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Stochastic nonlinear stabilization – I: A backstepping design[1]

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Abstract

While the current robust nonlinear control toolbox includes a number of methods for systems affine in *deterministic bounded* disturbances, the problem when the disturbance is *unbounded stochastic* noise has hardly been considered. We present a control design which achieves global asymptotic (Lyapunov) stability in probability for a class of strict-feedback nonlinear continuous-time systems driven by white noise. In a companion paper, we develop inverse optimal control laws for general stochastic systems affine in the noise input, and for strict-feedback systems. A reader of this paper needs no prior familiarity with techniques of stochastic control. © 1997 Published by Elsevier Science B.V.

Keywords: Stochastic nonlinear systems; Backstepping

1. Introduction

Despite major advances in robust stabilization of deterministic nonlinear systems achieved over the last few years and reported in [6, 10] and references therein, the stabilization problem for *stochastic* systems is yet to be addressed. While not as refined as their deterministic counterparts in [8], Lyapunov techniques for stability analysis of stochastic systems do exist, see, for example, the classical book of Khas'minskii [9] (see also [11]). Efforts toward (global) *stabilization* of stochastic nonlinear systems have been initiated in the work of Florchinger [3–5] who, among other things, extended the concept of control Lyapunov functions and Sontag's stabilization formula [16] to the stochastic setting. A breakthrough towards arriving at *constructive* methods for stabilization of broader classes of stochastic nonlinear systems came with the result of Pan and Başar [14], who derived a backstepping design for strict-feedback systems motivated by a risk-sensitive cost criterion [1, 7, 12, 15].

In this paper we design a backstepping control law which guarantees global asymptotic stability in probability. In contrast to the design of Pan and Başar [14], our design is fully systematic and its algorithm given in Table 1 can be directly coded in symbolic software.

In a companion paper [2], we develop inverse optimal control laws for general stochastic systems affine in the noise input, and for strict-feedback systems.

Before we start, we point out that we deal with nonlinear systems in which the *equilibrium at the origin* is preserved even in the presence of noise because the noise vector field is vanishing at the origin. This means

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that we exclude linear systems with *additive noise*, that is, the reader should not expect to see LQG controllers as a special case of our controllers. Clearly, in the presence of additive noise, which perturbs the equilibrium at the origin, one cannot set Lyapunov-type stabilization as a goal but, instead, some form of input/output stabilization (for example, achieving a bounded or minimal $\mathscr{H}_2$ norm with respect to the noise as the input). Control of stochastic nonlinear systems where the noise vector field does not vanish at the origin is the topic of our future study.

Another preparatory comment of potential interest to the reader with technical expertise in backstepping designs is that the Lyapunov function that we construct is not of the form $V=\sum z_i^2$ but of the form $V=\sum z_i^4$. The quartic form is employed in order to handle some special terms in the Lyapunov analysis which arise due to the Itô differentiation rule.

2. Preliminaries on stability in probability

Consider the nonlinear stochastic system

$$dx = f(x)\,dt + g(x)\,dw, \tag{2.1}$$

where $x\in\mathbb{R}^n$ is the state, w is an r-dimensional independent standard Wiener process, and $f:\mathbb{R}^n\to\mathbb{R}^n$ and $g:\mathbb{R}^n\to\mathbb{R}^{n\times r}$ are locally Lipschitz and satisfy

$$f(0)=0, \qquad g(0)=0. \tag{2.2}$$

Definition 2.1. The equilibrium $x=0$ of the system (2.1) is said to be globally asymptotically stable in probability if for any $t_0\geqslant 0$ and $\varepsilon>0$,

$$\lim_{x(t_0)\to 0} P\left\{\sup_{t\geqslant t_0}|x(t)|>\varepsilon\right\}=0, \tag{2.3}$$

and for any initial condition $x(t_0)$,

$$P\left\{\lim_{t\to\infty} x(t)=0\right\}=1. \tag{2.4}$$

Theorem 2.1 (Khas'minskii [9]). *Consider system* (2.1) *and suppose there exists a positive definite, radially unbounded, twice continuously differentiable function* $V(x)$ *such that the infinitesimal generator*

$$\mathscr{L}V(x)=\frac{\partial V}{\partial x}f+\frac{1}{2}\mathrm{Tr}\left\{g^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g\right\} \tag{2.5}$$

is negative definite. Then the equilibrium $x=0$ *of* (2.1) *is globally asymptotically stable in probability.*

