Inverse Optimal Design of Input-to-State Stabilizing Nonlinear Controllers

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***Abstract*—We show that input-to-state stabilizability (as defined by Sontag) is both *necessary and sufficient* for the solvability of a Hamilton–Jacobi–Isaacs equation associated with a meaningful differential game problem similar to, but more general than, the "nonlinear $\mathcal{H}_\infty$" problem. The significance of the result stems from the fact that constructive solutions to the input-to-state stabilization problem are available (presented in the paper) and that, as shown here, inverse optimal controllers possess margins on input-to-state stability against a certain class of input unmodeled dynamics. Rather than completion of squares, the main tools in our analysis are Legendre–Fenchel transformations and the general form of Young's inequality.**

***Index Terms*— Backstepping, control Lyapunov functions, input-to-state stability, nonlinear H_∞.**

I. Introduction

Significant advances achieved over the last few years in formulating the "nonlinear $\mathcal{H}_\infty$" control theory [2]–[4], [7], [12]–[18], [23], [27], [37], [39], [40], [41] have not yet penetrated into control applications because of difficulties associated with solving the Hamilton–Jacobi–Isaacs (HJI) partial differential equations. The need to solve the HJI equations can be avoided by using the inverse optimality approach, originated by Kalman and introduced into robust nonlinear control via Freeman's robust control Lyapunov functions (CLF's) [8], [9], [34]. In parallel to nonlinear $\mathcal{H}_\infty$, the framework of input-to-state stability (ISS) introduced by Sontag [34] has triggered efforts toward designing input-to-state stabilizing controllers [10], [22], [24], [25], [29], [31], [35], [38]. In this paper, we show that input-to-state stabilizability is both necessary and sufficient for the solvability of a differential game problem similar to, but more general than, the nonlinear $\mathcal{H}_\infty$ problem.

Next, we briefly describe the problem addressed in the paper. We consider the system of the form

$$\dot{x} = f(x) + g_1(x)d + g_2(x)u \tag{1}$$

which is said to be input-to-state stabilizable with respect to the disturbance d if there exists a control law which guarantees that

$$|x(t)| \leq \beta(|x(0)|, t) + \chi\left(\sup_{0\leq\tau\leq t} |d(\tau)|\right) \tag{2}$$

where β is a class $\mathcal{KL}$ function and χ is a class $\mathcal{K}$ function. In [24] we showed that an input-to-state stabilizing controller can be designed if and only if there exists an ISS-CLF. A virtually identical result was obtained by Sontag and Wang simultaneously in [36]. In Section III we show that the controller from [24] is *inverse optimal* with respect to the following differential game problem:

$$\inf_u \sup_d \int_0^\infty [l(x) + u^{\mathrm{T}} R_2(x) u - \gamma(|d|)] \tag{3}$$

where $l(x)$ is positive definite and radially unbounded, $|R_2(x)|$ is bounded away from zero, and $\gamma(|d|)$ is class $\mathcal{K}_\infty$. We also show that if a problem of the form (3) is solvable, then (1) is input-to-state stabilizable. Our results extend those of Freeman and Kokotović [10], where the disturbance had to obey a state-dependent bound and was not penalized in the cost functional (ISS was achieved by invoking a result of Sontag and Wang [35] on robustness of ISS systems to a certain class of state-dependent perturbations).

By the *inverse* (rather than a *direct*) differential game problem, we mean that we are searching for, not only a control law, but also functions $l(x)$, $R_2(x)$, and $\gamma(|d|)$ which must be meaningful in a well-defined sense. This problem is easier than the *direct* one in which l, R_2, and γ are given, and where one has to solve an HJI partial differential equation. To motivate our *inverse* approach, we show a simple example where the HJI equation is not only difficult to solve, but *impossible* to solve. Consider the scalar system $\dot{x} = u + x^2 d$ and the differential game problem $\inf_u \sup_d \int_0^\infty (x^2 + u^2 - \gamma^2 d^2)$, where $\gamma > 0$. The resulting HJI equation $(x^4/\gamma^2 - 1)(\partial V/\partial x)^2 = -4x^2$ is not solvable outside of the interval $x \in (-\sqrt{\gamma}, \sqrt{\gamma})$, and the optimal control law $u^* = -\gamma x/\sqrt{\gamma^2 - x^4}$ is not defined outside of this interval either. Contrary to the discouraging outcome of the *direct* problem, the *inverse* problem is solvable, and in the paper we show several solutions.

Another benefit of inverse optimality is that the controller remains input-to-state stabilizing in the presence of a certain class of input unmodeled dynamics which do not have to be small in the $\mathcal{H}_\infty$ sense, do not have to be linear, and do not even have to be ISS. Efforts on input unmodeled dynamics have been intensive over the last few years, starting with Krstić *et al.* [26] and followed by Jiang and Pomet [21], Praly and Wang [31], and Jiang and Mareels [20]. Sepulchre

Manuscript received November 8, 1996; revised July 16, 1997. Recommended by Associate Editor, A. J. van der Schaft. This work was supported in part by the National Science Foundation under Grant ECS-9624386, in part by the Air Force Office of Scientific Research under Grant F496209610223, and in part by a gift from the United Technologies Research Center.

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Publisher Item Identifier S 0018-9286(98)02075-3.

et al. [32] were the first in the nonlinear setting to quantify stability margins to input unmodeled dynamics. They did this by using inverse optimality and passivity concepts. Our result presented in Section IV is an extension of their result to the case with disturbances. Unfortunately, like in the linear case (even) without disturbances [1], the class of allowable input dynamics does not include those that increase relative degree (and thus reduce the control authority at higher frequencies) such as, e.g., $1/(1+\mu s)$, which are typical actuator dynamics.

If viewed as extensions of "nonlinear $\mathcal{H}_\infty$," the results of this paper indicate that the restriction to a quadratic penalty on the disturbance has been a major factor that has prevented *constructive* solutions in the existing nonlinear $\mathcal{H}_\infty$ literature. In Section V we explore the possibility of retaining a quadratic penalty on the disturbance by introducing *state-dependent weighting*:

$$\inf_u \sup_d \int_0^\infty [l(x) + u^{\mathrm{T}} R_2(x) u - d^{\mathrm{T}} R_1(x) d]. \tag{4}$$

We show that input-to-state stabilizability guarantees the existence of an inverse optimal solution with $R_1(x)$ *continuous* and taking *nonnegative* definite symmetric values. Unfortunately, there is no guarantee in general that $R_1(x)$ remains bounded as $|x| \to \infty$.

The constructive character of the results of the paper is illustrated in Sections VI–VIII. Since every ISS-CLF is a solution to a meaningful HJI equation, we proceed to show in Section VI how backstepping can be used to generate ISS-CLF's. Finally, in Sections VII and VIII we address strict feedback systems for which disturbance attenuation controllers have been constructed by Marino *et al.* [28], Isidori [14], Krstić *et al.* [25], and Pan and Başar [29], but without a cost on the control effort. Our solution is the first that puts penalty on control and is derived from an HJI equation.

All of the controllers designed in this paper guarantee not only disturbance attenuation of an $\mathcal{L}_2$ type (or similar) but also attenuation of persistent $(\mathcal{L}_\infty)$ disturbance, a goal not pursued in the nonlinear $\mathcal{H}_\infty$ literature.

II. ISS AND ISS-CLF'S

In this section we present preliminaries on ISS, stabilizability, and ISS-CLF's.

Let us consider first the nonlinear system

$$\dot{x} = f(x) + g_1(x) d \tag{5}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^r$ is the disturbance, and $f(0) = 0$. System (5) is said to be *input-to-state stable (ISS)* [34] if the following property is satisfied:

$$|x(t)| \leq \beta(|x(0)|, t) + \chi\left(\sup_{0\leq\tau\leq t} |d(\tau)|\right) \tag{6}$$

where β is a class $\mathcal{KL}$ function and χ is a class $\mathcal{K}$ function. A smooth positive definite radially unbounded function $V : \mathbb{R}^n \to R_+$ is referred to as an ISS-Lyapunov function for (5) if there exists a class $\mathcal{K}_\infty$ function ρ such that the following implication holds for all $x \neq 0$:

$$\begin{gathered} |x| \geq \rho(|d|) \\ \Downarrow \\ \frac{\partial V}{\partial x}[f(x) + g_1(x) d] < 0. \end{gathered} \tag{7}$$

It was proved by Sontag and Wang [35] that the characterization (6) is equivalent to the existence of an ISS-Lyapunov function. An estimate of the gain function χ in (6) that follows from (7) is $\chi = \alpha_1^{-1} \circ \alpha_2 \circ \rho$, where $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$.

Now consider the system which, in addition to the disturbance input d, also has a control input u

$$\dot{x} = f(x) + g_1(x) d + g_2(x) u \tag{8}$$

where $u \in \mathbb{R}^m$ and $f(0) = 0$. We say that (8) is *input-to-state stabilizable* if there exists a control law $u = \alpha(x)$ continuous away from the origin with $\alpha(0) = 0$, such that the closed-loop system is ISS with respect to d.

Definition 2.1: A smooth positive definite radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an ISS-CLF for (8) if there exists a class $\mathcal{K}_\infty$ function ρ such that the following implication holds for all $x \neq 0$ and all $d \in \mathbb{R}^r$:

$$\begin{gathered} |x| \geq \rho(|d|) \\ \Downarrow \\ \inf_{u \in \mathbb{R}^m} \{L_f V + L_{g_1} V d + L_{g_2} V u\} < 0. \end{gathered} \tag{9}$$

The following theorem establishes equivalence between input-to-state stabilizability and the existence of an ISS-CLF. It extends Sontag's theorem in [33] to systems affine in the disturbance.

Theorem 2.1 [24]: System (8) is input-to-state stabilizable if and only if there exists an ISS-CLF.

The proof of this theorem employs the result of the following lemma. The actual construction of a control law is presented in the proof of Theorem 5.2.

Lemma 2.1: A pair (V, ρ) satisfies Definition 2.1 if and only if

$$L_{g_2} V(x) = 0 \Rightarrow L_f V(x) + |L_{g_1} V(x)| \rho^{-1}(|x|) < 0, \quad \forall x \neq 0. \tag{10}$$

Proof (Necessity): By Definition 2.1, if $x \neq 0$ and $L_{g_2} V = 0$, then

$$\begin{gathered} |x| \geq \rho(|d|) \\ \Downarrow \\ L_f V + L_{g_1} V d < 0. \end{gathered} \tag{11}$$

Now consider the particular input

$$d = \frac{(L_{g_1} V)^{\mathrm{T}}}{|L_{g_1} V|} \rho^{-1}(|x|). \tag{12}$$

This input satisfies the upper part of the implication (11)

$$\rho(|d|) = |x|. \tag{13}$$

Therefore, substituting (12) into the lower part of (11), we conclude that, if $x \neq 0$ and $L_{g_2} V = 0$, then

$$L_f V + |L_{g_1} V| \rho^{-1}(|x|) < 0 \tag{14}$$

that is, (10) is satisfied for $x \neq 0$.

