Nonlinear control under wave actuator dynamics with time- and state-dependent moving boundary

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SUMMARY

We consider the stabilization of nonlinear ODE systems with actuator dynamics modeled by a wave PDE whose boundary is moving and is a function of time and of the ODE’s state. Such a problem is inspired by applications in oil drilling where the position of the drill bit is a state variable in the ODE modeling the friction-dominated drill bit dynamics while at the same time being the position of the moving boundary of the wave PDE that models the distributed torsional dynamics of the drillstring. For moving boundaries that depend only on time, we extend the global result recently developed by Bekiaris-Liberis and Krstić for constant boundaries. For moving boundaries that also depend on the ODE’s state, we develop a local result where the initial condition is restricted in such a way that it is ensured that the rate of movement of the boundary (both ‘leftward’ and ‘rightward’) is bounded by unity in closed-loop. For strict-feedback systems under wave actuator dynamics with moving boundaries, the predictor-based feedback laws are obtained explicitly. The feedback design is illustrated through an example. Copyright © 2013 John Wiley & Sons, Ltd.

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KEY WORDS: nonlinear control; distributed parameter systems; delay systems; partial differential equations; stabilization

1. INTRODUCTION

1.1. Motivation

A common type of instability in oil drilling is stick-slip oscillation [1], caused by friction between the drill bit and the rock resulting in torsional vibrations of the drill string, which can severely damage the drilling facilities. The torsional dynamics of a drill string are modeled as a wave PDE that governs the dynamics of the angular displacement of the drill string. The drill string PDE is coupled with a nonlinear ODE that describes the dynamics of the angular velocity of the drill bit at the bottom of the drill string [2].

Based on the linearization of its dynamics, a control method for the stabilization of the drilling instability is presented in [3]. Furthermore, motivated by the fact that the friction force is nonlinear, a general result for the compensation of wave PDE dynamics at the input of a general nonlinear ODE was presented in [4].

However, in the drilling application, the drill bit progresses through the rock—penetration into the rock is the very purpose of drilling. As a result, the length of the drill string increases with time. Hence, the domain over which the wave equation models the torsional dynamics of the drill string changes with time. It is in two manners that the length of the drill string (which equals the position of the drill bit relative to a reference position) changes with time. First, because the position of the

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drill bit is a component of the state of the ODE governing the drill bit dynamics, the domain length for the wave PDE is dependent on the state of the ODE. Second, the progression of the drill bit through the rock is the result of a ‘trajectory’ commanded to the drill bit position; hence, the length of the domain of the wave PDE depends on both the drill bit position tracking error and on the trajectory for the drill bit position as a function of time, namely, the domain length depends on the ODE state (in tracking error variables) and on time.

Motivated by this application, it is of interest to study the design for cascaded systems of a wave PDE and a nonlinear ODE with the domain length of the wave PDE depending on both ODE state and on time. We present such a design in this paper. We stop short of applying such a design to the drilling problem itself as there are additional modeling and control design issues arising in the drilling problem that are beyond the result of this paper. These issues have to do with the fact that the drilling problem is actually a tracking problem, the output of interest is the boundary velocity, which is also a state of the ODE, the boundary that moves is actually on the side of the wave PDE opposite from the control input, and the boundary condition of the wave PDE at the uncontrolled end is not of the homogeneous Neumann type.

1.2. Related literature

Various predictor-based techniques have been developed for the compensation of constant input delays in linear systems [5–8]. Recent extensions of these designs to linear systems with unknown plant parameters are given in [9], with an unknown actuator delay in [10], for time-varying input delays in [11, 12], and with distributed input delay in [13]. Over the last 10 years, many innovations have been made on developing control designs and stability analysis for nonlinear systems with state delays [14–18] and with input delays [19–22]. The approach for the boundary stabilization of the first-order hyperbolic PDEs and application to systems with delays is given in [23]. A strict Lyapunov function for hyperbolic systems of conservation laws that can be diagonalised with Riemann invariants is presented in [24]. Stabilization of a system of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input is dealt with in [25]. The problem of boundary stabilization for a quasilinear $2 \times 2$ system of first-order hyperbolic PDEs is considered in [26]. Stabilizing control design for a broad class of nonlinear PDEs is in [27]. The convergence of the feedback laws design in [27] is proved, and the stability properties of the closed-loop system are established in [28]. Predictor-based control for nonlinear systems with arbitrarily large input delays is presented in [29], with a Lyapunov functional provided for the stability analysis. Stabilization design for nonlinear systems with constant input delay by means of approximate predictors is obtained in [30]. The predictor-based control scheme is extended to nonlinear systems under sampled and constantly delayed measurements, and with inputs subject to constant delay and zero-order hold in [31].

Although there are many results for systems with constant input delays, the predictor-based control for long time-varying input delays, even for linear systems, is tackled in only a few studies [11, 12, 32, 33]. Predictor-based control for the compensation of time-varying input delay for nonlinear systems is introduced in [34] and for nonlinear systems with state-dependent input delay in [35], with an extension to systems with delayed states given in [36]. Robustness of nonlinear systems under input delay variations that depend on time and state is studied in [37]. Yet, there are no available results for predictor-based feedback design for nonlinear systems under complicated actuator dynamics with time- and state-dependent moving boundary. An explicit feedback law that compensates the wave PDE dynamics at the input of a linear time-invariant ODE and stabilizes the overall system is introduced in [38]. For a multi-input, linear system that compensates the wave PDE dynamics in its input is dealt with in [39]. The result of [38] is extended to general nonlinear ODEs in [4].

1.3. Problem considered in the paper

We consider nonlinear systems under wave actuator dynamics with time- and state-dependent moving boundary given by

$$
\dot{X}(t) = f(X(t), u(0, t))
$$

(1)
where $X \in \mathbb{R}^n$ is the state vector and $U$ is the scalar input to the entire system, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz with $f(0, 0) = 0$, $u(x, t)$ is the state of the PDE dynamics of the actuator governed by a wave equation, and $L : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ is continuously differentiable. The Neumann actuation choice $u_x(L(t, X(t)), t) = U(t)$ is pursued because this is a natural physical choice because $u_x(L(t, X(t)), t)$ corresponds to a force on the string’s boundary.

1.4. Results of the paper

Following [4], our design method is based on a preliminary transformation that converts the wave PDE dynamics into a $2 \times 2$ system of first-order transport PDEs that convect in opposite directions. Then, a predictor-based design for the transport PDE convecting towards the ODE is employed, compensating the PDE dynamics. An appropriate adjustment for the state- and time-dependent character of the moving boundary is made.

Stability analysis is conducted with infinite-dimensional backstepping transformations for the two transport PDE states and by constructing a Lyapunov functional. For moving boundaries that depend on both the ODE’s state and time, we develop a local result where the initial condition is restricted in such a way that it is ensured that the rate of movement of the boundary is bounded by unity in closed-loop. When the moving boundary depends on time only, a global result is achieved. For strict-feedforward systems under wave actuator dynamics with moving boundaries, the predictor-based feedback laws are obtained explicitly.

1.5. Structure of the paper

In Section 2, we present a control design. In Section 3, the standing assumptions are given. In Section 4, we present the proof of local stability for time- and state-dependent moving boundary. In Section 5, we give the proof of global stability for time-dependent moving boundary. In Section 6, we extend the results to strict-feedforward nonlinear systems. An illustrative example is given in Section 7. Finally, some conclusions are drawn in Section 8. Some proofs of the lemmas are in the Appendix.

1.6. Notation

We use the common definitions of class $\mathcal{K}, \mathcal{K}_\infty, \mathcal{KL}$ functions from [11]. For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. For a scalar function $u(\cdot, t) \in L^\infty[0, L(t, \chi)]$, $L(t, \chi) > 0$, $t \geq 0$, we denote with $\|u(t)\|_\infty$ its supremum norm, that is, $\|u(t)\|_\infty = \sup_{x \in [0, L(t, \chi)]} |u(x, t)|$, especially, $\|u(t)\|_{\infty 1} = \sup_{x \in [0, 1]} |u(x, t)|$. For a vector valued function $p \in L^\infty[0, L(t, \chi)]$, we denote with $\|p(t)\|_\infty$ its supremum norm, that is, $\|p(t)\|_\infty = \sup_{x \in [0, L(t, \chi)]} \sqrt{p_1(x, t)^2 + \cdots + p_n(x, t)^2}$. $\phi^{-1}(t)$ is the inverse of the function $\phi(t)$. $C^j[0, 1]$ is the space of functions that have continuous derivatives of order $j$. $\nabla$ stands for the gradient.

2. CONTROLLER DESIGN

For system (1)–(4), we convert the problem of the compensation of the wave PDE to a problem of the compensation of a $2 \times 2$ system of first-order transport equations. Let

$$\zeta(x, t) = u_t(x, t) + u_x(x, t),$$

where $X \in \mathbb{R}^n$ is the state vector and $U$ is the scalar input to the entire system, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz with $f(0, 0) = 0$, $u(x, t)$ is the state of the PDE dynamics of the actuator governed by a wave equation, and $L : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ is continuously differentiable. The Neumann actuation choice $u_x(L(t, X(t)), t) = U(t)$ is pursued because this is a natural physical choice because $u_x(L(t, X(t)), t)$ corresponds to a force on the string’s boundary.

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$$\zeta(x, t) = u_t(x, t) + u_x(x, t),$$
\[ \eta(x, t) = u_t(x, t) - u_x(x, t), \quad (6) \]

which in reverse gives
\[ u_t(x, t) = \frac{\zeta(x, t) + \eta(x, t)}{2}, \quad (7) \]
\[ u_x(x, t) = \frac{\zeta(x, t) - \eta(x, t)}{2}, \quad (8) \]

and denotes
\[ \xi(t) = u(0, t). \quad (9) \]

System (1)–(4) can be represented as
\[ \dot{Z}(t) = \varphi(Z(t), \zeta(0, t)), \quad (10) \]
\[ \zeta_t(x, t) = \zeta_x(x, t), \quad (11) \]
\[ \eta_t(x, t) = -\eta_x(x, t), \quad (12) \]
\[ \eta(0, t) = \zeta(0, t), \quad (13) \]
\[ \zeta(L(t, X(t)), t) = \eta(L(t, X(t)), t) + 2U(t), \quad (14) \]

where
\[ Z = \begin{bmatrix} X \\ \xi \end{bmatrix} \quad (15) \]

and
\[ \varphi(Z, v) = \begin{bmatrix} f(X, \xi) \\ v \end{bmatrix}. \quad (16) \]

First, let us discuss the well-posedness of system (10)–(14). Define a new set of coordinates
\[ \bar{t} = t, \quad (17) \]
\[ \bar{x} = \frac{x}{L(t, X(t))}, \quad (18) \]

one can find that
\[ \frac{\partial \chi(x, t)}{\partial t} = \frac{\partial \chi(x, t)}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial t} + \frac{\partial \chi(x, t)}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial t}, \quad (19) \]
\[ = \frac{\partial \chi(L \bar{x}, \bar{t})}{\partial \bar{t}} - \bar{x} L \frac{\partial \chi(L \bar{x}, \bar{t})}{\partial \bar{x}}, \]
\[ \frac{\partial \chi(x, t)}{\partial x} = \frac{\partial \chi(x, t)}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x} + \frac{\partial \chi(x, t)}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial x}, \quad (20) \]

where \( \chi(x, t) \) is a differential function, and re-write system (10)–(14) as follows:
\[ \dot{Z}(\bar{t}) = \varphi(Z(\bar{t}), \zeta(0, \bar{t})), \quad (21) \]
The most essential condition for this system to be well-posed is that the transport velocities have to have the correct sign; if not, the boundary conditions are not in the correct side of the equation. Thus, one obtains the conditions:

\[ \xi(1, \tilde{r}) = \eta(1, \tilde{r}) + 2U(\tilde{r}). \]

The condition is equivalent to the following condition:

\[ F_d : \quad |L_\xi(t, X(t)) + \nabla L(t, X(t)) f(X(t), \xi(t))| \leq d, \]

for all \( t \geq 0 \) and \( d \in [0, 1] \). We refer to \( F_d \) as the feasibility condition.

**Remark 1**

In this paper, we aim at system (10)–(14), not system (21)–(25), and design a predictor-based feedback law that compensates the wave dynamics in the input of the system. The reason is after backstepping transformation (59) and (60), system (10)–(14) will be transformed to system (63)–(67), and it is easy to prove the stability of this system. In addition, if one aims at system (21)–(25), it still need to consider the control law design of system (10)–(14) because system (21)–(25) is obtained by the coordinate transformation.

