Minimum-Seeking for CLFs: Universal Semiglobally Stabilizing Feedback Under Unknown Control Directions

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***Abstract*—Employing extremum seeking (ES) for seeking minima of control Lyapunov function (CLF) candidates, we develop 1) the first systematic design of ES controllers for unstable plants, 2) a simple non-model based universal feedback law that emulates, in an average sense, the "L_gV controllers" for stabilization with inverse optimality, and 3) a new strategy for stabilization of systems with unknown control directions, as an alternative to Nussbaum gain controllers that lack exponential stability, lack transient performance guarantees, and lack robustness to changes in the control direction. The stability analysis that underlies our designs is inspired by an analysis approach synthesized in a recent work by Dürr, Stankovic, and Johansson, which combines a Lie bracket averaging result of Gurvits and Li with a semiglobal practical stability result under small parametric perturbations by Moreau and Aeyels.**

***Index Terms*—Control Lyapunov functions (CLF), extremum seeking (ES), lie bracket averaging.**

I. Introduction

1) Motivation: Stabilization and control Lyapunov functions (CLF) have been seminal accomplishments in the control field since the introduction of Sontag's formula in 1989 [54] and followed by constructive developments of CLF's throughout the 1990s for systems with known models [34], [53], unknown parameters [33], [34], [58], uncertain nonlinearities [20], and stochastic and deterministic disturbances [33].

In the CLF theory, a central place is occupied by "L_gV controllers" (damping controllers) [33], [53], which are capable of ensuring not only stability but also inverse optimality, and of which a representative example is Sontag's universal controller [54].

In developing stabilizing controllers for uncertain systems, the most challenging class of uncertainties is the unknown control direction, also referred to as the case with an unknown sign of the high frequency gain. This problem has a history that precedes CLF's and goes to the early period of development of robust adaptive control. Posed in the early 1980s by Morse and first solved by Nussbaum [49], the problem of stabilization in the presence of unknown control direction has recently received increased attention in adaptive control of nonlinear systems but the classical parameter-adaptive solutions suffer from poor transient performance and fail to achieve exponential stability even in the absence of other uncertainties.

Extremum seeking (ES) is a control framework that, at a first glance, appears unrelated to both CLF's and stabilization under unknown control direction. ES has traditionally been developed as a methodology for optimizing steady states for stable systems, rather than for stabilizing unstable plants, experiencing a resurgence since its stability proof appeared in [36]. A first effort towards applying ES to unstable plants was presented in [67] but only for simple linear examples and only for problems where instability is an obstacle to achieving optimization, rather than stabilization being the goal. An extension of [67] is presented in [43].

A recent result by Dürr *et al.* [17], [18], which studies application of ES to autonomous vehicles performing source seeking, reveals a striking connection between ES and Lyapunov stabilization. First, it provides a link between Lie bracket averaging theory and Lypunov stability, introducing an analysis framework for guaranteeing semi-global stability of extremum seeking schemes, without a tradeoff between the rate of convergence and region of attraction. Second, it makes an observation that the unknown map, whose maximization or minimization is the goal in extremum seeking, is a Lyapunov function candidate in the study of stability using Lie bracket averaging.

In this paper we build on the connection between ES, Lyapunov functions, and Lie bracket averaging, to develop a general framework for stabilization of systems with unknown models using CLF's and L_gV-like controllers. In simple terms, we design ES controllers that achieve semi-global practical stabilization by seeking the minimum of a control Lyapunov function.

Since our approach does not require the knowledge of the control direction (for systems affine in control, the input vector field g is allowed to be unknown), as a byproduct of our effort in designing ES-based L_gV-like controllers we provide a new solution to the problem of stabilization of systems with unknown direction. The new solution guarantees exponential convergence and does not suffer from poor transients that are characteristic of solutions that employ Nussbaum gain techniques.

2) Summary of Literature on ES for Optimization of Stable Plants: The extremum seeking (ES) method, a real-time non-model-based optimization approach, has seen significant theoretical advances during the past decade, including the proof of local convergence [3], [5], [11], [51], [59], extension to semi-

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global convergence [57], development of scalar Newton-like algorithms [44], [48], inclusion of measurement noise [56], [56], extremum seeking with partial modeling information [1], [2], [15], [19], [23], and learning in noncooperative games [21], [55].

