Compensation of Wave Actuator Dynamics for Nonlinear Systems

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***Abstract*—The problem of stabilization of PDE-ODE cascades has been solved in the linear case for several PDE classes, whereas in the nonlinear case the problem has been solved only for the transport/delay PDE, namely for compensation of an arbitrary delay at the input of a nonlinear plant. Motivated by a specific engineering application in off-shore drilling, we solve the problem of stabilization of the cascade of a wave PDE with a general nonlinear ODE. Due to the presence of nonlinearities of arbitrary growth and the time-reversibility of the wave PDE, and due to the possibility of using arguments based on Lyapunov functionals or explicit solutions, several stability analysis approaches are possible. We present stability results in the $H_2 \times H_1$ and $C^1 \times C^0$ norms for general nonlinear ODEs, as well as in the $H_1 \times L_2$ norm for linear ODEs. We specialize our general design for wave PDE-ODE cascades to the case of a wave PDE whose uncontrolled end does not drive an ODE but is instead governed by a nonlinear Robin boundary condition (a "nonlinear spring," as in the friction law in drilling). This is the first global stabilization result for wave equations that incorporate non-collocated destabilizing nonlinearities of superlinear growth. We present two numerical examples, one with a nonlinear ODE and one with a nonlinear spring at the uncontrolled boundary of the wave PDE.**

***Index Terms*—Distributed parameter systems, nonlinear control systems, delay systems.**

I. Introduction

A common type of instability in oil drilling is the friction-induced stick-slip oscillation [8], which results in torsional vibrations of the drillstring and can severely damage the drilling facilities (see Fig. 1 from [26]). The torsion dynamics of a drillstring are modeled as a wave PDE (that describes the dynamics of the angular displacement of the drillstring) coupled with a nonlinear ordinary differential equation (ODE) that describes the dynamics of the bottom angular velocity of the drill bit [27]. A control approach based on linearization is presented in [26]. In this article we present a design for general nonlinear ODE plants with a wave PDE as its actuator dynamics. This design solves the oil drilling problem (globally) as a special case.

For linear systems, predictor feedback and backstepping have been successful in compensation of input delays [2], [7], [16], [21], [25], [40] as well as for the compensation of more complex input dynamics such as diffusive and wave PDEs [3], [4], [16], [24], [32]–[34]. For nonlinear systems, results are available for the compensation of input delays [10], [11], [17], [22], [23] but there exist no results for the *compensation* of more complex actuator dynamics in nonlinear systems, although there are results for the control of nonlinear PDEs using backstepping [19], [20], [36]–[39].

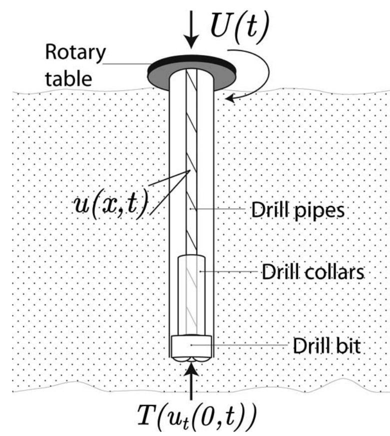

Fig. 1. A drillstring used in oil drilling. The angular displacement u of the drillstring is controlled through a torque U.

In this article, we consider finite-dimensional nonlinear plants which are controlled through a string and design a predictor-based feedback law that compensates the string (wave) dynamics in the input of the plant. Our design is based on a preliminary transformation which allows one to convert the wave PDE to a 2×2 system of first-order transport equations which convect in opposite directions (one towards the plant and one away from the plant), with an additional scalar state (Section II). We then introduce a backstepping transformation of the transport state which convects towards the plant. Backstepping transforms the original system to a "target system" for which we construct a Lyapunov functional. Due to the fact that the transformed 2×2 system of the first-order transport equations is not autonomous (the state of the plant is acting back on the actuator state through a nonlinear relation) we have to use in our Lyapunov functional the $H_2 \times H_1$ norm of the actuator state (Section III).

We also provide stability estimates in the (lower) $C^1 \times C^0$ norm of the actuator state. For proving stability in the $C^1 \times C^0$ norm we use two different techniques-one based on a Lyapunov construction and one based on estimates on the solutions of the closed-loop system. In the former case, a Lyapunov construction is possible after introducing a new backstepping transformation, of the transport PDE that convects in the opposite direction (i.e., away from the plant), that

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TABLE I
STABILITY RESULTS IN THE ARTICLE

Stability Result	Norm (u, u_t)	Extra Assumption	Target System	Proof Approach
Theorem 1	$H_2 \times H_1$	—	(34)–(38) (see Fig. 3)	Lyapunov/ISS
Theorem 2	$C^1 \times C^0$	ODE bkwd complete	(63)–(67) (see Fig. 4)	Lyapunov/ISS
Theorem 3	$C^1 \times C^0$	—	(10)–(14) (see Fig. 5)	w/ explicit sol's
Theorem 4 (linear)	$H_1 \times L_2$	—	(63)–(67) (see Fig. 4)	Lyapunov/ISS

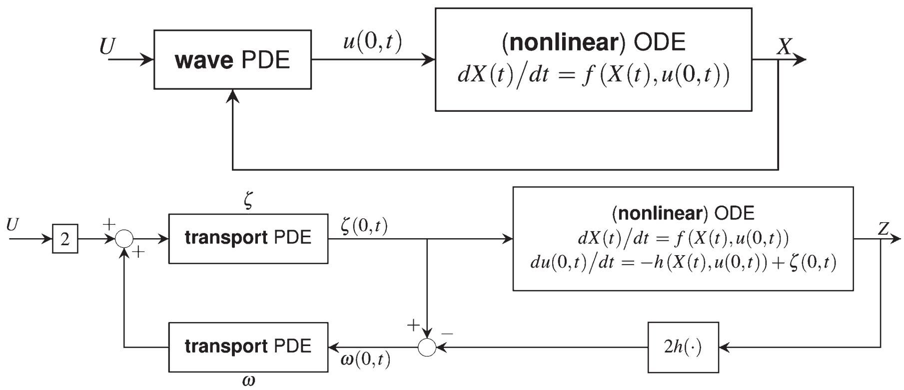

Fig. 2. Top: A nonlinear system with a wave PDE in the input as in (1)–(4). Bottom: The equivalent representation of the wave PDE/nonlinear ODE cascade using the change of variables (5), (6), as given in (10)–(14) and using the fact that $u_t(1,t) = U(t) + \omega(1,t)$.

converts the original system to a new target system. In this new target system the transformed 2×2 transport system is autonomous, and in particular, is a cascade of two transport PDEs with coupling only at the boundary (Section IV-A). Yet, this Lyapunov construction comes with one limitation-in order to guarantee the well-posedness and invertibility of the new backstepping transformation, one has to impose the assumption that both the open and closed-loop systems are not only forward but also backward complete. Our second, alternative proof, is based on constructing estimates on the solutions of the closed-loop system and exploiting the predictor nature of the feedback law (Section IV-B). We specialize our results to linear systems, for which we prove exponential stability in the (most desirable) $H_1 \times L_2$ norm of the actuator state (Section V).

Our results are new even when we remove the ODE part from the PDE-ODE cascade and consider the stabilization problem of the wave PDE whose uncontrolled end does not drive an ODE but is instead governed by a nonlinear spring. We specialize our general design to this case and design the first global stabilizing control law for wave equations that incorporate non-collocated destabilizing nonlinearities of superlinear growth (Section VI). We extend our design methodology to nonlinear systems with actuator dynamics that are governed by a wave PDE with anti-damping on the uncontrolled boundary (Section VII). Finally, we present two numerical examples. One with a nonlinear ODE and one with a nonlinear spring at the uncontrolled boundary of the wave PDE (Section VIII).

Notation: We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions from [13]. For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. For a scalar function $u \in L_\infty[0,1]$ we denote by $\|u(t)\|_\infty$ its supremum norm, i.e., $\|u(t)\|_\infty = \sup_{x\in[0,1]} |u(x,t)|$. For a scalar function $u \in H_1(0,1)$, we denote by $\|u(t)\|_{H_1}$ the norm $\|u(t)\|_{H_1} = (\int_0^1 u(x,t)^2 dx)^{1/2} + (\int_0^1 u_x(x,t)^2 dx)^{1/2}$. For any $c > 0$, we denote the weighted supremum norm of u by $\|u(t)\|_{c,\infty} = \sup_{x\in[0,1]} e^{c(1+x)}|u(x,t)|$ and the weighted H_1 norm by $\|u(t)\|_{c,H_1} = (\int_0^1 e^{c(1+x)} u(x,t)^2 dx)^{1/2} + (\int_0^1 e^{c(1+x)} u_x(x,t)^2 dx)^{1/2}$. For a vector valued function $p \in L_\infty[0,1]$ we denote by $\|p(t)\|_\infty$ its supremum norm, i.e., $\|p(t)\|_\infty = \sup_{x\in[0,1]} \sqrt{p_1(x,t)^2 + \ldots p_n(x,t)^2}$. We denote by $C^j(A)$ the space of functions that have continuous derivatives of order j on A.

II. CONTROLLER DESIGN

We consider the system

$$\dot{X}(t) = f\left(X(t), u(0,t)\right) \tag{1}$$
$$u_{tt}(x,t) = u_{xx}(x,t) \tag{2}$$
$$u_x(0,t) = h\left(X(t), u(0,t)\right) \tag{3}$$
$$u_x(1,t) = U(t) \tag{4}$$

where $X \in \mathbb{R}^n$, $U \in \mathbb{R}$, $t \in \mathbb{R}_+$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz with $f(0,0) = 0$, and $h : \mathbb{R}^{n+1} \to \mathbb{R}$ is continuously differentiable with $h(0,0) = 0$. Our controller design is based on converting the wave equation to a 2×2 system of first-order transport equations which convect in opposite directions (see Fig. 2). To achieve this we define the following transformations:

$$\zeta(x,t) = u_t(x,t) + u_x(x,t) \tag{5}$$
$$\omega(x,t) = u_t(x,t) - u_x(x,t) \tag{6}$$

together with their inverses given by

$$u_t(x,t) = \frac{\zeta(x,t) + \omega(x,t)}{2} \tag{7}$$
$$u_x(x,t) = \frac{\zeta(x,t) - \omega(x,t)}{2}. \tag{8}$$

Defining

$$\xi(t) = u(0,t) \tag{9}$$

and noting from (3) that $\zeta(0,t) = u_t(0,t) + h(X(t), \xi(t))$, system (1)–(4) is written as

$$\dot{Z}(t) = g\left(Z(t), \zeta(0,t)\right) \quad (10)$$
$$\omega_t(x,t) = -\omega_x(x,t) \quad (11)$$
$$\omega(0,t) = \zeta(0,t) - 2h\left(Z(t)\right) \quad (12)$$
$$\zeta_t(x,t) = \zeta_x(x,t) \quad (13)$$
$$\zeta(1,t) = U(t) + u_t(1,t) \quad (14)$$

where

$$Z = \begin{bmatrix} X \\ \xi \end{bmatrix} \quad (15)$$
$$g(Z,v) = \begin{bmatrix} f(X,\xi) \\ -h(X,\xi) + v \end{bmatrix}. \quad (16)$$

Our feedback design, that compensates the wave actuator dynamics, is based on applying the predictor approach to a nominal feedback law $\mu^* : \mathbb{R}^{n+1} \to \mathbb{R}$ that stabilizes the plant $\dot{Z} = g(Z,U)$ defined in (10), i.e., a nominal feedback law for the following system

$$\dot{X}(t) = f\left(X(t), \xi(t)\right) \quad (17)$$
$$\dot{\xi}(t) = -h\left(X(t), \xi(t)\right) + U(t). \quad (18)$$

Note that such a nominal control law for the augmented system (17), (18) can be constructed, using one step of backstepping, if there exists a control law κ that stabilizes the plant $\dot{X} = f(X,U)$, i.e., such that $\dot{X} = f(X, \kappa(X))$ is globally asymptotically stable. A choice of the feedback law μ^* is then

$$\mu^*\left(X(t), \xi(t)\right) = \mu\left(X(t), \xi(t)\right) + h\left(X(t), \xi(t)\right) \quad (19)$$
$$\mu\left(X(t), \xi(t)\right) = -c_1\left(\xi(t) - \kappa\left(X(t)\right)\right) + \frac{\partial \kappa\left(X(t)\right)}{\partial X} f\left(X(t), \xi(t)\right). \quad (20)$$

Noting that the input to the Z system is the delayed version of the signal $\zeta(1,t) = U(t) + u_t(1,t)$, we conclude that our control law has to employ the prediction of Z.