The feedback design that compensates the wave actuator dynamics is based on a nominal feedback law \( \mu : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) that stabilizes the following system:

\[ \dot{X}(t) = f(X(t), \xi(t)) \]

\[ \dot{\xi}(t) = \mu. \]

If there exists a control law \( \kappa \in C^1(\mathbb{R}^n; \mathbb{R}) \) that stabilizes the system \( \dot{X}(t) = f(X(t), U(t)) \), that is, if \( \dot{X}(t) = f(X(t), \kappa(X(t))) \) is globally asymptotically stable, then a feedback law \( \mu \) for system (29) and (30) can be constructed as

\[ \mu(X(t), \xi(t)) = -c_1(\xi(t) - \kappa(X(t))) + \frac{\partial \kappa(X(t))}{\partial X} f(X(t), \xi(t)). \]

Noting that the input to the \( Z \) system is the delayed signal \( \zeta(L(t, X(t)), t) + u_\tau(L(t, X(t)), t) \), we employ the prediction of \( Z \). The control law compensating the wave dynamics is given by

\[ U(t) = -\frac{1}{2} \eta(L(t, X(t)), t) - c_1 \left( p_2(L(t, X(t)), t) - \kappa(p_1(L(t, X(t)), t)) \right) \]

\[ + \frac{1}{2} \frac{\partial \kappa(p_1(L(t, X(t)), t))}{\partial p_1} f(p_1(L(t, X(t)), t), p_2(L(t, X(t)), t)) \]
where \( c_1 > 0 \) is arbitrary, and \( p_1(x, t) \in \mathbb{R}^n, p_2(x, t) \in \mathbb{R} \) are \( x \)-time-units-ahead predictions of \( X(t) \) and \( u(0, t) \) respectively, given by
\[
p_1(x, t) = \int_0^x f(p_1(y, t), p_2(y, t)) dy + X(t), \quad \text{for all } x \in [0, L(t, X(t))],
\]
(33)
\[
p_2(x, t) = u(x, t) + \int_0^x u_t(y, t) dy, \quad \text{for all } x \in [0, L(t, X(t))].
\]
(34)
The initial condition of (33) and (34) is given for \( t = 0 \) as
\[
p_1(x, 0) = \int_0^x f(p_1(y, 0), p_2(y, 0)) dy + X(0), \quad \text{for all } x \in [0, L(0, X(0))],
\]
(35)
\[
p_2(x, 0) = u(x, 0) + \int_0^x u_t(y, 0) dy, \quad \text{for all } x \in [0, L(0, X(0))].
\]
(36)

**Proposition 1**
The following holds:
\[
Z(t + L(t, X(t))) = \begin{bmatrix} p_1(L(t, X(t)), t) \\ p_2(L(t, X(t)), t) \end{bmatrix}, \quad \text{for all } t \geq 0.
\]
(37)

**Proof**
Because
\[
\int_t^{t+L(t, X(t))} \dot{Z}(\theta) d\theta = \int_t^{t+L(t, X(t))} \varphi(Z(\theta), \zeta(0, \theta)) d\theta,
\]
(38)
holds for all \( t \geq 0 \), it yields
\[
Z(t + L(t, X(t))) = Z(t) + \int_t^{t+L(t, X(t))} \varphi(Z(\theta), \zeta(0, \theta)) d\theta,
\]
(39)
for all \( t \geq 0 \). Let \( \theta = t + y \), we have
\[
Z(t + L(t, X(t))) = Z(t) + \int_0^{L(t, X(t))} \varphi(Z(t + y), \zeta(0, t + y)) dy,
\]
(40)
for all \( t \geq 0 \). With (11), it can be deduced that \( \zeta(0, t + y) = \zeta(y, t) \), and by (5), we have \( \zeta(y, t) = u_t(y, t) + u_y(y, t) \); so we have
\[
Z(t + L(t, X(t))) = Z(t) + \int_0^{L(t, X(t))} \varphi(Z(t + y), u_t(y, t) + u_y(y, t)) dy,
\]
(41)
for all \( t \geq 0 \). In view of \( \xi(t) = u(0, t) \) and by (15) and (16), we have
\[
Z(t + L(t, X(t))) = \begin{bmatrix} X(t) + \int_0^{L(t, X(t))} f(X(t + y), u(0, t + y)) dy \\ u(0, t) + \int_0^{L(t, X(t))} u_t(y, t) + u_y(y, t) dy \end{bmatrix}
\]
(42)
for all \( t \geq 0 \). Noting that \( p_1(y, t) \in \mathbb{R}^n, p_2(y, t) \in \mathbb{R} \) are \( y \)-time-units-ahead predictions of \( X(t) \) and \( u(0, t) \) respectively, it yields
\[
Z(t + L(t, X(t))) = \begin{bmatrix} X(t) + \int_0^{L(t, X(t))} f(p_1(y, t), p_2(y, t)) dy \\ u(L(t, X(t)), t) + \int_0^{L(t, X(t))} u_t(y, t) dy \end{bmatrix}
\]
(43)
for all \( t \geq 0 \). By (33) and (34), we have (37). The proof is complete.
3. STANDING ASSUMPTIONS

Assumption 1 (Extended plant is forward and backward complete)
The system \( \dot{Z} = \varphi(Z, v) \) is strongly forward complete and backward complete, that is, there exist smooth positive definite functions \( R_1 \) and \( R_2 \) and class \( K_\infty \) functions \( \alpha_1, \ldots, \alpha_6 \) such that

\[
\alpha_1(|Z|) \leq R_1(Z) \leq \alpha_2(|Z|) \tag{44}
\]

\[
\alpha_3(|Z|) \leq R_2(Z) \leq \alpha_4(|Z|) \tag{45}
\]

\[
- \frac{\partial R_1(Z)}{\partial Z} \varphi(Z, v) \leq R_1(Z) + \alpha_5(|v|) \tag{46}
\]

\[
- \frac{\partial R_2(Z)}{\partial Z} \varphi(Z, v) \leq R_2(Z) + \alpha_6(|v|) \tag{47}
\]

for all \( Z \in \mathbb{R}^{n+1} \) and for all \( v \in \mathbb{R} \).

The forward (backward) completeness implies that for every initial condition and every locally bounded input signal, the corresponding solution is defined for all \( t > 0 \) (\( t \leq 0 \)).

Assumption 2 (Extended closed-loop system is backward complete)
The system \( \dot{Z} = \varphi(Z, \mu(Z) + v) \) is strongly backward complete, that is, there exist a smooth positive definite function \( R_3 \) and class \( K_1 \) functions \( \alpha_7, \alpha_8, \alpha_9 \) such that

\[
\alpha_7(|Z|) \leq R_3(Z) \leq \alpha_8(|Z|), \tag{48}
\]

In the present case, both forward and backward completeness are required to account for the wave equation’s reflection at \( x = 0 \).

Assumption 3 (Closed-loop plant is input-to-state stable (ISS) to input perturbation)
The system \( \dot{X} = f(X, \kappa(X) + v) \) satisfies input-to-state stability property with respect to \( v \), and the function \( \kappa : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable with locally Lipschitz derivative \( \frac{\partial \kappa(X)}{\partial X} \) and satisfies \( \kappa(0) = 0 \).

4. LOCAL STABILITY FOR TIME- AND STATE-DEPENDENT MOVING BOUNDARY

With the assumptions on the nonlinear function \( f(X, w) \) stated later (4), the following holds:

\[
|f(X, w)| \leq \vartheta_1(|X| + |w|) \tag{50}
\]

for a class \( K_\infty \) function \( \vartheta_1 \). In addition, we make the following key assumption on the domain length.

Assumption 4 (State-dependent bounds on the domain length)
The domain length \( L(t, X) \in C^1(R^+ \times \mathbb{R}^n; R^+) \) is uniformly bounded and positive, that is, there exists a positive constant \( m \) such that

\[
0 < L(t, X) \leq m \tag{51}
\]

for all \( t \geq 0 \), and \( L_t(t, X), \nabla L(t, X) \) are Locally Lipschitz, and there exist class \( K_\infty \) functions \( \vartheta_2, \vartheta_3 \) such that

\[
|L_t(t, X)| \leq |L_t(0, 0)| + \vartheta_2(|X|), \tag{52}
\]

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\[ |\nabla L(t, X)| \leq |\nabla L(0, 0)| + \varphi_3(|X|), \quad (53) \]

for all \( t \geq 0 \).

In this section, we consider the condition on the solutions, which is given by (28), and denote

\[
\pi_0^* = \sup_{\theta \in L^{-1}(0, X(0))} \frac{1}{L(\theta, X(\theta))} > 0, \quad (54)
\]

\[
\pi_1^* = \sup_{\theta \in L^{-1}(0, X(0))} |L_\theta(\theta, X(\theta)) + \nabla L(\theta, X(\theta)) f(X(\theta), \xi(\theta))| < 1. \quad (55)
\]

In this section, we prove the following result:

**Theorem 1**

Consider system (1)–(4) together with the control law (32)–(34). Under the condition (28) and Assumptions 1–4, there exist a monotonically increasing function \( \psi_{\text{ROA}} \) and a class of KL function \( \beta \) such that for all initial conditions that are compatible with the feedback law (32) and that satisfy

\[
B_0(d): \quad \Omega(0) \leq \psi_{\text{ROA}}(d) \quad (56)
\]

for some \( 0 \leq d < 1 \), where

\[
\Omega(t) = |X(t)| + \|u(t)\|_\infty + \|u_1(t)\|_\infty + \|u_x(t)\|_\infty, \quad (57)
\]

there exists a unique solution to the closed-loop system with the ODE component \( X(t) \) that is continuously differentiable on \([0, \infty)\) and the PDE component \((u_x(x, t), u_1(x, t))\) that is continuously differentiable on \([0, L(t, X(t)) \times [0, \infty)\), and such that

\[
\Omega(t) \leq \bar{\beta}(\Omega(0), t) \quad \text{for all } t \geq 0. \quad (58)
\]

We prove the theorem by using a series of lemmas, which are given next.

**Lemma 1 (Backstepping transformation)**

The backstepping transformations of \( \zeta \) and \( \eta \) are defined as

\[
\omega(\zeta, t) = \zeta(L(t, X(t))\zeta, t) - \mu(q(L(t, X(t))\zeta, t)), \quad \text{for all } \zeta \in [0, 1], \quad (59)
\]

\[
\sigma(\zeta, t) = \eta(L(t, X(t))\zeta, t) - \mu(r(L(t, X(t))\zeta, t)), \quad \text{for all } \zeta \in [0, 1], \quad (60)
\]

where

\[
q(\zeta, t) = Z(t) + \int_0^\zeta \varphi(q(y, t), \zeta(y, t))dy, \quad \text{for all } \zeta \in [0, 1], \quad (61)
\]

\[
r(\zeta, t) = Z(t) - \int_0^\zeta \varphi(r(y, t), \eta(y, t))dy, \quad \text{for all } \zeta \in [0, 1], \quad (62)
\]

and \( \mu \) is defined in (31); the control law (32)–(34) transform system (10)–(14) to the target system given by

\[
\dot{Z}(t) = \varphi(Z(t), \mu(Z(t)) + \omega(0, t)) \quad (63)
\]

\[
\omega_t(\zeta, t) = \frac{1 + \zeta(L_1(t, X(t)) + \nabla L(t, X(t)) f(X(t), \xi(t)))}{L(t, X(t))} \omega_\zeta(\zeta, t) \quad (64)
\]

\[
\sigma_t(\zeta, t) = -\frac{1 - \zeta(L_1(t, X(t)) + \nabla L(t, X(t)) f(X(t), \xi(t)))}{L(t, X(t))} \sigma_\zeta(\zeta, t) \quad (65)
\]
\[ \sigma(0, t) = \omega(0, t) \]  
\[ \omega(1, t) = 0. \]  
\[ (66) \]
\[ (67) \]

**Proof**

Noting that \( q(0, t) = Z(t) \), by setting \( \zeta = 0 \) into (59), and by (10), we have (63). Setting \( \zeta = 0 \) into (59) and (60), and in view of \( q(0, t) = Z(t) \), \( r(0, t) = Z(t) \), and relation (13), we obtain (66). Next, we prove (64). From (61), it is easy to see that

\[ q_\zeta(\zeta, t) = \varphi(q(\zeta, t), \zeta(\zeta, t)) \]  
\[ (68) \]

