Stabilization of a System of $n+1$ Coupled First-Order Hyperbolic Linear PDEs With a Single Boundary Input

Florent Di Meglio, Rafael Vazquez, *Member, IEEE*, and Miroslav Krstic, *Fellow, IEEE*

***Abstract*—We solve the problem of stabilization of a class of linear first-order hyperbolic systems featuring n rightward convecting transport PDEs and one leftward convecting transport PDE. We design a controller, which requires a single control input applied on the leftward convecting PDE's right boundary, and an observer, which employs a single sensor on the same PDE's left boundary. We prove exponential stability of the origin of the resulting plant-observer-controller system in the spatial $\mathcal{L}^2$-sense.**

***Index Terms*— Control design, distributed parameters systems, observers.**

I. Introduction

WE INVESTIGATE boundary stabilization of a class of linear first-order hyperbolic systems of Partial Differential Equations (PDEs) on a finite space domain $x \in [0,1]$. Transport equations are predominant in modeling of traffic flow [2], heat exchangers [39], open channel flow [11], [14] or multiphase flow [15], [20], [22]. The coupling between states traveling in opposite directions, both in-domain and at the boundaries, may induce instability leading to undesirable behaviors. For example, oscillatory two-phase flow regimes occurring on oil and gas production systems directly result, in some cases, from these mechanisms [16]. The dynamics of most of these industrial systems are described by *nonlinear* transport equations. Control results for nonlinear first-order hyperbolic systems do exist in the literature: in [25] sufficient conditions on the structure of the control problem[1] for controllability and observability of such systems are given. In [11] and [38] control laws for a system of two coupled nonlinear PDEs are derived, whereas in [7], [10], [21], [30], [31] sufficient conditions for exponential stability are given for various classes of quasilinear first-order hyperbolic systems. However, there is, to our best knowledge, no boundary control design valid for the general case. The difficulty of handling the nonlinear equations motivates the study of linearized systems.

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F. Di Meglio and M. Krstic are with the Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093-0411 USA (e-mail: florent.di_meglio@mines-paristech.fr; krstic@ucsd.edu).

R. Vazquez is with the Department of Aerospace Engineering, Universidad de Sevilla, 41092 Sevilla, Spain (e-mail: rvazquez1@us.es).

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[1]More precisely, on the position and number of actuators and sensors according to the sign of the transport velocities.

Boundary stabilization of *linear* first-order hyperbolic systems has also received significant attention in the literature. Motivated by various industrial processes, many contributions focus on the stability of two-state heterodirectional systems (i.e., systems of two transport equations with opposite transport speeds). In [27] stabilization of such a system is investigated using a frequency domain approach while [1] focuses on the disturbance rejection problem. In [39] a control Lyapunov function is introduced to investigate stability of linear hyperbolic systems and provide a proof of the Rauch and Taylor theorem [32] in the particular case where the coupling coefficients matrix is symmetric. In [9] a similar control Lyapunov function is used to design stabilizing boundary feedback laws for two-state heterodirectional systems *with no in-domain coupling*. In these contributions, the control laws take the form of static output feedbacks applied at both boundaries. In [5] this result is extended to two-state heterodirectional systems with space-varying transport speeds and linear coupling. However, the coupling coefficients are required to satisfy a restrictive condition on their magnitude. The condition is necessary and sufficient for the existence of a stabilizing static output feedback law. It imposes that the solutions of an ODE depending on the coupling terms does not explode in finite time. Similarly, sufficient conditions for stability of general first-order hyperbolic systems, taking the form of upper bounds on the magnitude of the coupling terms are derived in, e.g., [19], in [3] for switched systems or in [29] for systems with nonlinear Lotka–Volterra coupling terms. Conversely, in this paper, we propose a controller that exponentially stabilizes the zero equilibrium regardless of the magnitude of the coupling coefficients, provided it remains finite on the spatial domain.

In [37] an observer-controller structure is proposed to stabilize two-state heterodirectional systems with the only restriction that the coupling coefficients are bounded in the $\mathcal{L}^\infty$-norm on the space domain. A full-state feedback law is designed, guaranteeing exponential stability of the zero equilibrium. Then, a collocated observer structure allows one to estimate the states from a boundary measurement. Both the controller and the observer schemes are based on the backstepping approach. Backstepping is a design tool primarily used for finite dimensional nonlinear systems [23] in *strict-feedback* form. The method has been extended to control and observer design for, e.g., parabolic PDEs [34], wave equations [8], delay systems [24] or beams [33]. Based on the same approach, a full-state feedback law is designed for a three-state first-order hyperbolic system representing gas-liquid flow in oil wells in [18].

In this paper, we propose to extend the control design of [37] to a broader class of systems. More precisely, we consider

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systems of $(n+1)$ linearly coupled transport equations, with space-varying transport speeds and coupling coefficients. One of the transport speeds remains strictly negative throughout the space domain, while the n others remain strictly positive. The state corresponding to the negative velocity is controlled at the right boundary ($x=1$). At the left boundary ($x=0$), reflexivity conditions ensure well-posedness of the system. The system is strongly underactuated, as only one state is controlled, whereas a possibly large number of states (determined by the value of n) are uncontrolled. Yet, we propose a control design that exponentially stabilizes the zero equilibrium *regardless of the (finite) magnitude of the coupling coefficients*. This result has already been published in [17], with less detailed proofs than in this contribution. Besides, we derive here an observer estimating the $(n+1)$ distributed states over the whole spatial domain from a single measurement of the controlled state at the left boundary ($x=0$). These are the main contributions of the paper.

Our approach is as follows. First, we design a full-state feedback law guaranteeing exponential stability of the zero equilibrium in the $\mathcal{L}^2$-norm. To do so, the system is mapped to a so-called *target* system with desirable stability properties using a Volterra transformation of the second kind. The target system is designed by removing from the original system the minimum amount of coupling terms required to ensure exponential stability. Then, a boundary observer is designed as a copy of the original system plus linear output error injection terms inside the domain, and direct output injection at the left boundary. Similarly to the control design, the observer error dynamics are mapped to a target system using a Volterra transformation of the second kind. The main technical difficulty of the paper lies in showing existence and invertibility of the transformations for both designs. For this purpose, the transformation kernels both for the observer and controller designs are shown to satisfy systems of first-order hyperbolic PDEs on triangular domains. After using the method of characteristics to transform these into integral equations, the method of successive approximations is used to prove well-posedness of the kernel equations, and thus the validity of the design.

The paper is organized as follows. In Section II we detail the problem statement and the notations. In Section III we derive the observer-controller structure using the backstepping method. The design is summarized in Section IV where we state the main result. Section V contains the main technical difficulty of the paper, namely the proof of existence of the backstepping transformation kernels. We illustrate our result in Section VI with numerical simulations, before describing some open problems in Section VII.

II. Problem Statement

For $n \geq 1$, we consider the following set of $(n+1)$ linear transport PDEs on the domain $\{t \in \mathbb{R}_+,\ x \in [0,1]\}$, for $i = 1,\dots,n$

$$u_t^i(t,x) + \lambda_i(x) u_x^i(t,x) = \sum_{j=1}^{n} \sigma_{i,j}(x) u^j(t,x) + \omega_i(x) v(t,x) \tag{1}$$

$$v_t(t,x) - \mu(x) v_x(t,x) = \sum_{j=1}^{n} \theta_j(x) u^j(t,x) \tag{2}$$

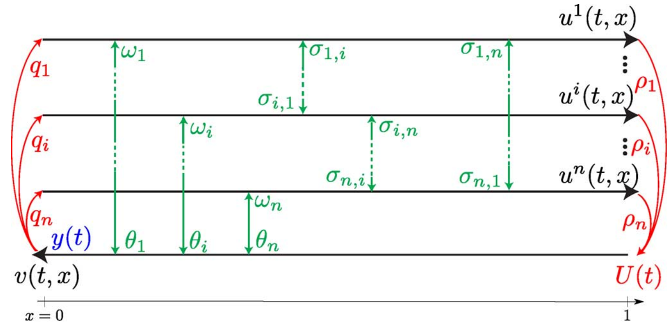


Fig. 1. Schematic view of the $(n+1)$-state hyperbolic system. The space dependence of the coupling coefficients has been omitted for the sake of clarity.

along with the following boundary conditions:

$$u^i(t,0) = q_i v(t,0), \qquad v(t,1) = \sum_{j=1}^{n} \rho_j u^j(t,1) + U(t) \tag{3}$$

$$v(t,0) = y(t) \tag{4}$$

where the u^i, $i=1,\dots,n$ and v are the distributed states, $U(t)$ is the control input and $y(t)$ the measured output. The transport velocities are assumed to satisfy the following inequalities:

$$\forall x \in [0,1] \qquad -\mu(x) < 0 < \lambda_1(x) < \dots < \lambda_n(x) \tag{5}$$

which indicate that the u^i states evolve left to right, whereas the v state evolves right to left. This setup is schematically depicted on Fig. 1. Along with the boundary conditions (3), Inequalities (5) ensure hyperbolicity and well-posedness of the mixed initial-boundary value problem (see, e.g., [25]).

Remark 1: There is no loss in generality in not considering a coupling term proportional to v in (2), since it can be canceled using a variable change (see, e.g., [4]).

Remark 2: As mentioned in Section I, results exist for small values of n. The case $n=1$ corresponding to an open channel flow problem is treated in [4] and [37]. A particular case of $n=2$, corresponding to gas-liquid flow in oil production systems is treated in [16]. Systems corresponding to higher values of n arise, e.g., when considering two-fluid models for gas-liquid flow [28].

Our goal is to find a feedback control law $U(t)$ that exponentially stabilizes the zero equilibrium of the system (1)–(3) using the sole measurement $y(t) = v(0,t)$. In the next section, we detail the control and observer design.

III. Controller and Observer Design

The observer-controller scheme consists in designing an observer and a controller separately, and using the observer estimates in the full-state feedback law. First, we design in Section III-A a stabilizing full-state feedback controller. Then, we design in Section III-B a boundary observer. Convergence of the full estimation and state error dynamics to zero is proved using a Lyapunov analysis.

A. Control Design

Following the backstepping approach, we design the control law by mapping System (1)–(3) to a so-called *target* system,

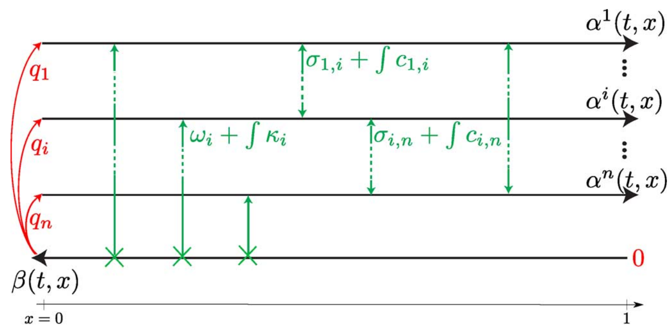


Fig. 2. Schematic view of the $(n+1)$-state target system. Only the θ_i, $i=1,\ldots,n$ coupling terms have been canceled and integral coupling terms have been added. For clarity purposes, the majority of coupling terms are omitted on this drawing.

described in Section III-A-1, using an integral transformation detailed in Section III-A-2.