ES has also been used in many diverse applications with unknown/uncertain systems, such as steering vehicles toward a source in GPS-denied environments [12], [13], [65], active flow control [6]–[8], [28], [31], [32], aeropropulsion [45], [61], cooling systems [38], [39], wind energy [14], photovoltaics [37], human exercise machines [66], optimizing the control of nonisothermal continuously stirred tank reactors [24], reducing the impact velocity of an electromechanical valve actuator [50], controlling Tokamak plasmas [9], [10], and enhancing mixing in magnetohydrodynamic channel flows [40].

3) Results of the Paper: In this paper we employ ES for semi-global stabilization of unstable and time-varying systems, in which control direction is not only unknown, but allowed to persistently change sign. Our design is inspired by a recent extremum seeking design by Dürr *et al.* [17], where an innovative combination of certain Lie bracket-based averaging results by Gurvits and Li [25]–[27] was combined with results of Moreau and Aeyels [46] on semiglobal practical stability for nonlinear systems with small parametric perturbations, to achieve semiglobal practical stabilization of the Nash equilibrium in certain games involving multiple mobile robots.

We briefly introduce our general idea next, for nonlinear systems affine in control

$$\dot{x} = f(x) + g(x)u. \tag{1}$$

We employ a nonlinear time-varying control law of the form

$$u = \sqrt{\omega}\left[\alpha\cos(\omega t) - k\sin(\omega t)V(x)\right] \tag{2}$$

where $V(x)$ is a CLF for (1).

The major differences between the controller (2) and typical ES schemes are the $\sqrt{\omega}$ gain on the perturbing high-frequency terms and the fact that we are minimizing a known function $V(x)$ for an unknown system $f(x)$, $g(x)$, which contains its own integrator and is possibly unstable, rather than for a system that is open-loop stable and fast, as in typical ES-schemes. The $\sqrt{\omega}$ term is a necessity for the application of Lie Bracket averaging and intuitively can be understood by the fact that a highly oscillatory term on average may have little or no influence on a system's overall dynamics unless its amplitude is appropriately increased. As in classical ES approaches we add a small perturbing term of the form (after integration) $(\alpha/\sqrt{\omega})\sin(\omega t)$, which we then mix with a large signal of the same frequency $\sqrt{\omega}\sin(\omega t)$ in order to extract an estimate of the gradient of $V(x)$. One would expect that induced oscillations of any form such as square or triangular waves could successfully achieve similar ES results, as long as care is taken to match frequency and amplitude relationships, which may require more analysis when using non-smooth perturbations with infinite frequency spectrum.

As stated, the controller (2) is, in fact, a minimum-seeking controller for $V(x)$. Under this controller, the Lie bracket average system is

$$\dot{\bar{x}} = f(\bar{x}) - k\alpha g(\bar{x})g^T(\bar{x})\left(\frac{\partial V(\bar{x})}{\partial \bar{x}}\right)^T. \tag{3}$$

This average system clearly displays that the ES algorithm introduces a gradient of the CLF V with respect to the state $\bar{x}$, multiplied by the unknown square positive semidefinite matrix $g(\bar{x})g^T(\bar{x})$. Under suitable conditions, the choice of a sufficiently high gain product $k\alpha > 0$ makes the gradient term dominate the term $f(\bar{x})$ and globally asymptotically stabilizes the average system (3) and semi-globally practically stabilizes the original system (1) for sufficiently high ω.

For linear time-varying plants, $\dot{x} = A(t)x + B(t)u$, the control law $u = \sqrt{\omega}[\alpha\cos(\omega t) - k\sin(\omega t)|x|^2]$, results in the Lie bracket average LTV system $\dot{\bar{x}} = [A(t) - k\alpha B(t)B^T(t)]\bar{x}$, which, under a persistency of excitation condition on the vector-valued function $B(t)$, is stabilized for sufficiently high $k\alpha > 0$.