(Sufficiency): For $|x| \geq \rho(|d|)$, using (10) we have

$$\begin{aligned}\inf_u \{L_f V + L_{g_1} V d + L_{g_2} V u\} &\leq \inf_u \{L_f V + |L_{g_1} V||d| + L_{g_2} V u\} \\ &\leq \inf_u \{L_f V + |L_{g_1} V|\rho^{-1}(|x|) + L_{g_2} V u\} \\ &< 0. \end{aligned} \tag{15}$$

This completes the proof. □

Corollary 2.1: System (5) is ISS if and only if there exist a smooth positive definite radially unbounded function $V(x)$ and a class $\mathcal{K}_\infty$ function ρ such that

$$L_f V(x) + |L_{g_1} V(x)|\rho^{-1}(|x|) < 0, \quad \forall x \neq 0. \tag{16}$$

III. Inverse Optimal Gain Assignment

Definition 3.1: The *inverse optimal gain assignment* problem for (8) is solvable if there exist a class $\mathcal{K}_\infty$ function γ whose derivative γ' is also a class $\mathcal{K}_\infty$ function, a matrix-valued function $R_2(x)$ such that $R_2(x) = R_2(x)^{\mathrm{T}} > 0$ for all x, positive definite radially unbounded functions $l(x)$ and $E(x)$, and a feedback law $u = \alpha(x)$ continuous away from the origin with $\alpha(0) = 0$, which minimizes the cost functional

$$J(u) = \sup_{d \in \mathcal{D}} \left\{ \lim_{t \to \infty} \left[E(x(t)) + \int_0^t (l(x) + u^{\mathrm{T}} R_2(x) u - \gamma(|d|))\, d\tau \right] \right\} \tag{17}$$

where $\mathcal{D}$ is the set of locally bounded functions of x.

The cost functional (17) puts penalty on the state and both the control and the disturbance. The state-dependent weight $R_2(x)$ on the control u is not allowed to vanish (and is, in fact, allowed to take infinite values in parts of the state space where the open-loop system is "well behaved" and zero control can be used). The penalty on the disturbance is allowed to be nonquadratic. (The purpose of the "terminal penalty" $E(x(t))$ is to avoid imposing an assumption that $x(t) \to 0$ as $t \to \infty$.)

In the next theorem we provide a sufficient condition for the solvability of the inverse optimal gain assignment problem. This theorem is followed by a result in Theorem 3.2, which shows how to construct a control law that satisfies the condition in Theorem 3.1 for any nonlinear system that is input-to-state stabilizable.

Before we start our developments, let us introduce the following notation: for a class $\mathcal{K}_\infty$ function γ whose derivative exists and is also a class $\mathcal{K}_\infty$ function, $\ell\gamma$ denotes the transform

$$\ell\gamma(r) = r(\gamma')^{-1}(r) - \gamma((\gamma')^{-1}(r)) \tag{18}$$

where $(\gamma')^{-1}(r)$ stands for the inverse function of $\frac{d\gamma(r)}{dr}$. Using integration by parts (Lemma A1-1), it is easy to show that $\ell\gamma$ is equal to the Legendre–Fenchel transform

$$\ell\gamma(r) = \int_0^r (\gamma')^{-1}(s)\, ds \tag{19}$$

which was brought into the control theory by Praly in [30].

Theorem 3.1: Consider the auxiliary system of (8)

$$\dot{x} = f(x) + g_1(x)\ell\gamma(2|L_{g_1} V|)\frac{(L_{g_1} V(x))^{\mathrm{T}}}{|L_{g_1} V|^2} + g_2(x) u \tag{20}$$

where $V(x)$ is a Lyapunov function candidate and γ is a class $\mathcal{K}_\infty$ function whose derivative γ' is also a class $\mathcal{K}_\infty$ function. Suppose that there exists a matrix-valued function $R_2(x) = R_2(x)^{\mathrm{T}} > 0$ such that the control law

$$u = \alpha(x) = -R_2(x)^{-1}(L_{g_2} V)^{\mathrm{T}} \tag{21}$$

globally asymptotically stabilizes (20) with respect to $V(x)$. Then, the control law

$$u = \alpha^*(x) = \beta\alpha(x) = -\beta R_2^{-1}(L_{g_2} V)^{\mathrm{T}} \tag{22}$$

with any $\beta \geq 2$ solves the inverse optimal gain assignment problem for (8) by minimizing the cost functional

$$J(u) = \sup_{d \in \mathcal{D}} \left\{ \lim_{t \to \infty} \left[2\beta V(x(t)) + \int_0^t \left(l(x) + u^{\mathrm{T}} R_2(x) u - \beta\lambda\gamma\left(\frac{|d|}{\lambda}\right)\right) d\tau \right] \right\} \tag{23}$$

for any $\lambda \in (0, 2]$, where

$$\begin{aligned} l(x) = &-2\beta[L_f V + \ell\gamma(2|L_{g_1} V|) - L_{g_2} V R_2^{-1}(L_{g_2} V)^{\mathrm{T}}] \\ &+ \beta(2-\lambda)\ell\gamma(2|L_{g_1} V|) + \beta(\beta-2) L_{g_2} \\ &\times V R_2^{-1}(L_{g_2} V)^{\mathrm{T}}. \end{aligned} \tag{24}$$

Proof: Since the control law (21) stabilizes (20), there exists a continuous positive definite function $W : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$L_f V + \ell\gamma(2|L_{g_1} V|) - L_{g_2} V R_2^{-1}(L_{g_2} V)^{\mathrm{T}} \leq -W(x). \tag{25}$$

We then have

$$\begin{aligned} l(x) \geq\ & 2\beta W(x) + \beta(2-\lambda)\ell\gamma(2|L_{g_1} V|) \\ &+ \beta(\beta-2) L_{g_2} V R_2^{-1}(L_{g_2} V)^{\mathrm{T}}. \end{aligned} \tag{26}$$

Since $\lambda \leq 2$, $\beta \geq 2$, $W(x)$ is positive definite, and $\ell\gamma$ is a class $\mathcal{K}_\infty$ function (Lemma A1-3), we conclude that $l(x)$ is also positive definite. Therefore, $J(u)$ defined in (23) is a meaningful cost functional that puts penalty on x, u, and d. Substituting $l(x)$ into (23), it follows that we have (27), as shown at the bottom of the next page, where

$$d^*(x) = \lambda(\gamma')^{-1}(2|L_{g_1} V|)\frac{(L_{g_1} V)^{\mathrm{T}}}{|L_{g_1} V|}. \tag{28}$$

By Lemma A1-4, $\Pi(d, d^*)$ can be rewritten as

$$\begin{aligned} \Pi(d, d^*) = &-\gamma\left(\frac{|d|}{\lambda}\right) - \ell\gamma\left(\gamma'\left(\frac{|d^*|}{\lambda}\right)\right) \\ &+ \gamma'\left(\frac{|d^*|}{\lambda}\right)\frac{(d^*)^{\mathrm{T}}}{|d^*|}\frac{d}{\lambda}. \end{aligned} \tag{29}$$

Then by Lemma A2 we have

$$\Pi(d,d^*) \le -\gamma\left(\frac{|d|}{\lambda}\right) - \ell\gamma\left(\gamma'\left(\frac{|d^*|}{\lambda}\right)\right) + \gamma\left(\frac{|d|}{\lambda}\right) + \ell\gamma\left(\gamma'\left(\frac{|d^*|}{\lambda}\right)\right) = 0 \tag{30}$$

and $\Pi(d,d^*) = 0$ if and only if $\frac{d}{\lambda} = (\gamma')^{-1}(\gamma'(\frac{|d^*|}{\lambda}))\frac{d^*}{|d^*|}$, that is

$$\Pi(d,d^*) = 0 \qquad \text{iff } d = d^*. \tag{31}$$

Thus

$$\sup_{d\in\mathcal{D}} \int_0^\infty \Pi(d,d^*)\,dt = 0 \tag{32}$$

and the "worst case" disturbance is given by (28). The minimum of (27) is reached with $u = \alpha^*$. Hence the control law (22) minimizes the cost functional (23). The value function of (17) is $J^*(x) = 2\beta V(x)$. □

The parameter $\beta \ge 2$ in the statement of Theorem 3.1 represents a design degree of freedom. The parameter λ (note that it parameterizes not only the penalty on the disturbance but also the penalty on the state, $l(x)$) indicates that the same control law is inverse optimal with respect to an entire family of different cost functionals.

Remark 3.1: Even though not explicit in the proof of Theorem 3.1, $V(x)$ solves the following family of HJI equations:

$$L_fV - \frac{\beta}{2}L_{g_2}VR_2(x)^{-1}(L_{g_2}V)^{\mathrm{T}} + \frac{\lambda}{2}\ell\gamma(2|L_{g_1}V|) + \frac{l(x)}{2\beta} = 0 \tag{33}$$

parameterized by $(\beta,\lambda) \in [2,\infty)\times(0,2]$. It is easily seen from the proof of the above theorem that for zero initial conditions, the achieved disturbance attenuation level is

$$\int_0^\infty [l(x) + u^{\mathrm{T}}R_2(x)u]\,dt \le \beta\lambda\int_0^\infty \gamma\left(\frac{|d|}{\lambda}\right)dt. \tag{34}$$

In the next theorem we design controllers that are inverse optimal in the sense of Definition 3.1. We emphasize that these controllers are not restricted to disturbances with $\int_0^\infty \gamma(|d|)\,dt < \infty$ because they achieve ISS and allow any bounded (and persistent) d.

Theorem 3.2: If (8) is input-to-state stabilizable, then the inverse optimal gain assignment problem is solvable.

Proof: By Theorem 2.1, there exist an ISS-CLF $V(x)$ and a class $\mathcal{K}_\infty$ function ρ such that (9) is satisfied. We now show that there exist a class $\mathcal{K}_\infty$ function γ and a control law $u = \alpha(x)$ of the form (21) such that the auxiliary system (20) is stabilized. To this end, we define the following Sontag-type control law $u = \alpha_s(x)$:

$$\alpha_s = \begin{cases} -\frac{\omega + \sqrt{\omega^2 + (L_{g_2}V(L_{g_2}V)^{\mathrm{T}})^2}}{L_{g_2}V(L_{g_2}V)^{\mathrm{T}}}(L_{g_2}V)^{\mathrm{T}}, & (L_{g_2}V)^T \ne 0 \\ 0, & (L_{g_2}V)^{\mathrm{T}} = 0 \end{cases} \tag{35}$$

where

$$\omega = L_fV + |L_{g_1}V|\rho^{-1}(|x|). \tag{36}$$

We first show that (35) is continuous in x on $\mathbb{R}^n \setminus \{0\}$. Sontag proved in [33] that the function (35) is smooth, provided its arguments ω and $L_{g_2}V$ are such that

$$L_{g_2}V = 0 \implies \omega < 0. \tag{37}$$

By Lemma 2.1, (37) is satisfied. Therefore, (35) is a smooth function of ω and $L_{g_2}V$ whenever $x \ne 0$. Since $\omega(x)$ is continuous and $L_{g_2}V(x)$ is smooth, the control law (35) is continuous for $x \ne 0$.