1) Target System: We want to map system (1)–(3) to the following target system, schematically depicted on Fig. 2

$$\alpha_t^i(t,x)+\lambda_i(x)\alpha_x^i(t,x)=\sum_{j=1}^n\sigma_{i,j}(x)\alpha^j(t,x)+\omega_i(x)\beta(t,x)+\sum_{j=1}^n\int_0^x c_{i,j}(x,\xi)\alpha^j(t,\xi)d\xi+\int_0^x\kappa_i(x,\xi)\beta(t,\xi)d\xi \quad (6)$$

$$\beta_t(t,x)-\mu(x)\beta_x(t,x)=0 \quad (7)$$

with the following boundary conditions:

$$\forall i=1,\ldots,n\quad \alpha^i(t,0)=q_i\beta(t,0)\qquad \beta(t,1)=0 \quad (8)$$

where the $c_{i,j}(\cdot)$ and $\kappa_i(\cdot)$, $i,j=1,\ldots,n$ are functions to be defined on the triangular domain

$$\mathcal{T}=\left\{(x,\xi)\in\mathbb{R}^2|0\le\xi\le x\le 1\right\}.$$

This system was designed as a copy of the original dynamics (1)–(3), from which the coupling terms appearing in (2) were removed. The integral coupling terms appearing in (6), determined by the $c_{i,j}$ and κ_i coefficients, are necessary to the control design but do not affect the stability of the target system, as stated in the following lemma.

Lemma 3.1: Under the following assumptions

$$\forall i,j=1,\ldots,n\ \lambda_i,\mu\in\mathcal{C}^1\left([0,1],\mathbb{R}_+^*\right),\ \sigma_{i,j},\omega_i\in\mathcal{C}\left([0,1]\right)$$
$$\alpha_0^i,\beta_0\in\mathcal{L}^2\left([0,1]\right),\quad c_{i,j},\kappa_i\in\mathcal{C}(\mathcal{T})$$

the equilibrium $(\alpha,\beta)^T\equiv(0,\ldots 0,0)^T$ of system (6), (7) with boundary conditions (8) and initial conditions $(\alpha^0,\beta^0)^T=(\alpha_1^0,\ldots\alpha_n^0,\beta^0)^T$ is exponentially stable in the $\mathcal{L}^2$ sense.

Proof: Consider the following candidate Lyapunov functional:

$$V(t)=\int_0^1 pe^{-\delta x}\sum_{i=1}^n\frac{\alpha_i(t,x)^2}{\lambda_i(x)}dx+\int_0^1\frac{1+x}{\mu(x)}\beta(t,x)^2dx \quad (9)$$

where $p>0$ and $\delta>0$ are analysis parameters to be determined. $\sqrt{V}$ is equivalent to the $\mathcal{L}^2$-norm. Differentiating V with respect to time and integrating by parts yields

$$\begin{aligned}\dot V(t)=&\left[-pe^{-\delta x}\sum_{i=1}^n\alpha_i(t,x)^2+(1+x)\beta(t,x)^2\right]_0^1-\int_0^1\beta(t,x)^2dx\\&-\int_0^1 pe^{-\delta x}\boldsymbol{\alpha}(t,x)^T\left[\delta I_{n\times n}+\boldsymbol{\lambda}^{-1}(x)\boldsymbol{\sigma}(x)\right]\boldsymbol{\alpha}(t,x)dx\\&+2\int_0^1\int_0^x pe^{-\delta x}\boldsymbol{\alpha}(t,x)^T\boldsymbol{\lambda}^{-1}(x)\mathbf{c}(x,\xi)\boldsymbol{\alpha}(t,\xi)d\xi dx\\&+2\int_0^1 pe^{-\delta x}\boldsymbol{\alpha}(t,x)^T\boldsymbol{\lambda}^{-1}(x)\boldsymbol{\omega}(x)\beta(t,x)dx\\&+2\int_0^1\int_0^x pe^{-\delta x}\boldsymbol{\alpha}(t,x)^T\boldsymbol{\lambda}^{-1}(x)\boldsymbol{\kappa}(x,\xi)\beta(t,\xi)d\xi dx.\end{aligned} \quad (10)$$

Consider $M>0$ and $\epsilon>0$ such that

$$\begin{aligned}&\forall i=1,\ldots,n\quad\forall j=1,\ldots,n\quad \|\sigma_{i,j}\|_\infty,\|\omega_i\|_\infty,\|c_{i,j}\|_\infty,\\&\qquad\qquad\|\kappa_i\|_\infty<M\\&\forall i=1,\ldots,n\quad\forall x\in[0,1]\qquad\lambda_i(x)>\epsilon\end{aligned}$$

This yields, after some computation

$$\begin{aligned}\dot V(t)\le&\left(p\sum_{i=1}^n q_i^2-1\right)\beta(t,0)^2-\int_0^1\left[1-\frac{pM}{\epsilon}-\frac{npM}{\delta\epsilon}\right]\beta(t,x)^2dx\\&-\int_0^1 pe^{-\delta x}\boldsymbol{\alpha}(t,x)^TP(x)\boldsymbol{\alpha}(t,x)dx\end{aligned} \quad (11)$$

with $P(x)=[(\delta-(3M/\epsilon)-(M/\delta\epsilon))I_{n\times n}+\boldsymbol{\lambda}^{-1}(x)\boldsymbol{\sigma}(x)]$. For a sufficiently large $\delta>0$, the matrix $P(x)$ is positive definite for all $x\in[0,1]$. Thus, picking

$$p<\min\left\{\frac{1}{\sum_{i=1}^n q_i^2},\frac{2\delta\epsilon}{M(\delta+n)}\right\} \quad (12)$$

concludes the proof. ∎

In order to map the original system (1)–(3) to the target system (6)–(8), we propose a Volterra transformation of the second kind. In the next section, we derive a set of PDEs satisfied by the transformation kernels.

2) Backstepping Transformation: We consider the following backstepping transformation:

$$\beta(t,x)=v(t,x)-\int_0^x\sum_{i=1}^n k^i(x,\xi)u^i(t,\xi)d\xi - \int_0^x k^{n+1}(x,\xi)v(t,\xi)d\xi. \quad (13)$$

Besides, we set

$$\forall i=1,\dots,n \quad \alpha^i\equiv u^i. \quad (14)$$

We now seek sufficient conditions on the functions k^i, $c_{i,j}$ and κ_i, $i,j=1,\dots,n$ such that transformation (13) maps System (1)–(3) to System (6)–(8). Differentiating (13) with respect to space, using the Leibniz rule, yields

$$\beta_x(x)=v_x(x)-\sum_{i=1}^n k^i(x,x)u^i(x)-k^{n+1}(x,x)v(x) - \int_0^x\sum_{i=1}^n k_x^i(x,\xi)u^i(\xi)d\xi-\int_0^x k_x^{n+1}(x,\xi)v(\xi)d\xi \quad (15)$$

while differentiating with respect to time, using (1), (2) and integrating by parts yields

$$\begin{aligned}\beta_t(x)=&\mu(x)v_x(x)+\sum_{i=1}^n\theta_i(x)u^i(x)+\sum_{i=1}^n k^i(x,x)\lambda_i(x)u^i(x)\\ &-k^{n+1}(x,x)\mu(x)v(x)-\sum_{i=1}^n k^i(x,0)\lambda_i(0)u^i(0)\\ &+k^{n+1}(x,0)\mu(0)v(0)\\ &+\int_0^x\Big[k_\xi^{n+1}(x,\xi)\mu(\xi)+k^{n+1}(x,\xi)\mu'(\xi)\\ &\qquad-\sum_{i=1}^n k^i(x,\xi)\omega_i(\xi)\Big]v(\xi)d\xi\\ &-\sum_{i=1}^n\int_0^x\Big[k_\xi^i(x,\xi)\lambda_i(\xi)+k^i(x,\xi)\lambda_i'(\xi)\\ &\qquad+\sum_{j=1}^n\sigma_{j,i}(\xi)k^j(x,\xi)\\ &\qquad+k^{n+1}(x,\xi)\theta_i(\xi)\Big]u^i(\xi)d\xi. \end{aligned} \quad (16)$$

Plugging (13)–(16) into (6), (7) and using (3) yields

$$\begin{aligned}0=&\sum_{i=1}^n\left[k^i(x,x)\lambda_i(x)+\mu(x)k^i(x,x)+\theta_i(x)\right]u^i(x)\\ &-\left[\sum_{i=1}^n q_i\lambda_i(0)k^i(x,0)-\mu(0)k^{n+1}(x,0)\right]v(0)\\ &+\int_0^x\Big[k_\xi^{n+1}(x,\xi)\mu(\xi)+\mu(x)k_x^{n+1}(x,\xi)+k^{n+1}(x,\xi)\mu'(\xi)\\ &\qquad-\sum_{i=1}^n k^i(x,\xi)\omega_i(\xi)\Big]v(\xi)d\xi\\ &-\sum_{i=1}^n\int_0^x\Big[k_\xi^i(x,\xi)\lambda_i(\xi)-\mu(x)k_x^i(x,\xi)+k^i(x,\xi)\lambda_i'(\xi)\\ &\qquad+\sum_{j=1}^n\sigma_{j,i}(\xi)k^j(x,\xi)+k^{n+1}(x,\xi)\theta_i(\xi)\Big]u^i(\xi)d\xi.\end{aligned} \quad (17)$$

Thus, a sufficient condition for the transformation to map the original system to the target system is that the kernels k^i, $i=1,\dots,n+1$ satisfy the following system of first-order hyperbolic PDEs

$$\begin{cases}\mu(x)k_x^1-\lambda_1(\xi)k_\xi^1=\lambda_1'(\xi)k^1+\sum_{i=1}^n\sigma_{i,1}(\xi)k^i+\theta_1(\xi)k^{n+1}\\ \vdots\\ \mu(x)k_x^n-\lambda_n(\xi)k_\xi^n=\lambda_n'(\xi)k^n+\sum_{i=1}^n\sigma_{i,n}(\xi)k^i+\theta_n(\xi)k^{n+1}\\ \mu(x)k_x^{n+1}+\mu(\xi)k_\xi^{n+1}=-\mu'(\xi)k^{n+1}+\sum_{i=1}^n\omega_i(\xi)k^i\end{cases} \quad (18)$$

with boundary conditions

$$\begin{cases}k^1(x,x)=-\frac{\theta_1(x)}{\lambda_1(x)+\mu(x)}\\ \vdots\\ k^n(x,x)=-\frac{\theta_n(x)}{\lambda_n(x)+\mu(x)}\\ \mu(0)k^{n+1}(x,0)=\sum_{i=1}^n q_i\lambda_i(0)k^i(x,0).\end{cases} \quad (19)$$

Besides, plugging (13), (14) into (6) and using (1) yields, for all $i=1,\dots,n$

$$\begin{aligned}0=&\sum_{i=1}^n\int_0^x\left[\omega_i(x)k^j(x,\xi)-c_{i,j}(x,\xi)+\int_\xi^x\kappa_i(x,s)k^j(s,\xi)ds\right]u^j(\xi)d\xi\\ &+\int_0^x\left[\omega_i(x)k^{n+1}(x,\xi)-\kappa_i(x,\xi)+\int_\xi^x\kappa_i(x,s)k^{n+1}(s,\xi)ds\right]v(\xi)d\xi.\end{aligned}$$

Thus, provided the $k^i(\cdot)$, $i=1,\dots,n+1$ exist and are sufficiently smooth, the coefficients κ_i, $i=1,\dots,n$ must be chosen to satisfy the following integral equations:

$$\kappa_i(x,\xi)=\omega_i(x)k^{n+1}(x,\xi)+\int_\xi^x\kappa_i(x,s)k^{n+1}(s,\xi)ds \quad (20)$$

and the coefficients $c_{i,j}(\cdot)$ must be chosen, for all $i,j = 1,\dots,n$, as

$$c_{i,j}(x,\xi) = \omega_i(x)k^j(x,\xi) + \int_\xi^x \kappa_i(x,s)k^j(s,\xi)ds. \tag{21}$$

The existence, uniqueness and continuity of solutions to System (18) with boundary conditions (19) is assessed in Theorem 5.3. Besides, the continuity (and thus, the boundedness) of the k^i, $i=1,\dots,n+1$ also implies the existence and continuity of the solutions to each of the n Volterra equations of the second kind (20) (see, e.g., [26, Th. 3.1, p. 30]). Therefore, the functions κ_i and $c_{i,j}$ [defined by (21)], for $i,j=1,\dots,n$ are continuous on $\mathcal{T}$, thus bounded, and therefore the assumptions of Lemma 3.1 are satisfied.