The predictor-based control law that compensates the wave dynamics is chosen as $U(t) = -u_t(1,t) + \mu^*(X(t+1), \xi(t+1))$ and is given by

$$U(t) = -u_t(1,t) - c_1\left(p_2(1,t) - \kappa\left(p_1(1,t)\right)\right) + \frac{\partial \kappa\left(p_1(1,t)\right)}{\partial p_1} \times f\left(p_1(1,t), p_2(1,t)\right) + h\left(p_1(1,t), p_2(1,t)\right) \quad (21)$$

where $c_1 > 0$ is arbitrary, and $p_1 \in \mathbb{R}^n$ and $p_2 \in \mathbb{R}$, the predictors of $X(t)$ and $u(0,t)$, respectively, are

$$p_1(x,t) = X(t) + \int_0^x f\left(p_1(y,t), p_2(y,t)\right) dy, \quad \text{for all } x \in [0,1] \quad (22)$$

$$p_2(x,t) = u(x,t) + \int_0^x u_t(y,t) dy - \int_0^x h\left(p_1(y,t), p_2(y,t)\right) dy, \ \forall x \in [0,1] \quad (23)$$

with initial conditions for all $x \in [0,1]$

$$p_1(x,0) = X(0) + \int_0^x f\left(p_1(y,0), p_2(y,0)\right) dy \quad (24)$$

$$p_2(x,0) = u(x,0) + \int_0^x u_t(y,0) dy - \int_0^x h\left(p_1(y,0), p_2(y,0)\right) dy. \quad (25)$$

The name "predictors" for p_1 and p_2 is chosen to emphasize that $p_1(1,t)$ and $p_2(1,t)$ are actually the 1-time unit ahead predictors of $X(t)$ and $u(0,t)$ respectively, i.e., it holds that $p_1(1,t) = X(t+1)$ and $p_2(1,t) = u(0,t+1)$. This fact is shown in the next section.[1] Note that the control law (21) is implementable (see [10], [11] for a discussion on the implementation of nonlinear predictors).

III. Lyapunov-Based Stability Analysis in the $H_2 \times H_1$ Norm

A. Statement of Main Stability Result

Assumption 1: The plant $\dot{Z} = g(Z,v)$ is strongly forward complete, that is, there exist a smooth positive definite function R_{f} and class $\mathcal{K}_\infty$ functions $\alpha_1 \ldots \alpha_3$ such that

$$\alpha_1\left(|Z|\right) \leq R_{\mathrm{f}}(Z) \leq \alpha_2\left(|Z|\right) \quad (26)$$
$$\frac{\partial R_{\mathrm{f}}(Z)}{\partial Z} g(Z,v) \leq R_{\mathrm{f}}(Z) + \alpha_3\left(|v|\right), \quad (27)$$

for all $Z \in \mathbb{R}^{n+1}$ and for all $v \in \mathbb{R}$.

Forward completeness implies that for every initial condition and every locally bounded input signal the corresponding solution is defined for all $t \geq 0$. *Strong* forward completeness differs from the *standard* forward completeness property [1] in that we assume that $g(0,0) = 0$ and hence, $R_{\mathrm{f}}(\cdot)$ is positive definite.

Assumption 2: The system $\dot{X} = f(X, \kappa(X) + v)$ is input-to-state stable (ISS) with respect to v and the function $\kappa : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable with $\kappa(0) = 0$.

Theorem 1: Consider the closed-loop system consisting of the plant (1)–(4) and the control law (21), (22), (23). Under Assumptions 1–2, for any initial condition $u(\cdot,0) \in H_2(0,1)$, $u_t(\cdot,0) \in H_1(0,1)$ which is compatible with the feedback law (21) and is such that $u_x(0,0) = h(X(0), u(0,0))$, the closed-loop system has a unique classical solution

$$X(t) \in C^1\left([0,\infty), \mathbb{R}^n\right) \quad (28)$$
$$\left(u(\cdot,t), u_t(\cdot,t)\right) \in C\left([0,\infty), H_2(0,1) \times H_1(0,1)\right) \cap C^1\left([0,\infty), H_1(0,1) \times L_2(0,1)\right) \quad (29)$$

[1] Another way to see this is as follows. Construct first the standard 1-time unit ahead predictor for Z satisfying (10) as $P(t) = Z(t) + \int_{t-1}^t g(P(\theta), \Xi(\theta))d\theta$, where $\Xi(t+x-1) = \zeta(x,t)$ (see [16]). Defining $P(t+x-1) = p(x,t)$, we rewrite the predictor as $p(1,t) = Z(t) + \int_0^1 g(p(x,t), \zeta(x,t))dx$. Using definitions (15), (16) and noting that $p_2(1,t) = u(0,t) + \int_0^1 u_x(x,t)dx + \int_0^1 u_t(x,t)dx - \int_0^1 h(p_1(x,t), p_2(x,t))dx$, we get after integrating u_x relations (22), (23) for $x = 1$.

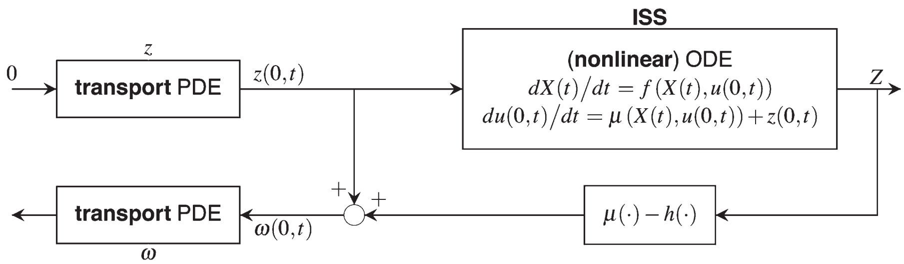

Fig. 3. Target system (34)–(38) used in the stability analysis of Theorem 1, where Z, g are defined in (15), (16), and μ^* in (19).

and there exist a class $\mathcal{KL}$ function β such that for all $t \geq 0$

$$\Omega(t) \leq \beta\left(\Omega(0), t\right) \quad (30)$$

$$\Omega(t) = |X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_{H_1} + \|u_x(t)\|_{H_1}. \quad (31)$$

B. Proof of Theorem 1

The proof of the theorem is based on a series of technical lemmas which are given next and whose proofs are provided in the Appendix A.

Lemma 1: The backstepping transformation of ζ defined as

$$z(x,t) = \zeta(x,t) - \mu\left(p(x,t)\right) - h\left(p(x,t)\right) \quad (32)$$

for all $x \in [0,1]$, where for all $x \in [0,1]$

$$p(x,t) = Z(t) + \int_0^x g\left(p(y,t), \zeta(y,t)\right) dy \quad (33)$$

and μ is defined in (20), and the control law (21)–(23) transform system (10)–(14) to the target system given by (see also Fig. 3)

$$\dot{Z}(t) = g\left(Z(t), \mu^*\left(Z(t)\right) + z(0,t)\right) \quad (34)$$

$$\omega_t(x,t) = -\omega_x(x,t) \quad (35)$$

$$\omega(0,t) = z(0,t) + \mu\left(Z(t)\right) - h\left(Z(t)\right) \quad (36)$$

$$z_t(x,t) = z_x(x,t) \quad (37)$$

$$z(1,t) = 0. \quad (38)$$

Lemma 2: The inverse backstepping transformation of ζ is defined for all $x \in [0,1]$ as

$$\zeta(x,t) = z(x,t) + \mu\left(\pi(x,t)\right) + h\left(\pi(x,t)\right), \quad (39)$$

where for all $x \in [0,1]$

$$\pi(x,t) = Z(t) + \int_0^x g\left(\pi(y,t), \mu\left(\pi(y,t)\right) + h\left(\pi(y,t)\right) + z(y,t)\right) dy. \quad (40)$$

Lemma 3: System $\dot{Z} = g(Z, \mu^*(Z) + v)$, where μ^* is given by (19), (20) is input-to-state stable with respect to v.

Lemma 4: There exists a class $\mathcal{KL}$ function β_3 such that the following holds:

$$\hat{\Xi}(t) \leq \beta_3\left(\hat{\Xi}(0), t\right), \quad \text{for all } t \geq 0 \quad (41)$$

$$\hat{\Xi}(t) = |Z(t)| + \|z(t)\|_{H_1} + \|\omega(t)\|_{H_1}. \quad (42)$$

Lemma 5: There exists a class $\mathcal{K}_\infty$ function ρ_1 such that

$$\|p(t)\|_\infty \leq \rho_1\left(|Z(t)| + \|\zeta(t)\|_\infty\right). \quad (43)$$

Lemma 6: There exists a class $\mathcal{K}_\infty$ function $\hat{\rho}_1$ such that

$$\|\pi(t)\|_\infty \leq \hat{\rho}_1\left(|Z(t)| + \|z(t)\|_\infty\right). \quad (44)$$

Lemma 7: There exist class $\mathcal{K}_\infty$ functions ρ_3, $\hat{\rho}_3$ such that

$$|Z(t)| + \|z(t)\|_{H_1} \leq \rho_3\left(|Z(t)| + \|\zeta(t)\|_{H_1}\right) \quad (45)$$

$$|Z(t)| + \|\zeta(t)\|_{H_1} \leq \hat{\rho}_3\left(|Z(t)| + \|z(t)\|_{H_1}\right). \quad (46)$$

Proof of Theorem 1: Combining Lemmas 4 and 7 we get

$$\Psi(t) \leq \beta^*\left(\Psi(0), t\right) \quad (47)$$

$$\Psi(t) = |Z(t)| + \|\zeta(t)\|_{H_1} + \|\omega(t)\|_{H_1}, \quad (48)$$

where the class $\mathcal{KL}$ function β^* is defined as $\beta^*(s,t) = \hat{\rho}_3(\beta_3(\rho_3(s) + s, t)) + \beta_3(\rho_3(s) + s, t)$. Using the fact that $Z(t) = \left[\begin{smallmatrix} X(t) \\ u(0,t) \end{smallmatrix}\right]$, we get

$$\Gamma(t) \leq \sqrt{2}\beta^*\left(\Gamma(0), t\right) \quad (49)$$

$$\Gamma(t) = |X(t)| + |u(0,t)| + \|\zeta(t)\|_{H_1} + \|\omega(t)\|_{H_1}. \quad (50)$$

Using the triangle inequality and (5)–(8), we get that

$$\|\zeta(t)\|_{H_1} + \|\omega(t)\|_{H_1} \leq 2\sqrt{2}\left(\|u_t(t)\|_{H_1} + \|u_x(t)\|_{H_1}\right) \quad (51)$$

$$\|u_t(t)\|_{H_1} + \|u_x(t)\|_{H_1} \leq \sqrt{2}\left(\|\zeta(t)\|_{H_1} + \|\omega(t)\|_{H_1}\right). \quad (52)$$

Therefore

$$\Xi(t) \leq 2\beta^*\left(2\sqrt{2}\Xi(0), t\right) \quad (53)$$

$$\Xi(t) = |X(t)| + |u(0,t)| + \|u_t(t)\|_{H_1} + \|u_x(t)\|_{H_1}. \quad (54)$$

Using the fact that $u(x,t) = u(0,t) + \int_0^x u_y(y,t)dy$, we get with the aid of the Cauchy-Schwartz inequality

$$\|u(t)\|_\infty \leq |u(0,t)| + \|u_x(t)\|_{H_1}. \quad (55)$$

With the fact that $|u(0,t)| \leq \|u(t)\|_\infty$ we get (30) with $\beta(s,t) = 4\beta^*(2\sqrt{2}s, t)$. Using (37) and (38), we get

$$z(x,t) = \begin{cases} z_0(t+x), & 0 \leq x+t < 1 \\ 0, & x+t \geq 1 \end{cases} \quad (56)$$

where the initial condition $z_0(x)$ is given by (32) with $t = 0$. Using relation (5), the fact that $u_t(\cdot, 0) \in H_1(0,1)$ and that