$$\begin{aligned}
J(u) &= \sup_{d\in\mathcal{D}}\left\{\lim_{t\to\infty}\left[2\beta V(x(t)) + \int_0^t\left(-2\beta L_fV\beta\lambda\ell\gamma(2|L_{g_1}V|) + \beta^2L_{g_2}VR_2^{-1}(L_{g_2}V)^{\mathrm{T}}\right.\right.\right.\\
&\qquad\left.\left.\left. +\ u^{\mathrm{T}}R_2u - \beta\lambda\gamma\left(\frac{|d|}{\lambda}\right)\right)d\tau\right]\right\}\\
&= \sup_{d\in\mathcal{D}}\left\{\lim_{t\to\infty}\left[2\beta V(x(t)) - 2\beta\int_0^t\left(L_fV + L_{g_1}Vd + L_{g_2}Vu\right)d\tau + \int_0^t\left(u^{\mathrm{T}}R_2u + 2\beta L_{g_2}Vu + \beta^2L_{g_2}VR_2^{-1}(L_{g_2}V)^{\mathrm{T}}\right)d\tau\right.\right.\\
&\qquad\left.\left. -\ \int_0^t\left(\beta\lambda\gamma\left(\frac{|d|}{\lambda}\right) - 2\beta L_{g_1}Vd + \beta\lambda\ell\gamma(2|L_{g_1}V|)\right)d\tau\right]\right\}\\
&= \sup_{d\in\mathcal{D}}\left\{\lim_{t\to\infty}\left[2\beta V(x(t)) - 2\beta\int_0^t dV + \int_0^t(u-\alpha^*)^{\mathrm{T}}R_2(u-\alpha^*)\,d\tau - \beta\int_0^t\left[\lambda\gamma\left(\frac{|d|}{\lambda}\right) - \lambda\gamma((\gamma')^{-1}(2|L_{g_1}V|))\right.\right.\right.\\
&\qquad\left.\left.\left. +\ 2(\lambda|L_{g_1}V|(\gamma')^{-1}(2|L_{g_1}V|) - L_{g_1}Vd)\right]d\tau\right]\right\} \quad \text{[by (18)]}\\
&= 2\beta V(x(0)) + \int_0^\infty(u-\alpha^*)^{\mathrm{T}}R_2(u-\alpha^*)\,dt\\
&\quad + \beta\lambda\sup_{d\in\mathcal{D}}\left\{\int_0^\infty\underbrace{\left[-\gamma\left(\frac{|d|}{\lambda}\right) + \gamma\left(\frac{|d^*|}{\lambda}\right) - \gamma'\left(\frac{|d^*|}{\lambda}\right)\frac{(d^*)^{\mathrm{T}}}{\lambda|d^*|}(d^*-d)\right]}_{\Pi(d,d^*)}dt\right\}
\end{aligned} \tag{27}$$

We then show that the control law $u = \frac{1}{2}\alpha_s(x)$ is an input-to-state stabilizing controller for (8). The derivative of V is

$$\begin{aligned}\dot{V}|_{u=\frac{\alpha_s}{2}} &= L_fV + L_{g_1}Vd - \frac{1}{2}\big(L_fV + \big|L_{g_1}V\big|\rho^{-1}(|x|)\big) \\ &\quad - \frac{1}{2}\sqrt{\omega^2 + \big(L_{g_2}V\big(L_{g_2}V\big)^{\mathrm{T}}\big)^2} \\ &= \frac{1}{2}\underbrace{\big(L_fV + \big|L_{g_1}V\big|\rho^{-1}(|x|)\big)}_{\omega} \\ &\quad - \frac{1}{2}\sqrt{\omega^2 + \big(L_{g_2}V\big(L_{g_2}V\big)^{\mathrm{T}}\big)^2} \\ &\quad + L_{g_1}Vd - \big|L_{g_1}V\big|\rho^{-1}(|x|) \\ &\le -W(x) - \big|L_{g_1}V\big|\big[\rho^{-1}(|x|) - |d|\big] \end{aligned} \tag{38}$$

where

$$W(x) = \frac{1}{2}\left[-\omega + \sqrt{\omega^2 + \big(L_{g_2}V\big(L_{g_2}V\big)^{\mathrm{T}}\big)^2}\,\right] \tag{39}$$

which is positive definite because of (37). Therefore, the control law $u = \frac{1}{2}\alpha_s(x)$ input-to-state stabilizes (8).

Next we show that there exists a class $\mathcal{K}_\infty$ function γ such that the control law $u = \frac{1}{2}\alpha_s(x)$ globally asymptotically stabilizes the auxiliary system (20) with respect to $V(x)$. From (38) it follows that

$$L_{f+g_2\frac{\alpha_s}{2}}V + \big|L_{g_1}V\big|\rho^{-1}(|x|) = -W(x). \tag{40}$$

Since $|L_{g_1}V(x)|$ vanishes at the origin $x = 0$, there exists a class $\mathcal{K}_\infty$ function π such that

$$\big|L_{g_1}V\big| \le \pi(|x|). \tag{41}$$

Since $\rho^{-1}\circ\pi^{-1}$ is in class $\mathcal{K}_\infty$, there exists a class $\mathcal{K}_\infty$ function ζ whose derivative ζ' is also a class $\mathcal{K}_\infty$ function, such that

$$\zeta(2r) \le r\rho^{-1}(\pi^{-1}(r)). \tag{42}$$

Let us define

$$\gamma = \ell\zeta. \tag{43}$$

From Lemma A1-2 it follows that $\ell\ell\zeta = \zeta$, which implies that

$$\ell\gamma(2r) \le r\rho^{-1}(\pi^{-1}(r)). \tag{44}$$

Then with (35) we have

$$\begin{aligned}&\dot{V}\big|_{(20)} \\ &= L_{f+g_2\frac{\alpha_s}{2}}V + \ell\gamma\big(2\big|L_{g_1}V\big|\big) \\ &\le L_{f+g_2\frac{\alpha_s}{2}}V + \big|L_{g_1}V\big|\rho^{-1}\circ\pi^{-1}\big(\big|L_{g_1}V\big|\big) \quad \text{[by (44)]} \\ &\le L_{f+g_2\frac{\alpha_s}{2}}V + \big|L_{g_1}V\big|\rho^{-1}(|x|) \quad \text{[by (41)]} \\ &= -W(x) \quad \text{[by (40)]}\end{aligned} \tag{45}$$

which means that (20) is globally asymptotically stabilized.

Since the control law $\frac{1}{2}\alpha_s(x)$ is of the form (21) with $R_2(x) = R_2(x)^{\mathrm{T}} > 0$ given by

$$R_2(x) = I\begin{cases}\frac{2L_{g_2}V(L_{g_2}V)^{\mathrm{T}}}{\omega+\sqrt{\omega^2+(L_{g_2}V(L_{g_2}V)^{\mathrm{T}})^2}}, & L_{g_2}V \ne 0 \\ \text{any positive real number}, & L_{g_2}V = 0\end{cases} \tag{46}$$

by Theorem 3.1, the control law $u = \alpha_s(x)$ is inverse optimal with respect to the cost functional (23) with the penalty on the state given by

$$\begin{aligned}l(x) &= 4\big[W(x) + \big|L_{g_1}V\big|\rho^{-1}(|x|) - \ell\gamma\big(2\big|L_{g_1}V\big|\big)\big] \ge 4W(x) \\ &= 2\left[-\omega + \sqrt{\omega^2 + \big(L_{g_2}V\big(L_{g_2}V\big)^{\mathrm{T}}\big)^2}\,\right].\end{aligned} \tag{47}$$

The function $l(x)$ is positive definite but not necessarily *radially unbounded*. We now modify the control law to achieve a new $\hat{l}(x)$ that is radially unbounded. Let us suppose that $W(x)$ is not radially unbounded. By following the procedure in [34, p. 440], we can find a continuous function $\eta'(\cdot)$ such that

$$\eta'(r) \ge 1, \qquad \forall r \ge 0 \tag{48}$$

and

$$\eta'(V(x))W(x) \text{ is radially unbounded.} \tag{49}$$

Let us introduce a new ISS-CLF $\hat{V}(x) = \int_0^{V(x)} \eta'(s)\,ds$ (which is positive definite and radially unbounded) and apply (35). The resulting penalty on control is

$$\hat{l}(x) \ge 4\hat{W}(x) = 2\left[-\hat{\omega} + \sqrt{\hat{\omega}^2 + \big(L_{g_2}\hat{V}\big(L_{g_2}\hat{V}\big)^{\mathrm{T}}\big)^2}\,\right] \tag{50}$$

where $L_{g_2}\hat{V} = \eta'(V)L_{g_2}V$ and $\hat{\omega} = L_f\hat{V} + |L_{g_1}\hat{V}|\rho^{-1}(|x|) = \eta'(V)\omega$. Thus (50) becomes

$$\begin{aligned}\hat{l}(x) &\ge 2\eta'(V)\left[-\omega + \sqrt{\omega^2 + (\eta'(V))^2\big(L_{g_2}V\big(L_{g_2}V\big)^{\mathrm{T}}\big)^2}\,\right] \\ &= 4\eta'(V)W(x) + 2\eta'(V) \\ &\quad \times\left[\sqrt{\omega^2 + (\eta'(V))^2\big(L_{g_2}V\big(L_{g_2}V\big)^{\mathrm{T}}\big)^2}\right. \\ &\quad \left. - \sqrt{\omega^2 + \big(L_{g_2}V\big(L_{g_2}V\big)^{\mathrm{T}}\big)^2}\,\right] \\ &\ge 4\eta'(V)W(x)\end{aligned} \tag{51}$$

which is radially unbounded. This completes the proof of Theorem 3.2. □

Remark 3.2: We point out that the control law (35) will be continuous not only away from the origin but also at the origin if and only if the ISS-CLF $V(x)$ satisfies the following *small control property* [33]: for any $\epsilon > 0$ there is a $\delta > 0$ such that, if $x \ne 0$ satisfies $\rho(|d|) \le |x| \le \delta$, then there is some u with $|u| \le \epsilon$ such that

$$L_{f+g_2u}V + L_{g_1}Vd < 0. \tag{52}$$

If there exists a control law $u = \alpha(x)$ continuous at the origin, which is input-to-state stabilizing with respect to an ISS-Lyapunov function $V(x)$, then $V(x)$ satisfies the small control property. □

The following example illustrates Theorem 3.2.

Example 3.1: Consider the system

$$\dot{x} = u + x^2 d. \tag{53}$$

Since the system is scalar, we take $V = \frac{1}{2}x^2$ and get $L_{g_1}V = x^3$ and $L_{g_2}V = x$. Picking the class $\mathcal{K}_\infty$ function ρ as $\rho(r) = r$, we have $\omega = |L_{g_1}V|\rho^{-1}(|x|) = x^4$, and the control law based on (35) is

$$u = \alpha_s(x) = -\left(x^2 + \sqrt{x^4+1}\right)x = -\frac{2}{R_2}L_{g_2}V \tag{54}$$

where

$$R_2(x) = \frac{2}{x^2+\sqrt{x^4+1}} > 0, \quad \forall x. \tag{55}$$

Now let us choose the class $\mathcal{K}_\infty$ function π as $\pi(r) = r^3$, from (44) we can take $\ell\gamma(2r) = r^{4/3}$, and from (19) we get $\gamma'(r) = \frac{27}{4}r^3$ and $\gamma(r) = \frac{27}{16}r^4$. The control $u = \frac{\alpha_s}{2}$ is stabilizing for the auxiliary system (20) of (53), which has the form

$$\dot{x} = u + x^3 \tag{56}$$

because the derivative of the Lyapunov function along the solutions of (56) is

$$\dot{V}\Big|_{u=\frac{\alpha_s}{2}} = -\frac{-x^2+\sqrt{x^4+1}}{2}x^2. \tag{57}$$

By Theorem 3.1, with $\beta = \lambda = 2$, the control $u = \alpha_s(x)$ (54) is optimal with respect to the cost functional

$$J(u) = \sup_d \left\{ \lim_{t\to\infty} \left[2x(t)^2 + \int_0^t \left(\frac{2x^2}{x^2+\sqrt{x^4+1}} + \frac{2u^2}{x^2+\sqrt{x^4+1}} - \frac{27}{64}d^4 \right) d\tau \right] \right\} \tag{58}$$

with a value function $J^*(x) = 2x^2$.