3) Inverse Transformation: To ensure that the target and the closed-loop systems have equivalent stability properties, transformation (13) has to be invertible. Since, for all $i=1,\dots,n$, $\alpha^i \equiv u^i$, transformation (13) rewrites

$$v(t,x) - \int_0^x k^{n+1}(x,\xi)v(t,\xi)d\xi = \Gamma(t,x) \tag{22}$$

with $\Gamma(t,x) = \beta(t,x) + \sum_{i=1}^n \int_0^x k^i(x,\xi)\alpha^i(t,\xi)d\xi$. Since k^{n+1} is continuous by Theorem 5.3, there exists a unique continuous inverse kernel l^{n+1} defined on $\mathcal{T}$ and such that (see, e.g., [35])

$$v(t,x) = \Gamma(t,x) + \int_0^x l^{n+1}(x,\xi)\Gamma(t,\xi)d\xi \tag{23}$$

which yields the following inverse transformation:

$$v(t,x) = \beta(t,x) + \sum_{i=1}^n \int_0^x l^i(x,\xi)\alpha^i(t,\xi)d\xi + \int_0^x l^{n+1}(x,\xi)\beta(t,\xi)d\xi \tag{24}$$

where, for each $i=1,\dots,n$, we have defined

$$l^i(x,\xi) = k^i(x,\xi) + \int_\xi^x k^i(x,\xi)l^{n+1}(\xi,s)ds. \tag{25}$$

4) Control Law: The control law is obtained by plugging the transformation (13) into (3). We now state the main result concerning the control design.

Theorem 3.2: Consider system (1), (2) with boundary conditions (3), initial conditions $(u_0^1,\dots,u_0^n,v_0)$ and the following control law:

$$U(t) = -\sum_{i=1}^n \rho_i u^i(t,1) + \int_0^1 \left[\sum_{i=1}^n k^i(1,\xi)u^i(t,\xi) + k^{n+1}(1,\xi)v(t,\xi)\right] d\xi \tag{26}$$

where, for $i=1,\dots,n+1$, the k^i satisfy System (18) with boundary conditions (19). Then, under the following assumptions

$$\forall i,j=1,\dots,n\ \lambda_i,\mu \in \mathcal{C}^1\left([0,1],\mathbb{R}_+^*\right),\ \sigma_{i,j},\omega_i,\theta_i \in \mathcal{C}([0,1])$$
$$\forall i=1,\dots,n \quad u_0^i, v_0 \in \mathcal{L}^2\left([0,1]\right)$$

the equilibrium $(u^1,\dots,u^n,v) \equiv 0$ is exponentially stable in the $\mathcal{L}^2$ sense.

Proof: We denote

$$w(t,x) = \left(\, u^1(t,x) \quad \cdots \quad u^n(t,x) \quad v(t,x)\,\right)^T$$
$$\gamma(t,x) = \left(\, \alpha^1(t,x) \quad \cdots \quad \alpha^n(t,x) \quad \beta(t,x)\,\right)^T$$
$$K(x,\xi) = \begin{pmatrix} & 0_{n\times(n+1)} & \\ k^1(x,\xi) & \cdots & k^{n+1}(x,\xi)\end{pmatrix}$$
$$L(x,\xi) = \begin{pmatrix} & 0_{n\times(n+1)} & \\ l^1(x,\xi) & \cdots & l^{n+1}(x,\xi)\end{pmatrix}.$$

Following Lemma 3.1, there exists $C>0$ and $\epsilon>0$ such that

$$\|\gamma(t,\cdot)\|_{\mathcal{L}^2} \le C\,\|\gamma(0,\cdot)\|_{\mathcal{L}^2}\, e^{-\epsilon t} \tag{27}$$

where

$$\|\gamma(t,\cdot)\|_{\mathcal{L}^2} \triangleq \sqrt{\int_0^1 \|\gamma(t,x)\|_2^2\, dx} \tag{28}$$

$$= \sqrt{\sum_{i=1}^n \int_0^1 \alpha^i(t,x)^2 dx + \int_0^1 \beta(t,x)^2 dx}. \tag{29}$$

By Theorem 5.3, the kernels k^i, l^i are continuous, therefore one can define the following upper bounds $K_\infty, L_\infty, > 0$

$$K_\infty^2 = \max_{(x,\xi)\in\mathcal{T}} |||K(x,\xi)|||_2^2, \qquad L_\infty^2 = \max_{(x,\xi)\in\mathcal{T}} |||L(x,\xi)|||_2^2$$

where $|||K(\cdot,\cdot)|||_2$ denotes the classical operator norm $|||K(\cdot,\cdot)|||_2 = \sup_{X\in\mathbb{R}^{n+1},\ \|X\|_2=1} \|K(\cdot,\cdot)X\|_2$. Besides, (24) yields

$$\begin{aligned}
\|w(t,\cdot)\|_{\mathcal{L}^2} &\le \|\gamma(t,\cdot)\|_{\mathcal{L}^2} + \left\| \int_0^{\cdot} K(\cdot,\xi)f(t,\xi)d\xi \right\|_{\mathcal{L}^2} \\
&\le \|\gamma(t,\cdot)\|_{\mathcal{L}^2} + \sqrt{\int_0^1 \left\| \int_0^x K(x,\xi)f(t,\xi)d\xi \right\|_2^2 dx} \\
&\le \|\gamma(t,\cdot)\|_{\mathcal{L}^2} + \sqrt{\int_0^1 \int_0^x \|K(x,\xi)f(t,\xi)\|_2^2\, d\xi dx} \\
&\le \|\gamma(t,\cdot)\|_{\mathcal{L}^2} + \sqrt{\int_0^1\int_0^x |||K(x,\xi)|||_2^2\, \|f(t,\xi)\|_2^2\, d\xi dx} \\
&\le (1+K_\infty)\,\|\gamma(t,\cdot)\|_{\mathcal{L}^2} \\
&\le C(1+K_\infty)(1+L_\infty)\,\|w(0,\cdot)\|_{\mathcal{L}^2}\, e^{-\epsilon t}
\end{aligned}$$

which concludes the proof. ■

The feedback control law defined by (26) requires the knowledge of the value of the states over the whole spatial domain. In practice, distributed measurements of all the states are rarely available, and these need to be estimated. In the next section, we propose an observer design reconstructing the distributed states from a single measurement of $v(t,0)$ at the left boundary.

B. Observer Design

In this section, we derive a boundary observer estimating the states of system (1)–(3) over the whole spatial domain using the measured output defined by (4). We design the observer as a copy of the plant plus output injection terms as follows:

$$\hat{u}_t^i(t,x) + \lambda_i(x)\hat{u}_x^i(t,x) = \sum_{j=1}^{n}\sigma_{i,j}(x)\hat{u}^j(t,x) + \omega_j(x)\hat{v}(t,x) - p_i(x)\left[y(t) - \hat{v}(t,0)\right] \quad (30)$$

$$\hat{v}_t(t,x) - \mu(x)\hat{v}_x(t,x) = \sum_{j=1}^{n}\theta_j(x)\hat{u}_j(t,x) - p_{n+1}(x)\left[y(t) - \hat{v}(t,0)\right] \quad (31)$$

along with the following boundary conditions:

$$\forall i = 1,\ldots,n \;\; \hat{u}^i(t,0) = q_i y(t), \;\; \hat{v}(t,1) = \sum_{j=1}^{n}\rho_j\hat{u}^j(t,1) + U(t). \quad (32)$$

Denoting $\tilde{u}(t,x) = \hat{u}(t,x) - u(t,x)$, this yields the following observer error dynamics:

$$\tilde{u}_t^i(t,x) + \lambda_i(x)\tilde{u}_x^i(t,x) = \sum_{j=1}^{n}\sigma_{i,j}(x)\tilde{u}^j(t,x) + \omega_i(x)\tilde{v}(t,x) + p_i(x)\tilde{v}(t,0) \quad (33)$$

$$\tilde{v}_t(t,x) - \mu(x)\tilde{v}_x(t,x) = \sum_{j=1}^{n}\theta_j(x)\tilde{u}_j(t,x) + p_{n+1}(x)\tilde{v}(t,0) \quad (34)$$

with boundary conditions

$$\forall i = 1,\ldots,n \quad \tilde{u}^i(t,0) = 0, \qquad \tilde{v}(t,1) = \sum_{j=1}^{n}\rho_j\tilde{u}^j(t,1). \quad (35)$$

To design the observer output injection gains, we follow the same method that was used to design the control feedback law. Thus, we map System (33)–(35) to an appropriate target system using a backstepping transformation.

1) Target System: The target system is designed, once again, by removing coupling terms that affect stability, as illustrated on Fig. 3. The equations are as follows, for $i = 1,\ldots,n$

$$\tilde{\alpha}_t^i(t,x) + \lambda_i(x)\tilde{\alpha}_x^i(t,x) = \sum_{j=1}^{n}\sigma_{i,j}(x)\tilde{\alpha}^j(t,x) + \sum_{j=1}^{n}\int_0^x g_{i,j}(x,\xi)\tilde{\alpha}^j(t,\xi)d\xi \quad (36)$$

$$\tilde{\beta}_t(t,x) - \mu(x)\tilde{\beta}_x(t,x) = \sum_{j=1}^{n}\theta_j(x)\tilde{\alpha}^j(t,x) + \sum_{j=1}^{n}\int_0^x h_j(x,\xi)\tilde{\alpha}^j(t,\xi)d\xi \quad (37)$$

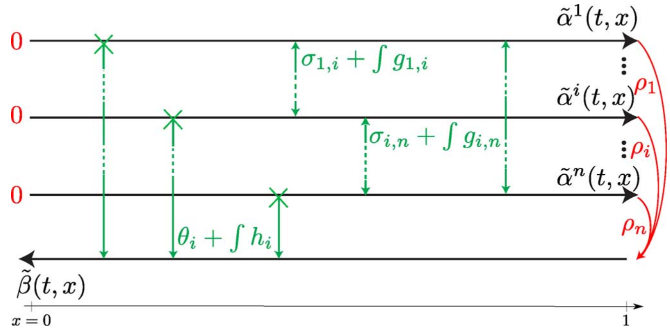


Fig. 3. Schematic representation of the observer error target system.

with boundary conditions

$$\forall i = 1,\ldots,n \quad \tilde{\alpha}^i(t,0) = 0, \qquad \tilde{\beta}(t,1) = \sum_{j=1}^{n}\rho_j\tilde{\alpha}^j(t,1) \quad (38)$$

where the $g_{i,j}(\cdot), h_j(\cdot)$ are functions to be determined on the triangular domain $\mathcal{T}$ The stability properties of this system, which is schematically depicted on Fig. 3, are stated in the following lemma.