In all our ES designs, the first step is the construction (or a guess) of a CLF $V(x)$, the second step is a choice of sufficiently high gains k and α of the ES algorithm, such that the product $k\alpha$ is large enough to stabilize the Lie bracket average system, and the third step is the choice of a sufficiently high ω to satisfy the requirements of the Lie bracket averaging theorem of Gurvits and Li [25]–[27] and of the semiglobal practical stability theorem of Moreau and Aeyels [46]. In the time-varying case, intuitively speaking, relative to the system dynamics, we must choose ω large enough such that the control oscillations are on a separate time-scale from the time-varying dynamics, $\omega \gg |\dot{A}(t)|, |\dot{B}(t)|$. The choice of ω is related to the region of attraction of our closed loop systems, with larger ω resulting in a smaller perturbing term $(\alpha/\sqrt{\omega})\sin(\omega t)$ and providing stability for larger initial conditions.

Although the Lie bracket averaging analysis leads to a sufficiency condition for the product $k\alpha$ to be high enough, the individual terms k and α play significantly different roles in the ES algorithm and have different influences on the required choice of ω. The choice of ω is less sensitive to α, because the dithering term, after integration, is of the order of $\alpha/\sqrt{\omega}$. The choice of ω is much more sensitive to the value of k, which is a control gain that increases the convergence rate as well as decreases the size of the residual convergence set. For example, once a stabilizing choice of $k\alpha$ is made and a large enough ω is chosen, if one doubles k while halving α and thereby maintaining a fixed product $k\alpha$, if ω is not increased the system may possibly have large overshoot and may even become unstable. If, on the other hand, one halves the value of k while doubling α, the same choice of ω will usually maintain stability.

Clearly, our approach is of a high-gain type in requiring that both $k\alpha$ and ω be large, furthermore we introduce fast oscillations into the system, which may become impractical, due to actuator capabilities, for very large choices of ω. When considering problems with an unknown control direction, unlike the approach by Nussbaum [49] and Mudgett and Morse [47] (which we refer to as the "MMN approach"), our approach is neither global nor asymptotic—it guarantees semiglobal exponential practical stability. As such, our results are robust to disturbances in the process, as well as to deterministic or random disturbances in the measurement of x that do not generate a component at or near the probing frequency in $V(x)$. Furthermore, unlike the MMN approach, we can not only handle unknown signs of the high frequency gains, but also signs that change with time. In particular, we can allow $B(t)$ to go through zero. The price we pay, besides the lack of globality and of perfect regulation to the origin, is that our high-gain choice of

$k\alpha$ requires that we know a lower bound on a mean value of $B(t)B^T(t)$ and upper bounds on mean-square values of $A(t)$ and $B(t)$.

The MMN controller is designed for the case of constant plant parameters. When it is applied to a system that is time-varying, and when the control coefficients are quickly varying with time, passing through zero, and alternating signs, such as when $B(t)$ contains sinusoids, then the MMN type controllers simply cannot keep up and repeatedly overshoot with greater and greater error magnitude. We illustrate this with an example. The controllers developed in this paper do not suffer from any such overshoots because the extremum seeking control scheme is by design operating on a faster time scale than the dynamics in the system's coefficients, and the system's behavior, as estimated by an averaged system, does not depend on the control coefficients' signs. Apart from the limited effort in [67], this paper provides the first results making extremum seeking applicable to unstable plants.