$u(\cdot,0) \in H_2(0,1)$, we conclude that $\zeta(\cdot,0) \in H_1(0,1)$, and hence, using the fact that p satisfies the following boundary value problem in x:

$$p_x(x,t) = g\left(p(x,t), \zeta(x,t)\right) \quad (57)$$

$$p(0,t) = Z(t) \quad (58)$$

and the Lipschitzness of g we conclude the existence and uniqueness of $p(x,0) \in C^1[0,1]$. Therefore, with (32) and the compatibility condition we get that $z_0 \in H_1(0,1)$ with $z_0(1) = 0$, and hence, with (34), (56) and the Lipschitzness of g and μ^* we conclude the existence and uniqueness of $(X(t), u(0,t)) \in C^1[0,\infty)$. The fact that $z_0 \in H_1(0,1)$ with $z_0(1) = 0$ and (56) guarantee the existence of $z \in C([0,\infty), H_1(0,1)) \cap C^1([0,\infty), L_2(0,1))$. The uniqueness of this solution follows from the uniqueness of the solution to (37), (38) (see Sections 2.1 and 2.3 in [5]). With the same arguments and using relation (6), relations (35), (36) and the fact that

$$\omega(x,t) = \begin{cases} \omega_0(x-t), & 0 \le t < x \\ z_0(t-x) + H\left(Z(t-x)\right), & 0 \le t-x < 1 \\ H\left(Z(t-x)\right) & t-x \ge 1 \end{cases} \quad (59)$$

$$H\left(Z(s)\right) = \mu\left(Z(s)\right) - h\left(Z(s)\right) \quad (60)$$

we get, with the compatibility condition of the control law, the compatibility condition $u_x(0,0) = h(X(0), u(0,0))$ and the fact that $(X(t), u(0,t)) \in C^1[0,\infty)$ the existence and uniqueness of $\omega \in C([0,\infty), H_1(0,1)) \cap C^1([0,\infty), L_2(0,1))$. With the inverse backstepping transformation (39), the fact that $p(x,t) = Z(t+x) \in C^1[0,\infty)$, which implies that $p(x,t) \in C^1([0,1] \times [0,\infty))$ and the fact that $\pi \equiv p$, we get the existence and uniqueness of $\zeta \in C([0,\infty), H_1(0,1)) \cap C^1([0,\infty), L_2(0,1))$. Therefore, using (7), (8) we conclude that there exists a unique solution $(u_t, u_x) \in C([0,\infty), H_1(0,1)) \cap C^1([0,\infty), L_2(0,1))$, and hence, with (30) that there exists a unique solution (29).

IV. Stability Analysis in the $C^1 \times C^0$ Norm

In this section, we provide two alternative stability results with estimates that incorporate only the L_∞ norm of u, u_x and u_t. The first result relies on the construction of a Lyapunov functional. Yet, this construction is based on the introduction of an additional backstepping transformation for the ω state. The well-posedness and the invertibility of this transformation requires an additional assumption that the system is not only forward, but also backward complete. The second result is based on stability estimates on the solutions of the closed-loop system, i.e., it does not employ a Lyapunov functional.

A. Lyapunov-Based Stability Analysis Under Backward Completeness Assumption

We introduce the following backstepping transformation for ω

$$w(x,t) = \omega(x,t) - \mu\left(r(x,t)\right) + h\left(r(x,t)\right) \quad (61)$$

for all $x \in [0,1]$, where r is the "reverse" predictor[2] of Z and is given for all $x \in [0,1]$ by

$$r(x,t) = Z(t) - \int_0^x g\left(r(y,t), 2h\left(r(y,t)\right) + \omega(y,t)\right) dy \quad (62)$$

and μ is defined in (20). Note that in the present case μ and h are allowed to be only locally Lipschitz, and κ only continuously differentiable functions. By combining the backstepping transformation (62) with (32) and the control law (21)–(23), system (10)–(14) is transformed to the new target system given by (see also Fig. 4)

$$\dot{Z}(t) = g\left(Z(t), \mu^*\left(Z(t)\right) + z(0,t)\right) \quad (63)$$

$$w_t(x,t) = -w_x(x,t) \quad (64)$$

$$w(0,t) = z(0,t) \quad (65)$$

$$z_t(x,t) = z_x(x,t) \quad (66)$$

$$z(1,t) = 0. \quad (67)$$

To see this, first apply Lemma 1 to get relations (66), (67). Relation (65) follows similarly by setting $x = 0$ into (61), (62) and using the fact that $z(0,t) = \zeta(0,t) - \mu(Z(t)) - h(Z(t))$. Relation (64) follows by the fact that r in (62) satisfies $r_t = -r_x$. Note that the new boundary condition for the transformed state $w(0,t)$ depends only on $z(0,t)$ and not on Z. This enables us to construct a Lyapunov functional by directly incorporating in the Lyapunov functional the L_∞ norm of both w and z. To see this define the following Lyapunov functional for the target system (63)–(67)

$$V(t) = S\left(Z(t)\right) + \frac{2}{c} \int_0^{\|v(t)\|_{c,\infty}} \frac{\alpha_7(r)}{r} dr \quad (68)$$

where S and α_7 are the dissipative Lyapunov function and one of the supply functions respectively, of the Lyapunov characterization of the input-to-state stability property of system $\dot{Z} = g(Z, \mu^*(Z) + v)$ (see (A.10), (A.11) in Appendix A), $c > 0$ is arbitrary and the new variable $v(x,t)$, $x \in [-1,1]$ is defined as

$$v(x,t) = \begin{cases} z(x,t), & \text{for all } x \in [0,1] \\ w^*(x,t), & \text{for all } x \in [-1,0] \end{cases} \quad (69)$$

where $w^*(x,t) = w(-x,t)$ and $\|v(t)\|_{c,\infty} = \sup_{x\in[-1,1]} e^{c(1+x)} |v(x,t)|$. Noting from (64)–(67) that for all $x \in [-1,1]$, $v_t(x,t) = v_x(x,t)$ and $v(1) = 0$, with similar calculations as in [17, Th. 5], we get

$$\frac{d\|v(t)\|_{c,\infty}}{dt} \le -c\|v(t)\|_{c,\infty}. \quad (70)$$

[2]The fact that r is the reverse predictor of Z is based on the fact that $r(x,t) = Z(t-x)$. The proof of this fact follows the same lines with the proof of the fact that $p(x,t) = Z(t+x)$ which is proved in Lemma 1. In other words, it is shown that $r(x,t) = Z(t-x)$ is the unique solution to the boundary value problem $r_x(x,t) = -g(r(x,t), 2h(r(x,t)) + \omega(x,t))$, $r(0,t) = Z(t)$ [which follows from (62)] using, (10) and the fact that the solution to (35) is $\omega(x,t) = G(t-x)$ for some function G.

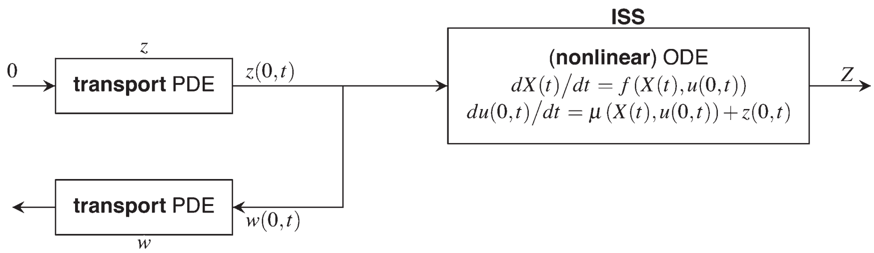

Fig. 4. Target system (63)–(67) used in the stability analysis of Theorem 2, where Z, g are defined in (15), (16), and μ^* in (19).

Following the calculations in the proof of Lemma 4, we arrive at $V(t) \leq \hat{\beta}_1(V(0), t)$ for a class $\mathcal{K}_\infty$ function $\hat{\beta}_1$, and hence, using the fact that $(1/2)(\|w(t)\|_\infty + \|z(t)\|_\infty) \leq \|v(t)\|_{c,\infty} \leq e^{2c}(\|w(t)\|_\infty + \|z(t)\|_\infty)$, we get for some class $\mathcal{K}_\infty$ function $\hat{\beta}_2$ that

$$\Lambda(t) \leq \hat{\beta}_2(\Lambda(0), t) \quad (71)$$

$$\Lambda(t) = |Z(t)| + \|w(t)\|_\infty + \|z(t)\|_\infty. \quad (72)$$

For proving stability in the original variables one has to relate the L_∞ norm of z and w with the norms of ζ and ω, i.e., one has to show that there exist class $\mathcal{K}_\infty$ functions θ_1, $\hat{\theta}_1$ such that

$$\Lambda(t) \leq \theta_1(|Z(t)| + \|\zeta(t)\|_\infty + \|\omega(t)\|_\infty) \quad (73)$$

$$\hat{\theta}_1(\Lambda(t)) \geq |Z(t)| + \|\zeta(t)\|_\infty + \|\omega(t)\|_\infty. \quad (74)$$

For proving bounds (73), (74) it is sufficient to prove that w and z are upper bounded by a class $\mathcal{K}_\infty$ function of Z, ω and by a class $\mathcal{K}_\infty$ function of Z, ζ respectively, and vice versa. For the case of z this is achieved with Lemmas 1, 5 and for ζ with Lemmas 2, 6. To relate the norm of w with the norm of ω and vice versa, one has to guarantee the boundness and invertibility of the backstepping transformation (61). For achieving this, we impose the following assumptions.

Assumption 3: The system $\dot{Z} = g(Z, 2h(Z) + v)$ is strongly backward complete, that is, there exist a smooth positive definite function R_{b} and class $\mathcal{K}_\infty$ functions $\alpha_{\mathrm{b},1} \ldots \alpha_{\mathrm{b},3}$ such that

$$\alpha_{\mathrm{b},1}(|Z|) \leq R_{\mathrm{b}}(Z) \leq \alpha_{\mathrm{b},2}(|Z|) \quad (75)$$

$$-\frac{\partial R_{\mathrm{b}}(Z)}{\partial Z} g(Z, 2h(Z)+v) \leq R_{\mathrm{b}}(Z) + \alpha_{\mathrm{b},3}(|v|), \quad (76)$$

for all $Z \in \mathbb{R}^{n+1}$ and for all $v \in \mathbb{R}$.

Assumption 4: The system $\dot{Z} = g(Z, \mu^*(Z) + v)$ is strongly backward complete with respect to v, that is, there exist a smooth positive definite function $R_{\mathrm{b,cl}}$ and class $\mathcal{K}_\infty$ functions $\alpha_{4,\mathrm{cl}}$, $\alpha_{5,\mathrm{cl}}$, $\alpha_{6,\mathrm{cl}}$ such that

$$\alpha_{4,\mathrm{cl}}(|Z|) \leq R_{\mathrm{b,cl}}(Z) \leq \alpha_{5,\mathrm{cl}}(|Z|) \quad (77)$$

$$-\frac{\partial R_{\mathrm{b,cl}}(Z)}{\partial Z} g(Z, \mu^*(Z)+v) \leq R_{\mathrm{b,cl}}(Z) + \alpha_{6,\mathrm{cl}}(|v|), \quad (78)$$

for all $Z \in \mathbb{R}^{n+1}$ and for all $v \in \mathbb{R}$.

The reader should notice that *backward* completeness implies that for every initial condition and every locally bounded input signal the corresponding solution is defined for all $t \leq 0$.

Theorem 2: Consider the closed-loop system consisting of the plant (1)–(4) and the control law (21), (22), (23). Under Assumptions 1–4, for any initial condition $u(\cdot, 0) \in C^1[0,1]$, $u_t(\cdot, 0) \in C[0,1]$ which is compatible with the feedback law (21) and is such that $u_x(0,0) = h(X(0), u(0,0))$, the closed-loop system has a unique solution

$$X(t) \in C^1([0,\infty), \mathbb{R}^n) \quad (79)$$

$$(u(\cdot,t), u_t(\cdot,t)) \in C\left([0,\infty), C^1[0,1] \times C[0,1]\right), \quad (80)$$

and there exist a class $\mathcal{KL}$ function $\hat{\beta}$ such that

$$\hat{\Omega}(t) \leq \hat{\beta}\left(\hat{\Omega}(0), t\right) \quad (81)$$

$$\hat{\Omega}(t) = |X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty, \quad (82)$$

for all $t \geq 0$.

