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Stochastic nonlinear stabilization – II: Inverse optimality[1]

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Abstract

After considering the stabilization of a specific class of stochastic nonlinear systems in a companion paper, in this second part, we address the classical question of when is a stabilizing (in probability) controller optimal and show that for every system with a *stochastic control Lyapunov function* it is possible to construct a controller which is optimal with respect to a meaningful cost functional. Then we return to the problem from Part I and design an optimal backstepping controller whose cost functional includes penalty on control effort and which has an infinite gain margin. © 1997 Published by Elsevier Science B.V.

Keywords: Stochastic nonlinear systems; Control Lyapunov functions; Backstepping

1. Introduction

After considering the stabilization of strict-feedback stochastic systems in a companion paper [2], in this paper we present the following new results: For general stochastic systems affine in the control and noise inputs, we design stabilizing control laws which are also optimal with respect to meaningful cost functionals. This result is a stochastic counterpart of the inverse optimality result of Freeman and Kokotović [4] for systems with deterministic uncertainties. Furthermore, we show how our backstepping design from [2], which achieves stability, can be redesigned to also achieve inverse optimality. In contrast to the design of Pan and Başar [9], our cost functional includes penalty on the control effort and has an infinite gain margin [10] (namely, the property that it remains stabilizing when multiplied by any constant no smaller than one), which is one of the main advantages of inverse optimality.

2. Stochastic control Lyapunov functions

Consider the system which, in addition to the noise input w, has a control input u:

$$dx = f(x)\,dt + g_1(x)\,dw + g_2(x)u\,dt, \tag{2.1}$$

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where $f(0)=0$, $g_1(0)=0$ and $u \in \mathbb{R}^m$. We say that the system is *globally asymptotically stabilizable in probability* if there exists a control law $u=\alpha(x)$ continuous away from the origin, with $\alpha(0)=0$, such that the equilibrium $x=0$ of the closed-loop system is globally asymptotically stable in probability.

Definition 2.1. A smooth positive-definite radially unbounded function $V:\mathbb{R}^n \to \mathbb{R}_+$ is called a *stochastic control Lyapunov function* (*sclf*) for system (2.1) if it satisfies

$$\inf_{u\in\mathbb{R}^m}\left\{L_f V + \frac{1}{2}\mathrm{Tr}\left\{g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right\} + L_{g_2}Vu\right\} < 0, \quad \forall x \neq 0. \tag{2.2}$$

The existence of an sclf guarantees global asymptotic stabilizability in probability, as shown in the following theorem whose proof we incorporate into the proof of Theorem 3.1 for completeness.

Theorem 2.1 (Florchinger [3]). *The system* (2.1) *is globally asymptotically stabilizable in probability if there exists an sclf.*

3. Inverse optimal stabilization in probability

Definition 3.1. The problem of *inverse optimal stabilization in probability* for system (2.1) is solvable if there exist a class $\mathscr{K}_\infty$ function γ_2 whose derivative γ_2' is also a class $\mathscr{K}_\infty$ function, a matrix-valued function $R_2(x)$ such that $R_2(x)=R_2(x)^{\mathrm{T}}>0$ for all x, a positive definite radially unbounded function $l(x)$, and a feedback control law $u=\alpha(x)$ continuous away from the origin with $\alpha(0)=0$, which guarantees global asymptotic stability in probability of the equilibrium $x=0$ and minimizes the cost functional

$$J(u)=E\left\{\int_0^\infty [l(x)+\gamma_2(|R_2(x)^{1/2}u|)]\,\mathrm{d}\tau\right\}. \tag{3.1}$$

In the language of stochastic risk-sensitive optimal control [1], the cost functional (3.1) would correspond to the risk-neutral case.

In the next theorem we extensively use the Legendre–Fenchel transform given as follows: For a class $\mathscr{K}_\infty$ function γ, whose derivative γ' is also a class $\mathscr{K}_\infty$ function, $\ell\gamma$ denotes

$$\ell\gamma(r)=\int_0^r(\gamma')^{-1}(s)\,\mathrm{d}s. \tag{3.2}$$

Theorem 3.1. *Consider the control law*

$$u=\alpha(x)=-R_2^{-1}(L_{g_2}V)^{\mathrm{T}}\frac{\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|)}{|L_{g_2}VR_2^{-1/2}|^2}, \tag{3.3}$$