If, instead, we choose the class $\mathcal{K}_\infty$ function ρ as $\rho(r) = r^{1/3}$, we will end up with a different controller as well as a different (quadratic with respect to d) cost functional. Now we have $\omega = |L_{g_1}V|\rho^{-1}(|x|) = x^6$ and the control law based on (35) becomes

$$u = \alpha_s(x) = -\left(x^4 + \sqrt{x^8+1}\right)x = -\frac{2}{R_2}L_{g_2}V \tag{59}$$

where

$$R_2(x) = \frac{2}{x^4+\sqrt{x^8+1}} > 0, \qquad \forall x. \tag{60}$$

We keep the class $\mathcal{K}_\infty$ function π the same as before, from (44) we can take $\ell\gamma(2r) = r^2$, and from (19) we get $\gamma'(r) = 2r$ and $\gamma(r) = r^2$. The control $u = \frac{\alpha_s}{2}$ is stabilizing for the auxiliary system (20) of (53), which has the form

$$\dot{x} = u + x^5 \tag{61}$$

because the time derivative of $V(x)$ along the solutions of (61) is

$$\dot{V}\Big|_{u=\frac{\alpha_s}{2}} = -\frac{-x^4+\sqrt{x^8+1}}{2}x^2. \tag{62}$$

Theorem 3.1 with $\beta = \lambda = 2$ then tells us that the control $u = \alpha_s(x)$ is optimal with respect to the cost functional

$$J(u) = \sup_d \left\{ \lim_{t\to\infty} \left[2x(t)^2 + \int_0^t \left(\frac{2x^2}{x^4+\sqrt{x^8+1}} + \frac{2u^2}{x^4+\sqrt{x^8+1}} - d^2 \right) d\tau \right] \right\} \tag{63}$$

with a value function $J^*(x) = 2x^2$. This example, where we are able to achieve inverse optimality with a quadratic penalty on both players—the control and the disturbance—motivates the developments in Section V.

Unfortunately, neither in (58) nor in (63) is $l(x)$ radially unbounded (it is only positive definite). In the proof of Theorem 3.2 we remedy this by redesigning the ISS-CLF and applying the Sontag formula with the new ISS-CLF. Fortunately, for this scalar system, it is easy to go a step further and show that controller (59), written as $u = -2L_{g_2}\hat{V}$, where

$$\hat{V}(x) = \int_0^{V(x)} \frac{4r^2+\sqrt{1+16r^4}}{2}\, dr \tag{64}$$

is optimal with respect to the cost functional

$$J(u) = \sup_d \left\{ \lim_{t\to\infty} \left[4\hat{V}(x(t)) + \int_0^t \left(x^2 + u^2 - \hat{\gamma}(|d|)\right) d\tau \right] \right\} \tag{65}$$

where $\hat{\gamma}(r) = 4\ell(\xi \circ \eta^{-1})(r/2)$, $\xi(r) = (r^4 + \sqrt{1+r^8})r^6/2$, and $\eta(r) = (r^4 + \sqrt{1+r^8})r^3$. From these expressions it is easy to see that the penalty $\hat{\gamma}(|d|)$ is quadratic near $d = 0$ and $\mathcal{O}(|d|^{10/3})$ as $|d| \to \infty$. A striking feature of the cost functional (65) is that it has unity weighting on control. In Section IV, we show that this can always be achieved for systems that are input-to-state stabilizable, and we derive *stability margins* associated with this property.

In Section I we stressed that all of the controllers we derive guarantee ISS, namely, guarantee bounded solutions for bounded disturbances. For example, (59) guarantees that $|x(t)| \le e^{-t}|x(0)| + \frac{1}{2}\|d\|_\infty$.

Next, we show that input-to-state stabilizability is not only sufficient but also necessary for the solvability of the inverse optimal gain assignment problem.

Theorem 3.3: If the inverse optimal gain assignment problem is solvable for (8), then (8) is input-to-state stabilizable.

Proof: We only sketch the proof. If the inverse optimal gain assignment problem is solvable, then the following HJI equation is satisfied:

$$L_fV - L_{g_2}VR_2^{-1}\left(L_{g_2}V\right)^{\mathrm{T}} + \ell\gamma\left(2\left|L_{g_1}V\right|\right) = -\frac{1}{4}l(x). \tag{66}$$

Then, along the solutions of (8) with a control law $u = -R_2(x)^{-1}(L_{g_2}V)^{\mathrm{T}}$ we have

$$\dot{V} = L_fV - L_{g_2}VR_2^{-1}\left(L_{g_2}V\right)^{\mathrm{T}} + L_{g_1}Vd. \tag{67}$$

By Lemma A2, we get

$$\dot{V} \le L_fV - L_{g_2}VR_2^{-1}\left(L_{g_2}V\right)^{\mathrm{T}} + \ell\gamma\left(2\left|L_{g_1}V\right|\right) + \gamma\left(\frac{|d|}{2}\right) \tag{68}$$

which with (66) results in

$$\dot{V} \leq -\frac{1}{4}l(x) + \gamma\left(\frac{|d|}{2}\right). \tag{69}$$

Since $l(x)$ is positive definite and radially unbounded, (8) with $u = -R_2(x)^{-1}(L_{g_2}V)^{\mathrm{T}}$ is ISS. □

By combining Theorem 3.2 and 3.3, we get the following result.

Corollary 3.1: The inverse optimal gain assignment problem for system (8) is solvable if and only if the system is input-to-state stabilizable.

IV. Stability Margins

The main benefit of inverse optimality is that the controller remains input-to-state stabilizing in the presence of a certain class of input uncertainties. In this section, we show that

1) to achieve these margins, it is sufficient to make $R_2(x) = I$;
2) $R_2(x) = I$ can be achieved for systems that are input-to-state stabilizable.

We first prove the latter statement and then characterize the margins.

Definition 4.1 (Small Control Property in the Sense of Janković et al.—SCPJ [19]): An ISS-CLF $V(x)$ is said to be an *ISS-CLF-SCPJ* if there exists a continuous control law $u = \alpha_c(x)$ such that, for all $x \neq 0$ and all $d \in \mathbb{R}^r$

$$|x| \geq \rho(|d|) \;\Downarrow\; L_fV + L_{g_1}Vd + L_{g_2}V\alpha_c(x) < 0 \tag{70}$$

and, in addition

$$\lim_{\varepsilon\to 0}\max_{|x|=\varepsilon}\frac{|\alpha_c(x)|}{|L_{g_2}V(x)|} < \infty. \tag{71}$$

Without (71), this is Sontag's small control property [33]. Property (71) is weaker than the requirement in Janković *et al.* [19] for rank $\{\frac{\partial}{\partial x}(\frac{\partial V}{\partial x}g_2)^{\mathrm{T}}(0)\} = \dim\{u\}$.

Theorem 4.1: If (8) has an ISS-CLF-SCPJ, then the inverse optimal gain assignment problem is solvable with $R_2(x) = I$.

Proof: The proof extends ideas from [32, pp. 104–105] and [19]. From the proof of Theorem 3.2 we know $V(x)$ satisfies the Isaacs equation

$$L_fV - L_{g_2}VR_2(x)^{-1}(L_{g_2}V)^{\mathrm{T}} + |L_{g_1}V|\rho^{-1}(|x|) = -W(x) \tag{72}$$

where $R_2(x)$ is defined in (46) and $W(x)$ is positive definite and radially unbounded. Our task in this proof is to show that there exist positive definite radially unbounded functions $\hat{V}(x)$ and $\hat{l}(x)$ and a class $\mathcal{K}_\infty$ function $\hat{\gamma}$ whose derivative $\hat{\gamma}'$ is also in class $\mathcal{K}_\infty$, such that the following Isaacs equation is satisfied:

$$L_f\hat{V} - \frac{\beta}{2}|L_{g_2}\hat{V}|^2 + \frac{\lambda}{2}\ell\hat{\gamma}(2|L_{g_1}\hat{V}|) = -\frac{\hat{l}(x)}{2\beta} \tag{73}$$

in which case, according to Theorem 3.1, the control law

$$u = -\beta(L_{g_2}\hat{V})^{\mathrm{T}}, \qquad \beta \geq 2 \tag{74}$$

solves the inverse optimal gain assignment problem with $\hat{R}_2(x) = I$. Since $V(x)$ is an ISS-CLF-SCPJ with some continuous $\alpha_c(x)$ such that

$$\underbrace{L_fV + |L_{g_1}V|\rho^{-1}(|x|)}_{\omega} + L_{g_2}V\alpha_c(x) < 0, \qquad \forall x \neq 0 \tag{75}$$

we have

$$|\omega| \leq |L_{g_2}V||\alpha_c(x)|, \qquad \text{for } \omega \geq 0. \tag{76}$$

Following the reasoning in [33, pp. 120–121], from (46) we get

$$R_2(x)^{-1} \leq \left(\frac{|\omega|}{|L_{g_2}V|^2} + \frac{1}{2}\right) \leq \left(\frac{|\alpha_c(x)|}{|L_{g_2}V(x)|} + \frac{1}{2}\right)I \tag{77}$$

for $\omega \geq 0$. For $\omega < 0$ we have $\omega + \sqrt{\omega^2 + (L_{g_2}V(L_{g_2}V)^{\mathrm{T}})^2} \leq L_{g_2}V(L_{g_2}V)^{\mathrm{T}}$ which yields $R_2(x)^{-1} \leq 1/2I$, so (77) holds for all $x \in \mathbb{R}^n$. The result in [33] (see also [14]), along with the continuity of $\omega(x)$ and $L_{g_2}V(x)$, implies that $R_2(x)^{-1}$ is continuous away from $x = 0$. This, along with (71) and (77), implies that there exists a continuous positive function $\varrho(V)$ such that

$$R_2(x)^{-1} \leq \varrho(V(x))I, \qquad \forall x \in \mathbb{R}^n. \tag{78}$$

(Such a function always exists since $V(x)$ is radially unbounded.) Consider

$$\hat{V}(x) = \int_0^{V(x)} \varrho(s)\,ds \tag{79}$$

which is positive definite, radially unbounded (due to the positiveness of $\varrho(\cdot)$), and $\mathcal{C}^1$. Multiplying (72) by $\varrho(V)$, we get

$$\begin{aligned} L_f\hat{V} &- \frac{\beta}{2}|L_{g_2}\hat{V}|^2 + |L_{g_1}\hat{V}|\rho^{-1}(|x|) \\ &= -\varrho(V)W(x) + L_{g_2}V\left(R_2^{-1} - \frac{\beta}{2}\varrho(V)I\right) \\ &\quad \times (L_{g_2}V)^{\mathrm{T}}\varrho(V) \leq -\varrho(V)W(x). \end{aligned} \tag{80}$$

Since $\varrho(V)L_{g_1}V(x)$ is continuous and vanishes at $x = 0$, there exists a class $\mathcal{K}_\infty$ function $\hat{\pi}$ such that $|\varrho(V)L_{g_1}V(x)| \leq \hat{\pi}(|x|)$. Similar to the proof of Theorem 3.2, let $\hat{\gamma} = \ell\hat{\zeta}$, where $\hat{\zeta}$ is a class $\mathcal{K}_\infty$ function with a class $\mathcal{K}_\infty$ derivative selected so that $\hat{\zeta}(2r) \leq r\rho^{-1}(\hat{\pi}^{-1}(r))$. Then

$$\begin{aligned} \ell\hat{\gamma}(2|L_{g_1}\hat{V}|) &\leq |L_{g_1}\hat{V}|\rho^{-1}(\hat{\pi}^{-1}(|L_{g_1}\hat{V}|)) \\ &\leq |L_{g_1}\hat{V}|\rho^{-1}(|x|). \end{aligned} \tag{81}$$

Substituting this into (80) yields

$$\begin{aligned} L_f\hat{V} &- \frac{\beta}{2}|L_{g_2}\hat{V}|^2 + \frac{\lambda}{2}\ell\hat{\gamma}(2|L_{g_1}\hat{V}|) \\ &\leq -\varrho(V)W(x) + \frac{\lambda}{2}\ell\hat{\gamma}(2|L_{g_1}\hat{V}|) \\ &\quad - |L_{g_1}\hat{V}|\rho^{-1}(|x|) \leq -\varrho(V)W(x). \end{aligned} \tag{82}$$

Thus $\hat{V}(x)$ satisfies the Isaacs equation (73) with $\hat{l}(x) \geq 2\beta\varrho(V)W(x)$, which is positive definite and radially unbounded. □

Next, we derive the stability margins. In order to characterize the class of allowable input uncertainties, we remind the reader of the definition of *strict passivity* [6].