Proof: See Appendix B. ■

B. Stability Analysis Without a Lyapunov Functional

Theorem 3: Consider the closed-loop system consisting of the plant (1)–(4) and the control law (21), (22), (23). Under Assumptions 1 and 2, for all initial conditions as in the statement of Theorem 2, there exist a unique solution (79), (80) and a class $\mathcal{KL}$ function $\hat{\eta}_3$ such that

$$\hat{\Omega}(t) \leq \hat{\eta}_3\left(\hat{\Omega}(0), t\right), \quad (83)$$

for all $t \geq 0$, where $\hat{\Omega}$ is defined in (82).

Proof: Using Lemma 3.5 from [9] and using the fact that $g(0,0) = 0$ which allows us to set $R = 0$, we get from (10) and the fact that $\zeta(x,t) = \zeta(x+t, 0)$ for all $t + x \leq 1$,

$$|Z(t)| \leq \psi(|Z(0)| + \|\zeta(0)\|_\infty), \quad \text{for all } t \leq 1, \quad (84)$$

for some class $\mathcal{K}_\infty$ function ψ. With relations (19), (20), (21), the fact that $p(1,t) = Z(t+1)$, the fact that for all $t \geq 1$, $\zeta(0,t) = u_t(1, t-1) + U(t-1)$ and since $\dot{Z} = g(Z, \mu^*(Z))$ is globally asymptotically stable we get for some class $\mathcal{KL}$ function σ that

$$|Z(t)| \leq \sigma(|Z(1)|, t-1), \quad \text{for all } t \geq 1. \quad (85)$$

Combining (84), (85) and assuming (with no generality loss) that $\sigma(s, 0) \geq s$ we arrive at

$$|Z(t)| \leq \hat{\sigma}(|Z(0)| + \|\zeta(0)\|_\infty, t), \quad \text{for all } t \geq 0, \quad (86)$$

where $\hat{\sigma}(s,t) = \sigma(\psi(|Z(0)| + \|\zeta(0)\|_\infty), \max\{0, t-1\})$. We estimate now $\|\zeta(t)\|_\infty$. Using (14), (21) it follows that $\zeta(1,t) = \mu^*(Z(t+1))$ (see also Fig. 5), and hence, since for all $t \geq 1$

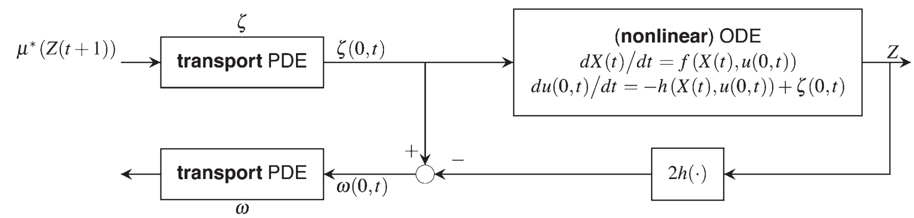

Fig. 5. The target system (10)–(14) used in the stability analysis of Theorem 3, where Z, g are defined in (15), (16), and μ^* in (19). Note that using the controller (21) the boundary condition for ζ is $\zeta(1,t)=\mu^*(p(1,t))$, where $p(1,t)$ is the 1-time unit ahead predictor of Z.

it holds that $\zeta(x,t)=\zeta(1,t+x-1)=\mu^*(Z(t+x))$, for all $x\in[0,1]$, we get from (84), (85) that for all $t\geq 1$

$$\|\zeta(t)\|_\infty \leq \hat{\psi}\left(\sigma\left(\psi\left(|Z(0)|+\|\zeta(0)\|_\infty\right),t-1\right)\right), \tag{87}$$

where we also used the fact that $|\mu^*(Z)|\leq\hat{\psi}(|Z|)$ for a class $\mathcal{K}_\infty$ function $\hat{\psi}$. Using the fact that for all $t\leq 1$, $\|\zeta(t)\|_\infty \leq \sup_{x\in[0,1-t]}|\zeta(x+t,0)|+\sup_{x\in[1-t,1]}|\zeta(1,t+x-1)|$ and estimates (84), (85) we arrive after some class $\mathcal{K}_\infty$ function majorizations at

$$\|\zeta(t)\|_\infty \leq \hat{\xi}\left(|Z(0)|+\|\zeta(0)\|_\infty\right), \quad \text{for all } t\leq 1, \tag{88}$$

for some class $\mathcal{K}_\infty$ function $\hat{\xi}$. Combining (86), (87), (88) we conclude that there exist a class $\mathcal{KL}$ function $\hat{\eta}$ such that for all $t\geq 0$

$$|Z(t)|+\|\zeta(t)\|_\infty \leq \hat{\eta}\left(|Z(0)|+\|\zeta(0)\|_\infty, t\right). \tag{89}$$

From relations (11), (12) it follows that for all $t\geq 1$, $\omega(x,t)=\zeta(0,t-x)-2h(Z(t-x))$, and hence, with the Lipschitzness of h we get that

$$\|\omega(t)\|_\infty \leq \sup_{x\in[0,1]}\|\zeta(t-x)\|_\infty + \hat{\psi}_1\left(\sup_{x\in[0,1]}|Z(t-x)|\right), \quad \text{for all } t\geq 1 \tag{90}$$

where the class $\mathcal{K}_\infty$ function $\hat{\psi}_1$ is such that $2|h(Z)|\leq\hat{\psi}_1(|Z|)$. Therefore, with (89) we get for all $t\geq 1$

$$\|\omega(t)\|_\infty \leq \hat{\beta}_1^*\left(|Z(0)|+\|\zeta(0)\|_\infty, t-1\right), \tag{91}$$

where the class $\mathcal{KL}$ function is defined as $\hat{\beta}_1^*=\hat{\eta}(s,t)+\hat{\psi}_1(\hat{\eta}(s),t)$. Moreover, for all $t\leq 1$ we have $\|\omega(t)\|_\infty \leq \sup_{x\in[t,1]}|\omega(x-t,0)|+\sup_{x\in[0,t]}|\omega(0,t-x)|$, and hence,

$$\|\omega(t)\|_\infty \leq \hat{\xi}_1\left(N(0)\right), \quad \text{for all } t\leq 1, \tag{92}$$

$$N(t)=|Z(t)|+\|\zeta(t)\|_\infty+\|\omega(t)\|_\infty, \tag{93}$$

where the class $\mathcal{K}_\infty$ function $\hat{\xi}_1$ is defined as $\hat{\xi}(s)=s+\hat{\eta}(s,0)+\hat{\psi}_1(\hat{\eta}(s),0)$. Similarly to the proof of (89) by combining (91), (92) we arrive at

$$\|\omega(t)\|_\infty \leq \hat{\eta}_1\left(N(0),t\right), \quad \text{for all } t\geq 0, \tag{94}$$

for some class $\mathcal{KL}$ function $\hat{\eta}_1$. Combining (89), (94) we get for all $t\geq 0$ that

$$N(t)\leq\hat{\eta}_2\left(N(0),t\right), \tag{95}$$

where $\hat{\eta}_2(s,t)=\hat{\eta}(s,t)+\hat{\eta}_1(s,t)$. Using the fact that $Z(t)=\left[\begin{smallmatrix}X(t)\\u(0,t)\end{smallmatrix}\right]$, definitions (5)–(8) and the triangle inequality we get bound (83) with $\hat{\eta}_3(s,t)=2\sqrt{2}\hat{\eta}_2(2s,t)$. Existence and uniqueness of a solution (79), (80) is proved by adapting the corresponding part in the proof of Theorem 1 to the case of the initial conditions in Theorem 3 (see also Appendix B). ■

Since the proof of Theorem 3 relies only on the fact that $\dot{Z}=g(Z,\mu^*(Z))$ is globally asymptotically stable one can relax Assumption 2 to only global asymptotic stabilizability of $\dot{Z}=g(Z,v)$. Yet, in this case one has to appropriately modify the control law (21) as $U(t)=-u_t(1,t)+\hat{\mu}^*(p(1,t))$, where $\hat{\mu}^*$ renders $\dot{Z}=g(Z,\hat{\mu}^*)$ globally asymptotically stable and can be explicitly constructed when a feedback law and a Lyapunov function are known for the system $\dot{X}=f(X,\kappa(X))$ (see, for example, [18], [29]).

V. Application to Linear Systems

In this section we specialize our results to the case of linear systems

$$\dot{X}(t)=AX(t)+Bu(0,t) \tag{96}$$

$$u_{tt}(x,t)=u_{xx}(x,t) \tag{97}$$

$$u_x(0,t)=\sigma u(0,t)+CX(t) \tag{98}$$

$$u_x(1,t)=U(t), \tag{99}$$

where A, B, C^{T} are matrices of dimension $n\times n$, $n\times 1$, $n\times 1$ respectively, and σ is a scalar.

Theorem 4: Consider the closed-loop system consisting of (96)–(99) together with the control law

$$U(t)=K^*\left(e^F\begin{bmatrix}X(t)\\u(0,t)\end{bmatrix}+F\int_0^1 e^{F(1-x)}Gu(x,t)dx + \int_0^1 e^{F(1-x)}Gu_t(x,t)dx\right) + K^*\left(-e^FGu(0,t)+Gu(1,t)\right)-u_t(1,t), \tag{100}$$

where

$$F=\begin{bmatrix}A & B\\ -C & -\sigma\end{bmatrix} \tag{101}$$

$$G=\begin{bmatrix}O_{n\times 1}\\ 1\end{bmatrix}, \tag{102}$$

$$K^*=\left[\,Kc_1+KA+C \quad -c_1+KB+\sigma\,\right], \tag{103}$$

and K renders the matrix $A+BK$ Hurwitz and $c_1>0$ is arbitrary. For all initial conditions $u(\cdot,0)\in H_1(0,1)$ and $u_t(\cdot,0)\in L_2(0,1)$ the closed-loop systems has a unique solution $(X(t), u(\cdot,t), u_t(\cdot,t))\in C([0,\infty),\mathbb{R}^n\times H_1(0,1)\times L_2(0,1))$, and there exist positive constants λ and ρ such that

$$\Omega(t)\leq \mu\Omega_{\mathrm{L}}(0)e^{-\lambda t}, \tag{104}$$

for all $t\geq 0$, where

$$\Omega(t)=|X(t)|^2+\int_0^1 u(x,t)^2dx + \int_0^1 u_x(x,t)^2+\int_0^1 u_t(x,t)^2dx. \tag{105}$$

Moreover, if the initial conditions are as in the statement of Theorem 1, then (28), (29) is the classical solution to the closed-loop system.

The proof of Theorem 4 is based on the target system (63)–(67) (see Fig. 4), since for linear systems both the open-loop and closed-loop systems are complete. Yet, we conduct a stability analysis directly for the linear plant, rather than just specializing the nonlinear results to the linear case. Such an analysis is important because it allows one to derive more explicit stability estimates. In addition, it leads to stability estimates that incorporate the $H_1\times L_2$ rather than the $C^1\times C^0$ norm of the actuator state. The lower order norm is less restrictive on the set of initial conditions and thus preferrable. Also, with such an analysis one gets exponential stability of the closed-loop system.

Proof: See Appendix C. ■

The stabilization problem for a wave PDE/linear ODE cascade, in the case of a homogenous boundary condition, i.e., when (98) holds with $\sigma=0$ and $C=0$, is also solved in [15]. Yet, the two approaches are different. The design in this article is based on backstepping transformations for the two transport PDE states, whereas [15] is based on backstepping transformations on the wave PDE state. Startlingly, the two approaches lead to the same control design as stated in the following proposition which we prove below.

Proposition 5.1: For the wave PDE/linear ODE cascade (96)–(99) with $\sigma=0$ and $C=0$, the control law (100) is identical to the control law (47) from [15] with the constants c_1 and c_0 replaced by 1 and c_1 respectively.

