectories of the vehicles converging toward the signal source for the each value of \( N \) are shown in Fig. 3 with the contour lines of the signal field superimposed. The average of the signal field measurements \( J \) for the three scenarios is shown in Fig. 3(d). From both the planar trajectories and the average cost value, we see that the convergence rate slows linearly as the number of vehicles is increased. Figure 4 is a zoomed in portion of \( U_1 \)’s trajectory to highlight the star-pattern that occurs on the average, which is similar to the star-pattern that occurs when deterministic extremum seeking is employed [29].

Figure 5 depicts the same content as Fig. 3 but with the vehicles initialized randomly about the point \((2,2)\). The vehicles converge to the source without issue when initialized in a small cluster, showing that convergence is not dependent on the vehicles’ initial formation. Again, one can see that the rate of convergence linearly decreases as the number of vehicles is increased. The time \( t^* \), which is when the inequality \( |J - Q^*| < 1 \) is first satisfied, is plotted in Fig. 6, showing the linear relationship between the number of vehicles in the swarm and \( t^* \). The time \( t^* \) is used as a settling time metric to approximate the vehicles’ convergence rate to \( Q^* = 1 \).

The scalar velocity command used by all the vehicles in Fig. 5(a) and the vehicles’ signal field measurements are shown in Fig. 7. Note how the cost \( J \) is monotonically increasing with occasional steplike increases that correspond to “spikes” in the velocity command. Not every vehicle, however, increases its signal field measurement during these steplike increases as seen at the 25 and 35 s marks in Fig. 7(b). After 50 s, the velocity command remains at its steady state levels with a mean value near zero.

![Fig. 3 Trajectories of the vehicle centers converging to a neighborhood of the source at (0,0) from an initial circle formation for (a) \( N = 5 \), (b) \( N = 20 \), and (c) \( N = 40 \) vehicles. The signal field’s contour lines are superimposed. The cost \( J \) for each scenario is shown in (d).](http://asmedigitalcollection.asme.org/)
When the number of vehicles in a swarm changes is increased significantly, retuning the controller parameters can mitigate this decrease in convergence rate, particularly since the higher number of vehicles/signal field measurements leads to more averaging for the cost value $J$, which allows for larger parameter values to be used. Figure 8 shows the same scenario as Fig. 3 but with the parameter $c$ increased from 150 to 250. Note the faster convergence compared to Fig. 3.

An alternative to using the average signal field measurement (10) to drive the forward velocity controller (8) is to use a weighted average where the centralized controller may assign weights based on the quality of the individual measurements, namely

$$J = \sum_{i=1}^{N} \lambda_i Q(x_i, y_i)$$

(44)

Fig. 5 Trajectories of the vehicle centers converging to a neighborhood of the source at (0, 0) from a random clustering about the point (2,2) for (a) $N = 5$, (b) $N = 20$, and (c) $N = 40$ vehicles. The signal field’s contour lines are superimposed. The average cost $J$ for each scenario is shown in (d).
where $\lambda_i > 0$ and $\sum \lambda_i = 1$. Figure 9 shows the planar trajectories and signal field measurements when using a weighted cost value for $N = 20$ vehicles initialized in a circular formation. The weights $\lambda_i$ are randomly selected with unicycles U6, U9, U15, and U17 assigned the smallest values. (U6 has the smallest weight overall.) Hence, these vehicles converge to the source at a much smaller rate than the other, more trusted vehicles. Note the difference in trajectories between Figs. 3(b) and 9(a).
5 Concluding Remarks

We have introduced a single stochastic forward velocity controller to steer \( N \) unicycles, with constant and distinct angular velocities, toward the maximum of an unknown signal field, extending the results of Ref. [33]. The controller is driven by the mean of the vehicles’ signal field measurements and stochastic excitation. The vehicles’ positions are not measured. To prove local exponential convergence, both almost surely and in probability, to a neighborhood of the maximum, we utilize the persistence of excitation property and the approximation results of stochastic averaging.