Using (11), it can be deduced that \( \zeta \) is a function of \( \zeta + t \), that is, \( \zeta(\zeta, t) = S(\zeta + t) \) for some function \( S \). Using \( \zeta + t \) in place of \( t \) in (10) and noting that \( \zeta(0, \zeta + t) = S(\zeta + t) \), we have

\[ \dot{Z}(\zeta + t) = \varphi(Z(\zeta + t), S(\zeta + t)). \]  
\[ (70) \]

By (68) and (70), it implies that \( q(\zeta, t) = Z(\zeta + t) \) is a solution to the boundary value problem (68) and (69). Because \( \varphi \) is locally Lipschitz, we have that \( q(\zeta, t) = Z(\zeta + t) \) is the unique solution to the (68) and (69). Hence, \( q_\zeta(\zeta, t) = q_t(\zeta, t) \), and \( \omega_\zeta(\zeta, t) = S(L(t, X(t))) \zeta + t) - \mu(Z(t, X(t))) \zeta + t) \), so we have (64). Analogously, from (62), we have

\[ r_\zeta(\zeta, t) = -\varphi(r(\zeta, t), \eta(\zeta, t)) \]  
\[ (71) \]

Using (12), it can be deduced that \( \eta \) is a function of \( t - \zeta \), that is, \( \eta(\zeta, t) = W(t - \zeta) \) for some function \( W \). Noting (13), then we have \( \zeta(0, t - \zeta) = \eta(0, t - \zeta) = W(t - \zeta) \). Using \( t - \zeta \) in place of \( t \) in (10), it implies

\[ \dot{Z}(t - \zeta) = \varphi(Z(t - \zeta), W(t - \zeta)). \]  
\[ (73) \]

By (71) and (73), then \( r_\zeta(\zeta, t) = Z(t - \zeta) \) is a solution to the boundary value problem (71) and (72). Because \( \varphi \) is locally Lipschitz, we have that \( r_\zeta(\zeta, t) = Z(t - \zeta) \) is the unique solution to (71) and (72). Hence, \( r_\zeta(\zeta, t) = -r_t(\zeta, t) \), and \( \varphi_\zeta(\zeta, t) = W(t - L(t, X(t))) \zeta - \mu(Z(t - L(t, X(t))) \zeta) \), and we have (65). Last, we prove (67). Noting that \( q(L(t, X(t))), t) = Z(L(t, X(t))) t) \), that is, \( q(L(t, X(t))), t) = \frac{1}{T}(p_1(L(t, X(t))), t))T, p_2(L(t, X(t))), t))T, with the control law (32)–(34), and relation (14) and (31), we have (67).

**Lemma 2 (Inverse backstepping transformation)**

The inverse backstepping transformations of \( \omega \) and \( \sigma \) are defined as

\[ \zeta(x, t) = \omega(L^{-1}(t, X(t))x, t) + \mu(\lambda(x, t)), \quad \text{for all } x \in [0, L(t, X(t))], \]  
\[ (74) \]

\[ \eta(x, t) = \sigma(L^{-1}(t, X(t))x, t) + \mu(\iota(x, t)), \quad \text{for all } x \in [0, L(t, X(t))], \]  
\[ (75) \]

where

\[ \lambda(x, t) = Z(t) + \int_0^x \varphi(\lambda(y, t), \omega(L^{-1}(t, X(t))y, t) + \mu(\lambda(y, t)))dy, \]  
\[ (76) \]

for all \( x \in [0, L(t, X(t))], \) and

\[ \iota(x, t) = Z(t) - \int_0^x \varphi(\iota(y, t), \sigma(L^{-1}(t, X(t))y, t) + \mu(\iota(y, t)))dy, \]  
\[ (77) \]

for all \( x \in [0, L(t, X(t))], \) and \( \mu \) is defined in (31), and the control law (32)–(34) transform the target system (63)–(67) to system (10)–(14).
Proof
In Appendix. □

Lemma 3 (Extended closed-loop system is ISS to input perturbation)
Under Assumption 3, the control law \( \mu \) given in (31) guarantees that the system
\[
\dot{Z} = \varphi(Z, \mu(Z) + v) = \begin{bmatrix} f(X, \xi) \\ \mu(Z) + v \end{bmatrix}
\]
(78)
is ISS from \( v \) to \( Z = [X^T, \xi]^T \).

Proof
In Appendix. □

Lemma 4 (Stability estimate for target system)
Consider system (63)–(67). Under the condition (28), and Assumptions 3 and 4, there exist a monotonically increasing function \( \gamma_d \) and a class \( K_L \) function \( \beta \), such that for all initial conditions that satisfy
\[
B_1(d) : \Omega_1(0) \subseteq \gamma_d(d)
\]
(79)
for some \( 0 \leq d < 1 \), where
\[
\Omega_1(t) = |X(t)| + |\xi(t)| + \|\omega(t)\|_{\infty} + \|\varpi(t)\|_{\infty},
\]
(80)
the following holds:
\[
\Omega_1(t) \leq \beta(\Omega_1(0), t), \quad \text{for all } t \geq 0.
\]
(81)

Proof
By (50), (52), and (53), we know if a solution satisfies
\[
|L_t(0, 0)| + \vartheta_2(|X|) + (|\nabla L(0, 0)| + \vartheta_3(|X|))\vartheta_1(|X| + |\xi|) \leq d, \quad t \geq 0
\]
(82)
for \( 0 \leq d < 1 \), then it also satisfies (28). The function \( \gamma_d \) is defined as
\[
\gamma_d(s) = |L_t(0, 0)| + \vartheta_2(s) + (|\nabla L(0, 0)| + \vartheta_3(s))\vartheta_1(s)
\]
(83)
and with \( \gamma_d \), we denote the inverse function of \( \gamma_d \). Note that \( \gamma_d \) is a monotonically increasing function.

Let the initial condition of system (63)–(67) be in the set \( B_1(d) : \Omega_1(0) \subseteq \gamma_d(d) \), for some \( 0 \leq d < 1 \), with \( \Omega_1(t) = |X(t)| + |\xi(t)| + \|\omega(t)\|_{\infty} + \|\varpi(t)\|_{\infty} \).

Under Assumption 3, from Lemma 3, there exist a smooth function \( S(Z) : R^{n+1} \rightarrow R^+ \) and class \( K_\infty \) functions \( \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13} \) such that
\[
\alpha_{10}(|Z|) \leq S(Z) \leq \alpha_{11}(|Z|),
\]
(84)
\[
\frac{\partial S(Z)}{\partial Z} \varphi(Z, \mu(Z) + \omega(0, t)) \leq -\alpha_{12}(|Z|) + \alpha_{13}(|\omega(0, t)|)
\]
(85)
where \( Z = [X^T, \xi]^T \). The new variable \( v(\zeta, t), \zeta \in [-1, 1] \) is defined as
\[
v(\zeta, t) = \begin{cases} 
\omega(\zeta, t), & \text{for all } \zeta \in [0, 1], \\
\varpi(-\zeta, t), & \text{for all } \zeta \in [-1, 0]. 
\end{cases}
\]
(86)

By (64), (65), and (67), we have \( v_t(\zeta, t) = \zeta \frac{d}{dt} \frac{L_t(X(t), \dot{X}(t))}{L_t(X(t))} \frac{d}{dt} + 1 \phi(\zeta, t) \) for all \( \zeta \in [-1, 1] \), and \( v(1, t) = 0 \). Let \( \Gamma(t) = \|v(t)\|_{c, \infty} \) denote the following norm:
\[
\Gamma(t) = \sup_{\zeta \in [-1, 1]} e^{\zeta(1+\zeta)} v(\zeta, t)
\]
(87)
\[
= \lim_{n \to \infty} \left( \int_{-1}^{1} e^{2nc(1+\zeta)} e^{2nc(1+\zeta)} d\zeta \right)^{\frac{1}{2n}}
\]
where \( c > 0 \) and \( n \) is a positive integer. The derivative of \( \Gamma(t) \) is given by

\[
\dot{\Gamma}(t) = \lim_{n \to \infty} \frac{d}{dt} \left( \int_{-1}^{1} e^{2nc(1+\xi)} v(\xi,t)^{2n} d\xi \right)^{\frac{1}{2n}}
\]

\[
= \lim_{n \to \infty} \frac{1}{2n} \left( \int_{-1}^{1} e^{2nc(1+\xi)} v(\xi,t)^{2n} d\xi \right)^{\frac{1}{2n} - 1} \times \left( \int_{-1}^{1} 2ne^{2nc(1+\xi)} v(\xi,t)^{2n-1} v_{t}(\xi,t) d\xi \right)
\]

\[
= \lim_{n \to \infty} \frac{1}{2n} \left( \int_{-1}^{1} e^{2nc(1+\xi)} v(\xi,t)^{2n} d\xi \right)^{\frac{1}{2n} - 1} \times \left( \int_{-1}^{1} 2ne^{2nc(1+\xi)} v(\xi,t)^{2n-1} \frac{\xi(L(t,X(t)) + \nabla L(t,X(t)) f(X(t),\xi(t))) + 1}{L(t,X(t))} v_{\xi}(\xi,t) d\xi \right).
\]

With integration by parts, we have

\[
\dot{\Gamma}(t) = -\lim_{n \to \infty} \frac{1}{2n} \left( \int_{-1}^{1} e^{2nc(1+\xi)} v(\xi,t)^{2n} d\xi \right)^{\frac{1}{2n} - 1} \times \left( \int_{-1}^{1} e^{2nc(1+\xi)} \Pi(\xi,t) v(\xi,t)^{2n} d\xi \right)
\]

where

\[
\Pi(\xi,t) = \frac{2nc(\xi(L(t,X(t)) + \nabla L(t,X(t)) f(X(t),\xi(t))) + 1)}{L(t,X(t))} + \frac{L(t,X(t)) + \nabla L(t,X(t)) f(X(t),\xi(t))}{L(t,X(t))}.
\]

Because this is a linear function of \( \xi \), it follows that it has a minimum of either \( \xi = -1 \) or \( \xi = 1 \). In view of \( -1 < L(t,X(t)) + \nabla L(t,X(t)) f(X(t),\xi(t)) < 1 \) and choosing \( c > \frac{3}{2(1-\pi_1^2)} \), we have

\[
\Pi(\xi,t) \geq \frac{2nc(-|L(t,X(t)) + \nabla L(t,X(t)) f(X(t),\xi(t))| + 1)}{L(t,X(t))} + \frac{L(t,X(t)) + \nabla L(t,X(t)) f(X(t),\xi(t))}{L(t,X(t))}
\]

\[
\geq 2\pi_0^n \left[ c(1 - \pi_1^2) - \frac{1}{2n} \right]
\]

(90)

By (89) and (90), we have

\[
\dot{\Gamma}(t) \leq -\pi_0^n \Gamma(t).
\]

(91)

Take a Lyapunov function as

\[
V(t) = S(Z(t)) + \frac{2}{2n} \int_{0}^{\Gamma(t)} \frac{\alpha_{13}(r)}{r} dr.
\]

(92)

Noting that \( |\omega(0,0)| \leq \sup_{\xi \in [0,1]} |\omega(\xi,t)| \leq \|v(t)\|_{c,\infty} \) and by (84), (85), (91), and (92), the derivative of \( V(t) \) along the solutions of system (63)–(67) satisfies

\[
\dot{V}(t) \leq -\alpha_{12}(|Z(t)|) - \alpha_{13}(\|v(t)\|_{c,\infty}).
\]

(93)

By (84), there exists a class \( \mathcal{K} \) function \( \Upsilon_1 \) such that \( \dot{V}(t) \leq -\Upsilon_1(V(t)) \). Using the comparison principle, there exists a class \( \mathcal{K} \mathcal{L} \) function \( \beta_1 \) such that \( V(t) \leq \beta_1(V(0),t) \) for all \( t \geq 0 \). With additional routine class \( \mathcal{K} \) calculation, using the definition (92), one can show that there exists a class \( \mathcal{K} \mathcal{L} \) function \( \beta_2 \) such that

\[
|Z(t)| + \|v(t)\|_{c,\infty} \leq \beta_2(|Z(0)| + \|v(0)\|_{c,\infty},t).
\]