4) Organization: In Section II we state definitions of stability used throughout the rest of the paper and provide a review of a Lie bracket averaging result of Gurvits and Li [25]–[27] in order to derive an averaged system of our original system. We then use the result of Moreau and Aeyels [46] to show that stability of the averaged system implies stability of the original system. In Section III we present a general framework for the design and analysis of stabilizing controllers for systems with unknown models by combining the ES approach with CLF's. In Section IV we present the first of our major results, for stabilization of unstable n-dimensional linear time-varying systems whose control vector coefficients may not only be of unknown sign but also of persistently changing sign. In Section V we consider unknown linear systems in strict-feedback form, as a representative of readily tractable but more notationally burdensome nonlinear systems in strict-feedback form, and design a stabilizing ES controller based on the backstepping approach [34], [58], which allows all the coefficients of the plant to be unknown, with only two mild conditions on bounds on the coefficients, which does not imply the knowledge of any of the coefficients' signs. In Section VI we present results for MIMO nonlinear systems with matched uncertainties and illustrate how to achieve stabilization for uncertain nonlinearities of arbitrary growth, which allows us, for example, to stabilize systems with polynomial nonlinearities without requiring the knowledge of the nonlinearities' polynomial order, using exponential feedback of the state's norm. Finally, in Section VII we illustrate the controllers' performance on examples that include an unstable two-dimensional time-varying system and a scalar unstable nonlinear systems. We also compare the performance of our controller with the MMN controller for a scalar time-varying system.

II. Background on Lie Bracket Averaging and Semiglobal Practical Asymptotic Stability

The Lie bracket averaging results of Gurvits and Li [25]–[27] apply to systems of the form

$$\dot{x} = \sum_{i=1}^{m} b_i(x) u_i^{\epsilon}(t,\theta) \tag{4}$$

where

$$u_i^{\epsilon}(t,\theta) = \bar{u}_i(t) + \frac{1}{\sqrt{\epsilon}} \hat{u}_i(t,\theta) \tag{5}$$

and $\hat{u}_i(t,\theta)$ is T-periodic in $\theta = t/\epsilon$, $T \in (0,\infty)$ and has zero average, $\int_0^T \hat{u}_i(t,\theta)d\theta = 0$. Along with the system (4), the following approximation model is considered:

$$\dot{z} = \sum_{i}^{m} b_i(z)\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}[b_i, b_j](z)\nu_{i,j}(t), \quad z(0) = x(0) \tag{6}$$

where

$$\nu_{i,j}(t) = \int_0^T \int_0^\theta \hat{u}_i(t,\tau)\hat{u}_j(t,\theta) d\tau d\theta \tag{7}$$

$$[b_i, b_j](z) = \frac{db_j(z)}{dz} b_i(z) - \frac{db_i(z)}{dz} b_j(z). \tag{8}$$

Noting that a T-periodic function is also nT-periodic, If we replace T with nT, $n \in \mathbb{N}$ and apply the arguments of the original proof [25]–[27] as well as [18, Lemma 2], the systems (4) and (6) then satisfy the following approximation result.

Lemma 1 ([4], [18], [25]–[27]): For period $T > 0$ and $n \in \mathbb{N}$, there exists $\epsilon^\star$ such that for all $\epsilon \in (0, \epsilon^\star)$, the trajectory of system (4) is within a $\Delta(nT, \epsilon)$-distance of the solution of system (6), namely

$$\max_{t\in[0,nT]} |x(t) - z(t)| = \|x - z\|_{C[0,nT]} \le \Delta(nT, \epsilon) \tag{9}$$

where $\Delta(nT, \epsilon) \to 0$ as $\epsilon \to 0$.

Before we can take advantage of these averaging results we make the following definitions as in Moreau and Aeyels [46]. In what follows, given a system:

$$\dot{x} = f(t,x) \tag{10}$$

$\psi(t, t_0, x_0)$ denotes the solution of (10) which passes through the point x_0 at time t_0.

Definition 1: Global Uniform Asymptotic Stability (GUAS): An equilibrium point of (10) is said to be GUAS if it satisfies the following three conditions:

Uniform Stability: For every $c_2 \in (0,\infty)$ there exists $c_1 \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$, $\|\psi(t,t_0,x_0)\| < c_2$ $\forall t \in [t_0,\infty)$.

Uniform Boundedness: For every $c_1 \in (0,\infty)$ there exists $c_2 \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$, $\|\psi(t,t_0,x_0)\| < c_2$ $\forall t \in [t_0,\infty)$.

Global Uniform Attractivity: For all $c_1, c_2 \in (0,\infty)$ there exists $\bar{T} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$, $\|\psi(t,t_0,x_0)\| < c_2$ $\forall t \in [t_0 + \bar{T},\infty)$.