where $V(x)$ is a Lyapunov function candidate, γ_2 is a class $\mathscr{K}_\infty$ function whose derivative is also a class $\mathscr{K}_\infty$ function, and $R_2(x)$ is a matrix-valued function such that $R_2(x)=R_2(x)^{\mathrm{T}}>0$. If the control law (3.3) *achieves global asymptotic stability in probability for the system* (2.1) *with respect to $V(x)$, then the control law*

$$u^*=\alpha^*(x)=-\frac{\beta}{2}R_2^{-1}(L_{g_2}V)^{\mathrm{T}}\frac{(\gamma_2')^{-1}(|L_{g_2}VR_2^{-1/2}|)}{|L_{g_2}VR_2^{-1/2}|}, \quad \beta\geqslant 2 \tag{3.4}$$

solves the problem of inverse optimal stabilization in probability for the system (2.1) *by minimizing the cost functional*

$$J(u)=E\left\{\int_0^\infty\left[l(x)+\beta^2\gamma_2\left(\frac{2}{\beta}|R_2^{1/2}u|\right)\right]\mathrm{d}\tau\right\}, \tag{3.5}$$

where

$$l(x) = 2\beta\left[\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|) - L_fV - \frac{1}{2}\mathrm{Tr}\left\{g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right\}\right] + \beta(\beta-2)\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|). \tag{3.6}$$

Proof. Since the control law (3.3) globally asymptotically stabilizes the system in probability, then there exists a continuous positive-definite function $W:\mathbb{R}^n \to \mathbb{R}_+$ such that

$$\begin{aligned}\mathscr{L}V|_{(3.3)} &= L_fV + \frac{1}{2}\mathrm{Tr}\left\{g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right\} + L_{g_2}V\alpha(x) \\ &= -\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|) + L_fV + \frac{1}{2}\mathrm{Tr}\left\{g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right\} \leqslant -W(x). \end{aligned} \tag{3.7}$$

Then we have

$$l(x) \geqslant 2\beta W(x) + \beta(\beta-2)\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|). \tag{3.8}$$

Since $W(x)$ is positive definite, $\beta \geqslant 2$, and $\ell\gamma_2$ is a class $\mathscr{K}_\infty$ function (Lemma A.1), $l(x)$ is positive definite. Therefore $J(u)$ is a meaningful cost functional.

Before we engage into proving that the control law (3.4) minimizes (3.5), we first show that it is stabilizing. With Lemma A.1 we get

$$\begin{aligned}\mathscr{L}V|_{(3.4)} &= L_fV + \frac{1}{2}\mathrm{Tr}\left\{g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right\} - \frac{\beta}{2}|L_{g_2}VR_2^{-1/2}|(\gamma_2')^{-1}(|L_{g_2}VR_2^{-1/2}|) \\ &= L_fV + \frac{1}{2}\mathrm{Tr}\left\{g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right\} - \frac{\beta}{2}[\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|) + \gamma_2((\gamma_2')^{-1}(|L_{g_2}VR_2^{-1/2}|))] \\ &\leqslant \mathscr{L}V|_{(3.3)} < 0, \quad \forall x \neq 0, \end{aligned} \tag{3.9}$$

which proves that (3.4) achieves global asymptotic stability in probability and, in particular, that $x(t) \to 0$ with probability 1.

Now we prove optimality. Recalling that the Itô differential of V is

$$\mathrm{d}V = \mathscr{L}V(x)\,\mathrm{d}t + \frac{\partial V}{\partial x}g_1(x)\,\mathrm{d}w, \tag{3.10}$$

according to the property of Itô's integral [8, Theorem 3.9], we get

$$E\left\{V(0) - V(t) + \int_0^t \mathscr{L}V(x(\tau))\,\mathrm{d}\tau\right\} = 0. \tag{3.11}$$

Then substituting $l(x)$ into $J(u)$, we have

$$\begin{aligned}J(u) &= E\left\{\int_0^\infty \left[l(x) + \beta^2\gamma_2\left(\frac{2}{\beta}|R_2^{1/2}u|\right)\right]\mathrm{d}\tau\right\} \\ &= 2\beta E\{V(x(0))\} + E\left\{\int_0^\infty \left[2\beta\mathscr{L}V|_{(2.1)} + l(x) + \beta^2\gamma_2\left(\frac{2}{\beta}|R_2^{1/2}u|\right)\right]\mathrm{d}\tau\right\} - 2\beta\lim_{t\to\infty}E\{V(x(t))\} \\ &= 2\beta E\{V(x(0))\} + E\left\{\int_0^\infty \left[\beta^2\gamma_2\left(\frac{2}{\beta}|R_2^{1/2}u|\right) + \beta^2\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|) + 2\beta L_{g_2}Vu\right]\mathrm{d}\tau\right\} \\ &\quad -2\beta\lim_{t\to\infty}E\{V(x(t))\}. \end{aligned} \tag{3.12}$$