(94)
It is easy to see that
\[ 0.5(\|\omega(t)\|_{\infty} + \|\varpi(t)\|_{\infty}) \leq \|v(t)\|_{c,\infty} \leq e^{2c}(\|\omega(t)\|_{\infty} + \|\varpi(t)\|_{\infty}), \] (95)
thus, we have
\[ |X(t)| + |\xi(t)| + \|\omega(t)\|_{\infty} + \|\varpi(t)\|_{\infty} \]
\[ \leq \sqrt{2}|Z(t)| + 2\|v(t)\|_{c,\infty} \]
\[ \leq 2\beta_2(\|Z(0)\| + \|v(0)\|_{c,\infty}, t) \]
\[ \leq 2\beta_2(e^{2c}(\|Z(0)\| + \|\omega(0)\|_{\infty} + \|\varpi(0)\|_{\infty}), t) \]
\[ \leq 2\beta_2(e^{2c}(\|X(0)\| + |\xi(0)| + \|\omega(0)\|_{\infty} + \|\varpi(0)\|_{\infty}), t). \] (96)

Let \( \beta(s,t) = 2\beta_2(e^{2c}s,t) \); with \( c > \frac{3}{2(1-\pi^2)} \), we have \( \Omega_1(t) \leq \beta(\Omega_1(0), t) \) for all \( t \geq 0 \). \( \square \)

**Lemma 5 (Bound on extended state predictor in terms of wave PDE state)**
There exist class \( C_\infty \) functions \( \gamma_1, \gamma_2 \) such that
\[ \sup_{0 \leq \xi \leq 1} |q(L(t,X(t))\xi,t)| \leq \gamma_1(|Z(t)| + \|\xi(t)\|_{\infty}), \] (97)
\[ \sup_{0 \leq \xi \leq 1} |r(L(t,X(t))\xi,t)| \leq \gamma_2(|Z(t)| + \|\eta(t)\|_{\infty}). \] (98)

**Proof**
Because \( q(\xi, t) \) satisfies the initial problem (68) and (69), using (46), we have
\[ \frac{\partial R_1(q(\xi, t))}{\partial \xi} \phi(q(\xi, t), \xi, t) \leq R_1(q(\xi, t)) + \alpha_5(|\xi(\xi, t)|). \] (99)

With (68), we have
\[ \frac{dR_1(q(\xi, t))}{d\xi} \leq R_1(q(\xi, t)) + \alpha_5(|\xi(\xi, t)|), \] (100)

it yields
\[ \frac{dR_1(q(L(t,X(t))\xi,t))}{d\xi} \leq L(t,X(t))R_1(q(L(t,X(t))\xi,t)) + L(t,X(t))\alpha_5(|\xi(L(t,X(t))\xi,t)|). \] (101)

It follows for all \( \zeta \in [0, 1] \) that
\[ R_1(q(L(t,X(t))\zeta,t)) \leq e^{L(t,X(t))\zeta} R_1(q(0,t)) \]
\[ + L(t,X(t)) \int_0^\zeta e^{L(t,X(t))(\zeta-y)} \alpha_5(|\xi(L(t,X(t))y,t)|)dy \]
\[ = e^{L(t,X(t))\zeta} R_1(Z(t)) \]
\[ + L(t,X(t)) \int_0^\zeta e^{L(t,X(t))(\zeta-y)} \alpha_5(|\xi(L(t,X(t))y,t)|)dy \]
\[ \leq e^{L(t,X(t))\zeta} R_1(Z(t)) + \left(e^{L(t,X(t))\zeta} - 1\right) \sup_{0 \leq \gamma \leq \zeta} \alpha_5(|\xi(L(t,X(t))\gamma,t)|). \] (102)

Using (44) and (51), we have that for all \( \zeta \in [0, 1] \)
\[ |q(L(t,X(t))\zeta,t)| \leq \alpha_1^{-1}(e^m(\alpha_2(|Z(t)|) + (e^m - 1)\alpha_5(|\xi(t)|_{\infty})) \] (103)
we obtain (97) with \( \gamma_1(s) = \alpha_1^{-1}(e^m(\alpha_2(s)) + (e^m - 1)\alpha_5(s)) \), using the same arguments as those previously discussed, we have \( \gamma_2(s) = \alpha_2^{-1}(e^m(\alpha_4(s)) + (e^m - 1)\alpha_6(s)) \). \( \square \)
Lemma 6 (Bound on backward predictor in terms of target PDE state)
Let the following system
\[
I_x(t) = -\varphi(I(x,t), \sigma(L^{-1}(t, X(t))x,t) + \mu(I(x,t)))
\]
\[
I(0, t) = Z(t)
\]
satisfy Assumption 2, and \(L(t, X(t))\) satisfy Assumption 4. Then there exists a class \(K_\infty\) function \(\gamma_3\) such that
\[
\|I(t)\|_\infty \leq \gamma_3(|Z(t)| + \|\sigma(t)\|_\infty).
\]

Proof
In Appendix.

Lemma 7 (Bound on forward predictor in terms of target PDE state)
Under Assumption 3 and \(L(t, X(t))\) satisfies Assumption 4, then there exists a class \(K_\infty\) function \(\gamma_4\) such that
\[
\|\lambda(t)\|_\infty \leq \gamma_4(|Z(t)| + \|\omega(t)\|_\infty).
\]

Proof
In Appendix.

Lemma 8 (Wave PDE state bounded by target PDE state)
Consider the system
\[
\lambda_x(x,t) = \varphi(\lambda(x,t), \omega(L^{-1}(t, X(t))x,t) + \mu(\lambda(x,t)))
\]
\[
I_x(t) = -\varphi(I(x,t), \sigma(L^{-1}(t, X(t))x,t) + \mu(I(x,t)))
\]
\[
I(0, t) = Z(t)
\]
\[
\lambda(0, t) = Z(t)
\]
and the output maps are (74) and (75). Suppose that Assumption 3 and Assumption 4 hold and the subsystem (109) satisfies Assumption 2, then there exists a class \(K_\infty\) function \(\gamma_5\) such that
\[
|Z(t)| + \|\xi(t)\|_\infty + \|\eta(t)\|_\infty \leq \gamma_5(|Z(t)| + \|\omega(t)\|_\infty + \|\sigma(t)\|_\infty).
\]

Proof
Under Assumption 3, we have that \(\mu\) is locally Lipschitz with \(\mu(0, 0) = 0\), and there exists a class \(K_\infty\) function \(\alpha^*\) such that
\[
|\mu(Z)| \leq \alpha^*(|Z|).
\]

With (74) and (113), we have
\[
\sup_{0 \leq x \leq L(t, X(t))} |\xi(x,t)| \leq \sup_{0 \leq x \leq L(t, X(t))} |\omega(L^{-1}(t, X(t))x,t)| + \alpha^* \left( \sup_{0 \leq x \leq L(t, X(t))} |\lambda(x,t)| \right)
\]

it yields
\[
\|\xi(t)\|_\infty \leq \|\omega(t)\|_\infty + \alpha^*(\|\lambda(t)\|_\infty).
\]

Analogously, by (75) and (113), we have
\[
\|\eta(t)\|_\infty \leq \|\sigma(t)\|_\infty + \alpha^*(\|I(t)\|_\infty).
\]
Using (106) and (107), we have
\[
|Z(t)| + \|\xi(t)\|_\infty + \|\eta(t)\|_\infty \leq |Z(t)| + \|\omega(t)\|_\infty + \alpha^*(\gamma_4(|Z(t)| + \|\omega(t)\|_\infty))
+ \|\sigma(t)\|_\infty + \alpha^*(\gamma_5(|Z(t)| + \|\sigma(t)\|_\infty)).
\] (117)

Let \(\gamma_5(s) = s + \alpha^*(\gamma_3(s)) + \alpha^*(\gamma_4(s))\), we have (112).

\[\square\]

**Lemma 9 (Target PDE State bounded by wave PDE state)**
Consider the system
\[
q(x,t) = \varphi(q(x,t),\xi(x,t))
\] (118)
\[
r_x(x,t) = -\varphi(r(x,t),\eta(x,t))
\] (119)
\[q(0,t) = Z(t)
\] (120)
\[r(0,t) = Z(t)
\] (121)
and the output maps are (59) and (60). Suppose that Assumption 3 and Assumption 4 hold and the subsystem (119) satisfies Assumption 2, then there exists a class \(\mathcal{K}_\infty\) function \(\gamma_6\) such that
\[
|Z(t)| + \|\omega(t)\|_\infty + \|\sigma(t)\|_\infty \leq \gamma_6(|Z(t)| + \|\xi(t)\|_\infty + \|\eta(t)\|_\infty).
\] (122)

**Proof**
With (59) and (113), we have
\[
\sup_{0 \leq \xi \leq 1} |\omega(\xi,t)| \leq \sup_{0 \leq \xi \leq 1} \xi(L(t(X(t))\xi,t)) + \alpha^* \left( \sup_{0 \leq \xi \leq 1} |q(L(t(X(t))\xi,t)| \right).
\] (123)

With (97), it yields
\[
\|\omega(t)\|_\infty \leq \|\xi(t)\|_\infty + \alpha^*(\gamma_1(|Z(t)| + \|\xi(t)\|_\infty)).
\] (124)

Analogously, with (60), (98), and (113), we have
\[
\|\sigma(t)\|_\infty \leq \|\eta(t)\|_\infty + \alpha^*(\gamma_2(|Z(t)| + \|\eta(t)\|_\infty)).
\] (125)

By (124) and (125), we have
\[
|Z(t)| + \|\omega(t)\|_\infty + \|\sigma(t)\|_\infty \leq \gamma_6(|Z(t)| + \|\xi(t)\|_\infty + \|\eta(t)\|_\infty)
+ \alpha^*(\gamma_1(|Z(t)| + \|\xi(t)\|_\infty))
+ \alpha^*(\gamma_2(|Z(t)| + \|\eta(t)\|_\infty)).
\] (126)

Let \(\gamma_6(s) = s + \alpha^*(\gamma_1(s)) + \alpha^*(\gamma_2(s))\), we have (122).

\[\square\]

**Proof of Theorem 1**
Using (5)–(8), it yields
\[
\|\xi(t)\|_\infty + \|\eta(t)\|_\infty \leq 2(\|u(t)\|_\infty + \|u_x(t)\|_\infty),
\] (127)
\[
\|u(t)\|_\infty + \|u_x(t)\|_\infty \leq \|\xi(t)\|_\infty + \|\eta(t)\|_\infty.
\] (128)

Using Lemma 4 and Lemma 8, we have
\[
|X(0)| + |u(0,0)| + |u_t(0)|_\infty + |u_x(0)|_\infty
\leq |X(0)| + |u(0,0)| + \|\xi(0)\|_\infty + \|\eta(0)\|_\infty
\leq \sqrt{2}|Z(0)| + \|\xi(0)\|_\infty + \|\eta(0)\|_\infty
\leq \sqrt{2}\gamma_5(|Z(0)| + \|\omega(0)\|_\infty + \|\sigma(0)\|_\infty)
\leq \sqrt{2}\gamma_5(|X(0)| + |u(0,0)| + \|\omega(0)\|_\infty + \|\sigma(0)\|_\infty)
\leq \sqrt{2}\gamma_5(\gamma_d(d))
\] (129)
where $\mathcal{P}_d$ is given by (79) and $Z(0) = [X(0)^T, u(0, 0)]^T$. Owing to

$$u(x, t) = u(0, t) + \int_0^x u_y(y, t)dy,$$

and with (51), we have

$$\|u(t)\|_\infty \leq \|u(0, t)\| + m\|u_x(t)\|_\infty$$

(130)

for $x \in [0, L(t, X(t))]$. By (129) and (130), we have

$$\begin{align*}
|X(0)| + \|u(0)\|_\infty + \|u_t(0)\|_\infty + \|u_x(0)\|_\infty \\
\leq (1 + m)(|X(0)| + \|u(0)\| + \|u_t(0)\| + \|u_x(0)\|_\infty)
\end{align*}$$

(131)

Thus, there exists a monotonically increasing function $\psi_{ROA}(d) = \sqrt{2}(1 + m)\gamma_5(\mathcal{P}_d(d))$. Let the initial condition of system (10)–(14) be in the set

$$B_0(d) : \Omega(0) \leq \psi_{ROA}(d),$$

(132)

where $\Omega(t) = |X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty$. It implies that the initial condition of system (10)–(14) satisfies (28). Using Lemma 4, Lemma 8, and Lemma 9, we have

$$\begin{align*}
|X(t)| + \|u(0, t)\|_\infty + \|\xi(t)\|_\infty + \|\eta(t)\|_\infty \\
\leq \sqrt{2}|Z(t)| + \|\xi(t)\|_\infty + \|\eta(t)\|_\infty \\
\leq \sqrt{2}\gamma_5(|Z(0)| + \|\omega(0)\|_\infty + \|\sigma(t)\|_\infty1) \\
\leq \sqrt{2}\gamma_5(|X(0)| + \|u(0, t)\| + \|\omega(0)\|_\infty + \|\sigma(t)\|_\infty1) \\
\leq \sqrt{2}\gamma_5(\beta(|X(0)| + \|u(0, 0)\|_\infty + \|\omega(0)\|_\infty + \|\sigma(0)\|_\infty1, t) \\
\leq \sqrt{2}\gamma_5(\beta(\sqrt{2}|Z(0)| + \|\omega(0)\|_\infty1 + \|\sigma(0)\|_\infty1, t) \\
\leq \sqrt{2}\gamma_5(\beta(\sqrt{2}\gamma_6(|Z(0)| + \|\xi(0)\| + \|\eta(0)\|_\infty), t)) \\
\leq \sqrt{2}\gamma_5(\beta(\sqrt{2}\gamma_6(2(|X(0)| + \|u(0, 0)\|_\infty + \|u_t(0)\|_\infty + \|u_x(0)\|_\infty)), t)) \\
\leq \sqrt{2}\gamma_5(\beta(\sqrt{2}\gamma_6(2s, t), t)).
\end{align*}$$

By (128), (130), and (133), we have

$$\begin{align*}
|X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty \\
\leq \sqrt{2}(1 + m)\gamma_5(\beta(\sqrt{2}\gamma_6(2(|X(0)| + \|u(0, 0)\|_\infty + \|u_t(0)\|_\infty + \|u_x(0)\|_\infty)), t)).
\end{align*}$$

(134)

Thus, there exists a class $\mathcal{KL}$ function $\tilde{\beta} = \sqrt{2}(1 + m)\gamma_5(\beta(\sqrt{2}\gamma_6(2s, t))$, such that

$$\Omega(t) \leq \tilde{\beta}(\Omega(0), t)$$

(135)

for all $t \geq 0$, with $\Omega(t) = |X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty$.