Proof: See Appendix D. ■

VI. Specialization to Wave PDE With Anti-Collocated Nonlinear Stiffness

Our results are new even for the case of a lone wave PDE (i.e., not coupled with an ODE) with anti-collocated nonlinear stiffness, i.e., the system

$$u_{tt}(x,t)=u_{xx}(x,t) \tag{106}$$
$$u_x(0,t)=h\left(u(0,t)\right) \tag{107}$$
$$u_x(1,t)=U(t). \tag{108}$$

The control law for (106)–(108) is derived from the original control law (21), (22), (23) as

$$U(t)=-u_t(1,t)-c_1p_2(1,t)+h\left(p_2(1,t)\right), \tag{109}$$

where for all $x\in[0,1]$

$$p_2(x,t)=u(x,t)+\int_0^x u_t(y,t)dy-\int_0^x h\left(p_2(y,t)\right)dy. \tag{110}$$

Under a forward completeness assumption for the system $\dot{\xi}=-h(\xi)+v$ with respect to input v, the stability analysis of the closed-loop system (106)–(108), (109), (110) is performed using the target system (34)–(38) with $g(\xi(t))=-c_1\xi(t)+z(0,t)$ and $\mu(\xi(t))=-c_1\xi(t)$.

VII. Extension to Wave Equations With Anti-Damping

The approach in this paper can be also applied to the case where the boundary term (3) is replaced by

$$u_x(0,t)=qu_t(0,t)+h\left(X(t),u(0,t)\right) \tag{111}$$

where $q\neq\pm 1$. To see this note that, with the preliminary transformations

$$\zeta(x,t)=\frac{1}{q+1}\left(u_t(x,t)+u_x(x,t)\right) \tag{112}$$
$$\omega(x,t)=\frac{1}{1-q}\left(u_t(x,t)-u_x(x,t)\right) \tag{113}$$

we transform (1), (2), (111), (4) to

$$\dot{X}(t)=f\left(X(t),\xi(t)\right) \tag{114}$$
$$\dot{\xi}(t)=\zeta(0,t)-\frac{1}{q+1}h\left(X(t),u(0,t)\right) \tag{115}$$
$$\omega_t(x,t)=-\omega_x(x,t) \tag{116}$$
$$\omega(0,t)=\zeta(0,t)-\left(\frac{1}{1+q}+\frac{1}{1-q}\right)\times h\left(X(t),u(0,t)\right) \tag{117}$$
$$\zeta_t(x,t)=\zeta_x(x,t) \tag{118}$$
$$\zeta(1,t)=\frac{1}{q+1}\left(U(t)+u_t(1,t)\right). \tag{119}$$

Then the backstepping transformation

$$z(x,t)=\zeta(x,t)-\mu\left(p(x,t)\right)-\frac{1}{1+q}h\left(p(x,t)\right) \tag{120}$$

where for all $x\in[0,1]$

$$p(x,t)=Z(t)+\int_0^x g\left(p(y,t),\zeta(y,t)\right)dy \tag{121}$$

and $g=\left[\begin{smallmatrix} f(X,\xi) \\ -(1/(1+q))h(X,\xi)+v\end{smallmatrix}\right]$, and the control law

$$U(t)=-u_t(1,t)+(1+q)\mu\left(p(1,t)\right)+h\left(p(1,t)\right) \tag{122}$$

transform system (114)–(119) to the target system

$$\dot{X}(t) = f\left(X(t), \xi(t)\right) \tag{123}$$
$$\dot{\xi}(t) = \mu\left(X(t), \xi(t)\right) + z(0,t) \tag{124}$$
$$\omega_t(x,t) = -\,\omega_x(x,t) \tag{125}$$
$$\omega(0,t) = z(0,t) + \mu\left(Z(t)\right) - \frac{1}{1-q} h\left(Z(t)\right) \tag{126}$$
$$z_t(x,t) = z_x(x,t) \tag{127}$$
$$z(1,t) = 0. \tag{128}$$

The same analysis as in Theorem 1 can be then used to prove the stability of the closed-loop system.

When $q<0$, even the stabilization problem of the wave equation alone is of interest, since in this case, all eigenvalues of the wave equation are located on the right-hand side of the complex plane (the open-loop system is anti-stable). When there is no ODE and $h=0$, by choosing $\mu(s) = -c_0 s$, $c_0 > 0$ (which stabilizes $\dot{\xi} = \mu$), controller (122) becomes $U(t) = -u_t(1,t) - c_0(1+q)p(1,t)$, where $p(1,t) = u(0,t) + \int_0^1 \zeta(x,t)dx$. Using (112) we get $p(1,t) = (1/(1+q)) \int_0^1 u_t(x,t)dx + (1/(1+q)) u(1,t) + (q/(1+q))u(0,t)$. Therefore, the control law is

$$U(t) = -qc_0 u(0,t) - c_0 u(1,t) - u_t(1,t) - c_0 \int_0^1 u_t(x,t)dx. \tag{129}$$

This is the same control law that was derived using a backstepping transformation of the state u in [28] [relation (29) for $c=1$].

VIII. Examples

Example 1: We consider the following system:

$$\dot{X}_1(t) = X_2(t) - X_2(t)^2 u(0,t) \tag{130}$$
$$\dot{X}_2(t) = u(0,t) \tag{131}$$
$$u_{tt}(x,t) = u_{xx}(x,t) \tag{132}$$
$$u_x(0,t) = 0 \tag{133}$$
$$u_x(1,t) = U(t). \tag{134}$$

System (130), (131) is in the strict-feedforward form, and hence, is forward complete with respect to the input $u(0,t)$. The nominal control law (i.e., in the case where $u(0,t) \equiv U(t)$)

$$U(t) = -X_1(t) - 2X_2(t) - \frac{1}{3} X_2(t)^3 \tag{135}$$

renders the closed-loop system input-to-state stable.[3] The control design that compensates the wave dynamics is

$$\begin{aligned} U(t) =& -u_t(1,t) - 2\left(p_3(1,t) + p_1(1,t) + 2p_2(1,t)\right) \\ & - \frac{2}{3} p_2(1,t)^3 - p_2(1,t) + p_2(1,t)^2 p_3(1,t) \\ & - \left(2 + p_2(1,t)^2\right) p_3(1,t) \end{aligned} \tag{136}$$

[3]This fact follows from the fact that the control law (135) can be written as $U = -\phi_1 - \phi_2$, where ϕ is the linearizing diffeomorphic transformation $\phi_1 = X_1 + X_2 + (1/3)X_2^3$, $\phi_2 = X_2$, which transforms system (130), (131) to $\dot{\phi}_1 = \phi_2 + U$, $\dot{\phi}_2 = U$ (see [14]).

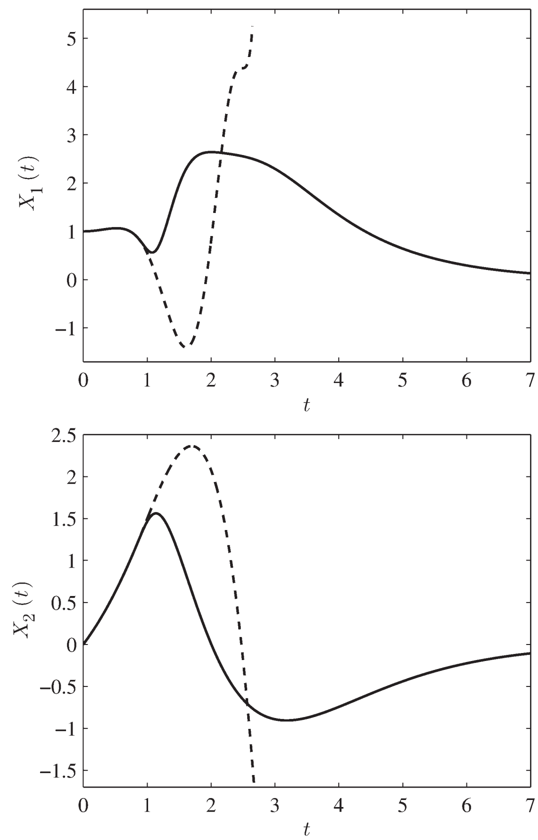

Fig. 6. Response of the states of the plant (130), (131) with the control law (136)–(139) (solid line) and with the nominal control law (135) (dashed line) for initial conditions as $X_1(0) = 1$, $X_2(0) = 0$ and $u(x,0) = u_t(x,0) = 1$, for all $x \in [0,1]$.

where

$$\begin{aligned} p_1(1,t) =& X_1(t) + X_2(t) + \int_0^1 (1-x)u(x,t)dx \\ & + \int_0^1 (1-x)^2 u_t(x,t)dx - \int_0^1 \\ & \times \left(X_2(t) + \int_0^x u(y,t)dy + \int_0^x (1-y)u_t(y,t)dy \right)^2 \\ & \times \left(u(x,t) + \int_0^x u_t(y,t)dy \right) dx \end{aligned} \tag{137}$$

$$\begin{aligned} p_2(1,t) =& X_2(t) + \int_0^1 u(x,t)dx \\ & + \int_0^1 (1-x)u_t(x,t)dx \end{aligned} \tag{138}$$

$$p_3(1,t) = u(1,t) + \int_0^1 u_t(x,t)dx. \tag{139}$$

We choose the initial conditions for the plant as $X_1(0) = 1$, $X_2(0) = 0$ and for the actuator state as $u(x,0) = u_t(x,0) = 1$, for all $x \in [0,1]$. In Fig. 6, we show the response of the states of

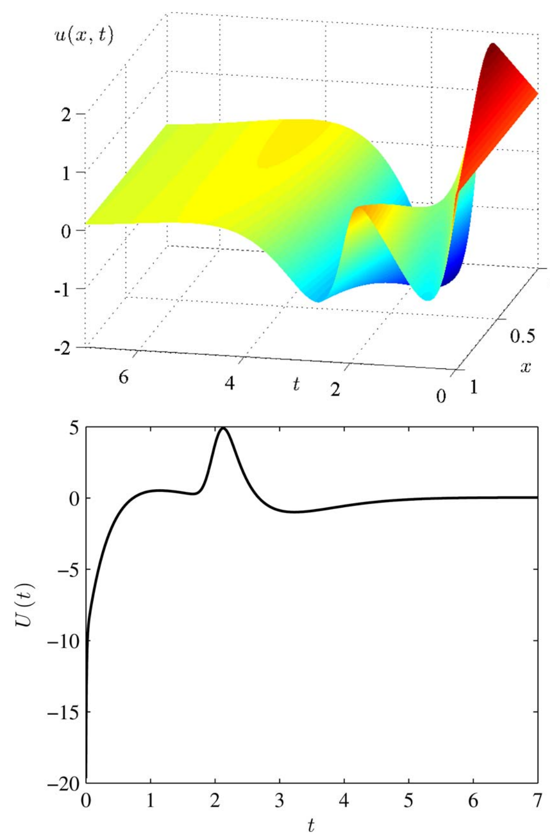

Fig. 7. Response of the actuator state (left) and the control effort (right) of the plant (130)–(134) with the control law (136)–(139) for initial conditions as $X_1(0) = 1$, $X_2(0) = 0$ and $u(x, 0) = u_t(x, 0) = 1$, for all $x \in [0, 1]$.