In conjunction with (10), we consider systems of the form

$$\dot{x} = f^\epsilon(t,x) \tag{11}$$

whose trajectories are denoted as $\phi^\epsilon(t,t_0,x_0)$.

Definition 2: Converging Trajectories Property: The systems (10) and (11) are said to satisfy the converging trajectories property if for every $\hat{T} \in (0,\infty)$ and compact set $K \subset \mathbb{R}^n$ satisfying $\{(t,t_0,x_0) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n : t \in [t_0, t_0+\hat{T}], x_0 \in K\} \subset \mathrm{Dom}\psi$, for every $d \in (0, \infty)$ there exists $\epsilon^\star$ such that for all $t_0 \in \mathbb{R}$, for all $x_0 \in K$ and for all $\epsilon \in (0, \epsilon^\star)$

$$\|\phi^\epsilon(t, t_0, x_0) - \psi(t, t_0, x_0)\| < d, \quad \forall t \in [t_0, t_0 + \hat{T}]. \tag{12}$$

We then define the following form of stability for system (11).

Definition 3: ϵ-Semiglobal Practical Uniform Asymptotic Stability (ϵ-SPUAS): An equilibrium point of (11) is said to be ϵ-SPUAS if it satisfies the following three conditions:

Uniform Stability: For every $c_2 \in (0, \infty)$ there exists $c_1 \in (0, \infty)$ and $\hat{\epsilon} \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\epsilon \in (0, \hat{\epsilon})$, $\|\phi^\epsilon(t, t_0, x_0)\| < c_2 \ \forall t \in [t_0, \infty)$.

Uniform Boundedness: For every $c_1 \in (0, \infty)$ there exists $c_2 \in (0, \infty)$ and $\hat{\epsilon} \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\epsilon \in (0, \hat{\epsilon})$, $\|\phi^\epsilon(t, t_0, x_0)\| < c_2 \ \forall t \in [t_0, \infty)$.

Global Uniform Attractivity: For all $c_1, c_2 \in (0, \infty)$ there exists $\bar{T} \in (0, \infty)$ and $\hat{\epsilon} \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\epsilon \in (0, \hat{\epsilon})$, $\|\phi^\epsilon(t, t_0, x_0)\| < c_2 \ \forall t \in [t_0 + \bar{T}, \infty)$.

With these definitions the following result of Moreau and Aeyels [46] is used in the analysis that follows.

Theorem 1 ([46]): If systems (11) and (10) satisfy the converging trajectories property and if the origin is a GUAS equilibrium point of (10), then the origin of (11) is ϵ-SPUAS.

Corollary 1: If the origin of system (6) is GUAS, then the origin of system (4) is ϵ-SPUAS.

Proof: Given any $\hat{T} > 0$, taking $n \in \mathbb{N}$ such that $nT > \hat{T}$. by Lemma 1 the solutions of (6) and (4) satisfy the converging trajectories property. Since the origin of (6) is GUAS, by Theorem 1, the origin of (4) is ϵ-SPUAS. ■

III. CLF-Based Extremum Seeking: The Basic Idea

Our interest is in stabilization of the origin of systems

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0 \tag{13}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$, and the vector fields f and g are unknown. Though our approach permits a time dependence in f and g, as long as they can be represented as sums of products of functions of x and functions of t, as required by the analysis methodology in Section II, for clarity we concentrate in this section on time-invariant f and g.

Consider a controller in the form

$$u = \sqrt{\omega}\left[\alpha\cos(\omega t) - k\sin(\omega t)V(x)\right] \tag{14}$$

where $\alpha, k > 0$ and the function V is soon to be discussed. The Lie bracket average of the system (13), (14) is given by

$$\dot{\bar{x}} = f(\bar{x}) - k\alpha g(\bar{x})\left(L_g V(\bar{x})\right)^T \tag{15}$$

where we use the standard Lie derivative notation $L_g V = (\partial V/\partial x)g$. The form of the system (15) motivates the following assumption.

Assumption 1 (Strong L_gV-Stabilizability): There exists a positive definite, radially unbounded, continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ and a constant $\beta > 0$ such that

$$L_f V - \beta(L_g V)^2 < 0, \quad \forall x \neq 0. \tag{16}$$

With Corollary 1 we establish the following result.