For any initial condition $u_x(x, 0) \in C^1[0, L(0, X(0))]$, $u_t(x, 0) \in C^1[0, L(0, X(0))]$ in the set $B_0(d)$ and in view of (5), we have $\xi(x, 0) \in C^1[0, L(0, X(0))]$, so $\xi(\xi, 0) \in C^1[0, 1]$. Using the ODEs (68) and (69) and the Lipschitzness of $\varphi$, we have the existence and uniqueness of $q(\xi, 0) \in C^1[0, 1]$. By (59) and the compatibility condition, we know $\omega(\xi, 0) \in C^1[0, 1]$.

With (64) and (67), we have

$$\omega(\xi, t) = \begin{cases} \\
\omega(0)(L(t, X(t))\xi(t) + t), & 0 \leq L(t, X(t))\xi + t < 1 \\
0, & L(t, X(t))\xi + t \geq 1
\end{cases}$$

(136)
where the initial condition $\omega_0(\zeta)$ is given by (59) with $t = 0$. Using (63) and the Lipschitz condition $\varphi$ and $\mu$, we have the existence and uniqueness of $(X(t), u(0, t)) \in C^1[0, \infty)$. The existence of $\omega(\zeta, t) \in C^1([0, 1] \times [0, \infty))$ comes from $\omega_0(\zeta) \in C^1[0, 1]$, and the compatibility condition and (136), and the uniqueness follows from the uniqueness of the solution to (64) and (67).

Using (65) and (66), we have

$$
\sigma(\zeta, t) = \begin{cases} 
\omega_0(L(t, X(t))\zeta - t), & 0 \leq L^{-1}(t, X(t))t < \zeta \\
\omega_0(t - L(t, X(t))\zeta), & 0 \leq t - L(t, X(t))\zeta < 1 \\
0, & t - L(t, X(t))\zeta > 1.
\end{cases}
\tag{137}
$$

With the similar arguments and using (6), the ODEs (71) and (72), we have the existence and uniqueness of

$$
\pi_0 = \sup_{\theta \geq L^{-1}(0)} \frac{1}{L(\theta)} > 0,
\tag{140}
$$

and

$$
\pi_1 = \sup_{\theta \geq L^{-1}(0)} \frac{|\dot{L}(\theta)|}{L(\theta)} < 1.
\tag{141}
$$

In this section, we prove the following result:

**Theorem 2**

Consider system (1)–(4) together with the control law (32)–(34). Under the condition (138), Assumptions 1–3 and Assumption 5, for any initial condition $u_x(\cdot, 0) \in C^1[0, L(0)], u_t(\cdot, 0) \in C^1[0, L(0)]$ that is compatible with the feedback law (32), the closed-loop system has a unique solution:

$$
X(t) \in C^1([0, \infty), R^n)
\tag{142}
$$

and there exists a class $\mathcal{KL}$ function $\overline{\beta}$ such that

$$
|X(t)| + \|u(t)\|_{\infty} + \|u_t(t)\|_{\infty} + \|u_x(t)\|_{\infty} \leq \overline{\beta}(X(0) + \|u(0)\|_{\infty} + \|u_t(0)\|_{\infty} + \|u_x(0)\|_{\infty}, t)
\tag{144}
$$

for all $t \geq 0$. 

---

5. GLOBAL STABILITY FOR TIME-DEPENDENT MOVING BOUNDARY

Throughout this section, we consider time-dependent prediction horizon $L(t)$, simplify the feasibility condition (28) as

$$
-1 < \dot{L}(t) < 1,
\tag{138}
$$

for all $t \geq 0$, and make the following assumption:

**Assumption 5 (Bound on the domain length)**

The prediction horizon $L(t)$ is bounded and positive, that is, there exists a positive constant $m_1$ such that

$$
0 < L(t) \leq m_1
\tag{139}
$$

for all $t \geq 0$. Denote

$$
\pi_0 = \sup_{\theta \geq L^{-1}(0)} \frac{1}{L(\theta)} > 0,
\tag{140}
$$

and

$$
\pi_1 = \sup_{\theta \geq L^{-1}(0)} \frac{|\dot{L}(\theta)|}{L(\theta)} < 1.
\tag{141}
$$

In this section, we prove the following result:

**Theorem 2**

Consider system (1)–(4) together with the control law (32)–(34). Under the condition (138), Assumptions 1–3 and Assumption 5, for any initial condition $u_x(\cdot, 0) \in C^1[0, L(0)], u_t(\cdot, 0) \in C^1[0, L(0)]$ that is compatible with the feedback law (32), the closed-loop system has a unique solution:

$$
X(t) \in C^1([0, \infty), R^n)
\tag{142}
$$

and there exists a class $\mathcal{KL}$ function $\overline{\beta}$ such that

$$
|X(t)| + \|u(t)\|_{\infty} + \|u_t(t)\|_{\infty} + \|u_x(t)\|_{\infty} \leq \overline{\beta}(X(0) + \|u(0)\|_{\infty} + \|u_t(0)\|_{\infty} + \|u_x(0)\|_{\infty}, t)
\tag{144}
$$

for all $t \geq 0$. 

---

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We prove the theorem by using a series of lemmas.

**Lemma 10 (Backstepping transformation)**
The backstepping transformations of $\tilde{\zeta}$ and $\tilde{\eta}$ are defined as

$$
\tilde{\omega}(\tilde{\zeta}, t) = \tilde{\zeta}(L(t)\tilde{\zeta}, t) - \mu(\tilde{q}(L(t)\tilde{\zeta}, t)), \quad \text{for all } \tilde{\zeta} \in [0, 1],
$$

$$
\tilde{\sigma}(\tilde{\zeta}, t) = \tilde{\eta}(L(t)\tilde{\zeta}, t) - \mu(\tilde{r}(L(t)\tilde{\zeta}, t)), \quad \text{for all } \tilde{\zeta} \in [0, 1],
$$

where

$$
\tilde{q}(\tilde{\zeta}, t) = Z(t) + \int_0^{\tilde{\zeta}} \varphi(\tilde{q}(y, t), \tilde{\zeta}(y, t))dy, \quad \text{for all } \tilde{\zeta} \in [0, 1],
$$

$$
\tilde{r}(\tilde{\zeta}, t) = Z(t) - \int_0^{\tilde{\zeta}} \varphi(\tilde{r}(y, t), \tilde{\eta}(y, t))dy, \quad \text{for all } \tilde{\zeta} \in [0, 1],
$$

and $\mu$ is defined in (31), and the control law (32)–(34) transform system (10)–(14) to the target system given by

$$
\dot{Z}(t) = \varphi(Z(t), \mu(Z(t)) + \tilde{\omega}(0, t))
$$

$$
\tilde{\omega}_{\tau}(\tilde{\zeta}, t) = \frac{1 + \tilde{\zeta}\hat{L}(t)}{L(t)} \tilde{\omega}_{\tilde{\xi}}(\tilde{\zeta}, t)
$$

$$
\tilde{\sigma}_{\tau}(\tilde{\zeta}, t) = -\frac{1 - \tilde{\zeta}\hat{L}(t)}{L(t)} \tilde{\sigma}_{\tilde{\xi}}(\tilde{\zeta}, t)
$$

$$
\tilde{\sigma}(0, t) = \tilde{\omega}(0, t)
$$

$$
\tilde{\omega}(1, t) = 0.
$$

**Proof**
In system (10)–(14), let $\tilde{\zeta}$ and $\tilde{\eta}$ be instead $\zeta$ and $\eta$, respectively. Using the arguments as in Lemma 1, one can prove it.

**Lemma 11 (Inverse backstepping transformation)**
The inverse backstepping transformations of $\tilde{\omega}$ and $\tilde{\sigma}$ are defined as

$$
\tilde{\zeta}(x, t) = \tilde{\omega}(L^{-1}(t)x, t) + \mu(\hat{\lambda}(x, t)), \quad \text{for all } x \in [0, L(t)],
$$

$$
\tilde{\eta}(x, t) = \tilde{\sigma}(L^{-1}(t)x, t) + \mu(\hat{i}(x, t)), \quad \text{for all } x \in [0, L(t)],
$$

where

$$
\hat{\lambda}(x, t) = Z(t) + \int_x^0 \varphi(\hat{\lambda}(y, t), \tilde{\omega}(L^{-1}(t)y, t) + \mu(\hat{\lambda}(y, t)))dy, \quad \text{for all } x \in [0, L(t)],
$$

$$
\hat{i}(x, t) = Z(t) - \int_0^x \varphi(\hat{i}(y, t), \tilde{\sigma}(L^{-1}(t)y, t) + \mu(\hat{i}(y, t)))dy, \quad \text{for all } x \in [0, L(t)],
$$

and $\mu$ is defined in (31), and the control law (32)–(34) transform the target system (149)–(153) to system (10)–(14).

**Proof**
Using the arguments as in Lemma 2.
Let system (149)–(153), under the condition (138) and Assumptions 3 and 5, there exists a class \( \mathcal{K}_L \) function \( \underline{\beta} \) such that the following holds:

\[
|Z(t)| + \|\overline{\omega}(t)\|_{\infty} + \|\overline{\varphi}(t)\|_{\infty} \leq \underline{\beta}(\|Z(0)\| + \|\overline{\omega}(0)\|_{\infty} + \|\overline{\varphi}(0)\|_{\infty}, t),
\]

for all \( t \geq 0 \).