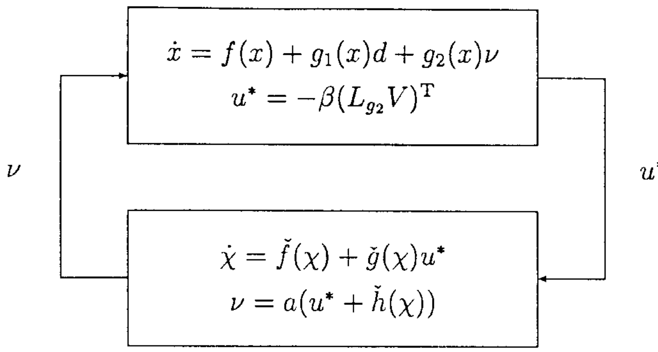

Fig. 1. The composite system (85) is ISS with respect to d.

Definition 4.2: The system

$$\begin{aligned} \dot{\chi} &= \check{f}(\chi) + \check{g}(\chi)\check{u} \\ \check{y} &= \check{h}(\chi) \end{aligned} \tag{83}$$

is said to be *strictly passive* if there exists a $\mathcal{C}^1$ positive definite radially unbounded (storage) function $\check{V}(\chi)$ and a class $\mathcal{K}_\infty$ (dissipation rate) function $\psi(\cdot)$ such that

$$\int_0^t \check{y}^{\mathrm{T}}\check{u}\,d\sigma \geq \check{V}(\chi(t)) - \check{V}(\chi(0)) + \int_0^t \psi(|\chi(\sigma)|)\,d\sigma \tag{84}$$

for all $\check{u} \in \mathcal{C}^0$, $\chi(0) \in \mathbb{R}^{\check{n}}$, $t \geq 0$.

Theorem 4.2: If a controller solves the inverse optimal gain assignment problem for (8) with $R_2(x) = I$, then it is input-to-state stabilizing for the system

$$\begin{aligned} &\dot{x} = f(x) + g_1(x)d + g_2(x)a(u + \check{y}) \\ &\dot{\chi} = \check{f}(\chi) + \check{g}(\chi)u, \qquad \check{y} = \check{h}(\chi) \end{aligned} \tag{85}$$

where $a \in [1/2, \infty)$ and the χ-system is strictly passive.

In simple words, an inverse optimal ISS controller remains ISS stabilizing through unmodeled dynamics of the form $a(I + \mathcal{P})$ where $\mathcal{P}$ is strictly passive, as depicted in Fig. 1.

Proof: From the assumptions of the theorem we know that there exist Lyapunov-type functions $V(x)$ and $\check{V}(\chi)$ such that

$$L_f V - \frac{\beta}{2}\left|L_{g_2}V\right|^2 = -\frac{l(x)}{2\beta} - \frac{\lambda}{2}\ell\gamma\left(2\left|L_{g_1}V\right|\right) \tag{86}$$

$$L_{\check{f}}\check{V} + L_{\check{g}}\check{V}u \leq -\psi(|\chi|) + \check{y}^{\mathrm{T}}u \tag{87}$$

with l, γ, and ψ as in Definitions 3.1 and 4.2. Consider the following candidate for a composite ISS-CLF:

$$V_c(x,\chi) = V(x) + \frac{a}{\beta}\check{V}(\chi). \tag{88}$$

Then the control law $u^* = -\beta(L_{g_2}V)^{\mathrm{T}}$ guarantees that

$$\begin{aligned} \dot{V}_c &= L_f V + L_{g_1}Vd + aL_{g_2}Vu^* \\ &\quad + aL_{g_2}V\check{y} + \frac{a}{\beta}\left(L_{\check{f}}\check{V} + L_{\check{g}}\check{V}u^*\right) \\ &\leq -\frac{l(x)}{2\beta} + \left(\frac{1}{2} - a\right)\beta\left|L_{g_2}V\right|^2 \\ &\quad - \frac{\lambda}{2}\ell\gamma\left(2\left|L_{g_1}V\right|\right) + L_{g_1}Vd + aL_{g_2}V\check{y} \\ &\quad - \frac{a}{\beta}\psi(|\chi|) + \frac{a}{\beta}\check{y}^{\mathrm{T}}\left(-\beta\left(L_{g_2}V\right)^{\mathrm{T}}\right). \end{aligned} \tag{89}$$

Since $a \geq 1/2$, it follows that

$$\dot{V}_c \leq -\frac{l(x)}{2\beta} - \frac{a}{\beta}\psi(|\chi|) - \frac{\lambda}{2}\ell\gamma\left(2\left|L_{g_1}V\right|\right) + \frac{\lambda}{2}2\left|L_{g_1}V\right|\frac{|d|}{\lambda}. \tag{90}$$

By applying Lemma A2 to the last term, we get

$$\dot{V}_c \leq -\frac{l(x)}{2\beta} - \frac{a}{\beta}\psi(|\chi|) + \gamma\left(\frac{|d|}{\lambda}\right). \tag{91}$$

Since $l(x)$ and $\psi(|\chi|)$ are both radially unbounded, by [35] the closed-loop system is ISS. □

Theorems 4.1 and 4.2 can be combined to obtain the following corollary.

Corollary 4.1: If (8) has an ISS-CLF-SCPJ, then there exists a control law that achieves ISS in the presence of input unmodeled dynamics of the form $a(I + \mathcal{P})$ with $a \geq 1/2$ and $\mathcal{P}$ strictly passive.

By setting $d = 0$, we recover the result in [32]. In the linear case, this result implies the standard result that inverse optimal controllers possess infinite gain margins and 60° phase margins [1].

V. Rapprochement with "Nonlinear $\mathcal{H}_\infty$"

Definition 5.1: The *inverse optimal* $\mathcal{H}_\infty$ problem for system (8) is solvable if there exist a continuous matrix-valued function $R_1(x)$ such that $R_1(x) = R_1(x)^{\mathrm{T}} \geq 0$ for all x, a matrix-valued function $R_2(x)$ such that $R_2(x) = R_2(x)^{\mathrm{T}} > 0$ for all x, positive definite radially unbounded functions $l(x)$ and $E(x)$, and a feedback law $u = \alpha(x)$ continuous away from the origin with $\alpha(0) = 0$, which minimizes the cost functional

$$\begin{aligned} J(u) = \sup_{d\in\mathcal{D}}\Bigg\{\lim_{t\to\infty}\Bigg[E(x(t)) + \int_0^t \big(l(x) \\ + u^{\mathrm{T}}R_2(x)u - d^{\mathrm{T}}R_1(x)d\big)\,d\tau\Bigg]\Bigg\} \end{aligned} \tag{92}$$

where $\mathcal{D}$ is the set of locally bounded functions of x.

In this definition we perpetuate the now common abuse of terminology where the term $\mathcal{H}_\infty$ is used both for $\mathcal{L}_2$ disturbance attenuation problems and for dynamic games, both for linear and for nonlinear systems. An important feature in Definition 5.1 is that the state-dependent weight $R_1(x)$ (not present in standard nonlinear $\mathcal{H}_\infty$ formulations) is required to take finite values for all finite values of the state and it may even be zero, hence, putting the disturbance in a more privileged position than in the standard nonlinear $\mathcal{H}_\infty$ results. We stress that there is nothing strange in $R_1(x)$ being zero at some or even all x because lower R_1 means better disturbance attenuation.

Theorem 5.1: Consider the auxiliary system of (8)

$$\dot{x} = f(x) + g_1(x)R_1(x)^{-1}\left(L_{g_1}V(x)\right)^{\mathrm{T}} + g_2(x)u \tag{93}$$

where $V(x)$ is a Lyapunov function candidate and $R_1(x) = R_1(x)^{\mathrm{T}} \geq 0$ is a continuous matrix-valued function such that $R_1(x)^{-1}(L_{g_1}V(x))^{\mathrm{T}}$ is locally bounded. Suppose that there exists a matrix-valued function $R_2(x) = R_2(x)^{\mathrm{T}} > 0$ such that the control law

$$u = \alpha(x) = -R_2(x)^{-1}\big(L_{g_2}V\big)^{\mathrm{T}} \tag{94}$$

globally asymptotically stabilizes (93) with respect to $V(x)$. Then the control law

$$u = \alpha^*(x) = \beta\alpha(x) = -\beta R_2^{-1}\big(L_{g_2}V\big)^{\mathrm{T}} \tag{95}$$

with any $\beta \geq 2$, solves the inverse optimal $\mathcal{H}_\infty$ problem for (8) by minimizing the cost functional

$$J(u) = \sup_{d\in\mathcal{D}}\left\{\lim_{t\to\infty}\left[2\beta V(x(t)) + \int_0^t \bigg(l(x) + u^{\mathrm{T}}R_2(x)u - \frac{\beta}{\lambda}d^{\mathrm{T}}R_1(x)d\bigg)\,d\tau\right]\right\} \tag{96}$$

for any $\lambda \in (0, 2]$, where

$$\begin{aligned} l(x) = {} & -2\beta\big[L_fV + L_{g_1}VR_1^{-1}\big(L_{g_1}V\big)^{\mathrm{T}} - L_{g_2}VR_2^{-1}\big(L_{g_2}V\big)^{\mathrm{T}}\big] \\ & + \beta(2-\lambda)L_{g_1}VR_1^{-1}\big(L_{g_1}V\big)^{\mathrm{T}} \\ & + \beta(\beta-2)L_{g_2}VR_2^{-1}\big(L_{g_2}V\big)^{\mathrm{T}}. \end{aligned} \tag{97}$$

Proof: Since the control law (94) stabilizes (93), there exists a continuous positive definite function $W : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$L_fV + L_{g_1}VR_1^{-1}\big(L_{g_1}V\big)^{\mathrm{T}} - L_{g_2}VR_2^{-1}\big(L_{g_2}V\big)^{\mathrm{T}} \leq -W(x) \tag{98}$$

and thus $l(x) \geq 2\beta W(x) + \beta(2-\lambda)L_{g_1}VR_1^{-1}(L_{g_1}V)^{\mathrm{T}} + \beta(\beta-2)L_{g_2}VR_2^{-1}(L_{g_2}V)^{\mathrm{T}}$, which is positive definite. By performing similar steps as in (27) and by completing squares

$$\begin{aligned} J(u) = {} & 2\beta V(x(0)) + \int_0^\infty (u-\alpha^*)^{\mathrm{T}}R_2(u-\alpha^*)\,dt \\ & - \frac{\beta}{\lambda}\inf_{d\in\mathcal{D}}\left\{\int_0^\infty (d-d^*)^{\mathrm{T}}R_1(d-d^*)\,dt\right\} \end{aligned} \tag{99}$$

where

$$d^* = \lambda R_1^{-1}\big(L_{g_1}V\big)^{\mathrm{T}}. \tag{100}$$

The "worst case" disturbance is $d = d^*$, and the minimum of (99) is reached with $u = \alpha^*$. The value function of (92) is $J^*(x) = 2\beta V(x)$. □

Remark 5.1: The function $V(x)$ solves the following family of HJI equations:

$$\begin{aligned} & L_fV - \frac{\beta}{2}L_{g_2}VR_2(x)^{-1}\big(L_{g_2}V\big)^{\mathrm{T}} \\ & + \frac{\lambda}{2}L_{g_1}VR_1(x)^{-1}\big(L_{g_1}V\big)^{\mathrm{T}} + \frac{l(x)}{2\beta} = 0 \end{aligned} \tag{101}$$

and the achieved disturbance attenuation level is

$$\int_0^\infty [l(x) + u^{\mathrm{T}}R_2(x)u]dt \leq \frac{\beta}{\lambda}\int_0^\infty [d^{\mathrm{T}}R_1(x)d]\,dt. \tag{102}$$

□

Theorem 5.2: If (8) is input-to-state stabilizable, then the inverse optimal $\mathcal{H}_\infty$ problem is solvable.