Lemma 3.3: Under the following assumptions

$$\forall i,j = 1,\ldots,n \;\; \lambda_i,\mu \in \mathcal{C}^1\left([0,1],\mathbb{R}_+^*\right), \;\; \sigma_{i,j},\theta_i \in \mathcal{C}\left([0,1]\right)$$
$$\alpha_0^i,\beta_0 \in \mathcal{L}^2\left([0,1]\right), \quad g_{i,j},h_i \in \mathcal{C}(\mathcal{T})$$

the equilibrium $\left(\tilde{\alpha}^1,\ldots,\tilde{\alpha}^n,\tilde{\beta}\right)^T \equiv (0,\ldots 0,0)^T$ of system (36), (37) with boundary conditions (38) and initial conditions $\left(\alpha_0^1,\ldots\alpha_0^n,\beta_0\right)^T$ is exponentially stable in the $\mathcal{L}^2$ sense.

Proof: Exponential stability can be shown using the following Lyapunov functional

$$V(t) = \int_0^1 pe^{-\delta x}\sum_{i=1}^{n}\frac{\tilde{\alpha}_i(t,x)^2}{\lambda_i(x)}dx + \int_0^1\frac{e^{\delta x}}{\mu(x)}\tilde{\beta}(t,x)^2dx \quad (39)$$

with carefully selected design parameters $p > 0$ and $\delta > 0$, similarly to the Proof of Lemma 3.1. A more intuitive approach consists in recognizing that, in (36)–(38), the $(\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n)$ subsystem (which is exponentially stable since it consists of homodirectional states only)[2] drives the β subsystem, itself exponentially stable for $\tilde{\alpha}_1 = \ldots = \tilde{\alpha}_n = 0$. Thus, the cascade system is exponentially stable. ■

We now seek a backstepping transformation mapping System (33)–(35) to the target system (36)–(38).

2) Backstepping Transformation: To map System (33)–(35) to (36)–(38), we consider a backstepping transformation of the form, for $i = 1,\ldots,n$

$$\tilde{u}^i(t,x) = \tilde{\alpha}^i(t,x) + \int_0^x m^i(x,\xi)\tilde{\beta}(t,\xi)d\xi \quad (40)$$

$$\tilde{v}(t,x) = \tilde{\beta}(t,x) + \int_0^x m^{n+1}(x,\xi)\tilde{\beta}(t,\xi)d\xi \quad (41)$$

where the kernels $m^i(\cdot), i = 1,\ldots,n+1$ are defined on the triangular domain $\mathcal{T}$. Similarly to the control design case, we look

[2] i.e., states evolving in the same direction.

for sufficient conditions on the kernels by differentiating (40) with respect to time and space, using (36)–(38) and plugging the result in (33)–(35). This yields, after some computation, the following system of hyperbolic PDEs on the triangular domain $\mathcal{T}$ (see (42), shown at the bottom of the page) with boundary conditions

$$\begin{cases} m^1(x,x) = \frac{\omega_1(x)}{\lambda_1(x)+\mu(x)} \\ \vdots \\ m^n(x,x) = \frac{\omega_n(x)}{\lambda_n(x)+\mu(x)} \\ m^{n+1}(1,\xi) = \sum_{j=1}^{n} \rho_j m^j(1,\xi). \end{cases} \tag{43}$$

Besides, the observer gains are given by

$$\forall i = 1,\dots,n+1 \quad p_i(x) = -\mu(0)m^i(x,0) \tag{44}$$

and the integral coupling coefficients are defined by the following equations, for $i,j = 1,\dots,n$

$$h_i(x,\xi) = -\theta_i(\xi)m^{n+1}(x,\xi) - \int_{\xi}^{x} m^{n+1}(x,s)h_i(s,\xi)ds \tag{45}$$

$$g_{i,j}(x,\xi) = -\theta_j(\xi)m^i(x,\xi) - \int_{\xi}^{x} m^i(x,s)h_j(s,\xi)ds. \tag{46}$$

System equation (42) with boundary conditions (43) has the same structure as the system satisfied by the controller kernels equations (18), (19). The existence, uniqueness and continuity of solutions to both kernel systems is assessed in Theorem 5.3.

3) Inverse Transformation: Transformation (41) is a scalar Volterra integral equation of the second type. Since m^{n+1} is continuous by Theorem 5.3, there exists a unique continuous inverse kernel $r^{n+1}(\cdot)$ (see, e.g., [35]) such that

$$\tilde{\beta}(t,x) = \tilde{v}(t,x) + \int_0^x r^{n+1}(x,\xi)\tilde{v}(t,\xi)d\xi \tag{47}$$

implicitly defined on $\mathcal{T}$ by

$$r^{n+1}(x,\xi) = m^{n+1}(x,\xi) + \int_{\xi}^{x} m^{n+1}(x,s)r^{n+1}(s,\xi)ds. \tag{48}$$

Besides, plugging (47) into (40) yields, for each $i = 1,\dots,n$

$$\begin{aligned} \tilde{\alpha}^i(t,x) &= \tilde{u}^i(t,x) - \int_0^x m^i(x,\xi)\tilde{v}(t,\xi)d\xi \\ &\quad - \int_0^x \int_0^{\xi} m^i(x,\xi)r^{n+1}(\xi,s)\tilde{v}(t,s)dsd\xi \\ &= \tilde{u}^i(t,x) - \int_0^x m^i(x,\xi)\tilde{v}(t,\xi)d\xi \\ &\quad - \int_0^x \tilde{v}(t,\xi)\int_{\xi}^{x} m^i(x,s)r^{n+1}(s,\xi)dsd\xi \\ \tilde{\alpha}^i(t,x) &= \tilde{u}^i(t,x) + \int_0^x r^i(x,\xi)\tilde{v}(t,\xi)d\xi \end{aligned}$$

where we denote, for $i = 1,\dots,n$

$$r^i(x,\xi) = -m^i(x,\xi) - \int_{\xi}^{x} m^i(x,s)r^{n+1}(s,\xi)ds.$$

4) Observer Gains and Main Observer Result: The main result regarding the observer design is summarized in the following theorem.

Theorem 3.4: Consider system (33), (34) with boundary conditions (35), initial conditions $(\tilde{u}_0^1,\dots,\tilde{u}_0^n,\tilde{v}_0)$ and the gains $p_i(\cdot)$, $i = 1,\dots,n$ defined by (44). Then, under the following assumptions:

$$\begin{aligned} &\forall i,j = 1,\dots,n \ \lambda_i,\mu \in \mathcal{C}^1\left([0,1],\mathbb{R}_+^*\right),\ \sigma_{i,j},\omega_i,\theta_i \in \mathcal{C}\left([0,1]\right) \\ &\forall i = 1,\dots,n \quad \tilde{u}_0^i,\tilde{v}_0 \in \mathcal{L}^2\left([0,1]\right) \end{aligned}$$

the equilibrium $\tilde{\alpha}^1 \equiv \cdots \equiv \tilde{\alpha}^n \equiv \tilde{\beta} \equiv 0$ is exponentially stable in the $\mathcal{L}^2$ sense.

The proof of this theorem is identical to the controller design case (Theorem 3.2), and is therefore omitted.

IV. Output Feedback Control Law and Main Result

We now state the main result of the paper.

$$\begin{cases} \lambda_1(x)m_x^1 - \mu(\xi)m_\xi^1 = \mu'(\xi)m^1 + \sum_{j=1}^{n}\sigma_{1,j}(x)m^j + \omega_1(x)m^{n+1} \\ \vdots \\ \lambda_n(x)m_x^n - \mu(\xi)m_\xi^n = \mu'(\xi)m^n + \sum_{j=1}^{n}\sigma_{n,j}(x)m^j + \omega_n(x)m^{n+1}\mu(x)m_x^{n+1} + \mu(\xi)m_\xi^{n+1} = -\mu'(\xi)m^{n+1} - \sum_{i=1}^{n}\theta_i(x)m^i \end{cases} \tag{42}$$

Theorem 4.1: Consider system (1)–(3), (30)–(32) with initial conditions $(u_0^1,\ldots,u_0^n,v_0,\hat{u}_0^1,\ldots,\hat{u}_0^n,\hat{v}_0)$, and the following control law:

$$U(t) = -\sum_{i=1}^{n}\rho_i\hat{u}^i(t,1) + \int_0^1\left[\sum_{i=1}^{n}k^i(1,\xi)\hat{u}_i(t,\xi)+k^{n+1}(1,\xi)\hat{v}(t,\xi)\right]d\xi \quad (49)$$

where, for $i = 1,\ldots,n+1$, the k^i satisfy system equation (18) with boundary conditions (19). Then, under the following assumptions

$$\forall i,j=1,\ldots,n\ \lambda_i,\mu\in\mathcal{C}^1\left([0,1],\mathbb{R}_+^*\right),\sigma_{i,j},\omega_i,\theta_i\in\mathcal{C}([0,1])$$
$$\forall i=1,\ldots,n \quad u_0^i,\hat{u}_0^i,v_0,\hat{v}_0\in\mathcal{L}^2([0,1])$$

the equilibrium $(u^1,\ldots,u^n,v,\hat{u}^1,\ldots,\hat{u}^n,\hat{v})\equiv 0$ is exponentially stable in the $\mathcal{L}^2$ sense.