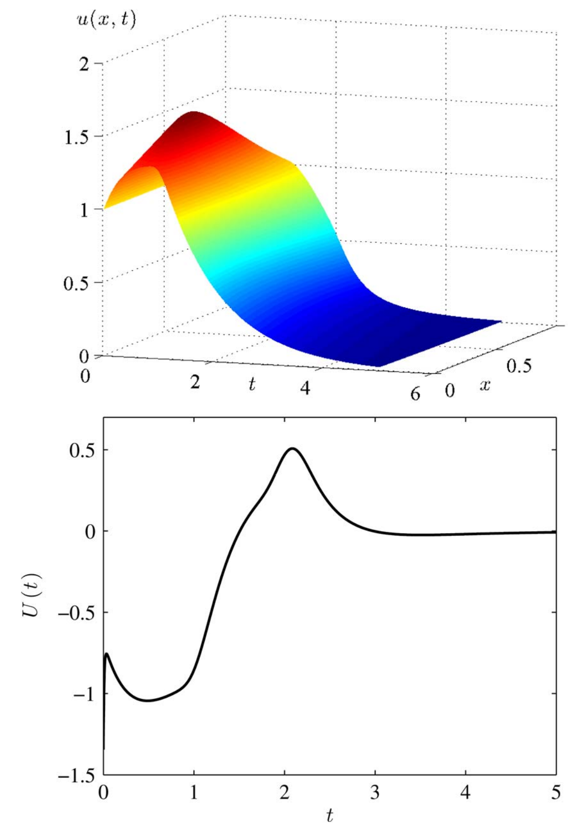

Fig. 8. Response of the state of the plant (106)–(108) with $h(u(0, t)) = u(0, t)^3 - u(0, t)$ (left), under the control law (140), (141) (right) for initial conditions $u(x, 0) = u_t(x, 0) = 1$, for all $x \in [0, 1]$.

the plant (130), (131) for the case of the uncompensated nominal control law (135) and the case of the proposed control law (136)–(139). As one can observe, in the latter case stabilization is achieved, whereas the states grow unbounded in the former case, in which a control law that does not take into account the wave dynamics is employed. In Fig. 7, we show the response of the actuator state and the control effort in the case of the proposed design (136)–(139). As one can observe, both the actuator state and control effort converge.

Example 2: In this example, we apply our developments for the stabilization of a wave PDE with anti-collocated nonlinear stiffness. We consider system (106)–(108) with $h(u(0, t)) = u(0, t)^3 - u(0, t)$. Hence, system $\dot{\xi}(t) = -\xi(t)^3 + \xi(t) + U(t)$ is forward complete. The predictor-based feedback law (109) is

$$U(t) = -u_t(1, t) - 2p_2(1, t) + p_2(1, t)^3 \tag{140}$$

where

$$p_2(1, t) = u(1, t) + \int_0^1 u_t(x, t)dx - \int_0^1 \left(p_2(x, t)^3 - p_2(x, t)\right) dx. \tag{141}$$

In Fig. 8, we show the response of the closed-loop system consisting of (106)–(108) with $h(u(0, t)) = u(0, t)^3 - u(0, t)$ and the control law (140), (141), and the control effort for initial conditions as $u(x, 0) = u_t(x, 0) = 1$ for all $x \in [0, 1]$. As one can observe the proposed control law achieves stabilization.

IX. Conclusions

We present a methodology for the compensation of wave dynamics in the actuation path of nonlinear systems by employing a predictor inspired design. The stability of the closed-loop system is established in two different norms of the actuator state and is based on Lyapunov functionals that we construct with the aid of novel backstepping transformations of the actuator state. This work raises the question of stabilization of other PDE/nonlinear ODE cascades, such as when the PDE is of diffusive type, in which case a conversion to a delay-like problem does not seem possible.

Appendix A

Proof of Lemma 1

Using the fact that $p(0, t) = Z(t)$, we get (34), (36) by setting $x = 0$ into (32) and using the definition (19). With (21)–(23), definitions (15), (16), and relation (14), we get (38). We prove next (37). From (33), we have that p satisfies the following boundary value problem in x:

$$p_x(x, t) = g\left(p(x, t), \zeta(x, t)\right) \tag{A.1}$$

$$p(0, t) = Z(t). \tag{A.2}$$

We now show that the unique solution to (A.1), (A.2) is

$$p(x, t) = Z(t + x) \tag{A.3}$$

and hence, $p_t(x,t) = p_x(x,t)$, which in turn implies, with the help of (13) and (32), that (37) holds. Using (13), we get that ζ is a function of $t+x$, i.e., $\zeta(x,t) = F(t+x)$, for some function F. Therefore, setting $t \to t+x$ in (10) and noting that $\zeta(0,t) = F(t)$, we get that

$$Z'(t+x) = g\left(Z(t+x), F(t+x)\right). \tag{A.4}$$

Using (A.1), we conclude that $p(x,t) = Z(t+x)$ is a solution to the boundary value problem (A.1), (A.2). Since g is locally Lipschitz we get that $p(x,t) = Z(t+x)$ is the unique solution to (A.1), (A.2).

Proof of Lemma 2

We first observe that $\pi(0,t) = Z(t) = p(0,t)$, and hence, π satisfies the following initial value problem:

$$\pi_x(x,t) = g\left(\pi(x,t), \mu\left(\pi(x,t)\right) + h\left(\pi(x,t)\right) + z(x,t)\right) \tag{A.5}$$

$$\pi(0,t) = Z(t). \tag{A.6}$$

Using (32) and (A.1) we conclude that p and π satisfy the same initial value problem. From the uniqueness of solutions we conclude that $p \equiv \pi$, and hence, $\pi_t(x,t) = \pi_x(x,t)$. Therefore, ζ in (39) satisfies (13).

Proof of Lemma 3

Defining the change of variables $y(t) = \xi(t) - \kappa(X(t))$ and rewriting the Z system in (X,y), we get with (19), (20) $\dot{X}(t) = f(X(t), \kappa(X(t)) + y(t))$, $\dot{y}(t) = -c_1 y(t) + v(t)$. Under Assumption 2 (the properties of κ) the function μ is continuously differentiable with $\mu(0,0) = 0$. Since h is also continuously differentiable and $h(0,0) = 0$, using (19) we conclude that μ^* is continuously differentiable as well with $\mu^*(0,0) = 0$. Lemma C.4 in [18] guarantees that (X,y) is input-to-state stable with respect to v, i.e., there exists a class $\mathcal{KL}$ function $\bar{\beta}$ and a class $\mathcal{K}$ function $\bar{\gamma}$ such that $|X(t)| + |y(t)| \leq \bar{\beta}(|X(0)| + |y(0)|, t) + \bar{\gamma}(\sup_{0\leq\tau\leq t}|v(\tau)|)$. Since κ is continuously differentiable with $\kappa(0) = 0$ there exist a class $\mathcal{K}_\infty$ function $\hat{\alpha}$ such that

$$|\kappa(X)| \leq \hat{\alpha}\left(|X|\right), \quad \text{for all } X \in \mathbb{R}^n \tag{A.7}$$

and hence, $|X(t)| + |\xi(t)| = |X(t)| + |\xi(t) - \kappa(X(t)) + \kappa(X(t))| \leq \bar{\alpha}(|X(t)| + |y(t)|)$, where the class $\mathcal{K}_\infty$ function $\bar{\alpha}$ is given by $\bar{\alpha}(s) = s + \hat{\alpha}(s)$. With the fact that $|y(t)| \leq |\xi(t)| + \hat{\alpha}(|X(t)|)$, we conclude that the Z system is input-to-state stable with respect to v with

$$\beta_Z(s,t) = \bar{\alpha}\left(2\bar{\beta}\left(\sqrt{2}s + \sqrt{2}\hat{\alpha}(s), t\right)\right) \tag{A.8}$$

$$\gamma_Z(s) = \bar{\alpha}\left(2\bar{\gamma}(s)\right). \tag{A.9}$$

Proof of Lemma 4

From Lemma 3 and [31], there exist a smooth function $S(Z): \mathbb{R}^{n+1} \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions α_4, α_5, α_6 and α_7 such that

$$\alpha_4\left(|Z|\right) \leq S(Z) \leq \alpha_5\left(|Z|\right) \tag{A.10}$$

$$\frac{\partial S(Z)}{\partial Z} g\left(Z, \mu^*(Z) + d\right) \leq -\alpha_6\left(|Z|\right) + \alpha_7\left(|d|\right) \tag{A.11}$$

for all $Z \in \mathbb{R}^{n+1}$ and for all $d \in \mathbb{R}$. Moreover, using (35)–(38), we also get that

$$\omega_{xt}(x,t) = -\omega_{xx}(x,t) \tag{A.12}$$

$$\omega_x(0,t) = -z_x(0,t) - \left(\frac{\partial \mu\left(Z(t)\right)}{\partial Z} - \frac{\partial h\left(Z(t)\right)}{\partial Z}\right) \times g\left(Z(t), \mu^*\left(Z(t)\right) + z(0,t)\right) \tag{A.13}$$

$$z_{xt}(x,t) = z_{xx}(x,t) \tag{A.14}$$

$$z_x(1,t) = 0. \tag{A.15}$$

We choose first a Lyapunov functional for the subsystem (34), (37), and (38) as

$$V_2(t) = S\left(Z(t)\right) + \frac{2}{c} \int_0^{\|z(t)\|_{c,H_1}} \frac{\alpha_7\left(\sqrt{2}r\right)}{r} dr \tag{A.16}$$

where $c > 0$ is arbitrary. Using (37), (38), (A.14), (A.15) and integration by parts, it follows that

$$\frac{d\|z(t)\|_{c,H_1}}{dt} \leq -c\|z(t)\|_{c,H_1}. \tag{A.17}$$

Using the fact that $|z(0,t)| \leq |z(x,t)| + \int_0^1 |z_x(x,t)|dx$ together with Young's and Cauchy-Scharwz inequalities, we get for all $x \in [0,1]$

$$z(0,t)^2 \leq 2z(x,t)^2 + 2\int_0^1 z_x(x,t)^2 dx \tag{A.18}$$

and hence, by integrating over [0,1] we get

$$|z(0,t)| \leq \sqrt{2}\|z(t)\|_{c,H_1}. \tag{A.19}$$

Therefore, using (A.10), (A.11), (A.19), (A.17) we get along the solutions of the target system (34)–(38)

$$\dot{V}_2(t) \leq -\alpha_6\left(|Z(t)|\right) - \alpha_7\left(\sqrt{2}\|z(t)\|_{c,H_1}\right). \tag{A.20}$$

Using (A.10) and (A.16), we conclude that there exists a class $\mathcal{K}_\infty$ function α_8 such that

$$\dot{V}_2(t) \leq -\alpha_8\left(V_2(t)\right). \tag{A.21}$$

Take now another Lyapunov functional as

$$V_1(t) = \int_0^1 e^{c(1-x)}\omega(x,t)^2 dx + \int_0^1 e^{c(1-x)}\omega_x(x,t)^2 dx + 2\int_0^1 e^{c(1+x)} z_x(x,t)^2 dx. \quad (A.22)$$

Taking the derivative of V_1 and using (35)–(38) together with (A.12)–(A.15), integration by parts, we get

$$\dot{V}_1(t) \le -cV_1(t) + e^c\left(\omega(0,t)^2 + \omega_x(0,t)^2\right) - 2e^c z_x(0,t)^2 \quad (A.23)$$

and hence, using (36), (A.16) together with Young's inequality we arrive at

$$\dot{V}_1(t) \le -cV_1(t) + G\left(Z(t), z(0,t)\right) \quad (A.24)$$

where

$$G\left(Z(t), z(0,t)\right) = 4e^c\mu\left(Z(t)\right)^2 + 4h\left(Z(t)\right)^2 + 2e^c z(0,t)^2 + 2e^c\left(\left(\frac{\partial\mu\left(Z(t)\right)}{\partial Z} + \frac{\partial h\left(Z(t)\right)}{\partial Z}\right) \times f\left(Z(t), \mu^*\left(Z(t)\right) + z(0,t)\right)\right)^2. \quad (A.25)$$

Using the fact that μ and h are continuously differentiable with $\mu(0) = h(0) = 0$ and f is locally Lipschitz with $f(0,0) = 0$ we get that there exists a class $\mathcal{K}_\infty$ function α_9 such that $|G(Z(t), z(0,t))| \le \alpha_9(|Z(t)| + |z(0,t)|)$, and hence, using (A.19) and relations (A.10), (A.16) that there exists a class $\mathcal{K}_\infty$ function α_{10} such that $|G(Z(t), z(0,t))| \le \alpha_{10}(V_2(t))$. Therefore,

$$\dot{V}_1(t) \le -cV_1(t) + \alpha_{10}\left(V_2(t)\right). \quad (A.26)$$

From [30] there exist smooth, $\mathcal{K}_\infty$ functions $\tilde{\rho}_1, \tilde{\rho}_2$ such that $W_1 = \tilde{\rho}_1(V_1)$ and $W_2 = \tilde{\rho}_2(V_2)$ satisfy

$$\dot{W}_1(t) \le -\tilde{\alpha}_1\left(V_1(t)\right) + \frac{1}{2}\tilde{\alpha}_2\left(V_2(t)\right) \quad (A.27)$$

$$\dot{W}_2(t) \le -\tilde{\alpha}_2\left(V_2(t)\right) \quad (A.28)$$

for some class $\mathcal{K}_\infty$ functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. Therefore

$$\dot{W}_1(t) + \dot{W}_2(t) \le -\tilde{\alpha}_3\left(W_1(t) + W_2(t)\right) \quad (A.29)$$

for some class $\mathcal{K}_\infty$ function $\tilde{\alpha}_3$. With the comparison principle and Lemma 4.4 in [13], there exists a class $\mathcal{KL}$ function $\tilde{\beta}_1$ such that $W(t) \le \tilde{\beta}_1(W(0), t)$, where $W(t) = W_1(t) + W_2(t)$, and hence,

$$V(t) \le \beta_1\left(V(0), t\right) \quad (A.30)$$

for some class $\mathcal{KL}$ function β_1, where

$$V(t) = V_1(t) + V_2(t). \quad (A.31)$$

Using (A.10) the definition of $V_2(t)$ in (A.16), together with the definition of $V_1(t)$ in (A.22) one can conclude that there exist class $\mathcal{K}_\infty$ functions α_{11}, α_{12} such that

$$V(t) \le \alpha_{11}\left(\|\omega(t)\|_{H_1} + \|z(t)\|_{H_1} + |Z(t)|\right) \quad (A.32)$$

$$\alpha_{12}\left(V(t)\right) \ge \|\omega(t)\|_{H_1} + \|z(t)\|_{H_1} + |Z(t)|. \quad (A.33)$$

Bound (41) then follows with $\beta_3(s,t) = \alpha_{12}(\beta_1(\alpha_{11}(s), t))$.