3. Stabilization in probability via backstepping

In this section we will deal with *strict-feedback* systems driven by white noise. This class of systems is given by the following nonlinear stochastic differential equations:

$$dx_i = x_{i+1}\,dt+\varphi_i(\bar{x}_i)^{\mathrm{T}}\,dw, \quad i=1,\dots,n-1, \tag{3.1}$$

$$dx_n = u\,dt+\varphi_n(\bar{x}_n)^{\mathrm{T}}\,dw, \tag{3.2}$$

where $\bar{x}_i=[x_1,\dots,x_i]^{\mathrm{T}}$, $\varphi_i(\bar{x}_i)$ are r-vector-valued smooth functions with $\varphi_i(0)=0$, and w is an independent r-dimensional standard Wiener process.

Remark 3.1. Even though the technique we present here is applicable to more general classes of systems dealt with in [10] and other references (block-strict feedback systems with zero dynamics, output-feedback systems, MIMO systems, systems with additional deterministic uncertainties, both static and dynamic, etc.), we restrict our attention to (3.1) and (3.2) for clarity.

In the standard backstepping method for deterministic systems [6] (where $\mathrm{d}w/\mathrm{d}t$ would be a bounded deterministic disturbance), a sequence of stabilizing functions $\alpha_i(\bar{x}_i)$ is constructed recursively to build a Lyapunov function of the form

$$V=\sum_{i=1}^{n}\frac{1}{2}z_i^2, \tag{3.3}$$

where the error variables z_i are given by $z_i=x_i-\alpha_{i-1}(\bar{x}_{i-1})$. The Lyapunov design for stochastic systems is *much* more difficult because of the term $\frac{1}{2}\mathrm{Tr}\{g^{\mathrm{T}}(\partial^2V/\partial x^2)g\}$ in (2.5). Pan and Başar [14] developed a design based on quadratic Lyapunov functions which they modified by state-dependent weighting on the z_i-variables, where the choice of the weighting functions is left to the designer. The synthesis we present here is more systematic and results in a simple design algorithm which can be coded in symbolic software. Our approach in dealing with the inadequacy of quadratic Lyapunov functions is not to introduce weighting, but to employ *quartic* Lyapunov functions

$$V=\sum_{i=1}^{n}\frac{1}{4}z_i^4. \tag{3.4}$$

The quartic form will allow us to handle the term $\frac{1}{2}\mathrm{Tr}\{g^{\mathrm{T}}(\partial^2V/\partial x^2)g\}$ more easily than in [14].

Our presentation of the backstepping procedure here is very concise: instead of introducing the stabilizing functions α_i in a step-by-step fashion, we derive them simultaneously. A reader who is a novice to the technique of backstepping is referred to [10].

We start by several important preparatory comments. Since $\varphi_i(0)=0$, the α_i's will vanish at $\bar{x}_i=0$, as well as at $\bar{z}_i=0$, where $\bar{z}_i=[z_1,\dots,z_i]^{\mathrm{T}}$. Thus, by the mean value theorem, $\alpha_i(\bar{x}_i)$ can be expressed as

$$\alpha_i(\bar{x}_i)=\sum_{l=1}^{i}z_l\alpha_{il}(\bar{x}_i), \tag{3.5}$$

where $\alpha_{il}(\bar{x}_i)$ are smooth functions. We can now write $\varphi_i(\bar{x}_i)$ as

$$\varphi_i(\bar{x}_i)=\sum_{k=1}^{i}x_k\varphi_{ik}(\bar{x}_i)=\sum_{k=1}^{i}(z_k+\alpha_{k-1})\varphi_{ik}(\bar{x}_i)=\sum_{k=1}^{i}z_k\psi_{ik}(\bar{x}_i), \tag{3.6}$$