Proof: The existence of the controller kernel coefficients verifying (18) with boundary conditions (19) is proved by applying Theorem 5.3 with, for all $i,j=1,\ldots,n$

$$F^i(x,\xi)=k^i(x,\xi), \qquad G(x,\xi)=k^{n+1}(x,\xi) \quad (50)$$
$$\bar{\mu}(x)=\mu(x), \qquad \bar{\lambda}_i(x)=\lambda_i(x) \quad (51)$$

and

$$a_i(x,\xi)=\theta_i(\xi), \qquad b_{i,j}(x,\xi)=\sigma_{j,i}(\xi)+\delta_{i,j}\lambda_i'(\xi)$$
$$d(x,\xi)=-\mu'(\xi), \qquad e_i(x,\xi)=-\omega_i(\xi),$$
$$f_i(x)=-\frac{\theta_i(x)}{\lambda_i(x)+\mu(x)}, \qquad g_i(x)=\frac{q_i\lambda_i(0)}{\mu(0)}$$

where $\delta_{i,j}$ is the Kroenecker symbol. Similarly, the existence of the observer kernel coefficients verifying (42) with boundary conditions (43) is proved by applying Theorem 5.3 with, for $i,j=1,\ldots,n$

$$F^i(x,\xi)=m^i(1-\xi,1-x), \quad G(x,\xi)=m^{n+1}(1-\xi,1-x),$$
$$\bar{\mu}(x)=\mu(1-x), \qquad \bar{\lambda}_i(x)=\lambda_i(1-x)$$

and

$$a_i(x,\xi)=\omega_i(1-\xi), \quad b_{i,j}(x,\xi)=\sigma_{j,i}(1-\xi)+\delta_{i,j}\mu'(1-x)$$
$$d(x,\xi)=-\mu'(1-x), \quad e_i(x,\xi)=-\theta_i(1-\xi),$$
$$f_i(x)=-\frac{\theta_i(1-x)}{\lambda_i(1-x)+\mu(1-x)}, \quad g_i(x)=\rho_i.$$

Therefore, the assumptions of Theorems 3.2 and 3.4 are satisfied. The end of the proof consists in combining the Lyapunov approaches of these two theorems to yield the final result, similarly to the proof of [36, Th. 1]. First, denoting

$$\hat{w}(t,x)=\left(\hat{u}^1(t,x) \quad \cdots \quad \hat{u}^n(t,x) \quad \hat{v}(t,x)\right)^T$$

and $\tilde{w}=\hat{w}-w$, we define

$$\hat{\gamma}(t,x)=\hat{w}(t,x)-\int_0^x K(x,\xi)\hat{w}(t,\xi)d\xi$$

and $\tilde{\gamma}=\hat{\gamma}-\gamma$. It is straightforward to show that the $(\hat{\gamma},\tilde{\gamma})$-system reads

$$\hat{\alpha}_t^i(t,x)+\lambda_i(x)\hat{\alpha}_x^i(t,x) = \sum_{j=1}^{n}\sigma_{i,j}(x)\hat{\alpha}^j(t,x)+\omega_i(x)\hat{\beta}(t,x)-p_i(x)\tilde{v}(t,0) + \sum_{j=1}^{n}\int_0^x c_{i,j}(x,\xi)\hat{\alpha}^j(t,\xi)d\xi+\int_0^x\kappa_i(x,\xi)\hat{\beta}(t,\xi)d\xi$$

$$\hat{\beta}_t(t,x)-\mu(x)\hat{\beta}_x(t,x) = -\left[p_{n+1}(x)-\int_0^x\sum_{i=1}^{n}k^i(x,\xi)p_i(\xi)d\xi\right]\tilde{v}(t,0)$$

$$\tilde{\alpha}_t^i(t,x)+\lambda_i(x)\tilde{\alpha}_x^i(t,x) = \sum_{j=1}^{n}\sigma_{i,j}(x)\tilde{\alpha}^j(t,x)+\sum_{j=1}^{n}\int_0^x g_{i,j}(x,\xi)\tilde{\alpha}^j(t,\xi)d\xi,$$

$$\tilde{\beta}_t(t,x)-\mu(x)\tilde{\beta}_x(t,x) = \sum_{j=1}^{n}\theta_j(x)\tilde{\alpha}^j(t,x)+\sum_{j=1}^{n}\int_0^x h_j(x,\xi)\tilde{\alpha}^j(t,\xi)d\xi$$

with boundary conditions

$$\forall i=1,\ldots,n \quad \hat{\alpha}^i(t,0)=q_i\hat{\beta}(t,0) \qquad \hat{\beta}(t,1)=0,$$
$$\tilde{\alpha}^i(t,0)=0, \qquad \tilde{\beta}(t,1)=\sum_{i}^{n}\rho_i\tilde{\alpha}^i(t,1).$$

Similarly to Lemmas 3.1 and 3.3, exponential stability of the zero equilibrium of this system can be proved using the following Lyapunov function:

$$V(t)=\int_0^1 pe^{-\delta x}\sum_{i=1}^{n}\frac{\hat{\alpha}_i(t,x)^2}{\lambda_i(x)}dx+\int_0^1 e^{\delta x}\frac{\hat{\beta}(t,x)^2}{\mu(x)}dx + \int_0^1 qe^{-\delta x}\sum_{i=1}^{n}\frac{\tilde{\alpha}_i(t,x)^2}{\lambda_i(x)}dx+\int_0^1 e^{\delta x}\frac{\tilde{\beta}(t,x)^2}{\mu(x)}dx$$

or simply by noticing that the $(\hat{\gamma},\tilde{\gamma})$-system consists of a cascade of the exponentially stable $\tilde{\gamma}$ system into the $\hat{\gamma}$ system, which is exponentially stable when unforced (i.e., with $\tilde{v}(t,0)\equiv 0$). Stability of the original $(\hat{w},\tilde{w})$-system follows by invertibility of the transformations, similarly to the proofs of Theorems 3.2 and 3.4. Finally, using the identity $w=\hat{w}-\tilde{w}$, exponentially stability of the zero equilibrium of (1)–(4) follows. ■

V. Well-Posedness of the Kernel Equations

In this section, we investigate the existence, uniqueness and continuity of the solution to system (18) with boundary conditions (19) and, equivalently, system (42) with boundary conditions (43). Because both systems have very similar structures, we study a generic system of linear first-order hyperbolic PDEs on a triangular domain. After giving some preliminary results, we convert the system of hyperbolic PDEs into integral equations, using the method of characteristics. Then, we use the method of successive approximations to construct a solution to the integral equations in the form of a converging series.

A. Preliminary Results

To convert hyperbolic PDEs into integral equations, one must define characteristic curves in the (t,x)—plane along which the equations are integrated. One way to define these is to give an implicit solution to the characteristics equations as is done in [37]. Here, we prefer to use the two following lemmas.

Lemma 5.1: Let $(y_0, z_0) \in \mathbb{R}$ be such that

$$0 \leq y_0 \leq z_0 \leq 1$$

and $h \in \mathcal{C}^1([0,1])$ be such that $\forall x \in [0,1]\ h(x) < 0$. Then, if y and z are the maximal solutions of the following Cauchy problems:

$$y'(s) = h(y(s)), \quad y(0) = y_0, \quad z'(s) = h(z(s)), \quad z(0) = z_0 \tag{52}$$

then, there exists $T > 0$ such that $y(T) = 0$ and $z(T) \geq 0$.

Lemma 5.2: Let $(y_0, z_0) \in \mathbb{R}$ be such that

$$0 \leq y_0 \leq z_0 \leq 1$$

and $h, g \in \mathcal{C}^1([0,1])$ be such that $\forall x \in [0,1]\ g(x) > 0$ and $h(x) < 0$. Then, if y and z are the maximal solutions of the following Cauchy problems:

$$y'(s) = g(y(s)), \quad y(0) = y_0, \quad z'(s) = h(z(s)), \quad z(0) = z_0$$

then there exists $T > 0$ such that $y(T) = z(T)$.

The proofs of these lemmas are trivial and are omitted here for brevity purposes. Their interpretation and usefulness will appear clearly in Section V-C. In the next section, we state the main theorem regarding the existence of the kernel coefficients.

B. Existence of the Kernels

We now investigate existence of the controller and observer kernels. We show well-posedness of the following generic hyperbolic $(n+1) \times (n+1)$ system on the triangular domain $\mathcal{T} = \{(x,\xi) : 0 \leq \xi \leq x \leq 1\}$. For $i = 1, \dots, n$, the system equations read

$$\bar{\mu}(x) F_x^i - \bar{\lambda}_i(\xi) F_\xi^i = a_i(x,\xi) G + \sum_{j=1}^{n} b_{i,j}(x,\xi) F^j \tag{53}$$

$$\bar{\mu}(x) G_x + \bar{\mu}(\xi) G_\xi = d(x,\xi) G + \sum_{i=1}^{n} e_i(x,\xi) F^i \tag{54}$$

with boundary conditions, for $i = 1, \dots, n$

$$\forall x \in [0,1] \quad F^i(x,x) = f_i(x) \tag{55}$$

$$\forall x \in [0,1] \quad G(x,0) = \sum_{i=1}^{n} g_i(x) F^i(x,0). \tag{56}$$

The following theorem discusses existence and uniqueness of the solutions to (53)–(56).

Theorem 5.3: Under the following assumptions

$$\forall i,j = 1,\dots,n \quad a_i, b_{i,j}, d, e_{i,j} \in \mathcal{C}(\mathcal{T}), \quad f_i, g_i \in \mathcal{C}([0,1])$$
$$\forall i = 1,\dots,n \quad \bar{\lambda}_i, \bar{\mu} \in \mathcal{C}^0\left([0,1], \mathbb{R}_+^*\right)$$

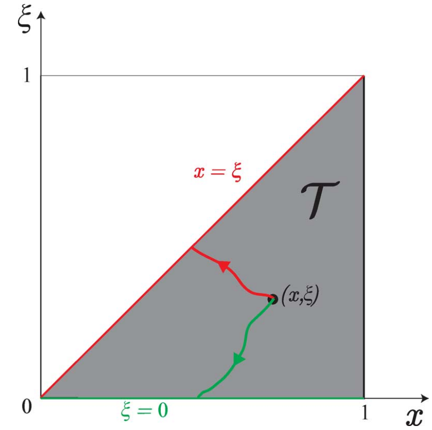


Fig. 4. Characteristic curves.

system equations (53)–(56) admit a unique continuous solution on $\mathcal{T}$.

The proof of Theorem 5.3 is contained in the next three sections. First, we transform system equations (53)–(56) into integral equations using the method of characteristics.

C. Transformation to Integral Equations

For (54), we define the characteristic curves (χ, ζ) along which the equations can be integrated as the solutions of the following Cauchy problems:

$$\begin{cases} \frac{d}{ds}\chi(x,\xi;s) = \bar{\mu}(\chi(x,\xi;s)), & s \in \left[0, s^F(x,\xi)\right], \\ \chi\left(x,\xi; s^F(x,\xi)\right) = x, & \chi(x,\xi;0) = \chi^0(x,\xi) \end{cases} \tag{57}$$

$$\begin{cases} \frac{d}{ds}\zeta(x,\xi;s) = \bar{\mu}(\xi(x,\xi;s)), & s \in \left[0, s^F(x,\xi)\right], \\ \zeta\left(x,\xi; s^F(x,\xi)\right) = \xi, & \zeta(x,\xi;0) = 0. \end{cases} \tag{58}$$

For each $(x,\xi) \in \mathcal{T}$, the existence of $s^F(x,\xi)$ such that there exists such solutions is proved by applying Lemma 5.1 with

$$h(x) = -\bar{\mu}(x), \qquad s^F(x,\xi) = T \tag{59}$$

$$\chi(x,\xi;s) = z\left(s^F(x,\xi) - s\right) \quad \zeta(x,\xi;s) = y\left(s^F(x,\xi) - s\right). \tag{60}$$

In other words, Lemma 5.1 ensures that, when solving the characteristic (57), (58) backward from a given point (x,ξ) in $\mathcal{T}$, one "hits" the boundary $\xi = 0$ depicted in green on Fig. 4. Similarly, for each equation of system (53), we define the characteristics curves (x_i, ξ_i) as the solutions of the following Cauchy problems:

$$\begin{cases} \frac{d}{ds}x_i(x,\xi;s) = \bar{\mu}(x_i(x,\xi;s)), & s \in \left[0, s_i^F(x,\xi)\right], \\ x_i\left(x,\xi; s_i^F(x,\xi)\right) = x, & x_i(x,\xi;0) = x_i^0(x,\xi) \end{cases} \tag{61}$$

$$\begin{cases} \frac{d}{ds}\xi_i(x,\xi;s) = -\bar{\lambda}_i(\xi_i(x,\xi;s)), & s \in \left[0, s_i^F(x,\xi)\right], \\ \xi_i\left(x,\xi; s_i^F(x,\xi)\right) = \xi, & \zeta(x,\xi;0) = x_i(x,\xi;0.) \end{cases} \tag{62}$$