Now we note that

$$\gamma_2'\left(\frac{2}{\beta}|R_2^{1/2}u^*|\right) = |L_{g_2}VR_2^{-1/2}|, \tag{3.13}$$

which yields

$$J(u) = 2\beta E\{V(x(0))\} + E\left\{\int_0^\infty \left[\beta^2\gamma_2\left(\left|\frac{2}{\beta}R_2^{1/2}u\right|\right) + \beta^2\ell\gamma_2\left(\gamma_2'\left(\left|\frac{2}{\beta}R_2^{1/2}u^*\right|\right)\right)\right.\right.$$
$$\left.\left. - 2\beta\gamma_2'\left(\left|\frac{2}{\beta}R_2^{1/2}u^*\right|\right)\frac{(2/\beta R_2^{1/2}u^*)^{\mathrm{T}}}{|2/\beta R_2^{1/2}u^*|}R_2^{1/2}u\right]\mathrm{d}\tau\right\} - 2\beta\lim_{t\to\infty}E\{V(x(t))\}. \tag{3.14}$$

With the general Young's inequality (Lemma A.2), we obtain

$$J(u) \geqslant 2\beta E\{V(x(0))\} + E\left\{\int_0^\infty \left[\beta^2\gamma_2\left(\left|\frac{2}{\beta}R_2^{1/2}u\right|\right) + \beta^2\ell\gamma_2\left(\gamma_2'\left(\left|\frac{2}{\beta}R_2^{1/2}u^*\right|\right)\right)\right.\right.$$
$$\left.\left. -\beta^2\gamma_2\left(\left|\frac{2}{\beta}R_2^{1/2}u\right|\right) - \beta^2\ell\gamma_2\left(\gamma_2'\left(\left|\frac{2}{\beta}R_2^{1/2}u^*\right|\right)\right)\right]\mathrm{d}\tau\right\} - 2\beta\lim_{t\to\infty}E\{V(x(t))\}$$
$$= 2\beta E\{V(x(0))\} - 2\beta\lim_{t\to\infty}E\{V(x(t))\}, \tag{3.15}$$

where the equality holds if and only if

$$\gamma_2'\left(\left|\frac{2}{\beta}R_2^{1/2}u^*\right|\right)\frac{(2/\beta R_2^{1/2}u^*)^{\mathrm{T}}}{|2/\beta R_2^{1/2}u^*|} = \gamma_2'\left(\left|\frac{2}{\beta}R_2^{1/2}u\right|\right)\frac{(2/\beta R_2^{1/2}u)^{\mathrm{T}}}{|2/\beta R_2^{1/2}u|}, \tag{3.16}$$

i.e. $u=u^*$. Since $u=u^*$ is stabilizing in probability, $\lim_{t\to\infty}E\{V(x(t))\}=0$, and thus

$$\arg\min_u J(u) = u^*, \tag{3.17}$$

$$\min_u J(u) = 2\beta E\{V(x(0))\}. \tag{3.18}$$

To satisfy the requirements of Definition 3.1, it only remains to prove that $\alpha^*(x)$ is continuous and $\alpha^*(0)=0$. Because g_2, R_2 and $\partial V/\partial x$ are continuous functions, and $(\gamma_2')^{-1}$ is a class $\mathscr{K}_\infty$ function, $\alpha^*(x)$ is continuous away from $L_{g_2}VR_2^{-1/2}=0$. If $|L_{g_2}VR_2^{-1/2}|\to 0$, continuity can be inferred from the fact that

$$\lim_{|L_{g_2}VR_2^{-1/2}|\to 0}|\alpha^*(x)| = \lim_{|L_{g_2}VR_2^{-1/2}|\to 0}\left\{\frac{\beta}{2}|R_2^{-1/2}||L_{g_2}VR_2^{-1/2}|\frac{(\gamma_2')^{-1}(|L_{g_2}VR_2^{-1/2}|)}{|L_{g_2}VR_2^{-1/2}|}\right\}$$
$$= \lim_{|L_{g_2}VR_2^{-1/2}|\to 0}\left\{\frac{\beta}{2}|R_2^{-1/2}|(\gamma_2')^{-1}(|L_{g_2}VR_2^{-1/2}|)\right\} = 0. \tag{3.19}$$