where $\varphi_{ik}(\bar{x}_i)$ and $\psi_{ik}(\bar{x}_i)$ are smooth functions. Then the system (3.1), (3.2) can be written as

$$\mathrm{d}x_i=x_{i+1}\,\mathrm{d}t+\left(\sum_{k=1}^{i}z_k\psi_{ik}(\bar{x}_i)^{\mathrm{T}}\right)\mathrm{d}w,\quad i=1,\dots,n-1, \tag{3.7}$$

$$\mathrm{d}x_n=u\,\mathrm{d}t+\left(\sum_{k=1}^{n}z_k\psi_{nk}(\bar{x}_n)^{\mathrm{T}}\right)\mathrm{d}w. \tag{3.8}$$

Now, we are ready to start the backstepping design procedure. According to Itô's differentiation rule [13], we have

$$\mathrm{d}z_i=\mathrm{d}(x_i-\alpha_{i-1})=\left(x_{i+1}-\sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1}-\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\varphi_q\right)\mathrm{d}t$$
$$+\left(\varphi_i^{\mathrm{T}}-\sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l^{\mathrm{T}}\right)\mathrm{d}w,\quad i=1,\dots,n, \tag{3.9}$$

where $x_{n+1}=u$. As we announced previously, we employ a Lyapunov function of the form

$$V(z)=\sum_{i=1}^{n}\frac{1}{4}z_i^4, \tag{3.10}$$

and engage in the process of selecting the functions $\alpha_i(\bar{x}_i)$ to make $\mathscr{L}V(x)$ negative definite. Along the solutions of (3.9), we have

$$\begin{aligned}\mathscr{L}V &= z_n^3\left(u - \sum_{l=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_l}x_{l+1} - \frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\varphi_q\right)\\ &\quad+\sum_{i=1}^{n-1}z_i^3\left(x_{i+1} - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} - \frac{1}{2}\sum_{m,n=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_m\partial x_n}\varphi_m^{\mathrm{T}}\varphi_n\right)\\ &\quad+\frac{3}{2}\sum_{i=1}^{n}z_i^2\left(\varphi_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l\right)^{\mathrm{T}}\left(\varphi_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l\right)\\ &= z_n^3\left(u - \sum_{l=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_l}x_{l+1} - \frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\varphi_q\right) + \sum_{i=1}^{n-1}z_i^3 z_{i+1}\\ &\quad+\sum_{i=1}^{n-1}z_i^3\left(\alpha_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} - \frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\varphi_q\right)\\ &\quad+\frac{3}{2}\sum_{i=1}^{n}z_i^2\left(\varphi_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l\right)^{\mathrm{T}}\left(\varphi_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l\right),\end{aligned}\tag{3.11}$$

where the second equality comes from substituting $x_i = z_i + \alpha_{i-1}$. The first and the third term can be handled by choosing u and α_i which cancel the summations. To handle the second and the fourth terms, we need Young's inequality [10, Eq. (2.253)]

$$xy \leqslant \frac{\varepsilon^p}{p}|x|^p + \frac{1}{q\varepsilon^q}|y|^q,\tag{3.12}$$

where $\varepsilon > 0$, the constants $p > 1$ and $q > 1$ satisfy $(p-1)(q-1) = 1$, and $(x, y) \in \mathbb{R}^2$. Applying (3.12) to the second term in (3.11), using $p = \frac{4}{3}$, $q = 4$ and $\varepsilon_i > 0$ for each i, we have

$$\sum_{i=1}^{n-1}z_i^3 z_{i+1} \leqslant \frac{3}{4}\sum_{i=1}^{n-1}\varepsilon_i^{4/3}z_i^4 + \sum_{i=2}^{n}\frac{1}{4\varepsilon_{i-1}^4}z_i^4.\tag{3.13}$$

Substituting (3.13) into (3.11), and incorporating into the first and the third terms, the terms originated in (3.13) are easily cancelled by u and α_i.