**Proof**

Under Assumption 3, from Lemma 3, there exist a smooth function \( S(Z) : \mathbb{R}^{n+1} \to \mathbb{R}^+ \) and class \( \mathcal{K}_\infty \) functions \( \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13} \) such that (84) and (85) hold. The new variable \( \varpi(\zeta, t), \zeta \in [-1, 1] \) is defined as

\[
\varpi(\zeta, t) = \begin{cases} 
\overline{\omega}(\zeta, t), & \text{for all } \zeta \in [0, 1], \\
\overline{\varphi}(\zeta, t), & \text{for all } \zeta \in [-1, 0]. 
\end{cases}
\]

By (150), (151), and (153), we have \( v_i(\zeta, t) = \frac{\zeta L(t) + 1}{L(t)} v_i(\zeta, t) \) for all \( \zeta \in [-1, 1] \) and \( \varpi(1, t) = 0 \). Let \( \Gamma(t) = \|\varpi(t)\|_{c, \infty} \) denote the following norm:

\[
\|\varpi(t)\|_{c, \infty} = \sup_{\zeta \in [-1, 1]} e^{(1+\zeta)} |v(\zeta, t)|
\]

\[
= \lim_{n \to \infty} \left( \int_{-1}^{1} e^{2nc(1+\zeta)} v(\zeta, t)^{2n} d\zeta \right)^{\frac{1}{2n}}
\]

where \( c > 0 \) and \( n \) is a positive integer. Under the condition (138) and Assumption 5, choosing \( c > \frac{3}{2(1-\pi_1)} \), with the similar calculations as in Lemma 4, we have that

\[
\dot{\Gamma}(t) = \frac{d}{dt} \|\varpi(t)\|_{c, \infty} \leq -\pi_0 \|\varpi(t)\|_{c, \infty}.
\]

Take a Lyapunov function as

\[
V(t) = S(Z(t)) + \frac{2}{\pi_0} \int_0^{\Gamma(t)} \frac{\alpha_{13}(r)}{r} dr.
\]

Noting that \( |\overline{\omega}(0, t)| \leq \sup_{\zeta \in [-1, 1]} |\overline{\omega}(\zeta, t)| \leq \|\varpi(t)\|_{c, \infty} \) and by (84), (85), (161), and (162), the derivative \( V(t) \) along the solutions of system (149)–(153) satisfies

\[
\dot{V}(t) \leq -\alpha_{12}(|Z(t)|) - \alpha_{13}(|\varpi(t)|_{c, \infty}.
\]

By (84), there exists a class \( \mathcal{K} \) function \( \overline{\gamma}_1 \) such that \( \dot{V}(t) \leq -\overline{\gamma}_1(V(t)) \). Using the comparison principle, there exists a class \( \mathcal{K}_L \) function \( \underline{\beta}_2 \) such that \( V(t) \leq \underline{\beta}_2(V(0), t) \) for all \( t \geq 0 \). With additional routine class \( \mathcal{K} \) calculation, using the definition (162), one can show that there exists a class \( \mathcal{K}_L \) function \( \underline{\beta}_2 \) such that

\[
|Z(t)| + \|\overline{\omega}(t)\|_{\infty} + \|\overline{\varphi}(t)\|_{\infty} \leq \underline{\beta}_2(\|Z(0)\| + \|\overline{\omega}(0)\|_{\infty} + \|\overline{\varphi}(0)\|_{\infty}, t).
\]

Using the arguments as in Lemma 4, we have

\[
|Z(t)| + \|\overline{\omega}(t)\|_{\infty} + \|\overline{\varphi}(t)\|_{\infty} \leq 2\underline{\beta}_2(e^{2c}|Z(0)| + \|\overline{\omega}(0)\|_{\infty} + \|\overline{\varphi}(0)\|_{\infty}, t).
\]

Let \( \underline{\beta}(s, t) = 2\underline{\beta}_2(e^{2c}s, t) \), we have (158).

**Lemma 13 (Bound on extended state predictor in terms of wave PDE state)**

Let system

\[
\dot{\overline{\varphi}}(\zeta, t) = \varphi \left( \dot{\overline{\varphi}}(\zeta, t), \overline{\varphi}(\zeta, t) \right)
\]
\[ \tilde{q}(0, t) = Z(t) \]  
(167)

satisfy strongly forward complete, and system
\[ \tilde{r}_x(\tilde{\xi}, t) = -\varphi(\tilde{r}(\tilde{\xi}, t), \tilde{\eta}(\tilde{\xi}, t)) \]  
(168)

\[ \tilde{r}(0, t) = Z(t) \]  
(169)

satisfy strongly backward complete, and \( L(t) \) satisfy Assumption 5; then there exist class \( K_\infty \) functions \( \gamma_1, \gamma_2 \) such that
\[
\sup_{0 \leq \xi \leq 1} |\tilde{q}(L(t)\tilde{\xi}, t)| \leq \gamma_1 \left( |Z(t)| + \|\tilde{\xi}(t)\|_\infty \right),
\]  
(170)

\[
\sup_{0 \leq \xi \leq 1} |\tilde{r}(L(t)\tilde{\xi}, t)| \leq \gamma_2 \left( |Z(t)| + \|\tilde{\eta}(t)\|_\infty \right).
\]  
(171)

**Proof**
Using the arguments as in Lemma 5.

**Lemma 14 (Bound on backward predictor in terms of target PDE state)**
Let the system
\[
\tilde{r}_x(x, t) = -\varphi(\tilde{r}(y, t), \tilde{\sigma}(L^{-1}(t)x, t) + \mu(\tilde{x}(x, t)))
\]  
(172)

\[ \tilde{r}(0, t) = Z(t) \]  
(173)

satisfy Assumption 2 and let \( L(t) \) satisfy Assumption 5. Then there exists a class \( K_\infty \) function \( \gamma_3 \) such that
\[
\|\tilde{r}(t)\|_\infty \leq \gamma_3(|Z(t)| + \|\tilde{\sigma}(t)\|_\infty_1).
\]  
(174)

**Proof**
Using the arguments as in Lemma 6.

**Lemma 15 (Wave PDE state bounded by target PDE state)**
Consider the system
\[
\tilde{\lambda}_x(x, t) = \varphi(\tilde{\lambda}(x, t), \tilde{\omega}(L^{-1}(t)x, t) + \mu(\tilde{\lambda}(x, t)))
\]  
(175)

\[
\tilde{r}_x(x, t) = -\varphi(\tilde{r}(y, t), \tilde{\omega}(L^{-1}(t)x, t) + \mu(x, t))
\]  
(176)

\[ \tilde{r}(0, t) = \tilde{\lambda}(0, t) = Z(t) \]  
(177)

and the output maps are (154) and (155). Suppose that Assumption 3 and Assumption 5 hold, and the subsystem (176) satisfies Assumption 2. Then there exists a class \( K_\infty \) function \( \gamma_5 \) such that
\[
|Z(t)| + \|\tilde{\xi}(t)\|_\infty + \|\tilde{\eta}(t)\|_\infty \leq \gamma_5(|Z(t)| + \|\tilde{\omega}(t)\|_\infty_1 + \|\tilde{\sigma}(t)\|_\infty_1).
\]  
(178)

**Proof**
Using the arguments as in Lemma 7.
Lemma 16 (Target PDE state bounded by wave PDE state)
Consider the system

\[ \tilde{q}_x(x,t) = \varphi \left( \tilde{q}(x,t), \tilde{\zeta}(x,t) \right) \]  

(179)

\[ \tilde{r}_x(x,t) = -\varphi \left( \tilde{r}(x,t), \tilde{\eta}(x,t) \right) \]  

(180)

\[ \tilde{q}(0,t) = \tilde{r}(0,t) = Z(t) \]  

(181)

and the output maps are (145) and (146). Suppose that Assumption 3 and Assumption 5 hold, and the subsystem (180) satisfies Assumption 2. Then there exists a class \( K_\infty \) function \( \nu_6 \) such that

\[ |Z(t)| + \| \tilde{\omega}(t) \|_{\infty} + \| \tilde{\sigma}(t) \|_{\infty} \leq \nu_6 \left( |Z(0)| + \| \tilde{\zeta}(0) \|_{\infty} + \| \tilde{\eta}(0) \|_{\infty} \right). \]  

(182)

**Proof**

Using the arguments as in Lemma 8.

**Proof of Theorem 2**

Using Lemma 12, Lemma 15, and Lemma 16, we have

\[ |Z(t)| + \| \tilde{\zeta}(t) \|_{\infty} + \| \tilde{\eta}(t) \|_{\infty} \leq \nu_5 \left( \beta \left( \nu_6 \left( |Z(0)| + \| \tilde{\zeta}(0) \|_{\infty} + \| \tilde{\eta}(0) \|_{\infty} \right), t \right) \right). \]  

(183)

Because \( Z(t) = [X^T(t), u(0, t)]^T \), then we have

\[ |X(t)| + |u(0, t)| + \| \tilde{\zeta}(t) \|_{\infty} + \| \tilde{\eta}(t) \|_{\infty} \leq \sqrt{2} \nu_5 \left( \beta \left( \nu_6 \left( |Z(0)| + \| \tilde{\zeta}(0) \|_{\infty} + \| \tilde{\eta}(0) \|_{\infty} \right), t \right) \right). \]  

(184)

Using (5)–(8), we have

\[ |X(t)| + |u(0, t)| + \| u_t(t) \|_{\infty} + \| u_x(t) \|_{\infty} \leq \sqrt{2} \nu_5 \left( \beta \left( \nu_6 \left( 2(|X(0)| + |u(0, 0)| + \| u_t(0) \|_{\infty} + \| u_x(0) \|_{\infty}) \right), t \right) \right). \]  

(185)

Owing to

\[ u(x, t) = u(0, t) + \int_0^x u_y(y, t) dy, \]

so we have

\[ \| u(t) \|_{\infty} \leq |u(0, t)| + m_1 \| u_x(t) \|_{\infty}, \]  

(186)

for \( x \in [0, L(t)]. \) By (185) and (186), we have

\[ |X(t)| + \| u(t) \|_{\infty} + \| u_t(t) \|_{\infty} + \| u_x(t) \|_{\infty} \leq \sqrt{2} (1 + m_1) \nu_5 \left( \beta \left( \nu_6 \left( 2(|X(0)| + |u(0)| + \| u_t(0) \|_{\infty} + \| u_x(0) \|_{\infty}) \right), t \right) \right). \]  

(187)

Let \( \beta = \sqrt{2} (1 + m_1) \nu_5 \left( \beta \left( \nu_6 \left( 2x \right), t \right) \right), \) we have (144).

For any initial condition \( u_x(x, 0) \in C^1[0, L(0)], u_t(x, 0) \in C^1[0, L(0)] \) and in view of (5), we have \( \tilde{\zeta}(x, 0) \in C^1[0, L(0)], \) so \( \tilde{\zeta}(\xi, 0) \in C^1[0, 1]. \) Using the ODEs (166) and (167), and the Lipschitzness of \( \varphi, \) we have the existence and uniqueness of \( \tilde{\zeta}(\xi, 0) \in C^1[0, 1]. \) By (145) and the compatibility condition, we know \( \tilde{\omega}(\tilde{\zeta}, 0) \in C^1[0, 1]. \)

With (150) and (153), we have

\[ \tilde{\omega}(\tilde{\zeta}, t) = \begin{cases} \tilde{\omega}_0(L(t)\tilde{\zeta} + t), & 0 \leq L(t)\tilde{\zeta} + t < 1 \\ 0, & L(t)\tilde{\zeta} + t \geq 1 \end{cases} \]  

(188)
where the initial condition $\widetilde{\omega}_0(\xi)$ is given by (145) with $t = 0$. Using (149) and the Lipschitz condition $\phi$ and $\mu$, we have the existence and uniqueness of $X(t), u(0, t) \in C^1[0, \infty)$. The existence of $\widetilde{\omega}(\xi, t) \in C^1([0, 1] \times [0, \infty))$ comes from $\widetilde{\omega}_0(\xi, t) \in C^1[0, 1]$, and the compatibility condition and (188), and the uniqueness follows from the uniqueness of the solution to (150) and (153).

Using (151) and (152), we have

$$
\widetilde{\omega}(\xi, t) = \begin{cases} 
\widetilde{\omega}_0(L(t) \xi - t), & 0 \leq L^{-1}(t) t < \xi \\
\widetilde{\omega}_0(t - L(t) \xi) - \overline{\omega}(\xi, t), & 0 \leq t - L(t) \xi < 1 \\
0, & t - L(t) \xi > 1.
\end{cases}
$$

(189)

With the similar arguments and using (6), the ODEs (168) and (169), we have the existence and uniqueness of $\widetilde{\omega}(\xi, t) \in C^1([0, 1] \times [0, \infty))$. From the inverse backstepping transformation (154) and (155), and $\lambda(x, t) = \overline{\gamma}(L(t) \xi, t), \overline{\gamma}(x, t) = \overline{\gamma}(L(t) \xi, t), \overline{\gamma}(x, t)$, with $x = L(t) \xi$, we have the existence and uniqueness of $\overline{\gamma}, \overline{\gamma} \in C^1([0, L(t)] \times [0, \infty))$. So by (7) and (8), there exists a unique solution $(u_1, u_2) \in C^1([0, L(t)] \times [0, \infty))$, and hence, there exists a unique solution (143).

6. STRICT-FEEDFORWARD NONLINEAR SYSTEMS

The strict-feedforward systems have an additional property that despite being nonlinear, they can be solved explicitly. The consequence of this is that the predictor state can be defined explicitly.