Proof: The proof is based on the same Sontag-type formula as that in the proof of Theorem 3.2. The main difference is that here we have to find a continuous matrix-valued function $R_1(x) = R_1(x)^{\mathrm{T}} \geq 0$ such that the control law $u = \frac{1}{2}\alpha_s(x)$ globally asymptotically stabilizes (93) with respect to $V(x)$. According to (40), we can select

$$R_1(x) = R_1(x)^{\mathrm{T}} = I\frac{\big|L_{g_1}V\big|}{\rho^{-1}(|x|)} > 0 \tag{103}$$

to get

$$\begin{aligned} & L_{f+g_2\frac{\alpha_s}{2}}V + L_{g_1}VR_1^{-1}\big(L_{g_1}V\big)^{\mathrm{T}} \\ & = L_{f+g_2\frac{\alpha_s}{2}}V + \big|L_{g_1}V\big|\rho^{-1}(|x|) = -W(x) \end{aligned} \tag{104}$$

which means that (93) is globally asymptotically stabilized. However, $R_1(x)$ given by (103) is not guaranteed to be bounded at the origin. Fortunately, a modified Lyapunov function

$$\hat{V}(x) = \int_0^{V(x)} \rho^{-1}\big(\alpha_2^{-1}(r)\big)\,dr \tag{105}$$

where $\alpha_2(\cdot)$ is a class $\mathcal{K}_\infty$ function such that $V(x) \leq \alpha_2(|x|)$, can be used to achieve a continuous $R_1(x)$. Let us denote

$$L_{g_2}\hat{V} = \rho^{-1}\big(\alpha_2^{-1}(V)\big)L_{g_2}V \tag{106}$$

$$\begin{aligned} \hat{\omega} & = L_f\hat{V} + \big|L_{g_1}\hat{V}\big|\rho^{-1}(|x|) \\ & = \rho^{-1}\big(\alpha_2^{-1}(V)\big)\omega. \end{aligned} \tag{107}$$

Since $\rho^{-1}\circ\alpha_2^{-1}$ is a class $\mathcal{K}_\infty$ function, we have

$$L_{g_2}\hat{V} = 0 \implies \hat{\omega} = 0 \tag{108}$$

which by Lemma 2.1 implies that $\hat{V}(x)$ is an ISS-CLF. Let us now design a new control law $u = \hat{\alpha}_s(x)$ of the form (35) with $L_{g_2}V$ and ω replaced by $L_{g_2}\hat{V}$ and $\hat{\omega}$. This control law satisfies [similar to (40)]

$$L_{f+g_2\frac{\hat{\alpha}_s}{2}}\hat{V} + \big|L_{g_1}\hat{V}\big|\rho^{-1}(|x|) = -\hat{W}(x) \tag{109}$$

where

$$\hat{W}(x) = \frac{1}{2}\left[-\hat{\omega} + \sqrt{\hat{\omega}^2 + \big(L_{g_2}\hat{V}\big(L_{g_2}\hat{V}\big)^{\mathrm{T}}\big)^2}\,\right] \tag{110}$$

is positive definite. Consider

$$\begin{aligned} \hat{R}_1(x) = \hat{R}_1(x)^{\mathrm{T}} & = I\frac{\big|L_{g_1}\hat{V}\big|}{\rho^{-1}(|x|)} = I\rho^{-1}\big(\alpha_2^{-1}(V)\big)\frac{\big|L_{g_1}V\big|}{\rho^{-1}(|x|)} \\ & \leq I\big|L_{g_1}V\big| \end{aligned} \tag{111}$$

which is continuous, and the auxiliary system

$$\dot{x} = f(x) + g_1(x)\hat{R}_1(x)^{-1}\big(L_{g_1}\hat{V}\big)^{\mathrm{T}} + g_2(x)u. \tag{112}$$

Under the feedback law $u = \frac{1}{2}\hat{\alpha}_s(x)$, the time derivative of $\hat{V}(x)$ along the solutions of (112) is

$$\begin{aligned} \dot{\hat{V}} & = L_{f+g_2\frac{\hat{\alpha}_s}{2}}\hat{V} + L_{g_1}\hat{V}\hat{R}_1^{-1}\big(L_{g_1}\hat{V}\big)^{\mathrm{T}} \\ & = L_{f+g_2\frac{\hat{\alpha}_s}{2}}\hat{V} + L_{g_1}\hat{V}\frac{\rho^{-1}(|x|)}{\rho^{-1}\big(\alpha_2^{-1}(V)\big)}\frac{\big(L_{g_1}\hat{V}\big)^{\mathrm{T}}}{\big|L_{g_1}V\big|} \\ & = L_{f+g_2\frac{\hat{\alpha}_s}{2}}\hat{V} + \big|L_{g_1}\hat{V}\big|\rho^{-1}(|x|) \\ & = -\hat{W}(x) \end{aligned} \tag{113}$$

which proves that $u = \frac{1}{2}\hat{\alpha}_s(x)$ stabilizes (112). Since $\frac{1}{2}\hat{\alpha}_s(x)$ is of the form (94), by Theorem 5.1, $u = \hat{\alpha}_s(x)$ solves the inverse optimal $\mathcal{H}_\infty$ problem. The radial unboundedness of $l(x)$ can be achieved as in Theorem 3.2. □

Example 5.1: Consider again (53). The control law (54) is optimal with respect to the cost functional

$$J(u) = \sup_d \left\{ \lim_{t\to\infty} \left[2x(t)^2 + \int_0^t \left(\frac{2x^2}{x^2+\sqrt{x^4+1}} + \frac{2u^2}{x^2+\sqrt{x^4+1}} - x^2 d^2 \right) d\tau \right] \right\}. \tag{114}$$

The control law in Example 5.1 achieved a quadratic penalty on the disturbance but with a state-dependent weight $R_1(x) = x^2$ which is radially unbounded. On one hand, the radial unboundedness of the weight should not be viewed as a disadvantage because the control law (54) guarantees boundedness of x for any bounded d. On the other hand, we see from Example 3.1 that it is possible to design a different control law (59) which achieves a quadratic penalty on d with a constant (and bounded!) weight [cf. (63)], thus achieving inverse optimality in the standard "nonlinear $\mathcal{H}_\infty$" sense.

This motivates us to attempt to design controllers which are inverse optimal in the sense of Definition 5.1 but with a weight $R_1(x)$ that is *bounded*, rather that just continuous. In the sequel, we sketch a modification to the proof of Theorem 5.2 which results in a bounded $R_1(x)$. We start by modifying (105) to

$$\hat{V}(x) = \int_0^{V(x)} \frac{\rho^{-1}\left(\alpha_2^{-1}(r)\right)}{1+\pi\left(\alpha_1^{-1}(r)\right)}\, dr \tag{115}$$

where π is defined in (41) and $\alpha_1(\cdot)$ is a class $\mathcal{K}_\infty$ function such that $V(x) \geq \alpha_1(|x|)$. Although $\hat{V}(x)$ is not guaranteed to be radially unbounded, it is positive definite and satisfies

$$L_{g_2}\hat{V} = 0 \implies L_f\hat{V} + \left|L_{g_1}\hat{V}\right|\rho^{-1}(|x|) < 0. \tag{116}$$

We treat it as a legitimate ISS-CLF and design a control law using the formula (35). The feedback law $u = \frac{1}{2}\hat{\alpha}_s(x)$ is stabilizing for the auxiliary system (112) with

$$\begin{aligned} \hat{R}_1(x) = \hat{R}_1(x)^{\mathrm{T}} &= I\frac{\left|L_{g_1}\hat{V}\right|}{\rho^{-1}(|x|)} \\ &= I\frac{\rho^{-1}\left(\alpha_2^{-1}(V)\right)}{1+\pi\left(\alpha_1^{-1}(V)\right)}\frac{\left|L_{g_1}V\right|}{\rho^{-1}(|x|)} < I. \end{aligned} \tag{117}$$

Thus $\hat{R}_1(x)$ is a bounded function. By Theorem 5.1, the control law $u = \hat{\alpha}_s(x)$ minimizes a cost functional of the form (96). However, the penalty on the state

$$\hat{l}(x) \geq 4\hat{W}(x) = 2\left[-\hat{\omega} + \sqrt{\hat{\omega}^2 + \left(L_{g_2}\hat{V}\left(L_{g_2}\hat{V}\right)^{\mathrm{T}}\right)^2}\,\right] \tag{118}$$

is not guaranteed to be radially unbounded because

$$L_{g_2}\hat{V} = \frac{\rho^{-1}\left(\alpha_2^{-1}(V)\right)}{1+\pi\left(\alpha_1^{-1}(V)\right)} L_{g_2}V$$

and

$$\hat{\omega} = \frac{\rho^{-1}\left(\alpha_2^{-1}(V)\right)}{1+\pi\left(\alpha_1^{-1}(V)\right)}\omega \tag{119}$$

contain the division by the class $\mathcal{K}_\infty$ function $\pi(\alpha_1^{-1}(V))$, which may make $\hat{l}(x)$ bounded. Moreover, at present it does not seem possible to systematically modify the Lyapunov function $V(x)$ to get a control law of the type (35) which would at the same time guarantee that $\hat{l}(x)$ is radially unbounded and $\hat{R}_1(x)$ is bounded. Nevertheless, $\hat{l}(x)$ in (118) is positive definite, which ensures that x is penalized, although large values of x may be tolerated.

VI. Inverse Optimality via Backstepping

In the last two sections we showed that both the inverse optimal gain assignment problem and the inverse optimal $\mathcal{H}_\infty$ problem reduce to the problem of finding an ISS-CLF. In this section, we show that integrator backstepping can be used for systematically constructing ISS-CLF's.

Lemma 6.1: If the system

$$\dot{x} = f(x) + g_1(x)d + g_2(x)u \tag{120}$$

is input-to-state stabilizable with a *smooth* control law $u = \alpha(x)$, then the augmented system

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)d + g_2(x)\xi \\ \dot{\xi} &= u \end{aligned} \tag{121}$$

is also input-to-state stabilizable with a *smooth* control law.

An outline of the proof of this lemma, originally given in [24] in the context of modular adaptive nonlinear stabilization, is provided next for completeness.