Since $\partial V(0)/\partial x=0$, $L_{g_2}V(0)=0$ and we have $\alpha^*(0)=0$. □

Remark 3.1. Even though not explicit in the proof of Theorem 3.1, $V(x)$ solves the following family of *Hamilton–Jacobi–Bellman* equations parameterized by $\beta\in[2,\infty)$:

$$L_fV + \frac{1}{2}\mathrm{Tr}\left\{g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right\} - \frac{\beta}{2}\ell\gamma_2(|L_{g_2}VR_2^{-1/2}|) + \frac{l(x)}{2\beta} = 0. \tag{3.20}$$

In the next corollary we design controllers which are inverse optimal in the sense of Definition 2.1.

Corollary 3.1. *If the system* (2.1) *has an sclf, then the problem of inverse optimal stabilization in probability is solvable.*

Proof. Let $\gamma_2(r)=\frac{1}{4}r^2$, $\beta=2$, and

$$R_2(x)=I\begin{cases} -\dfrac{2L_{g_2}V(L_{g_2}V)^{\mathrm T}}{\omega+\sqrt{\omega^2+(L_{g_2}V(L_{g_2}V)^{\mathrm T})^2}}, & L_{g_2}V\neq 0,\\ \text{any positive number}, & L_{g_2}V=0. \end{cases} \tag{3.21}$$

Then, with $(\gamma_2')^{-1}(r)=2r$, the optimal control candidate (3.4) is given by Sontag's formula [11]

$$u^*=\alpha_s(x)=\begin{cases} -\dfrac{\omega+\sqrt{\omega^2+(L_{g_2}V(L_{g_2}V)^{\mathrm T})^2}}{L_{g_2}V(L_{g_2}V)^{\mathrm T}}(L_{g_2}V)^{\mathrm T}, & L_{g_2}V\neq 0,\\ 0, & L_{g_2}V=0, \end{cases} \tag{3.22}$$

where

$$\omega=L_fV+\frac{1}{2}\mathrm{Tr}\left\{g_1^{\mathrm T}\frac{\partial^2 V}{\partial x^2}g_1\right\}. \tag{3.23}$$

According to [11], this formula is smooth away from the origin because (2.2) implies that, when $x\neq 0$,

$$L_{g_2}V=0\Rightarrow \omega<0. \tag{3.24}$$

We note that the control law $\alpha_s(x)$ will also be continuous at the origin if and only if the sclf $V(x)$ satisfies the *small control property* [11] (the property that there exist *some* control law continuous at the origin which is stabilizing with respect to $V(x)$).

To prove that $\alpha_s(x)$ is inverse optimal, we prove that the control law (3.3) is stabilizing. Since $\ell\gamma_2(r)=r^2$, (3.3) becomes $u=\frac{1}{2}\alpha_s(x)$. Then the infinitesimal generator of the system (2.1) with the control law $u=\frac{1}{2}\alpha_s(x)$ is

$$\mathscr{L}V=-\frac{1}{2}\left[-\omega+\sqrt{\omega^2+(L_{g_2}V(L_{g_2}V)^{\mathrm T})^2}\right], \tag{3.25}$$

which is negative definite because of (3.24). □

4. Inverse optimal stabilization via backstepping

The controller given in Table 1 in [2] is stabilizing but is not inverse optimal because it is not of the form (3.4). In this section we will redesign the stabilizing functions $\alpha_i(x)$ to get an inverse optimal control law.

To design a control law in the form (3.4), we first note from $V=\sum_{i=1}^n \frac{1}{4}z_i^4$ that for system

$$\mathrm{d}x_i=x_{i+1}\,\mathrm{d}t+\varphi_i(\bar{x}_i)^{\mathrm T}\,\mathrm{d}w,\quad i=1,\dots,n-1, \tag{4.1}$$

$$\mathrm{d}x_n=u\,\mathrm{d}t+\varphi_n(\bar{x}_n)^{\mathrm T}\,\mathrm{d}w, \tag{4.2}$$

$L_{g_2}V=z_n^3$ and, since u is a scalar input, $R_2(x)$ is sought as a scalar positive function. The following lemma is instrumental in motivating the inverse optimal design.