Again, for each $(x,\xi) \in \mathcal{T}$ and each $i = 1,\dots,n$, the existence of $s_i^F(x,\xi)$ such that there exists such solutions is proved by applying Lemma 5.2 with

$$h(x) = -\bar{\mu}(x), \quad g(x) = \bar{\lambda}_i(x), \quad s_i^F(x,\xi) = T \tag{63}$$

$$x_i(x,\xi;s) = z\left(s_i^F(x,\xi) - s\right)$$
$$\xi_i(x,\xi;s) = y\left(s_i^F(x,\xi) - s\right). \tag{64}$$

Again, Lemma 5.2 ensures that, when solving the characteristics equations backward from a given point (x,ξ) in $\mathcal{T}$, one "hits" the boundary $x=\xi$ depicted in red on Fig. 4. Integrating (53) along there respective characteristic lines defined by (61), (62), between 0 and $s_i^F(x,\xi)$ yields, for all $i=1,\dots,n$

$$\begin{aligned} F^i(x,\xi) - F^i\left(x_i^0(x,\xi),0\right) &= \int_0^{s_i^F(x,\xi)} [a_i(x_i(x,\xi;s),\xi_i(x,\xi;s))G(x_i(x,\xi;s),\xi_i(x,\xi;s)) \\ &\quad + \sum_{j=1}^n b_{i,j}\left(x_i(x,\xi;s),\xi_i(x,\xi;s)\right) F^j \\ &\quad \times \left(x_i(x,\xi;s),\xi_i(x,\xi;s)\right)]\, ds. \end{aligned} \tag{65}$$

Using the boundary conditions (55) yields

$$\begin{aligned} F^i(x,\xi) =& f_i\left(x_i^0(x,\xi)\right) \\ &+ \int_0^{s_i^F(x,\xi)} [a_i\left(x_i(x,\xi;s),\xi_i(x,\xi;s)\right) G \\ &\quad \times \left(x_i(x,\xi;s),\xi_i(x,\xi;s)\right) \\ &\quad + \sum_{j=1}^n b_{i,j}\left(x_i(x,\xi;s),\xi_i(x,\xi;s)\right) F^j \\ &\quad \times \left(x_i(x,\xi;s),\xi_i(x,\xi;s)\right)]\, ds. \end{aligned} \tag{66}$$

Similarly, integrating (54) along the characteristic lines defined by (57), (58) between 0 and $s^F(x,\xi)$ and using boundary conditions (55) yields, for all $(x,\xi)\in\mathcal{T}$ and $i=1,\dots,n$

$$\begin{aligned} G(x,\xi) &= \sum_{i=1}^n g_i\left(\chi^0(x,\xi)\right) F^i\left(\chi^0(x,\xi),0\right) \\ &+ \int_0^{s^F(x,\xi)} [d(\chi(x,\xi;s),\zeta(x,\xi;s))\, G(\chi(x,\xi;s),z_i(x,\xi;s)) \\ &+ \sum_{i=1}^n e_i(\chi(x,\xi;s),\zeta(x,\xi;s)) F^i(\chi(x,\xi;s),z_i(x,\xi;s))] ds. \end{aligned} \tag{67}$$

Using the expression of F^i given by (66) to express $F^i(\chi^0(x,\xi),0)$ in (67) yields

$$\begin{aligned} G(x,\xi) &= \sum_{i=1}^n g_i\left(\chi^0(x,\xi)\right) \\ &\times \Bigg[f_i\left(x_i^0\left(\chi^0(x,\xi),0\right)\right) + \int_0^{s_i^F} \left(\chi^0(x,\xi),0\right) \\ &\quad \times [a_i\left(x_i\left(\chi^0(x,\xi),0;s\right),\xi_i\left(\chi^0(x,\xi),0;s\right)\right) \\ &\quad \cdot G\left(x_i\left(\chi^0(x,\xi),0;s\right),\xi_i\left(\chi^0(x,\xi),0;s\right)\right) \\ &\quad + \sum_{j=1}^n b_{i,j}\left(x_i\left(\chi^0(x,\xi),0;s\right),\xi_i\left(\chi^0(x,\xi),0;s\right)\right) \\ &\quad \cdot F^j\left(x_i\left(\chi^0(x,\xi),0;s\right),\xi_i\left(\chi^0(x,\xi),0;s\right)\right)]\, ds] \\ &+ \int_0^{s^F(x,\xi)} [d\left(\chi(x,\xi;s),\zeta(x,\xi;s)\right) G(\chi(x,\xi;s),\zeta(x,\xi;s)) \\ &\quad + \sum_{i=1}^n e_i\left(\chi(x,\xi;s),\zeta(x,\xi;s)\right) F^i \\ &\quad \times \left(\chi(x,\xi;s),z_i(x,\xi;s)\right)]\, ds. \end{aligned} \tag{68}$$

In the next section, we solve (66), for $i=1,\dots,n$ and (68) using the method of successive approximations.

D. Solution of the Integral Equations via a Successive Approximation Series

The successive approximation method can be used to solve the integral equations. Define first, for $i=1,\dots,n$

$$\begin{aligned} \varphi_i(x,\xi) &= f_i\left(x_i^0(x,\xi)\right) \\ \psi(x,\xi) &= \sum_{i=1}^n g_i\left(\chi^0(x,\xi)\right) f_i\left(x_i^0\left(\chi^0(x,\xi),0\right)\right). \end{aligned}$$

Besides, denoting

$$\mathbf{H} = \begin{bmatrix} F^1 & \cdots & F^n & G \end{bmatrix}^T \tag{69}$$

$$\boldsymbol{\phi}(x,\xi) = \begin{bmatrix} \varphi_1(x,\xi) & \cdots & \varphi_n(x,\xi) & \psi \end{bmatrix}^T \tag{70}$$

we define the following functionals acting on $\mathbf{H}$:

$$\begin{aligned} \Phi_i[\mathbf{H}](x,\xi) &= \int_0^{s_i^F(x,\xi)} [a_i\left(x_i(x,\xi;s),\xi_i(x,\xi;s)\right) G \\ &\quad \times \left(x_i(x,\xi;s),\xi_i(x,\xi;s)\right) \\ &\quad + \sum_{j=1}^n b_{i,j}\left(x_i(x,\xi;s),\xi_i(x,\xi;s)\right) F^j \\ &\quad \times \left(x_i(x,\xi;s),\xi_i(x,\xi;s)\right)]\, ds \end{aligned} \tag{71}$$

$$\begin{aligned} \Psi[\mathbf{H}](x,\xi) &= \sum_{i=1}^n g_i\left(\chi^0(x,\xi)\right) \cdot \int_0^{s_i^F\left(\chi^0(x,\xi),0\right)} \\ &\times [a_i\left(x_i\left(\chi^0(x,\xi),0;s\right),\xi_i\left(\chi^0(x,\xi),0;s\right)\right) \\ &\quad \cdot G\left(x_i\left(\chi^0(x,\xi),0;s\right),\xi_i\left(\chi^0(x,\xi),0;s\right)\right) \\ &\quad + \sum_{j=1}^n b_{i,j}\left(x_i\left(\chi^0(x,\xi),0;s\right),\xi_i\left(\chi^0(x,\xi),0;s\right)\right) \\ &\quad \cdot F^j\left(x_i\left(\chi^0(x,\xi),0;s\right),\xi_i\left(\chi^0(x,\xi),0;s\right)\right)]\, ds \\ &+ \int_0^{s^F(x,\xi)} [d\left(\chi(x,\xi;s),\zeta(x,\xi;s)\right) G \\ &\quad \times \left(\chi(x,\xi;s),\zeta(x,\xi;s)\right) \\ &\quad + \sum_{i=1}^n e_i\left(\chi(x,\xi;s),\zeta(x,\xi;s)\right) F^i \\ &\quad \times \left(\chi(x,\xi;s),z_i(x,\xi;s)\right)]\, ds. \end{aligned} \tag{72}$$

Define then the following sequence:

$$\begin{aligned}\mathbf{H}^0(x,\xi) &= 0,\\ \mathbf{H}^m(x,\xi) &= \boldsymbol{\phi}(x,\xi) + \boldsymbol{\Phi}[\mathbf{H}^{m-1}](x,\xi)\\ &= \begin{bmatrix}\phi_1(x,\xi)+\Phi_1[\mathbf{H}^{m-1}](x,\xi)\\ \vdots\\ \phi_n(x,\xi)+\Phi_n[\mathbf{H}^{m-1}](x,\xi)\\ \psi(x,\xi)+\Psi[\mathbf{H}^{m-1}](x,\xi)\end{bmatrix}\\ &= \begin{bmatrix}H_1^m(x,\xi)\\ \vdots\\ H_n^m(x,\xi)\\ H_{n+1}^m(x,\xi)\end{bmatrix}.\end{aligned} \tag{73}$$

Finally, define for $m \geq 1$ the increment $\Delta\mathbf{H}^m = \mathbf{H}^m - \mathbf{H}^{m-1}$, with $\Delta\mathbf{H}^0 = \boldsymbol{\phi}$ by definition. Since the functional $\boldsymbol{\Phi}$ is linear, the following equation $\Delta\mathbf{H}^m(x,\xi) = \boldsymbol{\Phi}[\mathbf{H}^{m-1}](x,\xi)$ holds.

If the limit exists, then $\mathbf{H} = \lim_{m\to+\infty} \mathbf{H}^m(x,\xi)$ is a solution of the integral equations, and thus solves the original hyperbolic system. Using the definition of $\Delta\mathbf{H}^m$, it follows that if the sum $\sum_{m=0}^{+\infty}\Delta\mathbf{H}^m(x,\xi)$ is finite, then

$$\mathbf{H}(x,\xi) = \sum_{m=0}^{+\infty}\Delta\mathbf{H}^m(x,\xi). \tag{74}$$

In the next section, we prove convergence of the series.