Theorem 2: For given V and β, denote by $\mathcal{S}(V, \beta)$ the class of all systems (13) for which Assumption 1 is satisfied. Under the control law (14) all the systems in $\mathcal{S}(V, \beta)$ are $1/\omega$-SPUAS for all $k\alpha \geq \beta$.

It is relevant to explore the special case of linear systems

$$\dot{x} = Ax + bu \tag{17}$$

with control

$$u = \sqrt{\omega}\left[\alpha\cos(\omega t) - k\sin(\omega t)x^T P x\right] \tag{18}$$

where P is a positive definite and symmetric matrix. The Lie bracket average of the system (17), (18) is given by

$$\dot{\bar{x}} = (A - k\alpha b b^T P)\bar{x}. \tag{19}$$

Hence, the linear analog of Assumption 1 is that there exists a positive definite and symmetric *control Lyapunov matrix* (clm) P and a positive constant β such that

$$PA + A^T P - 2\beta P b b^T P < 0. \tag{20}$$

Corollary 2: For given P and β, denote by $\Sigma(P, \beta)$ the class of all pairs (A, b) for which (20) is satisfied. Under the control law (18) all the systems (17) in $\Sigma(P, \beta)$ are $1/\omega$-SPUAS for all $k\alpha \geq \beta$.

A. Is Assumption 1 Equivalent to Stabilizability?

It is well known that a system (13) with smooth f and g is stabilizable by feedback continuous at the origin and smooth away from the origin if and only if there exists a control Lyapunov function (CLF) with a suitable "small control property" (scp) [54], namely, a positive definite radially unbounded function W with the properties that $L_g W = 0 \Rightarrow L_f W < 0$ and $L_f W + L_g W \alpha_c < 0$ whenever $x \neq 0$, for some continuous function α_c.

Assumption 1 is somewhat stronger than mere stabilizability. For example, for the system

$$\dot{x} = x^3 + x^2 u \tag{21}$$

which is stabilizable by simple smooth feedback $u = -2x$, no function V exists that satisfies (16) for some $\beta > 0$, and yet $W = x^2/2$ is a CLF with an scp.

However, Assumption 1 is satisfied for any stabilizable system whose CLF W satisfies not only the CLF condition $L_g W = 0 \Rightarrow L_f W < 0$ but also a *strong small control property* (sscp) that

$$\lim_{\varepsilon \to 0} \max_{\substack{L_f W(x) > 0 \\ \text{and } |x| = \varepsilon}} \frac{L_f W(x)}{\left(L_g W(x)\right)^2} < \infty. \tag{22}$$

Under condition (22), it can be shown, by slightly modifying the proof in [35, (75)–(80)], that Assumption 1 is satisfied for any $\beta \geq 1$ by a new CLF V constructed as

$$V = \int_0^W \rho(r)dr \tag{23}$$

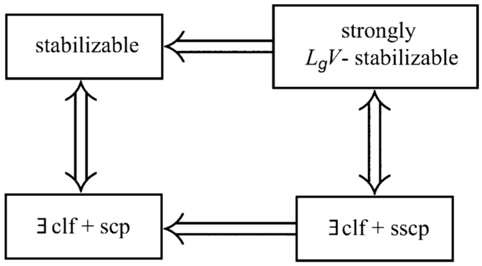

Fig. 1. Existence of a CLF with sscp is equivalent to the system being strongly L_gV-stabilizable, which implies that the system is stabilizable. It is well known [54] that stabilizability is equivalent to the existence of a CLF with scp, therefore, as shown above, existence of a CLF with sscp implies existence of a CLF with scp.

where

$$\rho(r) = 1 + 2 \sup_{x:V(x)\leq r} \frac{L_fV + \sqrt{(L_fV)^2 + (L_gV)^4}}{(L_gV)^2}. \tag{24}$$

In simple terms, a system is strongly L_gV-stabilizable if it has a CLF with a sscp. The converse is also true and follows trivially by noting that from (16) it follows that $L_gV = 0 \Rightarrow L_fV < 0$ whenever $x \neq 0$ and thus $\lim_{\varepsilon\to 0} \max_{\substack{L_fV(x)>0 \\ \text{and } |x|=\varepsilon}} (L_fV(x)/(L_gV(x))^2) < \beta < \infty$.