Lemma 4.1. *If there exists a continuous positive function $M(x)$ such that the control law*

$$u=\alpha(x)=-M(x)z_n \tag{4.3}$$

globally asymptotically stabilizes system (2.1) *in probability with respect to the Lyapunov function $V=\sum_{i=1}^n \frac{1}{4}z_i^4$, then the control law*

$$u^*=\alpha^*(x)=\tfrac{2}{3}\beta\alpha(x),\quad \beta\geqslant 2 \tag{4.4}$$

solves the problem of inverse optimal stabilization in probability.

Proof. Let

$$\gamma_2(r) = \tfrac{1}{4}r^4, \tag{4.5}$$

$$R_2 = (\tfrac{4}{3}M)^{-3/2}. \tag{4.6}$$

Then the control laws (3.3) and (3.4) are given, respectively, as

$$u = \alpha(x) = -\tfrac{3}{4}R_2^{-2/3}z_n, \tag{4.7}$$

$$u^* = \alpha^*(x) = -\tfrac{1}{2}\beta R_2^{-2/3}z_n. \tag{4.8}$$

By dividing the last two expressions, we get (4.4), and the lemma follows by Theorem 3.1. □

From the lemma it follows that we should seek a stabilizing control law in the form of (4.3). If we consider carefully the parenthesis including u in (3.21) in [2], every term except the second, the third, and the sixth has z_n as a factor. We now deal with the three terms one at a time. The second term yields

$$\begin{aligned}
&-z_n^3\sum_{l=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_l}x_{l+1}\\
&= -z_n^3\sum_{l=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_l}\left(z_{l+1}+\sum_{k=1}^{l}z_k\alpha_{lk}\right)\\
&= -z_n^4\frac{\partial\alpha_{n-1}}{\partial x_{n-1}} - \sum_{l=1}^{n-2}z_n^3\frac{\partial\alpha_{n-1}}{\partial x_l}z_{l+1} - \sum_{l=1}^{n-1}z_n^3\frac{\partial\alpha_{n-1}}{\partial x_l}\sum_{k=1}^{l}z_k\alpha_{lk}\\
&\leqslant z_n^4\left(\frac{1}{2}+\frac{1}{2}\left(\frac{\partial\alpha_{n-1}}{\partial x_{n-1}}\right)^2\right) + \sum_{l=1}^{n-2}\left[\frac{3}{4}z_n^4\left(\delta_l\frac{\partial\alpha_{n-1}}{\partial x_l}\right)^{4/3} + \frac{1}{4\delta_l^4}z_{l+1}^4\right]\\
&\quad + \sum_{l=1}^{n-1}\sum_{k=1}^{l}\left[\frac{3}{4}z_n^4\left(\delta_{lk}\frac{\partial\alpha_{n-1}}{\partial x_l}\alpha_{lk}\right)^{4/3} + \frac{1}{4\delta_{lk}^4}z_k^4\right]\\
&= z_n^4\left[\left(\frac{1}{2}+\frac{1}{2}\left(\frac{\partial\alpha_{n-1}}{\partial x_{n-1}}\right)^2\right) + \frac{3}{4}\sum_{l=1}^{n-2}\left(\delta_l\frac{\partial\alpha_{n-1}}{\partial x_l}\right)^{4/3} + \frac{3}{4}\sum_{l=1}^{n-1}\sum_{k=1}^{l}\left(\delta_{lk}\frac{\partial\alpha_{n-1}}{\partial x_l}\alpha_{lk}\right)^{4/3}\right]\\
&\quad + \sum_{l=1}^{n-1}\sum_{k=1}^{l}\frac{1}{4\delta_{lk}^4}z_k^4 + \sum_{l=2}^{n-1}\frac{1}{4\delta_{l-1}^4}z_l^4\\
&= z_n^4\left[\left(\frac{1}{2}+\frac{1}{2}\left(\frac{\partial\alpha_{n-1}}{\partial x_{n-1}}\right)^2\right) + \frac{3}{4}\sum_{l=1}^{n-2}\left(\delta_l\frac{\partial\alpha_{n-1}}{\partial x_l}\right)^{4/3} + \frac{3}{4}\sum_{l=1}^{n-1}\sum_{k=1}^{l}\left(\delta_{lk}\frac{\partial\alpha_{n-1}}{\partial x_l}\alpha_{lk}\right)^{4/3}\right]\\
&\quad + \sum_{k=1}^{n-1}z_k^4\sum_{l=k}^{n-1}\frac{1}{4\delta_{lk}^4} + \sum_{l=2}^{n-1}\frac{1}{4\delta_{l-1}^4}z_l^4,
\end{aligned} \tag{4.9}$$