Proof: Let us denote the input-to-state stabilizing controller by $u = \alpha(x)$. It was proven by Sontag and Wang [35] that a system is ISS if and only if there exist a smooth positive definite radially unbounded function $V(x)$ and class $\mathcal{K}_\infty$ functions μ and ν such that the following "dissipation" inequality holds:

$$L_{f+g_2\alpha}V + L_{g_1}Vd \leq -\mu(|x|) + \nu(|d|). \tag{122}$$

We now show that the control law

$$\begin{aligned} u = \bar{\alpha}(x,\xi) &= -L_{g_2}V - (\xi - \alpha) \\ &\quad + L_{f+g_2\xi}\alpha - \left|L_{g_1}\alpha\right|^2(\xi-\alpha) \end{aligned} \tag{123}$$

achieves input-to-state stabilization of (121) with respect to an ISS-CLF

$$\bar{V}(x,\xi) = V(x) + \frac{1}{2}(\xi - \alpha(x))^2. \tag{124}$$

Toward this end, consider

$$\begin{aligned} \dot{\bar{V}} &= L_{f+g_2\alpha}V + L_{g_1}Vd \\ &\quad + (\xi-\alpha)\left[u + L_{g_2}V - L_{f+g_2\xi}\alpha - \left(L_{g_1}\alpha\right)d\right]. \end{aligned} \tag{125}$$

By substituting (122) and (123) we get

$$\begin{aligned}\dot{\bar{V}} &\le -\mu(|x|) + \nu(|d|) - (\xi-\alpha)^2 \\ &\quad - \left|L_{g_1}\alpha\right|^2(\xi-\alpha)^2 - (\xi-\alpha)\left(L_{g_1}\alpha\right)d \\ &\le -\mu(|x|) + \nu(|d|) - (\xi-\alpha)^2 + \frac{1}{4}|d|^2. \qquad (126)\end{aligned}$$

Denoting $\bar{\nu}(r) = \nu(r) + \frac{1}{4}r^2$ and picking a class $\mathcal{K}_\infty$ function $\bar{\mu}(r) \le \min\{\mu(r), r^2\}$, we get

$$\begin{aligned}\dot{\bar{V}} &\le -\bar{\mu}\left(\left|\begin{bmatrix} x \\ \xi - \alpha(x)\end{bmatrix}\right|\right) + \bar{\nu}(|d|) \\ &\le -\tilde{\mu}\left(\left|\begin{bmatrix} x \\ \xi \end{bmatrix}\right|\right) + \bar{\nu}(|d|) \qquad (127)\end{aligned}$$

where $\tilde{\mu} \in \mathcal{K}_\infty$. Thus, the control law (123) achieves input-to-state stabilization of (121). To see that $\bar{V}(x)$ is an ISS-CLF, we choose $\rho = \tilde{\mu}^{-1} \circ 2\bar{\nu}$. □

Theorem 6.1: Under the conditions of Lemma 6.1, both the inverse optimal gain assignment and $\mathcal{H}_\infty$ problems are solvable for (121) with control laws which are continuous everywhere.

Proof: The proof is immediate by combining Lemma 6.1 with Theorems 3.2 and 5.2. The continuity at the origin follows from the fact that in Lemma 6.1 we found a smooth input-to-state stabilizing control law, which implies that $\bar{V}(x,\xi)$ satisfies a small control property, and therefore the Sontag-type controllers in Theorem 3.2 and 5.2 are continuous at the origin. □

A recursive application of Lemma 6.1, combined with Theorems 3.2 and 5.2, leads to the following result for a representative class of strict-feedback systems [25].

Corollary 6.1: Both the inverse optimal gain assignment and $\mathcal{H}_\infty$ problems are solvable for the following system:

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \varphi_i(x_1,\cdots,x_i)^{\mathrm{T}} d, \qquad i = 1,\cdots,n-1 \\ \dot{x}_n &= u + \varphi_n(x_1,\cdots,x_n)^{\mathrm{T}} d. \qquad (128)\end{aligned}$$

VII. Design for Strict-Feedback Systems

Since the control laws for strict-feedback systems (128) suggested by Corollary 6.1 are based on a Sontag-type formula and are typically nonsmooth (especially at the origin), in this section we design inverse optimal control laws that are *smooth everywhere*. In addition, they achieve a quadratic penalty on the disturbance with a constant weight function.

From Theorems 3.1 and 5.1, it follows that in order to solve the inverse optimal gain assignment and the inverse optimal $\mathcal{H}_\infty$ problems, it suffices to find a stabilizing controller of the form (94) for the auxiliary systems (20) and (93), respectively. For the auxiliary system (20), we choose

$$\gamma(r) = \frac{1}{\kappa}r^2 \qquad (129)$$

where κ is an arbitrary positive constant. This amounts to selecting the weight function in the auxiliary system (93) to be a constant

$$R_1(x) = \frac{1}{\kappa}. \qquad (130)$$

With these choices, the auxiliary systems (20) and (93) take the same form

$$\dot{x} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ 0 \end{bmatrix} + \kappa g_1 \left(L_{g_1}V\right)^{\mathrm{T}} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \qquad (131)$$

where $g_1 = [\varphi_1, \varphi_2, \cdots, \varphi_n]^{\mathrm{T}}$.

First, we search for an ISS-CLF for (128). Repeated application of Lemma 6.1 gives an ISS-CLF

$$\begin{aligned} V &= \frac{1}{2}\sum_{i=1}^{n} z_i^2 \\ z_i &= x_i - \alpha_{i-1}(x_1,\cdots,x_i), \quad i = 1,\cdots,n+1 \end{aligned} \qquad (132)$$

where α_i's are to be determined. For notational convenience we define $z_0 := 0$, $\alpha_0 := 0$, $x_{n+1} = 0$, and $z_{n+1} = 0$. We then have

$$\begin{aligned}\left(L_{g_1}V\right)^{\mathrm{T}} &= \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}\varphi_j = \sum_{j=1}^{n}\left(z_j - \sum_{k=j+1}^{n} \frac{\partial \alpha_{k-1}}{\partial x_j} z_k\right) \\ \varphi_j &= \sum_{j=1}^{n} w_j z_j \end{aligned} \qquad (133)$$

$$\begin{aligned}\frac{\partial V}{\partial x}\begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ 0 \end{bmatrix} &= \sum_{j=1}^{n} \frac{\partial V}{\partial x_j} x_{j+1} \\ &= \sum_{j=1}^{n}\left(x_{j+1} - \sum_{k=1}^{j-1}\frac{\partial \alpha_{j-1}}{\partial x_k} x_{k+1}\right) z_j \qquad (134)\end{aligned}$$

where

$$w_j(\bar{x}_j) = \varphi_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k}\varphi_k. \qquad (135)$$

The functions $\alpha_1,\cdots,\alpha_{n-1}$ are sought to make V defined in (132) a CLF for (131). The derivative of V along the solutions of (131) is

$$\begin{aligned}\dot{V} &= z_n u + \kappa\left(\sum_{k=1}^{n} w_k z_k\right)^{\mathrm{T}}\left(\sum_{i=1}^{n} w_i z_i\right) \\ &\quad + \sum_{i=1}^{n}\left(x_{i+1} - \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_k}x_{k+1}\right)z_i \\ &= z_n u + \kappa|w_n|^2 z_n^2 + \kappa z_n\left(\sum_{k=1}^{n-1} w_k^{\mathrm{T}} z_k\right)w_n \\ &\quad - z_n\sum_{k=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_k}x_{k+1} + \kappa\sum_{i=1}^{n-1}\left(\sum_{k=i+1}^{n} w_k^{\mathrm{T}} z_k\right)w_i z_i \\ &\quad + z_n z_{n-1} + \sum_{i=1}^{n-1}\left[z_{i-1} + \alpha_i + \kappa\left(\sum_{k=1}^{i} w_k^{\mathrm{T}} z_k\right)w_i \right. \\ &\quad \left. - \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_k}x_{k+1}\right]z_i. \qquad (136)\end{aligned}$$

The choice

$$\alpha_i = -z_{i-1} - c_i z_i - \kappa |w_i|^2 z_i - 2\kappa \left(\sum_{k=1}^{i-1} w_k^{\mathrm{T}} z_k \right) w_i + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} \tag{137}$$

where $c_i > 0$, results in

$$\dot{V} = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n \left[u + z_{n-1} + \kappa |w_n|^2 z_n + 2\kappa \left(\sum_{k=1}^{n-1} w_k^{\mathrm{T}} z_k \right) w_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} \right]. \tag{138}$$

In the derivation of (138), we have used the equality

$$\kappa \sum_{i=1}^{n-1} \left(\sum_{k=i+1}^{n} w_k^{\mathrm{T}} z_k \right) w_i z_i - \kappa \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i-1} w_k^{\mathrm{T}} z_k \right) w_i z_i = \kappa z_n \left(\sum_{k=1}^{n-1} w_k^{\mathrm{T}} z_k \right) w_n. \tag{139}$$

We are now at a position to choose the control u. We may choose u such that all the terms inside the bracket in (138) are cancelled and the bracketed term multiplying z_n is equal to $-c_n z_n^2$ as in [25], but the controller designed in that way is not guaranteed to be inverse optimal. In order for a controller to be inverse optimal, according to Theorem 3.1 or Theorem 5.1, it should be of the form

$$u = \alpha_n(x) = -R_2(x)^{-1} \left(L_{g_2} V \right)^{\mathrm{T}} \tag{140}$$

where $R_2(x) = R_2(x)^{\mathrm{T}} > 0$, for all x. In light of (128) and (132), (140) simplifies to

$$u = \alpha_n(x) = -R_2(x)^{-1} z_n \tag{141}$$

i.e., we must choose α_n with z_n as a factor.

Since $x_{k+1} = z_{k+1} + \alpha_k$, $k = 1, \cdots, n-1$, and each α_k vanishes at $z = 0$, there exist smooth functions ϕ_k, $k = 1, \cdots, n$, such that

$$-\sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} = \sum_{k=1}^{n} \phi_k z_k. \tag{142}$$

Thus (138) becomes

$$\dot{V} = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n u + \kappa |w_n|^2 z_n^2 + z_n \sum_{k=1}^{n} \Phi_k z_k \tag{143}$$

where

$$\begin{aligned} \Phi_k &= 2\kappa w_k^{\mathrm{T}} w_n + \phi_k, \quad k = 1, \cdots, n-2 \\ \Phi_{n-1} &= 1 + 2\kappa w_{n-1}^{\mathrm{T}} w_n + \phi_{n-1} \\ \Phi_n &= \phi_n. \end{aligned} \tag{144}$$

A control law of the form (141) with

$$R_2(x) = \left(c_n + \kappa |w_n|^2 + \sum_{k=1}^{n} \frac{\Phi_k^2}{2c_k} \right)^{-1} > 0, \quad c_n > 0 \tag{145}$$

results in

$$\dot{V} = -\frac{1}{2} \sum_{k=1}^{n} c_k z_k^2 - \frac{1}{2} \sum_{k=1}^{n} c_k \left(z_k - \frac{\Phi_k}{c_k} z_n \right)^2. \tag{146}$$

By Theorems 3.1 and 5.1, the inverse optimal gain assignment and $\mathcal{H}_\infty$ problems are solved with the feedback control law

$$u = \alpha_n^*(x) = \beta \alpha_n(x), \qquad \beta \geq 2. \tag{147}$$

Remark 7.1: We point out that the choice of α_i in (137) is the same as in [29], but the control u is chosen differently. While the controller in [29] cancels all the terms inside the bracket in (138), our controller does not. As a result, the controller in [29] achieves only attenuation of the effect of the disturbances on z, while our controller achieves optimality which includes a penalty on u.