Fig. 1 shows relations between stabilizability and Assumption 1 by highlighting that both assumptions are equivalent to the existence of a CLF, but with different small control properties.

Though violated for the example (21), condition (22) is satisfied for many systems, including all systems in strict-feedback and strict-feedforward forms. Hence Assumption 1 is far from being overly restrictive, despite not being equivalent to stabilizability by continuous control.

In the linear case, the inequality (20) is simply a Riccati inequality and by no means appears to be a restrictive condition. However, when (A, b) are unknown, the designer can only guess a P, rather than solving (20) for a given matrix on the right-hand side of the inequality. As we shall see next, simple guesses will often violate (20). However, as we demonstrate in the rest of the paper, good guesses for a clm are available for some non-trivial classes of systems with unknown model parameters, including unknown control direction.

B. Is Assumption 1 Reasonable for Systems With Unknown Models?

Given how hard it is to find a CLF when f and g are known, how can the designer have V and β that satisfy (16) when f and g are unknown?

For instance, for the scalar example $\dot{x} = f(x) + u$ with $f(x) = x^3$, the CLF $V = x^2$ violates Assumption 1, though the CLF $V = x^4$ verifies the assumption. In Section VI we present an approach that allows the designer to construct a CLF that verifies Assumption 1 despite not knowing f.

For the second-order linear example with $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, which is completely controllable, a simple clm $P = I$ violates (20) since $PA + A^TP - 2\beta Pbb^TP = \begin{bmatrix} 2 & 1 \\ 1 & -\beta \end{bmatrix}$ cannot be made negative definite for any $\beta > 0$. Yet, as we shall see in Section V, a more complicated, valid clm P that does not require exact knowledge of A and b can be constructed which, in conjunction with the injection of a periodic perturbation and the averaging principle, ensures stabilization.

IV. Linear Time-Varying Systems With Unknown and Possibly Alternating Control Directions

The problem of stabilizing a linear time-invariant system is difficult when the system is unstable and the coefficients of the control input have unknown signs, even when the system's dynamic order is one. This problem was posed by Morse and solved by Nussbaum [49], whose solution was refined and simplified by Mudgett and Morse in adaptive control of general linear systems [47]. The work of Nussbaum, Mudgett, and Morse inspired a wide range of extensions utilizing the so called Nussbaum-type gain technique, achieving stabilization of first-order nonlinear systems [42], parametric-strict-feedback nonlinear systems [64], and output-feedback nonlinear systems with unknown control directions [16]. The Nussbaum technique was also utilized to allow for control direction uncertainty in the backstepping [34] approach for strict-feedback systems [68]. The Nussbaum-gain techniques were also extended to n-th order single input nonlinear strict-feedback systems with time-varying control coefficient of unknown but constant sign in both continuous [52], [62], [63] and discrete time [22]. Robust control is designed in [30] for systems where the control coefficient is allowed to change sign and pass through zero but it is required that the nominal system be stable.

Nussbaum-gain type controllers guarantee global stability without any a priori knowledge on the system parameters. However, they suffer from a large initial overshoot when the initial guess of the sign of the control coefficient is incorrect. Furthermore, these controllers are tuning-based, involving dynamic feedback even in the case of full-state measurement, and lacking exponential stability. Consequently, these controllers are prone to parameter drift and robustness problems, as all other conventional adaptive controllers.

While the mentioned Nussbaum-gain extensions allow for control coefficient uncertainty and even for time-varying control coefficients, the control coefficient is always assumed to remain non-zero for all time, namely, it is never allowed to pass through zero and change sign. We remove this "control coefficient sign constancy" restriction in this section, for linear-time-varying plants. In fact our result allows the control coefficient to oscillate, passing through zero persistently. Another advantage of our approach over existing approaches is that our control law is very simple and given as feedback of the norm of the state with time-varying coefficients.