E. Proof of Convergence of the Successive Approximation Series

First define

$$\begin{aligned}M_\lambda &= \max_{x\in[0,1],i=1,\dots,n}\left\{\frac{1}{\bar{\lambda}_i(x)},\frac{1}{\bar{\mu}(x)}\right\}\\ \bar{a} &= \max_{(x,\xi)\in\mathcal{T},i=1,\dots,n}|a_i(x,\xi)| &(75)\\ \bar{b} &= \max_{(x,\xi)\in\mathcal{T},i,j=1,\dots,n}|b_{i,j}(x,\xi)|\\ \bar{d} &= \max_{(x,\xi)\in\mathcal{T},i=1,\dots,n}|d_i(x,\xi)| &(76)\\ \bar{e} &= \max_{(x,\xi)\in\mathcal{T},i,j=1,\dots,n}|e_{i,j}(x,\xi)|\\ \bar{g} &= \max\left\{1,\max_{\substack{(x,\xi)\in\mathcal{T},\\ i=1,\dots,n}}|g_i(x,\xi)|\right\} &(77)\\ \bar{\phi} &= \max_{\substack{(x,\xi)\in\mathcal{T},\\ i=1,\dots,n}}\{|\phi_i(x,\xi)|,|\psi_i(x,\xi)|\}\\ M &= n\bar{g}(\bar{a}+n\bar{b})+\bar{d}+n\bar{e}. &(78)\end{aligned}$$

Lemma 5.4: For $i = 1,\dots,n$, $p \geq 1$, $(x,\xi) \in \mathcal{T}$, and $s_i^F(x,\xi), \zeta_i^F(x,\xi), x_i(x,\xi,\cdot), \chi_i(c,\xi,\cdot)$ defined as in (57), (58), (61), (62), the following inequalities holds:

$$\int_0^{s_i^F(x,\xi)} x_i^m(x,\xi;s)ds \leq M_\lambda\frac{x^{m+1}}{m+1} \tag{79}$$

$$\int_0^{\zeta_i^F(x,\xi)} \chi^m(x,\xi;s)ds \leq M_\lambda\frac{x^{m+1}}{m+1}. \tag{80}$$

Proof: We first prove (79). Consider the following change of integration variable $\varsigma = x_i(x,\xi;s)$. Then,

$$d\varsigma = \frac{d}{ds}x_i(x,\xi;s)ds = \bar{\mu}\left(x_i(x,\xi;s)\right)ds. \tag{81}$$

Thus, the left-hand-side of (79) rewrites

$$\int_0^{s_i^F(x,\xi)} x_i^m(x,\xi;s)ds = \int_{x_i^0(x,\xi)}^{x}\frac{\varsigma^m}{\bar{\lambda}(\varsigma)}d\varsigma \leq M_\lambda\int_0^x\varsigma^m d\varsigma = M_\lambda\frac{x^{m+1}}{m+1}.$$

Inequality (80) is proved the same way using change of integration variable $\varsigma = \chi(x,\xi;s)$. ∎

Lemma 5.5: For $m \geq 1$, assume that, for

$$\forall (x,\xi)\in\mathcal{T},\ \forall i=1,\dots,n\quad \left|\Delta F^i(x,\xi)\right| \leq \bar{\phi}\frac{M^m x^m}{m!} \tag{82}$$

$$\text{and}\quad |\Delta G(x,\xi)| \leq \bar{\phi}\frac{M^m x^m}{m!} \tag{83}$$

then, it follows that

$$\forall (x,\xi)\in\mathcal{T},\ \forall i=1,\dots,n\quad |\Phi_i[\Delta\mathbf{H}](x,\xi)| \leq \bar{\phi}\frac{M^{m+1}x^{m+1}}{(m+1)!} \tag{84}$$

$$\text{and}\quad |\Psi[\Delta\mathbf{H}](x,\xi)| \leq \bar{\phi}\frac{M^{m+1}x^{m+1}}{(m+1)!}. \tag{85}$$

Proof: Assume that (83) holds. Then, for all $i = 1,\dots,n$ and $(x,\xi)\in\mathcal{T}$ one has, using the expression of Φ_i given by (71) and the inequality (83)

$$\begin{aligned}|\Phi_i[\Delta\mathbf{H}](x,\xi)| \leq \bar{a}\int_0^{s_i^F(x,\xi)}\bar{\phi}M^m\frac{x_i(x,\xi;s)^m}{m!}ds\\ +\sum_{j=1}^n\bar{b}\int_0^{s_i^F(x,\xi)}\bar{\phi}M^m\frac{x_i(x,\xi;s)^m}{m!}ds.\end{aligned}$$

Using Lemma 5.4, this yields

$$\begin{aligned}|\Phi_i[\Delta\mathbf{H}](x,\xi)| \leq \bar{a}\bar{\phi}M^m\frac{x_i(x,\xi;s)^{m+1}}{(m+1)!}\\ +n\bar{b}\bar{\phi}M^m\frac{x_i(x,\xi;s)^{m+1}}{(m+1)!}.\end{aligned}$$

Thus, using the fact that $x_i(x,\xi;s) \leq x$, this yields

$$\begin{aligned}|\Phi_i[\Delta\mathbf{H}](x,\xi)| &\leq \bar{\phi}[\bar{a}+n\bar{b}]M^m\frac{x^{m+1}}{(m+1)!}\\ &\leq \bar{\phi}M^{m+1}\frac{x^{m+1}}{(m+1)!}\end{aligned}$$

using the definition of M given by (78). Similarly, using the expression of Ψ given by (72) and the inequality (83), we have

$$\begin{aligned} &|\Psi[\Delta\mathbf{H}](x,\xi)| \\ &\leq \sum_{i=1}^{n}\bar{g}\int_0^{s_i^F(\chi^0(x,\xi),0)}\left[(\bar{a}+n\bar{b})\bar{\phi}\frac{M^m x_i\left(\chi^0(x,\xi),0;s\right)^m}{m!}\right]ds \\ &\quad+\int_0^{s^F(x,\xi)}(\bar{d}+n\bar{e})\bar{\phi}\frac{M^m\chi(x,\xi;s)^m}{(m)!}ds. \end{aligned}$$

Using Lemma 5.4, this yields

$$\begin{aligned} |\Psi[\Delta\mathbf{H}](x,\xi)| &\leq n\bar{g}(\bar{a}+n\bar{b})\bar{\phi}\frac{M^m\chi^0(x,\xi)^{m+1}}{(m+1)!} \\ &\quad+(\bar{d}+n\bar{e})\bar{\phi}\frac{M^m x^{m+1}}{(m+1)!} \\ &\leq \bar{\phi}\left[n\bar{g}(\bar{a}+n\bar{b})+\bar{d}+n\bar{e}\right]\frac{M^m x^{m+1}}{(m+1)!} \\ &\leq \bar{\phi}M^{m+1}\frac{x^{m+1}}{(m+1)!} \end{aligned}$$

which concludes the proof.

Finally, we prove that (74) converges.

Proposition 5.6: Consider the sequence $\mathbf{H}^m$, $m \geq 0$ defined by (73) and, for $i=1,\ldots,n+1$, $\Delta\mathbf{H}^m = \mathbf{H}^{m+1}-\mathbf{H}^m$. Then the following series normally converges on $\mathcal{T}$ and we have the upper bound

$$\forall(x,\xi)\in\mathcal{T}\quad \sum_{m=0}^{+\infty}\sup_{(x,\xi)\in\mathcal{T}}|\Delta\mathbf{H}_i^m(x,\xi)| \leq \bar{\phi}e^M.$$

Proof: The result follows if we show that for all $m \geq 0$, one has

$$\forall i=1,\ldots,m+1\quad |\Delta\mathbf{H}_i^m(x,\xi)| \leq \bar{\phi}\frac{M^m x^m}{m!} \leq \bar{\phi}\frac{M^m}{m!}. \tag{86}$$

We prove this result by induction. For $m=0$, it follows directly from the fact that $\Delta\mathbf{H}^0 = \phi$ and the definition of $\bar{\phi}$ given by (78). Assume that (86) holds for $m \geq 1$. Then, for $i=1,\ldots n$, one has by definition of ΔH_i^{m+1} and using Lemma 5.5

$$\begin{aligned} \left|\Delta H_i^{m+1}(x,\xi)\right| &= |\Phi_i[\Delta\mathbf{H}^m](x,\xi)| \\ &\leq \bar{\phi}\frac{M^{m+1}x^{m+1}}{(m+1)!}. \end{aligned}$$

Similarly one has, by definition of ΔH_{n+1}^{m+1} and using the same lemma

$$\begin{aligned} \left|\Delta H_{n+1}^{m+1}(x,\xi)\right| &= |\Psi[\Delta\mathbf{H}^m](x,\xi)| \\ &\leq \bar{\phi}\frac{M^{m+1}x^{m+1}}{(m+1)!} \end{aligned}$$

which concludes the proof. ■

The last result to prove is continuity of the sum (74). First, it is straightforward to show by induction that for all $m \in \mathbb{N}$, $\Delta\mathbf{H}^m$ is continuous on $\mathcal{T}$. Indeed $\boldsymbol{\phi}$ is continuous on $\mathcal{T}$ as a composition of continuous functions. Then, if $\Delta\mathbf{H}^m$ is continuous for some $m \in \mathbb{N}$, then $\Delta\mathbf{H}^{m+1}$ is continuous as the integral (with continuous limits of integration) of continuous functions times $\Delta\mathbf{H}^m$ composed with continuous functions. Finally, the normal convergence proved in Proposition 5.6 ensures continuity of the solutions on $\mathcal{T}$. The proof of uniqueness of the solutions, which directly relies on the linearity of the kernel equations, is identical to the one in [12]. For this reason and brevity purposes, we will not detail it here.

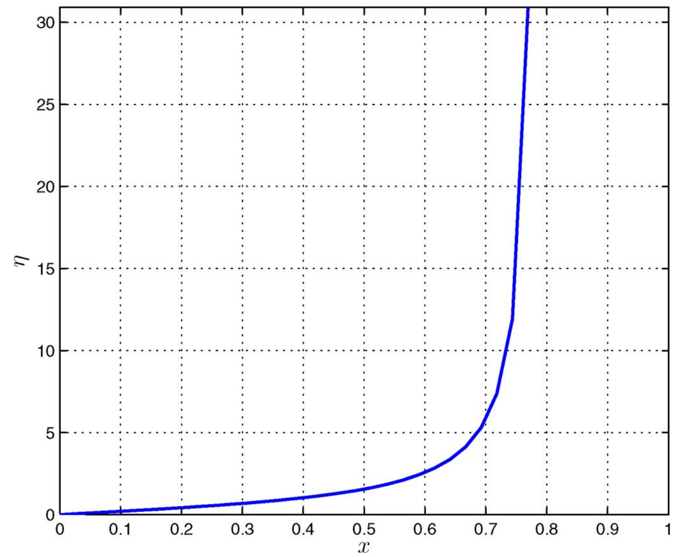


Fig. 5. Maximal solution of the Cauchy problem (87). The solution ceases to exist after $x = \pi/4$.