However, the last term in (3.11) remains difficult to handle. We try to arrange it in the form

$$\sum_{i=1}^{n} z_i^{\gamma}\eta_i(\bar{x}_i), \quad \gamma \geqslant 3,\tag{3.14}$$

so that it also can be combined into the first and the third terms in (3.11), and be cancelled by α_i and u. Substituting (3.6) into the last term in (3.11) yields

$$\begin{aligned}&\frac{3}{2}\sum_{i=1}^{n}z_i^2\left(\varphi_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l\right)^{\mathrm{T}}\left(\varphi_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l\right)\\ &\quad= \frac{3}{2}\sum_{i=1}^{n}z_i^2\left(\sum_{k=1}^{i}z_k\psi_{ik} - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\sum_{k=1}^{l}z_k\psi_{lk}\right)^{\mathrm{T}}\left(\sum_{k=1}^{i}z_k\psi_{ik} - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\sum_{k=1}^{l}z_k\psi_{lk}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{2}\sum_{i=1}^{n} z_i^2 \left(\sum_{k=1}^{i} z_k\psi_{ik} - \sum_{k=1}^{i-1}\sum_{l=k}^{i-1} z_k \frac{\partial\alpha_{i-1}}{\partial x_l}\psi_{lk}\right)^{\mathrm{T}} \left(\sum_{k=1}^{i} z_k\psi_{ik} - \sum_{k=1}^{i-1}\sum_{l=k}^{i-1} z_k \frac{\partial\alpha_{i-1}}{\partial x_l}\psi_{lk}\right) \\
&= \frac{3}{2}\sum_{i=1}^{n} \left\{ z_i^2 \left[\sum_{k=1}^{i} z_k \left(\psi_{ik} - \sum_{l=k}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_l}\psi_{lk}\right)\right]^{\mathrm{T}} \left[\sum_{k=1}^{i} z_k \left(\psi_{ik} - \sum_{l=k}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_l}\psi_{lk}\right)\right]\right\} \\
&= \frac{3}{2}\sum_{i=1}^{n} z_i^2 \left(\sum_{k=1}^{i} z_k\beta_{ik}\right)^{\mathrm{T}} \left(\sum_{k=1}^{i} z_k\beta_{ik}\right),
\end{aligned}
\tag{3.15}
$$

where $\partial\alpha_{i-1}/\partial x_i = 0$ and

$$
\beta_{ik}(\bar{x}_i) = \psi_{ik} - \sum_{l=k}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_l}\psi_{lk}, \quad k = 1,\ldots,i.
\tag{3.16}
$$

The second equality in (3.15) comes from changing the order of summations. It is obvious that β_{ik} depend only on $\bar{x}_i$. The next major step is to separate different z_i from each other, so every term can be handled by the proper α_i. Hence, we rearrange (3.15) as

$$
\begin{aligned}
&\frac{3}{2}\sum_{i=1}^{n} z_i^2 \left(\sum_{k=1}^{i} z_k\beta_{ik}\right)^{\mathrm{T}} \left(\sum_{k=1}^{i} z_k\beta_{ik}\right) \\
&\quad = \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{r} \left[z_i^4\beta_{iij}^2 + 2z_i^3\beta_{iij}\sum_{k=1}^{i-1} z_k\beta_{ikj} + z_i^2 \left(\sum_{k=1}^{i-1} z_k\beta_{ikj}\right)^2\right] \\
&\quad = \frac{3}{2} z_n^4\beta_{nn}^{\mathrm{T}}\beta_{nn} + 3z_n^3\beta_{nn}^{\mathrm{T}}\sum_{k=1}^{n-1} z_k\beta_{nk} + \frac{3}{2}\sum_{i=1}^{n-1}\left(z_i^4\beta_{ii}^{\mathrm{T}}\beta_{ii} + 2z_i^3\beta_{ii}^{\mathrm{T}}\sum_{k=1}^{i-1} z_k\beta_{ik}\right) + \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{r} z_i^2 \left(\sum_{k=1}^{i-1} z_k\beta_{ikj}\right)^2,
\end{aligned}
\tag{3.17}
$$