where the inequality is obtained by applying Young's inequality,

$$-z_n^3\frac{\partial\alpha_{n-1}}{\partial x_l}z_{l+1} \leqslant \frac{3}{4}z_n^4\left(\delta_l\frac{\partial\alpha_{n-1}}{\partial x_l}\right)^{4/3} + \frac{1}{4\delta_l^4}z_{l+1}^4, \tag{4.10}$$

$$z_n^3 \frac{\partial \alpha_{n-1}}{\partial x_l} z_k \alpha_{lk} \leqslant \frac{3}{4} z_n^4 \left(\delta_{lk} \frac{\partial \alpha_{n-1}}{\partial x_l} \alpha_{lk} \right)^{4/3} + \frac{1}{4\delta_{lk}^4} z_k^4 \tag{4.11}$$

$$-\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} \leqslant \frac{1}{2} + \frac{1}{2} \left(\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} \right)^2, \tag{4.12}$$

and the last equation comes from changing the summation order in the second term. As to the third and the sixth terms, in the first parenthesis in (3.21) in [2], we have

$$\begin{aligned}
&-\frac{1}{2} z_n^3 \sum_{p,q=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q} \varphi_p^{\mathrm{T}} \varphi_q \\
&= -\frac{1}{2} z_n^3 \sum_{p,q=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q} \left(\sum_{k=1}^{p} z_k \psi_{pk} \right)^{\mathrm{T}} \left(\sum_{l=1}^{q} z_l \psi_{ql} \right) \\
&= -\frac{1}{2} \sum_{p,q=1}^{n-1} \sum_{k=1}^{p} \sum_{l=1}^{q} \left(z_n^3 \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q} \psi_{pk}^{\mathrm{T}} \psi_{ql} \right) z_k z_l \\
&\leqslant \frac{1}{2} \sum_{p,q=1}^{n-1} \sum_{k=1}^{p} \sum_{l=1}^{q} \left[\frac{3}{4} \delta_{pqkl}^{4/3} z_n^4 \left(\frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q} \psi_{pk}^{\mathrm{T}} \psi_{ql} z_k \right)^{4/3} + \frac{1}{4\delta_{pqkl}^4} z_l^4 \right] \\
&\leqslant \frac{3}{8} z_n^4 \sum_{p,q=1}^{n-1} \sum_{k=1}^{p} \sum_{l=1}^{q} \left(\delta_{pqkl} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q} \psi_{pk}^{\mathrm{T}} \psi_{ql} z_k \right)^{4/3} + \frac{1}{2} \sum_{p,q=1}^{n-1} \sum_{k=1}^{p} \sum_{l=1}^{q} \frac{1}{4\delta_{pqkl}^4} z_l^4 \\
&= \frac{3}{8} z_n^4 \sum_{p,q=1}^{n-1} \sum_{k=1}^{p} \sum_{l=1}^{q} \left(\delta_{pqkl} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q} \psi_{pk}^{\mathrm{T}} \psi_{ql} z_k \right)^{4/3} + \frac{1}{2} \sum_{l=1}^{n-1} z_l^4 \sum_{p=1}^{n-1} \sum_{q=l}^{n-1} \sum_{k=1}^{p} \frac{1}{4\delta_{pqkl}^4}
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
3 z_n^3 \beta_{nn}^{\mathrm{T}} \sum_{k=1}^{n-1} z_k \beta_{nk} &= 3 \sum_{k=1}^{n-1} (z_n^3 \beta_{nn}^{\mathrm{T}} \beta_{nk}) z_k \\
&\leqslant 3 \sum_{k=1}^{n-1} \left[\frac{3}{4} (\delta_k \beta_{nn}^{\mathrm{T}} \beta_{nk})^{4/3} z_n^4 + \frac{1}{4\delta_k^4} z_k^4 \right] = \frac{9}{4} z_n^4 \sum_{k=1}^{n-1} (\delta_k \beta_{nn}^{\mathrm{T}} \beta_{nk})^{4/3} + \frac{3}{4} \sum_{k=1}^{n-1} \frac{1}{\delta_k^4} z_k^4.
\end{aligned} \tag{4.14}$$