Consider the class of strict-feedforward systems

$$
\dot{X}_1(t) = X_2(t) + \psi_1(X_2(t), \ldots, X_n(t)) + \Lambda_1(X_2(t), X_3(t), \ldots, X_n(t))u(0, t)
$$

(190)

$$
\vdots
$$

$$
\dot{X}_{n-2}(t) = X_{n-1}(t) + \psi_{n-2}(X_{n-1}(t), X_n(t)) + \Lambda_{n-2}(X_{n-1}(t), X_n(t))u(0, t)
$$

(191)

$$
\dot{X}_{n-1}(t) = X_n(t) + \Lambda_{n-1}(X_n(t))u(0, t)
$$

(192)

$$
X_n(t) = u(0, t)
$$

(193)

$$
\dot{u}_x(x, t) = u_{xx}(x, t)
$$

(194)

$$
u_x(0, t) = 0
$$

(195)

$$
\dot{u}_x(L(t), X(t)), t) = U(t).
$$

(196)

The subsystem (190)–(194) can be rewritten in short as follows:

$$
\dot{X}_i(t) = X_{i+1}(t) + \psi_i(X_{i+1}(t)) + \Lambda_i(X_{i+1}(t))u(0, t)
$$

(198)

where $i = 1, 2, \ldots, n, X_j = [X_j, X_{j+1}, \ldots, X_n]^T, X_{n+1}(t) = u(0, t), \psi_i(X_1, 0, \ldots, 0) = 0, \Lambda_1 = 1, \Lambda_i(0) = 0, (\partial \psi_i(0)/\partial X_j) = 0$ for $i = 1, 2, \ldots, n - 1, j = i + 1, \ldots, n$.

The nominal control design for system (190)–(194) is given by the following recursive procedure [40, 41]. Let

$$
\theta_{n+1} = 0, \quad \phi_{n+1} = 0.
$$

(199)

For $i = n, n - 1, \ldots, 2, 1$, the designer needs to calculate

$$
\begin{align*}
h_i(X_i) &= X_i - \theta_{i+1}(X_{i+1}) \\
\Lambda_i &= \sum_{j=i+1}^{n} \phi_i X_j
\end{align*}
$$

(200)

$$
w_i(X_{i+1}) = \Lambda_i - \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}}{\partial X_j} \Lambda_j - \frac{\partial \phi_{i+1}}{\partial X_j}
$$

(201)
\[ \theta_i(X_i) = \theta_{i+1} - w_i h_i \]  

(202)

\[ \theta_i(X_i) = - \int_0^\infty \left[ \xi_i^{[j]}(\tau, X_i) + \psi_{j-1} \left( \xi_i^{[j]}(\tau, X_i) \right) + \Lambda_{j-1} \left( \xi_i^{[j]}(\tau, X_i) \right) q_i \left( \xi_i^{[j]}(\tau, X_i) \right) \right] d\tau, \]  

(203)

where the notation in the integrand of (203) refers to the solutions of the (sub)system(s)

\[ \frac{d}{dt} \xi_i^{[j]} = \xi_i^{[j+1]} + \psi_j \left( \xi_i^{[j]} \right) + \Lambda_j \left( \xi_i^{[j]} \right) q_i \left( \xi_i^{[j]} \right) \]  

(204)

for \( j = i, i + 1, \ldots, n \), at time \( \tau \), starting from the initial condition \( X_i \). The control law for \( u(0, t) = U(t) \) is given by

\[ U(t) = \varphi_1(X(t)). \]  

(205)

We consider local stability for time- and state-dependent moving boundary. Suppose that there exists a class \( \mathcal{K}_\infty \) function \( \vartheta_4 \), such that

\[ \sum_{i=1}^n \left( X_{i+1}(t) + \psi_i(X_{i+1}(t)) + \Lambda_i(X_{i+1}(t))u(0, t) \right)^2 \leq \vartheta_4 \left( \sum_{i=1}^n X_i^2(t) + |u(0, t)| \right) \]  

(206)

for all \( t \geq 0 \), and prediction horizon \( L(t, X(t)) \) satisfies (28) and Assumption 4. The predictor-based feedback law is obtained from (205) as

\[ U(t) = - \frac{1}{2} \eta(L(t, X(t)), t) - \frac{c_1}{2} \left( p_{n+1}(L(t, X(t)), t) - \varphi_1(\pi(L(t, X(t)), t)) \right) \]

\[ + \frac{1}{2} \frac{\partial \varphi_1(\pi(L(t, X(t)), t))}{\partial \pi} f(\pi(L(t, X(t)), t), p_{n+1}(L(t, X(t)), t)) \]  

(207)

where \( c_1 > 0 \) is arbitrary, and \( \pi = [p_1, p_2, \ldots, p_n]^T \in \mathbb{R}^n \) and \( p_{n+1} \in \mathbb{R} \), the predictors of \( X(t) \) and \( u(0, t) \) respectively, are given by

\[ p_i(L(t, X(t), t) = \int_0^{L(t, X(t))} [p_{i+1}(y, t) + \psi_i(p_{i+1}(y, t), \ldots, p_n(y, t)) \]

\[ + \Lambda_i(p_i(y, t), \ldots, p_n(y, t))u(y, t)] dy + X_i \]  

(208)

for all \( i = 1, 2, \ldots, n \); and

\[ p_{n+1}(L(t, X(t), t) = u(L(t, X(t), t) + \int_0^{L(t, X(t))} u_f(y, t) dy. \]  

(209)

The feasibility condition (28) is given by

\[ F_d : |L_1(t, X(t)) + \sum_{i=1}^n \nabla L_i(t, X(t))X_{i+1}(t) + \psi_i(X_{i+1}(t)) + \Lambda_i(X_{i+1}(t))u(0, t)| \leq d \]  

(210)

for \( d \in [0, 1] \), and with \( \nabla L(t, X(t)) = [\nabla L_1(t, X(t)), \nabla L_2(t, X(t)), \ldots, \nabla L_n(t, X(t))] \),

\[ \nabla L_i(t, X(t)) = \frac{\partial L_i(X(t))}{\partial X_i}, \] \( X(t) = [X_1(t) \cdots X_n(t)]^T \).

Next, we consider global stability for system (190)–(194) with time-dependent moving boundary

\[ u_{tt}(x, t) = u_{xx}(x, t) \]  

(211)

\[ u_x(0, t) = 0 \]  

(212)

\[ u_x(L(t), t) = U(t). \]  

(213)
Suppose that prediction horizon $L(t)$ satisfies (138) and Assumption 5. The control law compensating the wave dynamics is obtained from (205) as

$$U(t) = \frac{1}{2} \eta(L(t), t) - \frac{c_1}{2} (p_{n+1}(L(t), t) - \varphi_1(\pi(L(t)), t))) + \frac{1}{2} \frac{\partial \varphi_1(\pi(L(t)), t)}{\partial \pi} f(\pi(L(t), t), p_{n+1}(L(t), t))$$

(214)

where $c_1 > 0$ is arbitrary, and $\pi = [p_1, p_2, \cdots, p_n]^T \in \mathbb{R}^n$ and $p_{n+1} \in \mathbb{R}$; the predictors of $X(t)$ and $u(0, t)$ respectively, are given by

$$p_i(L(t), t) = \int_0^{L(t)} [p_{i+1}(y, t) + \psi_i(p_{i+1}(y, t), \cdots, p_n(y, t))
+n\lambda_i(p_{i+1}(y, t), \cdots, p_n(y, t))u(y, t)] dy + X_i(t)$$

(215)

for all $i = 1, 2, \cdots, n$, and

$$p_{n+1}(L(t), t) = u(L(t), t) + \int_0^{L(t)} u_t(y, t) dy.$$ 

(216)

7. EXAMPLE

Consider the cascade of a wave (string) equation and a second-order system given by

$$\dot{X}_1(t) = X_2(t) - X_2^2(t)u(0, t)$$

(217)

$$\dot{X}_2(t) = u(0, t)$$

(218)

$$u_{tt}(x, t) = u_{xx}(x, t)$$

(219)

$$u_x(0, t) = 0$$

(220)

$$u_x(L(t), \dot{X}(t), t) = U(t)$$

(221)

where $X_1, X_2 \in \mathbb{R}$ are the state vector, $U$ is the scalar input to the entire system, $u(x, t)$ is the state of the PDE dynamics of the actuator governed by a wave equation, and $L : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuously differentiable.

Noting that system (217) is in the strict-feedforward form, and hence, is forward and backward complete with respect to the input $u(0, t)$, that is, system 217 satisfies Assumption 1. The nominal control law (that is, the control law in the case where $u(0, t) = U(t)$) yields that the following system

$$\dot{X}_1(t) = X_2(t) - X_2^2(t)(U(t) + v(t))$$

(223)

$$\dot{X}_2(t) = U(t) + v(t)$$

(224)

with $v(t) \in \mathbb{R}$ is ISS and backward complete.

First, we consider local stability for time- and state-dependent moving boundary.

Because

$$|f(X(t), u(0, t)))|^2 = (X_2(t) - X_2^2(t)u(0, t))^2 + u^2(0, t)$$

$$= X_2^2(t) (1 - 2X_2(t)u(0, t) + X_2^2(t)u^2(0, t)) + u^2(0, t)$$

(225)

$$\leq X_2^2(t) \left(1 + X_2^2(t) \right) \left(1 + u^2(0, t) \right) + u^2(0, t)$$

there exists a class $K_{\infty}$ function $\vartheta_1(x) = |x|\sqrt{1 + (1 + |x|^2)^2}$ such that $|f(X(t), u(0, t))| \leq \vartheta_1(|X(t)| + |u(0, t)|)$. Assume now $L(t, X(t))$ as

$$L(t, X(t)) = \frac{1 + X_1^2(t) + X_2^2(t) + t}{2(1 + 2(X_1^2(t) + X_2^2(t) + t))},$$

we have

$$0 < L(t, X(t)) \leq \frac{1}{2},$$

$$|L_t(t, X(t))| \leq \frac{1}{2},$$

$$|\nabla L(t, X(t))| \leq |X(t)|,$$

for all $t \geq 0$.

The control law that compensates the wave dynamics is

$$U(t) = -\frac{1}{2} \eta(L(t, X(t)), t) - p_3(L(t, X(t)), t) - p_1(L(t, X(t)), t) - 2p_2(L(t, X(t)), t)$$

$$- \frac{1}{3} p_2(L(t, X(t)), t) - \frac{1}{2} p_2(L(t, X(t)), t) - p_3(L(t, X(t)), t)$$

where

$$p_1(L(t, X(t)), t) = \int_0^{L(t, X(t))} (L(t, X(t)) - x)u(x, t)dx + \int_0^{L(t, X(t))} (L(t, X(t)) - x)^2 u_t(x, t)dx$$

$$- \int_0^{L(t, X(t))} \left( X_2(t) + \int_0^X u(y, t)dy + \int_0^X (L(t, X(t)) - y)u_t(y, t)dy \right)^2$$

$$\times (u(x, t) + \int_0^x u_t(y, t)dy)dx + X_1(t) + X_2(t),$$

$$p_2(L(t, X(t)), t) = \int_0^{L(t, X(t))} u(x, t)dx + \int_0^{L(t, X(t))} (L(t, X(t)) - x)u_t(x, t)dx + X_2(t),$$

$$p_3(L(t, X(t)), t) = u(L(t, X(t)), t) + \int_0^{L(t, X(t))} u_t(x, t)dx.$$

The feasibility condition of the controller (230)–(233) is

$$|(1 + 2X_1(t)X_2(t) - 2X_1(t)X_2^2(t))u(0, t) + 2X_2(t)u(0, t)|$$

$$\leq 2d \left( 1 + 2 \left( X_1^2(t) + X_2^2(t) + t \right) \right)^2$$

with $0 \leq d < 1$.

In Figure 1, the responses of the states of system (217)–(221) for the case of the proposed control law (230)–(233) and the case of the uncompensated nominal control law (222) are shown. One can observe, in the former case, that the stabilization is achieved, whereas in the latter case, the states grow unbounded. In Figure 2, the responses of the actuator state and the proposed control law (230)–(233) are shown. One can observe that both the actuator state and the control law converge to zero. For any initial state that satisfies the feasibility condition (234), it has a similar simulative result.

Next, we consider global stability for system (217)–(218) with time-dependent moving boundary

$$u_{tt}(x, t) = u_{xx}(x, t)$$

Figure 1. The response of the states of system (217)–(221) with the control law (230)–(233) (solid line) and with the nominal control law (222) (dashdot line) for initial conditions as $X_1(0) = 1$, $X_2(0) = 1$ and $u_x(x,0) = 1$, $u_t(x,0) = 1$ for all $x \in [0, L(0, X(0))]$.