where β_{ikj} is the jth component of the vector β_{ik}. The first three terms in (3.17) are already in the desired form. Now we concentrate on the last term in (3.17):

$$
\begin{aligned}
&\frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{r} z_i^2 \left(\sum_{k=1}^{i-1} z_k\beta_{ikj}\right)^2 = \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{r} z_i^2 \left(\sum_{k=1}^{i-1} z_k\beta_{ikj}\right)\left(\sum_{l=1}^{i-1} z_l\beta_{ilj}\right) \\
&\quad = \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1} z_i^2\beta_{ikj}\beta_{ilj}z_k z_l \leqslant \frac{3}{4}\sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}\left(\frac{1}{\varepsilon_{ikl}^2} z_i^4\beta_{ikj}^2\beta_{ilj}^2 + \varepsilon_{ikl}^2 z_k^2 z_l^2\right) \\
&\quad \leqslant \frac{3}{4}\sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1} \frac{1}{\varepsilon_{ikl}^2} z_i^4\beta_{ikj}^2\beta_{ilj}^2 + \frac{3}{4}\sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{k=1}^{i-1} z_k^4 \left(\sum_{l=1}^{i-1}\varepsilon_{ikl}^2\right) \\
&\quad = \frac{3}{4} z_n^4 \left(\sum_{j=1}^{r}\sum_{k=1}^{n-1}\sum_{l=1}^{n-1} \frac{1}{\varepsilon_{nkl}^2}\beta_{nkj}^2\beta_{nlj}^2\right) + \frac{3}{4}\sum_{i=1}^{n-1} z_i^4 \left(\sum_{j=1}^{r}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1} \frac{1}{\varepsilon_{ikl}^2}\beta_{ikj}^2\beta_{ilj}^2\right) \\
&\qquad + \frac{3r}{4}\sum_{i=1}^{n-1} z_i^4 \left(\sum_{k=i+1}^{n}\sum_{l=1}^{k-1}\varepsilon_{kil}^2\right).
\end{aligned}
\tag{3.18}
$$

The inequalities above are obtained by applying Young's inequalities:

$$z_i^2\beta_{ikj}\beta_{ilj}z_kz_l \leqslant \frac{1}{2\varepsilon_{ikl}^2}z_i^4\beta_{ikj}^2\beta_{ilj}^2 + \frac{\varepsilon_{ikl}^2}{2}z_k^2z_l^2 \tag{3.19}$$

$$z_k^2z_l^2 \leqslant \tfrac{1}{2}z_k^4 + \tfrac{1}{2}z_l^4, \tag{3.20}$$

where $\varepsilon_{ikl} = \varepsilon_{ilk}$. The final expression in (3.18) is obtained by separating the z_n from the other z_i's in the first term and changing the order of summations in the second term.