Substituting (4.9), (4.13) and (4.14) into (3.21) in [2], we have

$$\begin{aligned}
\mathscr{L}V \leqslant z_n^3 \Bigg\{ & u + z_n \left[\left(\frac{1}{2} + \frac{1}{2} \left(\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} \right)^2 \right) + \frac{3}{4} \sum_{l=1}^{n-2} \left(\delta_l \frac{\partial \alpha_{n-1}}{\partial x_l} \right)^{4/3} + \frac{3}{4} \sum_{l=1}^{n-1} \sum_{k=1}^{l} \left(\delta_{lk} \frac{\partial \alpha_{n-1}}{\partial x_l} \alpha_{lk} \right)^{4/3} \right] \\
& + \frac{3}{8} z_n \sum_{p,q=1}^{n-1} \sum_{k=1}^{p} \sum_{l=1}^{q} \left(\delta_{pqkl} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q} \psi_{pk}^{\mathrm{T}} \psi_{ql} z_k \right)^{4/3} + \frac{1}{4\varepsilon_{n-1}^4} z_n + \frac{3}{2} z_n \beta_{nn}^{\mathrm{T}} \beta_{nn} \\
& + \frac{9}{4} z_n \sum_{k=1}^{n-1} (\delta_k \beta_{nn}^{\mathrm{T}} \beta_{nk})^{4/3} + \frac{3}{4} z_n \sum_{j=1}^{r} \sum_{k=1}^{n-1} \sum_{l=1}^{n-1} \frac{1}{\varepsilon_{nkl}^2} \beta_{nkj}^2 \beta_{nlj}^2 \Bigg\}
\end{aligned}$$

$$+z_1^3\left(\alpha_1+z_1\sum_{l=1}^{n-1}\frac{1}{4\delta_{l1}^4}+\frac{3}{4}\varepsilon_1^{4/3}z_1+\frac{1}{2}z_1\sum_{p=1}^{n-1}\sum_{q=1}^{n-1}\sum_{k=1}^{p}\frac{1}{4\delta_{pqk1}^4}+\frac{3}{4}\frac{1}{\delta_1^4}z_1\right.$$
$$\left.+\frac{3}{2}z_1\beta_{11}^{\mathrm{T}}\beta_{11}+\frac{3r}{4}z_1\sum_{k=2}^{n}\sum_{l=1}^{k-1}\varepsilon_{k1l}^2\right)$$
$$+\sum_{i=2}^{n-1}z_i^3\left(\alpha_i+z_i\sum_{l=i}^{n-1}\frac{1}{4\delta_{li}^4}+\frac{1}{4\delta_{i-1}^4}z_i+\frac{1}{2}z_i\sum_{p=1}^{n-1}\sum_{q=i}^{n-1}\sum_{k=1}^{p}\frac{1}{4\delta_{pqki}^4}+\frac{3}{4}\frac{1}{\delta_i^4}z_i\right.$$
$$-\sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1}-\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\varphi_q+\frac{3}{4}\varepsilon_i^{4/3}z_i+\frac{1}{4\varepsilon_{i-1}^4}z_i+\frac{3}{2}z_i\beta_{ii}^{\mathrm{T}}\beta_{ii}$$
$$\left.+3\beta_{ii}^{\mathrm{T}}\sum_{k=1}^{i-1}z_k\beta_{ik}+\frac{3}{4}z_i\sum_{j=1}^{r}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}\frac{1}{\varepsilon_{ikl}^2}\beta_{ikj}^2\beta_{ilj}^2+\frac{3r}{4}z_i\sum_{k=i+1}^{n}\sum_{l=1}^{k-1}\varepsilon_{kil}^2\right). \tag{4.15}$$