Proof of Lemma 5

We prove bound (43) using relations (26), (27) of Assumption 1 and the fact that p satisfies the initial value problem in x (A.1), (A.2). Using (27) we get

$$\frac{\partial R_f\left(p(x,t)\right)}{\partial p} g\left(p(x,t), \zeta(x,t)\right) \le R_f\left(p(x,t)\right) + \alpha_3\left(|\zeta(x,t)|\right). \quad (A.34)$$

With (A.1), we get

$$\frac{dR_f\left(p(x,t)\right)}{dx} \le R_f\left(p(x,t)\right) + \alpha_3\left(|\zeta(x,t)|\right). \quad (A.35)$$

With the comparison principle and (A.2), we arrive at

$$R_f\left(p(x,t)\right) \le e^x R_f\left(Z(t)\right) + \int_0^x e^{x-y}\alpha_3\left(|\zeta(y,t)|\right) dy. \quad (A.36)$$

Using (26), we get for all $x \in [0,1]$

$$|p(x,t)| \le \alpha_1^{-1}\left(e\left(\alpha_2\left(|Z(t)|\right) + \alpha_3\left(\|\zeta(t)\|_\infty\right)\right)\right). \quad (A.37)$$

Hence, we get (43) with $\rho_2(s) = \alpha_1^{-1}(e(\alpha_2(s) + \alpha_3(s)))$.

Proof of Lemma 6

For proving bound (44). we use Lemma 3. Since π satisfies (A.5), (A.6) using (A.8), (A.9) we get

$$|\pi(x,t)| \le \beta_Z\left(|Z(t)|, x\right) + \gamma_Z\left(\sup_{0\le y\le x}|z(y,t)|\right), \quad (A.38)$$

and hence, bound (44) holds with $\hat{\rho}_1(s) = \beta_Z(s,0) + \gamma_Z(s)$.

Proof of Lemma 7

Since μ, h are continuously differentiable with $\mu(0,0) = h(0,0) = 0$ it holds that

$$|\mu(Z)| + |h(Z)| \le \alpha^*\left(|Z|\right). \quad (A.39)$$

for some class $\mathcal{K}_\infty$ function α^*. Therefore, using (32) we get that $\|z(t)\|_\infty \le \|\zeta(t)\|_\infty + \alpha^*(\|p(t)\|_\infty)$, and hence, with Lemma 5 that

$$|Z(t)| + \|z(t)\|_\infty \le \rho_2\left(|Z(t)| + \|\zeta(t)\|_\infty\right) \quad (A.40)$$

with $\rho_2(s) = s + \alpha^*(\rho_1(s))$. Moreover, from (32) we get

$$z_x(x,t) = \zeta_x(x,t) - \frac{\partial\mu^*\left(p(x,t)\right)}{\partial p} g\left(p(x,t), \zeta(x,t)\right) \quad (A.41)$$

and hence, using the fact that μ^* is continuously differentiable and g locally Lipschitz with $g(0,0)=0$ there exists a class $\mathcal{K}_\infty$ function α_1^* such that

$$|z_x(x,t)| \leq |\zeta_x(x,t)| + \alpha_1^*\left(\|p(t)\|_\infty + \|\zeta(t)\|_\infty\right). \quad \text{(A.42)}$$

Therefore

$$\int_0^1 z_x(x,t)^2 dx \leq 2\int_0^1 \zeta_x(x,t)^2 dx + 2\alpha_2^*\left(|Z(t)| + \|\zeta(t)\|_\infty\right) \quad \text{(A.43)}$$

where $\alpha_2^*(s)={\alpha_1^*}^2(s+\rho_1(s))$, and hence, $\|z(t)\|_\infty + (\int_0^1 z_x(x,t)^2 dx)^{(1/2)} \leq \rho_3^*(|Z(t)| + \|\zeta(t)\|_\infty + \int_0^1 \zeta_x(x,t)^2 dx)$, where $\rho_3^*(s) = \rho_2(s) + \sqrt{2s} + \sqrt{2\alpha_2^*(s)}$. For any $v \in H_1(0,1)$ it holds that $|v(0,t)| \leq |v(x,t)| + \int_0^1 |v_x(x,t)|dx$, and hence, with Young's and Cauchy-Schwartz's inequalities, and by integrating over [0,1] we get that $|v(0,t)| \leq \sqrt{2}\|v(t)\|_{H_1}$. Since also $|v(x,t)| \leq |v(0,t)| + \int_0^1 |v_x(x,t)|dx$, we arrive at

$$\|v(t)\|_\infty \leq \left(\sqrt{2}+1\right)\|v(t)\|_{H_1}. \quad \text{(A.44)}$$

Using the fact that for any function $v \in H_1(0,1)$, $(\int_0^1 v(x,t)^2 dx)^{(1/2)} \leq \|v(t)\|_\infty$ bound (45) is proved with $\rho_3(s) = \rho_3^*(\sqrt{2}s + 2s)$. Analogously, bound (46) is proved using (39), (A.39) and Lemma 6, with $\hat{\rho}_3(s) = \hat{\rho}_3^*((\sqrt{2}+2)s)$, where $\hat{\rho}_3^*(s) = \hat{\rho}_2(s) + \sqrt{2}s + \sqrt{2}\alpha_3^*(s)$ and $\hat{\rho}_2(s) = s + \alpha^*(\hat{\rho}_1(s))$, $\alpha_3^*(s) = \alpha_1^*(s + \hat{\rho}_1(s) + \alpha^*(\hat{\rho}_1(s)))$.

Appendix B

Proof of Theorem 2

We start the proof of Theorem 2 by defining the inverse backstepping transformation of (61), for all $x \in [0,1]$ as

$$\omega(x,t) = w(x,t) + \mu\left(\rho(x,t)\right) - h\left(\rho(x,t)\right) \quad \text{(B1)}$$

where for all $x \in [0,1]$

$$\rho(x,t) = Z(t) - \int_0^x g\left(\rho(y,t), \mu\left(\rho(y,t)\right) + h\left(\rho(y,t)\right) + w(y,t)\right) dy. \quad \text{(B.2)}$$

To see this, we first note that r in (62) satisfies the following boundary value problem:

$$r_x(x,t) = -g\left(r(x,t), 2h\left(r(x,t)\right) + \omega(x,t)\right) \quad \text{(B.3)}$$

$$r(0,t) = Z(t) \quad \text{(B.4)}$$

and hence, by combining (B.2) and (B.1), it follows that ρ and r satisfy the same boundary value problem. With the uniqueness of solutions it follows that $r \equiv \rho$, and hence, $\rho_t = -\rho_x$, which in turn implies that ω defined in (B.1) satisfies (11). The fact that ω satisfies (12) follows by combining (32) with (B.1) and the fact that $p(0,t) = \rho(0,t) = Z(t)$. Similarly to the proof of Lemma 7, using (61) and (B.1) we conclude that in order to prove that w and ω are upper bounded by a class $\mathcal{K}_\infty$ function of Z, ω and of Z, w respectively, it is sufficient to show that the following holds for some class $\mathcal{K}_\infty$ functions $\rho_{\mathrm{r},1}$, $\hat{\rho}_{\mathrm{r},1}$

$$\|r(t)\|_\infty \leq \rho_{\mathrm{r},1}\left(|Z(t)| + \|\omega(t)\|_\infty\right) \quad \text{(B.5)}$$

$$\|\rho(t)\|_\infty \leq \hat{\rho}_{\mathrm{r},1}\left(|Z(t)| + \|w(t)\|_\infty\right) \quad \text{(B.6)}$$

We prove first bound (B.6). Using (19), we get from (B.2) that

$$\rho_x(x,t) = -g\left(\rho(x,t), \mu^*\left(\rho(x,t)\right) + w(x,t)\right) \quad \text{(B.7)}$$

$$\rho(0,t) = Z(t). \quad \text{(B.8)}$$

Under Assumption 4, we get $-(\partial R_{\mathrm{b,cl}}(\rho(x,t))/\partial\rho) g(\rho(x,t), \mu^*(\rho(x,t)) + w(x,t)) \leq R_{\mathrm{b,cl}}(\rho(x,t)) + \alpha_{6,\mathrm{cl}}(|w(x,t)|)$. Using (B.7), we get

$$\frac{dR_{\mathrm{b,cl}}\left(\rho(x,t)\right)}{dx} \leq R_{\mathrm{b,cl}}\left(\rho(x,t)\right) + \alpha_{6,\mathrm{cl}}\left(|w(x,t)|\right) \quad \text{(B.9)}$$

and hence, with the comparison principle and (B.8)

$$R_{\mathrm{b,cl}}\left(\rho(x,t)\right) \leq e^x R_{\mathrm{b,cl}}\left(Z(t)\right) + \int_0^x e^{x-y}\alpha_{6,\mathrm{cl}}\left(|w(y,t)|\right) dy. \quad \text{(B.10)}$$

Hence, using (77) we get bound (B.6) with some class $\mathcal{K}_\infty$ function $\hat{\rho}_{\mathrm{r},1}$. Under Assumption 3 and using similar arguments we get bound (B.5) using (B.3), (B.4). Bound (73) then follows by combining (32), (61), the fact that μ, h are locally Lipschitz with $\mu(0) = h(0) = 0$ and (43), (B.5). Similarly, bound (74) follows by combining (39), (B.1) and (6), (B.6). Combining estimate (71) with (73), (74), we get

$$\Pi(t) \leq \hat{\beta}_3\left(\Pi(0), t\right) \quad \text{(B.11)}$$

$$\Pi(t) = |Z(t)| + \|\zeta(t)\|_\infty + \|\omega(t)\|_\infty \quad \text{(B.12)}$$

for some class $\mathcal{KL}$ function $\hat{\beta}_3$. Using the facts that $Z(t) = \left[\begin{smallmatrix} X(t) \\ u(0,t) \end{smallmatrix}\right]$ and $\|\zeta(t)\|_\infty + \|\omega(t)\|_\infty \leq 2(\|u_t(t)\|_\infty + \|u_x(t)\|_\infty)$, $\|u_t(t)\|_\infty + \|u_x(t)\|_\infty \leq \|\zeta(t)\|_\infty + \|\omega(t)\|_\infty$ [which follow from the triangle inequality and definitions (5)–(8)], we get $|X(t)| + |u(0,t)| + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty \leq \sqrt{2}\hat{\beta}_3(2(|X(0)| + |u(0,0)| + \|u_t(0)\|_\infty + \|u_x(0)\|_\infty), t)$. Using the fact that $\|u(t)\|_\infty \leq |u(0,t)| + \|u_x(t)\|_\infty$ we get (83) with $\hat{\eta}_3(s,t) = 2\sqrt{2}\hat{\beta}_3(2s,t)$.