Now, substituting (3.13), (3.17) and (3.18) into (3.11), we get

$$\begin{aligned}
\mathscr{L}V \leqslant{}& z_n^3\left(u - \sum_{l=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_l}x_{l+1} - \frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\varphi_q\right) + \frac{3}{4}\sum_{i=1}^{n-1}\varepsilon_i^{4/3}z_i^4 + \sum_{i=2}^{n}\frac{1}{4\varepsilon_{i-1}^4}z_i^4 \\
&+ \sum_{i=1}^{n-1}z_i^3\left(\alpha_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} - \frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\varphi_q\right) \\
&+ \frac{3}{2}z_n^4\beta_{nn}^{\mathrm{T}}\beta_{nn} + 3z_n^3\beta_{nn}^{\mathrm{T}}\sum_{k=1}^{n-1}z_k\beta_{nk} + \frac{3}{2}\sum_{i=1}^{n-1}\left(z_i^4\beta_{ii}^{\mathrm{T}}\beta_{ii} + 2z_i^3\beta_{ii}^{\mathrm{T}}\sum_{k=1}^{i-1}z_k\beta_{ik}\right) \\
&+ \frac{3}{4}z_n^4\sum_{j=1}^{r}\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\frac{1}{\varepsilon_{nkl}^2}\beta_{nkj}^2\beta_{nlj}^2 + \frac{3}{4}\sum_{i=1}^{n-1}z_i^4\sum_{j=1}^{r}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}\frac{1}{\varepsilon_{ikl}^2}\beta_{ikj}^2\beta_{ilj}^2 + \frac{3r}{4}\sum_{i=1}^{n-1}z_i^4\sum_{k=i+1}^{n}\sum_{l=1}^{k-1}\varepsilon_{kil}^2 \\
={}& z_n^3\left(u - \sum_{l=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_l}x_{l+1} - \frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\varphi_q + \frac{1}{4\varepsilon_{n-1}^4}z_n + \frac{3}{2}z_n\beta_{nn}^{\mathrm{T}}\beta_{nn}\right. \\
&\left.+ 3\beta_{nn}^{\mathrm{T}}\sum_{k=1}^{n-1}z_k\beta_{nk} + \frac{3}{4}z_n\sum_{j=1}^{r}\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\frac{1}{\varepsilon_{nkl}^2}\beta_{nkj}^2\beta_{nlj}^2\right) \\
&+ z_1^3\left(\alpha_1 + \frac{3}{4}\varepsilon_1^{4/3}z_1 + \frac{3}{2}z_1\beta_{11}^{\mathrm{T}}\beta_{11} + \frac{3r}{4}z_1\sum_{k=2}^{n}\sum_{l=1}^{k-1}\varepsilon_{k1l}^2\right) \\
&+ \sum_{i=2}^{n-1}z_i^3\left(\alpha_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} - \frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\varphi_q + \frac{3}{4}\varepsilon_i^{4/3}z_i + \frac{1}{4\varepsilon_{i-1}^4}z_i\right. \\
&\left.+ \frac{3}{2}z_i\beta_{ii}^{\mathrm{T}}\beta_{ii} + 3\beta_{ii}^{\mathrm{T}}\sum_{k=1}^{i-1}z_k\beta_{ik} + \frac{3}{4}z_i\sum_{j=1}^{r}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}\frac{1}{\varepsilon_{ikl}^2}\beta_{ikj}^2\beta_{ilj}^2 + \frac{3r}{4}z_i\sum_{k=i+1}^{n}\sum_{l=1}^{k-1}\varepsilon_{kil}^2\right).
\end{aligned} \tag{3.21}$$

At this point, all the terms can be cancelled by u and α_i. If we choose them as

$$\alpha_1 = -c_1z_1 - \frac{3}{4}\varepsilon_1^{4/3}z_1 - \frac{3}{2}z_1\beta_{11}^{\mathrm{T}}\beta_{11} - \frac{3r}{4}z_1\sum_{k=2}^{n}\sum_{l=1}^{k-1}\varepsilon_{k1l}^2, \tag{3.22}$$

$$\begin{aligned}
\alpha_i ={}& -c_iz_i + \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} + \frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\varphi_q - \frac{3}{4}\varepsilon_i^{4/3}z_i - \frac{1}{4\varepsilon_{i-1}^4}z_i - \frac{3}{2}z_i\beta_{ii}^{\mathrm{T}}\beta_{ii} \\
&-3\beta_{ii}^{\mathrm{T}}\sum_{k=1}^{i-1}z_k\beta_{ik} - \frac{3}{4}z_i\sum_{j=1}^{r}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}\frac{1}{\varepsilon_{ikl}^2}\beta_{ikj}^2\beta_{ilj}^2 - \frac{3r}{4}z_i\sum_{k=i+1}^{n}\sum_{l=1}^{k-1}\varepsilon_{kil}^2,
\end{aligned} \tag{3.23}$$