Remark 7.2: The choice of α_i as in (137) is not unique. In fact, the ISS-CLF framework provides more flexibility in choosing the α_i's. For example, another choice is the following: by using Lemma A3, we can rewrite (136) as

$$\begin{aligned} \dot{V} = {} & z_n u + \kappa \sum_{i=1}^{n} |w_i z_i|^2 + \sum_{i=1}^{n} \left(x_{i+1} - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} \right) z_i - \kappa \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left| \sqrt{\frac{\kappa_i}{\kappa_j}} w_i z_i - \sqrt{\frac{\kappa_j}{\kappa_i}} w_j z_j \right|^2 \\ = {} & z_n u + \kappa_n |w_n|^2 z_n^2 + z_n z_{n-1} - z_n \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} - \kappa \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left| \sqrt{\frac{\kappa_i}{\kappa_j}} w_i z_i - \sqrt{\frac{\kappa_j}{\kappa_i}} w_j z_j \right|^2 \\ & + \sum_{i=1}^{n-1} \left[z_{i-1} + \alpha_i + \kappa_i |w_i|^2 z_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} \right] z_i \end{aligned} \tag{148}$$

where $\kappa = \left(\sum_{i=1}^{n} \frac{1}{\kappa_i} \right)^{-1}$, $\kappa_i > 0$. The choice

$$\alpha_i = -z_{i-1} - c_i z_i - \kappa_i |w_i|^2 z_i + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} \tag{149}$$

results in

$$\dot{V} = -\sum_{k=1}^{n-1} c_k z_k^2 - \kappa \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left| \sqrt{\frac{\kappa_i}{\kappa_j}} w_i z_i - \sqrt{\frac{\kappa_j}{\kappa_i}} w_j z_j \right|^2 + z_n u + \kappa_n |w_n|^2 z_n^2 + z_n \sum_{k=1}^{n} \Psi_k z_k \tag{150}$$

where

$$\begin{aligned} \Psi_k &= \phi_k, \quad k = 1, \cdots, n-2 \\ \Psi_{n-1} &= 1 + \phi_{n-1} \\ \Psi_n &= \phi_n. \end{aligned} \tag{151}$$

Instead of (146), a control law of the form (141) with

$$R_2(x) = \left(c_n + \kappa |w_n|^2 + \sum_{k=1}^{n} \frac{\Psi_k^2}{2c_k} \right)^{-1} > 0 \tag{152}$$

results in

$$\dot{V} = -\frac{1}{2}\sum_{k=1}^{n} c_k z_k^2 - \frac{1}{2}\sum_{k=1}^{n} c_k\left(z_k - \frac{\Psi_k}{c_k}z_n\right)^2 - \kappa \sum_{\substack{i,j=1\\ i\neq j}}^{n}\left|\sqrt{\frac{\kappa_i}{\kappa_j}}w_i z_i - \sqrt{\frac{\kappa_j}{\kappa_i}}w_j z_j\right|^2. \tag{153}$$

By Theorems 3.1 and 5.1, the inverse optimal gain assignment and $\mathcal{H}_\infty$ problems are solved with the feedback control law $u = \alpha_n^*(x) = \beta\alpha_n(x),\ \beta \geq 2$. The design in this remark is similar to those in [28] and [25] for steps $i = 1, \cdots, n-1$, but different at step n where the new design selects control of the form (141) instead of cancelling the nonlinearities.

VIII. Performance Estimates

We now give performance bounds on the error state z and control u for the inverse optimal controller designed in Section VII. The $\mathcal{L}_2$ bound that we present is the first bound in the literature that incorporates the control u. Previous bounds without u were given in [28] and [25].

Theorem 8.1: In the closed-loop system (128), (141), the following inequalities hold:

1)
$$\int_0^\infty \left(2\sum_{k=1}^{n} c_k z_k^2 + \frac{u^2}{c_n + \kappa|w_n|^2 + \sum_{k=1}^{n}\frac{\Phi_k^2}{2c_k}}\right)dt \leq \frac{1}{\kappa}\|d\|_2^2 + 2|z(0)|^2 \tag{154}$$

2)
$$|z(t)| \leq \frac{1}{\sqrt{2c\kappa}}\sup_{0\leq\tau\leq t}|d(\tau)| + |z(0)|e^{-ct/2} \tag{155}$$

where $c = \min_{1\leq i\leq n} c_i$.

Proof i): According to Theorems 3.1 and 5.1, the control law $u = \alpha^*(x)$ is optimal with respect to the cost functional $(\beta = \lambda = 2)$

$$J(u) = \sup_d\left\{\lim_{t\to\infty}\left[4V(x(t)) + \int_0^t\left(2\sum_{k=1}^{n} c_k z_k^2 + 2\sum_{k=1}^{n} c_k\left(z_k - \frac{\Phi_k}{c_k}z_n\right)^2 + \frac{u^2}{c_n + \kappa|w_n|^2 + \sum_{k=1}^{n}\frac{\Phi_k^2}{2c_k}} - \frac{1}{\kappa}|d|^2\right)d\tau\right]\right\} \tag{156}$$

with a value function

$$J^* = 2|z|^2. \tag{157}$$

Therefore

$$\begin{aligned}\int_0^\infty &\left[2\sum_{k=1}^{n} c_k z_k^2 + \frac{u^2}{c_n + \kappa|w_n|^2 + \sum_{k=1}^{n}\frac{\Phi_k^2}{2c_k}} - \frac{1}{\kappa}|d|^2\right]dt\\ &\leq \int_0^\infty\left[2\sum_{k=1}^{n} c_k z_k^2 + 2\sum_{k=1}^{n} c_k\left(z_k - \frac{\Phi_k}{c_k}z_n\right)^2 + \frac{u^2}{c_n + \kappa|w_n|^2 + \sum_{k=1}^{n}\frac{\Phi_k^2}{2c_k}} - \frac{1}{\kappa}|d|^2\right]dt\\ &\leq J^* = 2|z(0)|^2\end{aligned} \tag{158}$$

which yields (154).

ii): Differentiating $\frac{1}{2}|z|^2$ along the solutions of (128), noting (146), we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}|z|^2\right) &= -\frac{1}{2}\sum_{k=1}^{n} c_k z_k^2 - \frac{1}{2}\sum_{k=1}^{n} c_k\left(z_k - \frac{\Phi_k}{c_k}z_n\right)^2 - \kappa\left|\sum_{k=1}^{n} w_k z_k\right|^2 + \left(\sum_{k=1}^{n} w_k z_k\right)^{\mathrm{T}} d\\ &\leq -\frac{c}{2}|z|^2 + \frac{1}{4\kappa}|d|^2.\end{aligned} \tag{159}$$

By the comparison principle, we get (155). □

Remark 8.1: If the control law is chosen as in Remark 7.2, the cost functional is (160), as shown at the bottom of the page. Instead of (154), we have the performance bound

$$\int_0^\infty \left(2\sum_{k=1}^{n} c_k z_k^2 + \frac{u^2}{c_n + \kappa|w_n|^2 + \sum_{k=1}^{n}\frac{\Psi_k^2}{2c_k}}\right)dt \leq \frac{1}{\kappa}\|d\|_2^2 + 2|z(0)|^2. \tag{161}$$

The control law from Remark 7.2 also achieves an ISS bound as in (155).

IX. Conclusion

We showed that it is possible to solve a meaningful HJI equation (and, therefore, solve an inverse optimal $\mathcal{H}_\infty$-like problem) if and only if a system is input-to-state stabilizable. Our results indicate that—for nonlinear systems—it is crucial

$$J(u) = \sup_d\left\{\lim_{t\to\infty}\left[4V(x(t)) + 2\int_0^t\left(\sum_{k=1}^{n} c_k z_k^2 + \sum_{k=1}^{n} c_k\left(z_k - \frac{\Psi_k}{c_k}z_n\right)^2 + 2\kappa\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left|\sqrt{\frac{\kappa_i}{\kappa_j}}w_i z_i - \sqrt{\frac{\kappa_j}{\kappa_i}}w_j z_j\right|^2 + \frac{u^2}{2\left(c_n + \kappa|w_n|^2 + \sum_{k=1}^{n}\frac{\Psi_k^2}{2c_k}\right)} - \frac{1}{2\kappa}|d|^2\right)d\tau\right]\right\} \tag{160}$$

to move away from quadratic cost functionals, and in particular, from the quadratic penalty on the disturbance. The benefits of the inverse optimal approach are that it is constructive and it guarantees stability margins against some input unmodeled dynamics.

APPENDIX

Lemma A1: If γ and its derivative γ' are class $\mathcal{K}_\infty$ functions, then the Legendre–Fenchel transform satisfies the following properties:

1)
$$\ell\gamma(r) = r(\gamma')^{-1}(r) - \gamma((\gamma')^{-1}(r)) = \int_0^r (\gamma')^{-1}(s)\,ds \tag{162}$$

2)
$$\ell\ell\gamma = \gamma \tag{163}$$

3)
$$\ell\gamma \text{ is a class } \mathcal{K}_\infty \text{ function} \tag{164}$$

4)
$$\ell\gamma(\gamma'(r)) = r\gamma'(r) - \gamma(r). \tag{165}$$

Proof:

1) Integrating by parts, we get

$$\begin{aligned}\int_0^r (\gamma')^{-1}(s)\,ds &= r(\gamma')^{-1}(r) - \int_0^r s d((\gamma')^{-1}(s)) \\ &= r(\gamma')^{-1}(r) - \int_0^r \gamma'((\gamma')^{-1}(s)) d((\gamma')^{-1}(s)) \\ &= r(\gamma')^{-1}(r) - \int_0^r d(\gamma((\gamma')^{-1}(s)))\end{aligned} \tag{166}$$

which completes the proof.

2) Immediate by differentiating the second expression in 1), and then inverting and integrating the result.

3) Obvious from the second expression in 1) because $(\gamma')^{-1}$ is a class $\mathcal{K}_\infty$ function.

4) Follows by direct substitution into (162). □

Lemma A2 (Young's Inequality [11, Th. 156]): For any two vectors x and y, the following holds:

$$x^{\mathrm{T}} y \le \gamma(|x|) + \ell\gamma(|y|) \tag{167}$$

and the equality is achieved if and only if

$$y = \gamma'(|x|)\frac{x}{|x|}, \qquad \text{that is, for } x = (\gamma')^{-1}(|y|)\frac{y}{|y|}. \tag{168}$$

Lemma A3: For vectors $y_1, \cdots, y_n$, the following identity holds:

$$\kappa\left|\sum_{i=1}^n y_i\right|^2 = \sum_{i=1}^n \kappa_i |y_i|^2 - \kappa \sum_{\substack{i,j=1\\ i\neq j}}^n \left|\sqrt{\frac{\kappa_i}{\kappa_j}} y_i - \sqrt{\frac{\kappa_j}{\kappa_i}} y_j\right|^2 \tag{169}$$

where $\kappa = (\sum_{i=1}^n \frac{1}{\kappa_i})^{-1}$, $\kappa_i > 0$.

Proof: Since

$$\left|\sum_{i=1}^n y_i\right|^2 = \sum_{i=1}^n |y_i|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^n 2y_i^{\mathrm{T}} y_j \tag{170}$$

and

$$2y_i^{\mathrm{T}} y_j = \frac{\kappa_i}{\kappa_j}|y_i|^2 + \frac{\kappa_j}{\kappa_i}|y_j|^2 - \left|\sqrt{\frac{\kappa_i}{\kappa_j}} y_i - \sqrt{\frac{\kappa_j}{\kappa_i}} y_j\right|^2 \tag{171}$$

we have that

$$\left|\sum_{i=1}^n y_i\right|^2 = \sum_{i=1}^n |y_i|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{\kappa_i}{\kappa_j}|y_i|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{\kappa_j}{\kappa_i}|y_j|^2 - \sum_{\substack{i,j=1\\ i\neq j}}^n \left|\sqrt{\frac{\kappa_i}{\kappa_j}} y_i - \sqrt{\frac{\kappa_j}{\kappa_i}} y_j\right|^2. \tag{172}$$

Noting that

$$\sum_{i=1}^n |y_i|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{\kappa_i}{\kappa_j}|y_i|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{\kappa_j}{\kappa_i}|y_j|^2 = \left(\sum_{i=1}^n \frac{1}{\kappa_i}\right)\sum_{i=1}^n \kappa_i |y_i|^2 \tag{173}$$

we get

$$\left|\sum_{i=1}^n y_i\right|^2 = \left(\sum_{i=1}^n \frac{1}{\kappa_i}\right)\sum_{i=1}^n \kappa_i |y_i|^2 - \sum_{\substack{i,j=1\\ i\neq j}}^n \left|\sqrt{\frac{\kappa_i}{\kappa_j}} y_i - \sqrt{\frac{\kappa_j}{\kappa_i}} y_j\right|^2 \tag{174}$$

which is (169). □

ACKNOWLEDGMENT

The authors would like to thank P. Kokotović for discussions that led to the results in Section IV.

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