Using relations (37), (38), we get that

$$z(x,t) = \begin{cases} z_0(t+x), & 0 \leq x+t < 1 \\ 0, & x+t \geq 1 \end{cases} \quad \text{(B.13)}$$

where the initial condition $z_0(x)$ is given by (32) with $t=0$. Using relation (5), the fact that $u_t(\cdot,0) \in C[0,1]$ and that $u(\cdot,0) \in C^1[0,1]$, we conclude that $\zeta(\cdot,0) \in C[0,1]$, and hence, using the fact that p satisfies the ODE in x (57), (58) and the Lipschitzness of g we conclude the existence and uniqueness of $p(x,0) \in C^1[0,1]$. Therefore, with (32) and the compatibility condition we get that $z_0 \in C[0,1]$, and hence, with (34), (B.13) and the Lipschitzness of g and μ^* we conclude the existence and uniqueness of $(X(t), u(0,t)) \in C^1[0,\infty)$. Using the fact that $z_0 \in C[0,1]$, the compatibility condition and (B.13) guarantee the existence of $z \in C([0,1]\times[0,\infty))$. The uniqueness of this solution follows from the uniqueness of the solution to (37), (38) (see Sections 2.1 and 2.3 in [5]). With the same arguments and using relation (6), the ODE (B.3), (B.4), relations (35), (36) and the fact that

$$w(x,t) = \begin{cases} w_0(x-t), & 0 \le t < x \\ z_0(t-x), & 0 \le t-x < 1 \\ 0 & t-x \ge 1 \end{cases} \tag{B.14}$$

we get, with the compatibility condition and $u_x(0,0) = h(X(0), u(0,0))$, the existence and uniqueness of $w \in C([0,1]\times[0,\infty))$. With the inverse backstepping transformations (B.1), (39) and the facts that $\pi(x,t) = p(x,t) = Z(t+x) \in C^1([0,\infty])$, which implies that $\pi(x,t) \in C^1([0,1]\times[0,\infty))$, and that $\rho(x,t) = r(x,t) = Z(t-x) \in C^1([-1,\infty))$ (since $\rho(x,0) = Z(-x)$ in (B.2) satisfies for all $x \in [0,1]$ $\rho_x(x,0) = -g(\rho(x,0), \mu^*(\rho(x,0)) + w_0(x))$), which implies that $\rho(x,t) \in C^1([0,1]\times[0,\infty))$, we get the existence and uniqueness of $\zeta, \omega \in C([0,1]\times[0,\infty))$. Therefore, using (7), (8) we conclude that there exists a unique solution $(u_t, u_x) \in C([0,1]\times[0,\infty))$, and hence, that there exists a unique solution (80).

Appendix C

Proof of Theorem 4

As in the case of nonlinear systems, the control law that stabilizes system (96)–(99) is based on a predictor feedback design that stabilizes the augmented system

$$\dot{X}(t) = AX(t) + B\xi(t) \tag{C.1}$$

$$\dot{\xi}(t) = -CX(t) - \sigma\xi(t) + \zeta(0,t). \tag{C.2}$$

The predictor states p_1 and p_2 defined in (22) and in (23) respectively satisfy the following ODE in x

$$\frac{\partial p(x,t)}{\partial x} = Fp(x,t) + G\left(u_t(x,t) + u_x(x,t)\right) \tag{C.3}$$

$$p(0,t) = \begin{bmatrix} X(t) \\ u(0,t) \end{bmatrix}, \tag{C.4}$$

where $p = \left[\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right]$ and where $O_{n\times 1}$ denotes an $n\times 1$ matrix with zero entries. The initial value problem (C.3), (C.4) can be solved explicitly as

$$p(x,t) = e^{Fx}X(t) + \int_0^x e^{F(x-y)}G \times \left(u_t(y,t) + u_y(y,t)\right) dy. \tag{C.5}$$

It follows from definitions (101), (102), (C.5) that the control law (21) for the special case of linear systems can be written in compact form as (100). The Lyapunov functional for the case of linear systems is

$$V(t) = Z^{\mathrm{T}}(t)PZ(t) + \left(\frac{2\,|PG|}{\lambda_{\min}(Q)} + 1\right)\int_0^1 e^{c(1+x)} \times z(x,t)^2 dx + \int_0^1 e^{c(1-x)} w(x,t)^2 dx, \tag{C.6}$$

where $P = P^{\mathrm{T}} > 0$ and $Q = Q^{\mathrm{T}} > 0$ satisfy

$$(F + GK^*)^{\mathrm{T}} P + P(F + GK^*) = -Q, \tag{C.7}$$

$Z(t) = \left[\begin{smallmatrix} X(t) \\ u(0,t) \end{smallmatrix}\right]$ and z, w satisfy (64)–(67). Taking the derivative of (C.6) along the target system (63)–(67) and using Young's inequality together with integration by parts we obtain

$$\dot{V}(t) \le -\frac{\lambda_{\min}(Q)}{2}\,|Z(t)|^2 - c\int_0^1 e^{c(1+x)} z(x,t)^2 dx - c\int_0^1 e^{c(1-x)} w(x,t)^2 dx. \tag{C.8}$$

Therefore,

$$\dot{V}(t) \le -\lambda V(t), \tag{C.9}$$

with $\lambda = \min\{\lambda_{\min}(Q), c\}/(\lambda_{\max}(P) + (2|PG|/\lambda_{\min}(Q)))$. Using the linearity of the backstepping transformations we get that

$$\Phi_1(t) \le M_1\Phi_2(t) \tag{C.10}$$

$$\Phi_2(t) \le M_2\Phi_1(t), \tag{C.11}$$

$$\Phi_1(t) = |Z(t)| + \int_0^1 w(x,t)^2 dx + \int_0^1 z(x,t)^2 dx \tag{C.12}$$

$$\Phi_2(t) = |Z(t)| + \int_0^1 \zeta(x,t)^2 dx + \int_0^1 \omega(x,t)^2 dx, \tag{C.13}$$

for some positive constants M_1 and M_2. With relations (5), (6), and the facts that $\int_0^1 u(x,t)^2 dx \le 2u(0,t)^2 + 4\int_0^1 u_x(x,t)^2 dx$ and that $u(0,t)^2 \le 2\int_0^1 u(x,t)^2 dx + 2\int_0^1 u_x(x,t)^2 dx$ we get estimate (104). The rest of the proof for the case in which the initial conditions are as in Theorem 1 is almost identical to Theorem 1 with the difference that we use (B.14) instead of (59) together with the inverse transformation (B.1) and the fact that $\rho(x,t) \in C^1([0,1]\times[0,\infty))$ (see also Appendix B). For the case in which $(u(\cdot,0), u_t(\cdot,0)) \in H_1(0,1)\times L_2(0,1)$ without the compatibility conditions then with similar arguments one can show that $p(x,0), r(x,0) \in C[0,1]$ and hence with (32), (61) and (B.13), (B.14) that $(w(\cdot,t), z(\cdot,t)) \in C([0,\infty), L_2(0,1)\times L_2(0,1))$ which also implies that $Z(t) = (X(t), u(0,t)) \in C[0,\infty)$. The facts $\pi(x,t) = Z(t+x)$ and $\rho(x,t) = Z(t-x)$ and the inverse transformations (B.1),

(39) guarantee that $(\zeta(\cdot,t),\omega(\cdot,t)) \in C([0,\infty), L_2(0,1) \times L_2(0,1))$, and hence, with (5), (6) that there exists a solution as in the statement of Theorem 4.

Appendix D

Proof of Proposition 5.1

When (98) holds with $\sigma = 0$ and $C = 0$, the control law (100) takes the form

$$U(t) = -u_t(1,t) + (-c_1 + KB)\,u(1,t) + \int_0^1 \times \left(Kc_1 \times \int_x^1 e^{A(1-y)}dyB - c_1 + Ke^{A(1-x)}B \right) \times u_t(x,t)dx + K(c_1+A)\int_0^1 e^{A(1-x)}Bu(x,t)dx + K(c_1+A)e^{A}X(t). \quad \text{(D.1)}$$

We recall next that the control law (47) derived in [15] is (for $D = 1$)

$$U(t) = (-\bar{c}_0 + \bar{c}_1 KB)u(1,t) - \bar{c}_1 u_t(1,t) + \int_0^1 p(1-y)u(y,t)dy + \int_0^1 q(1-y)u_t(y,t)dy + \pi(1)X(t) \quad \text{(D.2)}$$

where

$$p(s) = \mu'(s) + \bar{c}_0\mu(s) + \bar{c}_1\left(m''(s) + \bar{c}_0 m'(s)\right) \quad \text{(D.3)}$$

$$q(s) = m'(s) + \bar{c}_0 m(s) + \bar{c}_1\left(\mu(s) + \bar{c}_0\int_0^s \mu(\xi)d\xi - \bar{c}_0\right) \quad \text{(D.4)}$$

$$\pi(x) = \gamma'(x) + \gamma(x)(\bar{c}_0 I + \bar{c}_1 A) + \bar{c}_1\bar{c}_0\int_0^x \gamma(\xi)d\xi A \quad \text{(D.5)}$$

and

$$\gamma(x) = KM(x) \quad \text{(D.6)}$$

$$M(x) = [\,I \quad 0\,]\mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix}x}\begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \text{(D.7)}$$

$$m(s) = \int_0^s \gamma(\xi)Bd\xi, \quad \text{(D.8)}$$

$$\mu(s) = \int_0^s \gamma(\xi)ABd\xi. \quad \text{(D.9)}$$

For proving that the control law (D.1) is identical to (D.2) for $\bar{c}_0 = c_1$ and $\bar{c}_1 = 1$, we have to show that

$$\pi(x) = K(c_1 + A)e^{Ax} \quad \text{(D.10)}$$

$$p(x) = \pi(x)B \quad \text{(D.11)}$$

$$q(x) = K\left(e^{Ax} + c_1\int_0^x e^{Ay}dy\right)B - c_1. \quad \text{(D.12)}$$

To show that (D.10) holds, we first observe that $\pi(0) = K(c_1 + A)$ since from (D.6), (D.7) it follows that $\gamma'(0) = 0$. Hence, it remains to show that $\pi'(x) = K(c_1 + A)Ae^{Ax}$, i.e.,

$$(c_1+A)Ae^{Ax} = M(x)A^2 + M'(x)(c_1+A) + c_1M(x)A \quad \text{(D.13)}$$

where we also used the fact that $M''(x) = M(x)A^2$. By expanding M in Taylor series, we get that

$$M'(x) = A\sum_{i=0}^{\infty}\frac{(Ax)^{2i+1}}{(2i+1)!} \quad \text{(D.14)}$$

$$M(x)A = A\sum_{i=0}^{\infty}\frac{(Ax)^{2i}}{(2i)!}. \quad \text{(D.15)}$$

Using the fact that $e^{Ax} = \sum_{i=0}^{\infty}((Ax)^i/i!)$, we conclude that (D.13) holds, and hence, that (D.10) holds. We show next that (D.11) holds. Using (D.4), (D.8), (D.9), it follows that

$$p(x) = \left(\gamma'(x) + \gamma(x)(c_1 + A) + c_1\int_0^x \gamma(y)dyA\right)B \quad \text{(D.16)}$$

and hence, with (D.5) we conclude that (D.11) holds. To show (D.12), we first rewrite (D.4) as

$$q(x) = \left(\gamma(x) + c_1\int_0^x \gamma(y)dy + \int_0^x \gamma(y)dyA + c_1 + \int_0^x\int_0^y \gamma(r)drdyA\right)B - c_1. \quad \text{(D.17)}$$

Using (D.16), we get that

$$c_1 + q(x) = \int_0^x p(y)dy + \gamma(0)B, \quad \text{(D.18)}$$

and hence with (D.10), (D.11), we arrive at

$$c_1 + q(x) = \left(Kc_1\int_0^x e^{Ay}dy + KA\int_0^x e^{Ay}dy\right)B + \gamma(0)B. \quad \text{(D.19)}$$

The proof is complete by noting that $A\int_0^x e^{Ay}dy = e^{Ax} - I$ and $\gamma(0) = K$.

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