This approach alleviates the need for complex computers/controllers onboard the vehicles, which may be either cost prohibitive or physically infeasible as is the case for today’s nano/microscale motors, and requires 1/Nth the communication bandwidth of a fully decentralized controller since the same control signal is sent to all the agents, which is particularly important in underwater applications where bandwidth is limited. Having a common control signal broadcast to all agents is also more “secure” because an eavesdropper cannot infer any information regarding any individual agent from the control signal. Implementation of the controller, however, does require the collection of sensor data from all the agents, either by direct communication or by centralized monitoring of the agents.

To explore fully the tradeoffs between fully decentralized and centralized source seeking control, a future extension is to analyze the scenario where each vehicle employs its own stochastic excitation and its controller is driven by the mean of its neighboring agents’ signal field measurements. Each velocity controller would require only local rather than global information. As a consequence of the local controller approach, the vehicles would no longer travel at the same speed, which may have implications on the overall convergence properties of the group.

Other possible extensions to this work include the following: characterizing the agents’ collective convergence rate when weighting the signal field measurements according to a vehicle’s assumed sensor noise/quality (as mentioned in Sec. 4), determining the communication and/or centralized controller requirements needed to identify local extrema to guarantee that the agents converge to the global optimum, and applying this approach to vehicles/agents with more detailed actuation models to realize the forward-velocity command.

Acknowledgment

The research was supported by the National Science Foundation of China under grant 61174043 and also by Jiangsu NSF under grant BK2011582.

Appendix

Stochastic Averaging

Consider the system

\[
\frac{dX_t^i}{dt} = a(X_t^i, Y_t^j), \quad X_{0}^i = x
\]  
(A1)

where \( X_t^i \in \mathbb{R}^n, Y_t \in \mathbb{R}^m \) is a time homogeneous continuous Markov process defined on a complete probability space \( (\Omega, \mathcal{F}, P) \), where \( \Omega \) is the sample space, \( \mathcal{F} \) is the \( \sigma \)-field, and \( P \) is the probability measure. The initial condition \( X_0^i = x \) is deterministic. \( a \) is a small parameter in \( (0, \varepsilon_0) \) with fixed \( \varepsilon_0 > 0 \). Let \( S_y \subseteq \mathbb{R}^m \) be the living space of the perturbation process \( (Y_t, t \geq 0) \) and note that \( S_y \) may be a proper (e.g., compact) subset of \( \mathbb{R}^m \).

Consider the following assumptions:

**Assumption 1.** The vector field \( a(x, y) \) is a continuous function of \((x, y)\), and for any \( x \in \mathbb{R}^n \), it is a bounded function of \( y \). Further it satisfies the locally Lipschitz condition in \( x \in \mathbb{R}^n \) uniformly in \( y \in S_y \), i.e., for any compact subset \( D \subset \mathbb{R}^n \), there is a constant \( k_D \) such that for all \( x', x'' \in D \) and all \( y \in S_y \),

\[
|a(x', y) - a(x'', y)| \leq k_D |x' - x''|.
\]

**Assumption 2.** The perturbation process \((Y_t, t \geq 0)\) is ergodic with invariant distribution \( \mu \).

Under Assumption 2, we obtain the average system of system (A1) as follows:

\[
\frac{dX_t}{dt} = \bar{a}(X_t), \quad X_{0} = x
\]  
(A2)

where

\[
\bar{a}(x) = \int_{S_y} a(x, y)\mu(dy)
\]  
(A3)

**Assumption 3.** For any \( x \in \mathbb{R}^n \) and the perturbation process \((Y_t, t \geq 0)\), system (A1) has a unique (almost surely) continuous solution on \([0, \infty)\).

**Assumption 4.** The average system (A2) has a solution on \([0, \infty)\).

We have the following approximation result.

**Theorem 2.** Consider system (A1) under Assumptions 1–4. Then

(i) for any \( \delta > 0 \) [5]

\[
\liminf_{t \to 0} \{ \begin{array}{c}
|x_t - \bar{x}| > \delta  \\
\end{array} \} \Rightarrow +\infty, \quad a.s.
\]

(ii) there exists a function \( T(u) : (0, \varepsilon_0) \to \mathbb{N} \) such that for any \( \delta > 0 \)

\[
\lim_{t \to 0} P \left\{ \sup_{0 < t < T(u)} |X_t - \bar{X_t}| > \delta \right\} = 0
\]

where

\[
\lim_{t \to 0} T(u) = +\infty
\]

References