VI. Numerical Simulation

Several industrial processes are modeled by first-order hyperbolic system of the form studied here, such as irrigation channels [6] and oil and gas wells [16]. Applying our control design to such processes is the subject of future research requiring a complete physical treatment that is far beyond the space available in this paper. Rather, to illustrate our result, we implement the observer-controller scheme on a particularly challenging toy problem corresponding to $n = 2$. We design a system such that the zero equilibrium *cannot* be exponentially stabilized by a static output feedback law. Indeed, we choose the in-domain coupling coefficients as follows:

$$\begin{pmatrix}\sigma_{1,1} & \sigma_{1,2} & \omega_1\\ \sigma_{2,1} & \sigma_{2,2} & \omega_2\\ \theta_1 & \theta_2 & 0\end{pmatrix} = \begin{pmatrix}0 & 2 & 4\\ 0 & 0 & 2\\ 0 & 2 & 0\end{pmatrix}$$

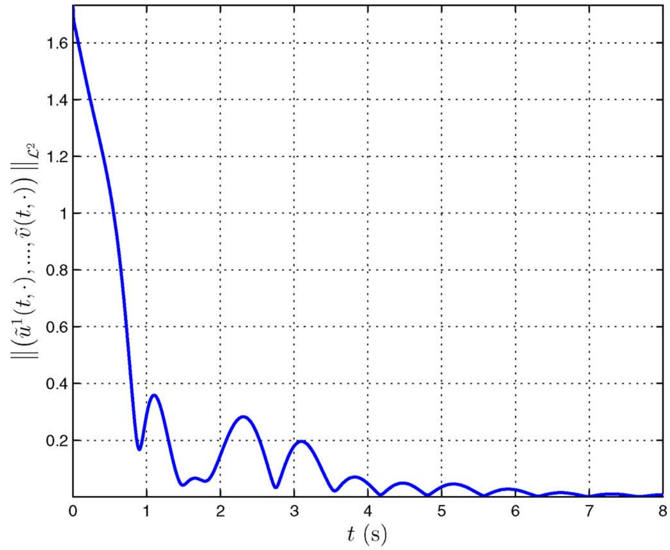
the boundary coefficients as follows:

$$q_1 = 1,\qquad q_2 = 1.2,\qquad \rho_1 = 0,\qquad \rho_2 = -0.8$$

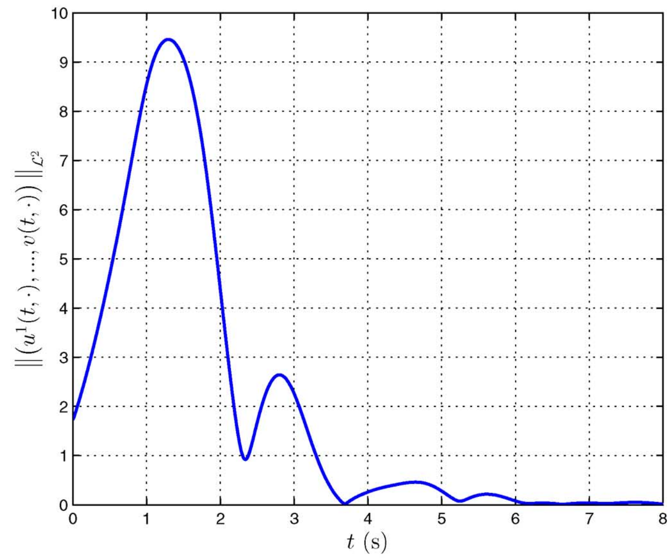
and the transport speeds as

$$\lambda_1 = \lambda_2 = \mu = 1.$$

This system is a cascade of the (u^2,v)–system into the u^1 system. As proved by [5, Th. 1], the (u^2,v)—system cannot be



(a)



(b)

Fig. 6. Time evolution of (a) the $\mathcal{L}^2$ norm of the observation error $\|(\tilde{u}^1(t,\cdot),\tilde{u}^2(t,\cdot),\tilde{v}(t,\cdot))\|_{\mathcal{L}^2}$ and (b) the $\mathcal{L}^2$ norm of the state $\|(u^1(t,\cdot),u^2(t,\cdot),v(t,\cdot))\|_{\mathcal{L}^2}$.

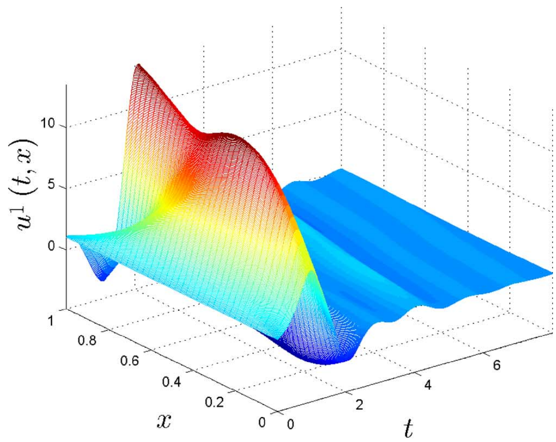
stabilized by a static output feedback control law, because the solution η to the following Cauchy problem on [0, 1]:

$$\eta' = \left|\frac{\omega_2}{\lambda_2} + \frac{\theta_2}{\mu}\eta^2\right|, \qquad \eta(0) = 0 \tag{87}$$

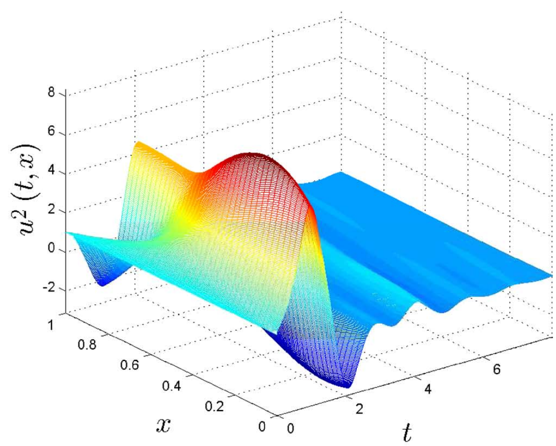
given by

$$\eta(x) = \tan(2x)$$

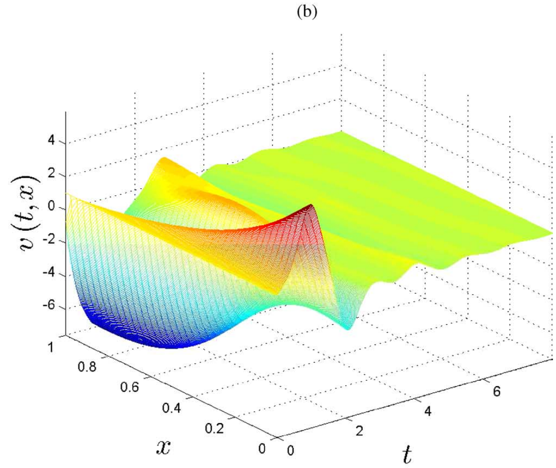
goes to infinity as x approches $\pi/4$ as depicted in Fig. 5. Equations (1)–(3) are discretized in space and time using an implicit characteristic scheme [13]. The time step is $\Delta t = 0.01$ and the spatial domain [0, 1] is divided in 40 intervals of equal length. Figs. 6–8(a) picture simulation results where the control law (49) is implemented to stabilize the zero equilibrium. The kernels are computed by solving, using a similar scheme, the linear



(a)



(b)



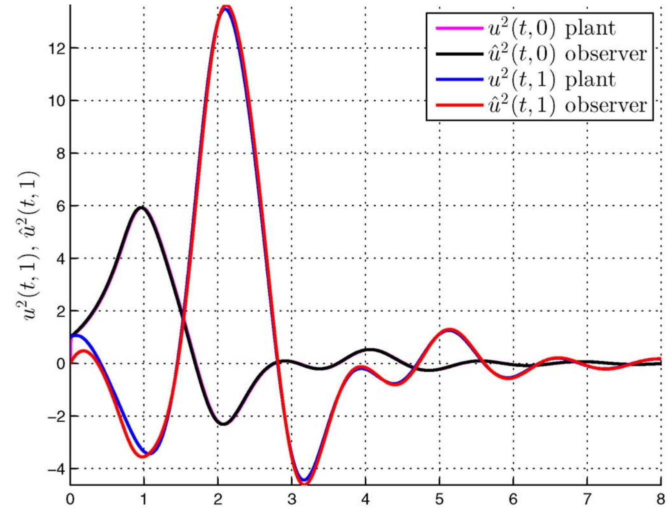
(c)

Fig. 7. Time evolution of the state profiles (a) $u^1(t,x)$, (b) $u^2(t,x)$, and (c) $v(t,x)$.

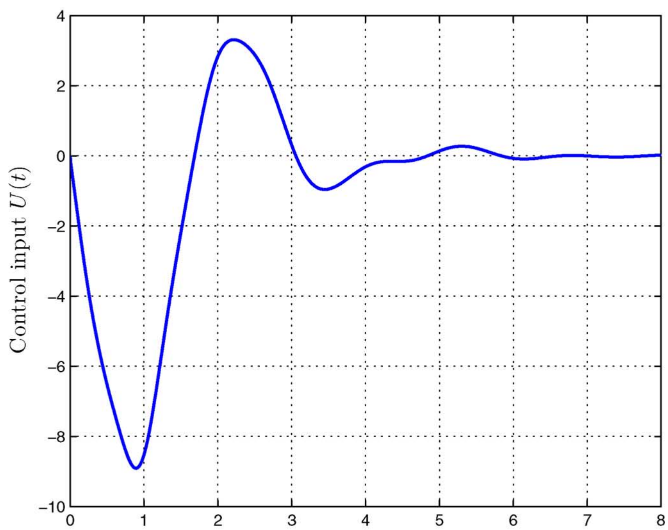
systems (18), (19), and (42), (43) respectively. Simulations results are plotted on Figs. 6–8. As expected by Theorem 4.1, both the $\mathcal{L}^2$ norms of the observer error and plant state, plotted on Fig. 6, asymptotically tend to zero as time goes to infinity after an initial overshoot.

VII. Open Problems

In the presented design, the observer and controller can be considered duals, since they lead to similar systems of equations for the backstepping transformation kernels. Unfortunately, in



(a)



(b)

Fig. 8. Comparison of the plant and observer state at (a) both boundaries and (b) control input.

a lot of applications, the design of a collocated observer is of greater interest, e.g., in the case of multiphase flow control in oil production systems [18]. Such an observer is more difficult to derive using the backstepping method, even when considering measurements of the n states "exiting" at the controlled boundary. The reason for this is the impossibility to add lower triangular integral coupling terms in the target system [as in (36), (37)] when the sensor is located at $x = 1$, because the backstepping transformation is usually *upper* triangular in that case (see, e.g., [37]). For the same reason, the generalization of the observer and controller designs to systems with n positive and m negative transport speeds remains an open problem for $m \geq 2$.

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Florent Di Meglio received the Ph.D. degree in mathematics and control from MINES ParisTech, Paris, France in 2011.

He is currently an Assistant Professor at the Systems and Control Center at MINES ParisTech. His interest include distributed parameters systems, nonlinear control and estimation, and signal processing with applications to process control, multiphase flow, and oil and gas drilling.

Rafael Vazquez (S'05–M'08) received the M.S. and Ph.D. degrees in aerospace engineering from the University of California, San Diego, CA, USA, and degrees in electrical engineering and mathematics from the University of Seville, Seville, Spain.

He is currently an Associate Professor in the Aerospace Engineering Department, University of Seville. His research interests include control theory, distributed parameter systems, and optimization, with applications to flow control, air traffic management, UAVs, and orbital mechanics. He is coauthor of the book *Control of Turbulent and Magnetohydrodynamic Channel Flows* (Birkhauser, 2007).

Miroslav Krstic (S'92–M'95–SM'99–F'02) holds the Daniel L. Alspach Endowed Chair and is the Founding Director of the Cymer Center for Control Systems and Dynamics, University of California—San Diego (UCSD), La Jolla, CA, USA. . He also serves as Associate Vice Chancellor for Research at UCSD.

Dr. Krstic is a recipient of the PECASE, NSF Career, and ONR Young Investigator Awards, as well as the Axelby and Schuck Paper Prizes. He was the first recipient of the UCSD Research Award in the area of engineering (immediately following the Nobel laureate in Chemistry Roger Tsien) and has held the Russell Severance Springer Distinguished Visiting Professorship at UC Berkeley and the Harold W. Sorenson Distinguished Professorship at UCSD. He is a Fellow of IFAC and serves as Senior Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and *Automatica*. He has served as Vice President of the IEEE Control Systems Society and Chair of the IEEE CSS Fellow Committee. He has coauthored nine books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.