If δ_{li}, δ_{pqil}, and δ_i are chosen as

$$\sum_{l=1}^{n-1}\frac{1}{4\delta_{l1}^4}+\frac{1}{2}\sum_{p=1}^{n-1}\sum_{q=1}^{n-1}\sum_{k=1}^{p}\frac{1}{4\delta_{pqk1}^4}-\frac{3}{4\delta_1^4}=\frac{c_1}{2}, \tag{4.16}$$

$$\sum_{l=i}^{n-1}\frac{1}{4\delta_{li}^4}-\frac{1}{4\delta_{i-1}^4}-\frac{1}{2}\sum_{p=1}^{n-1}\sum_{q=i}^{n-1}\sum_{k=1}^{p}\frac{1}{4\delta_{pqki}^4}-\frac{3}{4\delta_i^4}=\frac{c_i}{2}, \tag{4.17}$$

where c_1 and c_i are those in [2, (3.22) and (3.23)], and

$$u=-M(x)z_n, \tag{4.18}$$

$$M(x)=c_n+\frac{1}{4\varepsilon_{n-1}^4}+\frac{3}{2}\beta_{nn}^{\mathrm{T}}\beta_{nn}+\frac{3}{4}\sum_{j=1}^{r}\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\frac{1}{\varepsilon_{nkl}^2}\beta_{nkj}^2\beta_{nlj}^2$$
$$+\left(\frac{1}{2}+\frac{1}{2}\left(\frac{\partial\alpha_{n-1}}{\partial x_{n-1}}\right)^2\right)+\frac{3}{4}\sum_{l=1}^{n-2}\left(\delta_l\frac{\partial\alpha_{n-1}}{\partial x_l}\right)^{4/3}+\frac{3}{4}\sum_{l=1}^{n-1}\sum_{k=1}^{l}\left(\delta_{lk}\frac{\partial\alpha_{n-1}}{\partial x_l}\alpha_{lk}\right)^{4/3}$$
$$+\frac{3}{8}\sum_{p,q=1}^{n-1}\sum_{k=1}^{p}\sum_{l=1}^{q}\left(\delta_{pqkl}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\psi_{pk}^{\mathrm{T}}\psi_{ql}z_k\right)^{4/3}+\frac{9}{4}\sum_{k=1}^{n-1}(\delta_k\beta_{nn}^{\mathrm{T}}\beta_{nk})^{4/3}, \tag{4.19}$$

where $c_i>0$, $i=1,\dots,n$, and $M(x)$ is a positive function, with (4.16)–(4.18), we get

$$\mathscr{L}V\leqslant-\frac{1}{2}\sum_{i=1}^{n}c_iz_i^4. \tag{4.20}$$

Thus, according to Lemma 4.1, we achieve not only global asymptotic stability in probability, but also inverse optimality.

Theorem 4.1. *The control law*

$$u^*=-\tfrac{2}{3}\beta M(x)z_n,\quad \beta\geqslant 2 \tag{4.21}$$

guarantees that the equilibrium at the origin of the system (2.1) *is globally asymptotically stable in probability and also minimizes the cost functional*

$$J(u)=E\left\{\int_0^{\infty}\left[l(x)+\frac{27}{16\beta^2}M(x)^{-3}u^4\right]\mathrm{d}\tau\right\}, \tag{4.22}$$

for some positive definite radially unbounded function $l(x)$ *parameterized by* β.

Remark 4.1. We point out that the *quartic* form of the penalty on u in (4.22) is due to the *quartic* nature of the sclf.

Appendix

Lemma A.1 (Krstić and Li [7]). *If* γ *and its derivative* γ' *are class* $\mathscr{K}_\infty$ *functions, then the Legendre–Fenchel transform satisfies the following properties*:

$$\ell\gamma(r)=r(\gamma')^{-1}(r)-\gamma((\gamma')^{-1}(r))=\int_0^r(\gamma')^{-1}(s)\,\mathrm{d}s, \tag{A.1}$$

$$\ell\ell\gamma=\gamma, \tag{A.2}$$

$$\ell\gamma \text{ is a class } \mathscr{K}_\infty \text{ function}, \tag{A.3}$$

$$\ell\gamma(\gamma'(r))=r\gamma'(r)-\gamma(r). \tag{A.4}$$

Lemma A.2 (Young's inequality [5, Theorem 156]). *For any two vectors* x *and* y, *the following holds*:

$$x^{\mathrm{T}}y\leqslant\gamma(|x|)+\ell\gamma(|y|), \tag{A.5}$$

and the equality is achieved if and only if

$$y=\gamma'(|x|)\frac{x}{|x|},\ \text{i.e. for } x=(\gamma')^{-1}(|y|)\frac{y}{|y|}. \tag{A.6}$$

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