Figure 2. The response of the actuator state (left) of system (217)–(221) with the control law (230)–(233) and the control law (right) for initial conditions as $X_1(0) = 1$, $X_2(0) = 1$ and $u_x(x,0) = 1$, $u_t(x,0) = 1$ for all $x \in [0, L(0, X(0))]$.

$$u_x(0,t) = 0$$

$$u_x(L(t),t) = U(t).$$

Let $L(t)$ be given as

$$L(t) = \frac{1 + t}{2(1 + 2t)},$$

in which case we have $0 < L(t) \leq \frac{1}{2}$ and $-1 < \dot{L}(t) = -\frac{1}{2(1 + 2t)^2} < 1$, for all $t \geq 0$. The control law that compensates the wave dynamics is

$$U(t) = -\frac{1}{2} \eta(L(t),t) - p_3(L(t),t) - p_1(L(t),t) - 2p_2(L(t),t)$$

$$- \frac{1}{3} p_2^2(L(t),t) - \frac{1}{2} p_2(L(t),t) - p_3(L(t),t).$$
where

\[
p_1(L(t), t) = \int_0^{L(t)} (L(t) - x)u(x, t)\, dx + \int_0^{L(t)} (L(t) - x)^2u_t(x, t)\, dx
- \int_0^{L(t)} \left(X_2(t) + \int_0^x u(y, t)\, dy + \int_0^x (L(t) - y)u_t(y, t)\, dy\right)^2 \times \left(u(x, t) + \int_0^x u_t(y, t)\, dy\right)\, dx + X_1(t) + X_2(t),
\]

\[
p_2(L(t), t) = \int_0^{L(t)} u(x, t)\, dx + \int_0^{L(t)} (L(t) - x)u_t(x, t)\, dx + X_2(t),
\]

\[
p_3(L(t), t) = u(L(t), t) + \int_0^{L(t)} u_t(x, t)\, dx.
\]

In Figure 3, the responses of the states of system (217)–(218) and (235)–(237) with the control law (239)–(242) and the case of the uncompensated nominal control law (222) are shown. One can observe, in the former case, that the stabilization is achieved, whereas in the latter case, the states grow unbounded. In Figure 4, the responses of the actuator state and the proposed control law (239)–(242) are shown. One can observe that both the actuator state and the control law converge to zero. For any initial state \(x(0) \in \mathbb{R}^2, u_x(x, 0) \in \mathbb{R}, u_t(x, 0) \in \mathbb{R}\), it has a similar simulative result.

Figure 3. The response of the states of system (217)–(218) and (235)–(237) with the control law (239)–(242) (solid line) and with the nominal control law (222) (dashdot line) for initial conditions as \(X_1(0) = 1, X_2(0) = 2\) and \(u_x(x, 0) = 3, u_t(x, 0) = 2\) for all \(x \in [0, L(0)]\).

Figure 4. The response of the actuator state (left) of system (217)–(218) and (235)–(237) with the control law (239)–(242) (solid line) and with the control law (right) for initial conditions as \(X_1(0) = 1, X_2(0) = 2\) and \(u_x(x, 0) = 3, u_t(x, 0) = 2\) for all \(x \in [0, L(0)]\).
8. CONCLUSIONS

We introduce and solve stabilization problems for nonlinear systems under wave actuator dynamics with a time- and state-dependent moving boundary. The stability of the closed-loop system is proved using a Lyapunov functional, which is constructed by introducing two infinite-dimensional backstepping transformations of the actuator state. Predictor-based feedback law is designed explicitly. For moving boundaries that depend on the ODE’s state, we develop a local result where the initial condition is restricted in such a way that it is ensured that the rate of movement of the boundary is bounded by unity in closed-loop. For moving boundaries that depend only on time, global stability of the closed-loop system is ensured. In addition, for strict-feedforward systems under wave actuator dynamics with moving boundaries, the predictor-based feedback laws are obtained explicitly.

APPENDIX

Here, we give the proof of Lemma 2, Lemma 3, Lemma 6, and Lemma 7.

The proof of Lemma 2
Noting that \( \lambda(0,t) = Z(t) = \iota(0,t) \), setting \( x = 0 \) in (74), from (63), we have (10). It is easy to see that \( \lambda \) satisfies the following initial value problem:

\[
\begin{align*}
\dot{\lambda}_x(x,t) &= \varphi(\lambda(x,t), \omega(L^{-1}(t, X(t))x, t) + \mu(\lambda(x,t))) \\
\lambda(0,t) &= Z(t).
\end{align*}
\]

Using (59), (68), and (69), we can deduce that \( \lambda(x,t) = Z(x + t) \) is the unique solution to (243) and (244). Hence, \( \lambda_x(x,t) = \lambda_x(x,t) \). Using (64), it can be deduced that \( \omega(\xi, t) \) is a function of \( L(t, X(t)) \xi + t \), that is, \( \omega(\xi, t) = G_1(L(t, X(t)) \xi + t) \) for some function \( G_1 \), it yields \( \omega(L^{-1}(t, X(t))x, t) = G_1(x + t) \), then \( \omega(\dot{x}(L^{-1}(t, X(t))x, t) = \omega_0(L^{-1}(t, X(t))x, t) \). Thus, we have(11). Analogously, it can be deduced that \( \iota(x,t) = Z(t - x) \). From (65), it implies that \( \sigma(\xi, t) \) is a function of \( t - L(t, X(t)) \xi \), that is, \( \sigma(\xi, t) = G_2(t - L(t, X(t)) \xi) \) for some function \( G_2 \), it yields \( \sigma(L^{-1}(t, X(t))x, t) = G_2(t - x) \), then \( \sigma(\dot{x}(L^{-1}(t, X(t))x, t) = -\sigma_0(L^{-1}(t, X(t))x, t) \), so \( \eta(x,t) \) defined in (75) satisfies (12). Setting \( x = 0 \) in (74) and (75), and with the help of (66), we obtain (13). Setting \( x = L(t, X(t)) \) in (74), noting that \( \lambda(L(t, X(t)), t) = Z(t + L(t, X(t))) \), and with the control law (31)–(34), we have (14).

The proof of Lemma 3
Let \( y(t) = \dot{x}(t) - \kappa(X(t)) \). Noting that \( \dot{x}(t) = \mu(Z) + v \), we have

\[
\begin{align*}
\dot{X}(t) &= f(X(t), \kappa(X(t)) + y(t)) \\
\dot{y}(t) &= -c_1 y(t) + v(t).
\end{align*}
\]

Under Assumption 3, the \( X(t) \)-subsystem satisfies input-to-state stability property with respect to \( y(t) \). In addition, \( y(t) \)-subsystem is ISS with respect to \( v(t) \); by Lemma C.2 in [11], we know that system (245) and (246) is ISS with respect to \( v(t) \). Thus, there exist a class \( \mathcal{K} \mathcal{L} \) function \( \overline{\beta} \) and a class \( \mathcal{K} \) function \( \overline{\gamma} \) such that

\[
\begin{align*}
|X(t)| + |y(t)| \leq \overline{\beta}(|X(0)| + |y(0)|, t) + \overline{\gamma} \left( \sup_{0 \leq \delta \leq t} |v(\delta)| \right).
\end{align*}
\]

Because the function \( \kappa \) is continuously differentiable with \( \kappa(0) = 0 \), there exists a class \( \mathcal{K}_\infty \) function \( \gamma \) such that \( |\kappa(X)| \leq \gamma(|X|) \) and for all \( X(t) \in \mathbb{R}^n \), it can be deduced that

\[
\begin{align*}
|X(t)| + |\dot{x}(t)| &\leq |X(t)| + |y(t)| + |\kappa(X(t))| \\
&\leq |X(t)| + |y(t)| + \gamma(|X(t)|) \\
&\leq \overline{\alpha}(|X(t)| + |y(t)|)
\end{align*}
\]
where \( \alpha(s) = s + \gamma(s) \). Further, we have

\[
|X(t)| + |\xi(t)| \leq \alpha(|X(t)| + |y(t)|)
\]

\[
\leq \alpha \left( 2 \left( \beta \left( \alpha \left( \sqrt{2} |Z(0)| \right), t \right) \right) + \alpha \left( \sup_{0 \leq \delta \leq t} |v(\delta)| \right) \right)
\]

\[
\leq \alpha \left( 2 \left( \beta \left( \alpha \left( \sqrt{2} |Z(0)| \right), t \right) \right) + \alpha \left( \sup_{0 \leq \delta \leq t} |v(\delta)| \right) \right) + \alpha \left( \sup_{0 \leq \delta \leq t} |v(\delta)| \right).
\]

(249)

It yields

\[
|Z(t)| \leq |X(t)| + |\xi(t)| \leq \alpha \left( 2 \left( \beta \left( \alpha \left( \sqrt{2} |Z(0)| \right), t \right) \right) + \alpha \left( \sup_{0 \leq \delta \leq t} |v(\delta)| \right) \right) + \alpha \left( \sup_{0 \leq \delta \leq t} |v(\delta)| \right).
\]

(250)

Thus, there exist a class \( KL \) function

\[
\hat{\beta}(s, t) = \alpha \left( 2 \left( \beta \left( \alpha \left( \sqrt{2} |Z(0)| \right), t \right) \right) \right)
\]

and a class \( K \infty \) function

\[
\hat{\gamma}(s) = \alpha \left( \sup_{0 \leq \delta \leq t} |v(\delta)| \right).
\]

(252)

such that

\[
|Z(t)| \leq \hat{\beta}(|Z(0)|, t) + \hat{\gamma} \left( \sup_{0 \leq \delta \leq t} |v(\delta)| \right).
\]

\[
\square
\]

The proof of Lemma 6

Noting that \( \tau \) satisfies the initial value problem (104) and (105) and using (49) in Assumption 2, we have

\[
- \frac{\partial R_3(t(x, t))}{\partial t} \varphi \left( t(y, t), \sigma(L^{-1}(t, X(t))x, t) + \mu(t(x, t)) \right)
\]

\[
\leq R_3(t(x, t)) + \alpha_0(\|\sigma(L^{-1}(t, X(t))x, t)\|).
\]

(253)

With (104), we have

\[
\frac{dR_3(t(x, t))}{dx} \leq R_3(t(x, t)) + \alpha_0(\|\sigma(L^{-1}(t, X(t))x, t)\|),
\]

(254)

it follows for all \( x \in [0, L(t, X(t))] \) that

\[
R_3(t(x, t)) \leq e^{X} R_3(t(0, t)) + \int_{0}^{X} e^{X-y} \alpha_0(\|\sigma(L^{-1}(t, X(t))y, t)\|)dy
\]

\[
= e^{X} R_3(Z(t)) + \int_{0}^{X} e^{X-y} \alpha_0(\|\sigma(L^{-1}(t, X(t))y, t)\|)dy
\]

\[
\leq e^{L(t, X(t))} R_3(Z(t)) + \left( e^{L(t, X(t))} - 1 \right) \sup_{0 \leq y \leq L(t, X(t))} \alpha_0(\|\sigma(L^{-1}(t, X(t))y, t)\|)
\]

\[
\leq e^{L(t, X(t))} R_3(Z(t)) + \left( e^{L(t, X(t))} - 1 \right) \alpha_0(\|\sigma(t)\|_\infty).
\]

(255)

Using (51) in Assumption 4, we have

\[
R_5(t(x, t)) \leq e^{m} R_3(Z(t)) + (e^{m} - 1) \alpha_9(\|\sigma(t)\|_\infty).
\]

(256)
With (48), we have that for all \( x \in [0, L(t, X(t))] \)
\[
|t(x, t)| \leq \alpha_7^{-1}(e^m(\alpha_8(|Z(t)|)) + (e^m - 1)\alpha_9(\|\varpi(t)\|_{\infty})).
\]  
(257)

So we have (106) with \( \gamma_3 = \alpha_7^{-1}(e^m(\alpha_8(s)) + (e^m - 1)\alpha_9(s)) \).

The proof of Lemma 7

Noting that \( \lambda \) satisfies the initial value problem (243) and (244), from Lemma 3, we have that for all \( x \in [0, L(t, X(t))] \)
\[
|\lambda(x, t)| \leq \hat{\beta}(|Z(t)|, t) + \hat{\gamma} \left( \sup_{0 \leq y \leq x} |\alpha(L^{-1}(t, X(t))y, t)| \right),
\]  
(258)

so we have
\[
\|\lambda(t)\|_{\infty} \leq \hat{\beta}(|Z(t)|, t) + \hat{\gamma}(\|\omega(t)\|_{\infty}).
\]  
(259)

Denote \( \gamma_4(s) = \hat{\beta}(s, 0) + \hat{\gamma}(s) \) where \( \hat{\beta}, \hat{\gamma} \) are given by (251) and (252), respectively, we have (107).

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