Table 1
Backstepping design for system (3.1), (3.2)

$$z_i = x_i - \alpha_{i-1} \tag{3.26}$$

$$\psi_{ik} = \sum_{l=k}^{i} \alpha_{l-1,k}\varphi_{il}, \quad k = 1,\ldots,i \tag{3.27}$$

$$\beta_{ik} = \psi_{ik} - \sum_{l=k}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\psi_{lk}, \quad k = 1,\ldots,i \tag{3.28}$$

$$\alpha_{ik} = \frac{1}{2}\sum_{p=k}^{i-1} \psi_{pk}^{\mathrm{T}} \sum_{q=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q} \sum_{j=1}^{q} z_j \psi_{qj} + \sum_{l=k-1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\alpha_{lk} - 3\beta_{ii}^{\mathrm{T}}\beta_{ik}, \quad k = 1,\ldots,i-1 \tag{3.29}$$

$$\alpha_{ii} = -\left(c_i + \frac{3}{4}\varepsilon_i^{4/3} + \frac{1}{4\varepsilon_{i-1}^4} + \frac{3}{2}\beta_{ii}^{\mathrm{T}}\beta_{ii} + \frac{3r}{4}\sum_{k=i+1}^{n}\sum_{l=1}^{k-1}\varepsilon_{kil}^2 + \frac{3}{4}\sum_{j=1}^{r}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}\frac{1}{\varepsilon_{ikl}^2}\beta_{ikj}^2\beta_{ilj}^2\right) + \frac{\partial \alpha_{i-1}}{\partial x_{i-1}} \tag{3.30}$$

$$\alpha_{i,i+1} = 1 \tag{3.31}$$

$$\alpha_i = \sum_{k=1}^{i} z_k \alpha_{ik} \quad i = 1,\ldots,n \tag{3.32}$$

Control law:

$$u = \alpha_n \tag{3.33}$$

$$u = -c_n z_n + \sum_{l=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_l} x_{l+1} + \frac{1}{2}\sum_{p,q=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q}\varphi_p^{\mathrm{T}}\varphi_q - \frac{1}{4\varepsilon_{n-1}^4} z_n - \frac{3}{2} z_n \beta_{nn}^{\mathrm{T}}\beta_{nn}$$
$$-3\beta_{nn}^{\mathrm{T}}\sum_{k=1}^{n-1} z_k \beta_{nk} - \frac{3}{4} z_n \sum_{j=1}^{r}\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\frac{1}{\varepsilon_{nkl}^2}\beta_{nkj}^2\beta_{nlj}^2, \tag{3.24}$$

where $c_i > 0$, then the infinitesimal generator of the system (3.9) is negative definite:

$$\mathcal{L}V \leqslant -\sum_{i=1}^{n} c_i z_i^4. \tag{3.25}$$

At the beginning of this section we stated that we will derive a systematic design procedure which can be coded using symbolic software. By collecting expressions (3.5), (3.6), (3.16), (3.22), (3.23), and (3.24), and by performing necessary rearrangements, we get the algorithm given in Table 1. The only input the user needs to provide at the start of the algorithm are functions φ_{ik} for the factorization of the system nonlinearities, $\varphi_i = \sum_{k=1}^{i} x_k \varphi_{ik}$.

With (3.25), we have the following stability result for the design procedure in Table 1.

Theorem 3.1. *The equilibrium at the origin of the closed-loop stochastic system* (3.1), (3.2), (3.33) *is globally asymptotically stable in probability. Furthermore, the following estimate of the fourth-moment exponential stability is guaranteed:*

$$E\{|z(t)|_4^4\} \leqslant |z(0)|_4^4 e^{-4ct}, \tag{3.34}$$

where $|z|_4 = (\sum_i z_i^4)^{1/4}$ *and* $c = \min_i c_i$.

Proof. The first part of the theorem follows from (3.25) by Theorem 2.1. As to the second part, we just give the main point of the proof. According to [9, Lemma 3.1, p. 82], we have

$$E\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{4}\sum z_i^4\right]\right\}=E\{\mathscr{L}V\}, \tag{3.35}$$

and then achieve the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E\left\{\sum z_i^4\right\}\leqslant -4cE\left\{\sum z_i^4\right\}, \tag{3.36}$$

which gives (3.34). □

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