Before we state our results we introduce the notation

$$\langle Z\rangle_\Delta(s) \triangleq \frac{1}{\Delta}\int_s^{s+\Delta} Z(\tau)d\tau \tag{25}$$

for $Z : \mathbb{R} \to \mathbb{R}$, and note that, for any column vector $B, BB^T \leq |B|^2 I$.

Our main result for general n-th order LTV systems is given in Theorem 3. However, for clarity, we first present a simpler result for a scalar LTV case in Proposition 1, which is not a corollary to Theorem 3 but is proved under less restrictive conditions.

Proposition 1: Consider the system

$$\dot{x} = a(t)x + b(t)u \tag{26}$$
$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)x^2 \tag{27}$$

and let there exist $\Delta > 0$, $\beta_0 > 0$, $\bar{a} > 0$, and $T > 0$ such that $a(t)$ and $b(t)$ satisfy

$$\frac{1}{\Delta}\int_{s}^{s+\Delta} b^2(\tau)d\tau \geq \beta_0, \quad \forall\, s \geq T \tag{28}$$

$$\langle |a| \rangle_{\Delta}(s) \leq \bar{a}, \quad \forall\, s \geq T. \tag{29}$$

If

$$k\alpha > \frac{\bar{a}}{\beta_0} \tag{30}$$

then the origin of (26), (27) is $1/\omega$-SPUAS with a lower bound on the average decay rate given by

$$\gamma_r = k\alpha\beta_0 - \bar{a} > 0. \tag{31}$$

Proof: System (26), (27) in closed loop form is

$$\dot{x} = a(t)x + b(t)\alpha\sqrt{\omega}\cos\omega t - b(t)k\sqrt{\omega}\sin(\omega t)x^2 \tag{32}$$

which has a Lie bracket average

$$\dot{\bar{x}} = \left[a(t) - k\alpha b^2(t)\right]\bar{x}. \tag{33}$$

If $k\alpha > \bar{a}/\beta_0$ we have from (28) that

$$k\alpha\int_{s}^{s+\Delta} b^2(\tau)d\tau > \Delta\bar{a}. \tag{34}$$

Therefore, for any $s \geq T$, $N \in \mathbb{N}$ the integral

$$\begin{aligned}
&\int_{s}^{s+N\Delta}\left[a(\tau) - k\alpha b^2(\tau)\right]d\tau \\
&= \sum_{j=0}^{N-1}\int_{s+j\Delta}^{s+(j+1)\Delta}\left[a(\tau) - k\alpha b^2(\tau)\right]d\tau \\
&= \sum_{j=0}^{N-1}\left[\int_{s+j\Delta}^{s+(j+1)\Delta} a(\tau)d\tau - \int_{s+j\Delta}^{s+(j+1)\Delta} k\alpha b^2(\tau)d\tau\right]
\end{aligned}$$

is, by application of (28), (29) and (30), bounded by

$$\begin{aligned}
\int_{s}^{s+N\Delta}\left[a(\tau)-k\alpha b^2(\tau)\right]d\tau &\leq \sum_{j=0}^{N-1}\left[\Delta\bar{a}-\Delta k\alpha\beta_0\right] \\
&= \sum_{j=0}^{N-1}(-\Delta\gamma_r) = -N\Delta\gamma_r < 0
\end{aligned} \tag{35}$$

i.e.

$$\langle -k\alpha b^2\rangle_{N\Delta}(s) \leq -\gamma_r \tag{36}$$

where $\gamma_r > 0$ is defined in (31). Hence, for any $s \geq T$, $N \in \mathbb{N}$

$$\begin{aligned}
|\bar{x}(s+N\Delta)| &= |\bar{x}(s)|\, e^{\int_s^{s+N\Delta}\left[a(\tau)-k\alpha b^2(\tau)\right]d\tau} \\
&< |\bar{x}(s)|\, e^{-N\Delta\gamma_r}.
\end{aligned} \tag{37}$$

Because $\gamma_r > 0$ the state $\bar{x}(t)$ converges to zero. To study the convergence rate, for any $t \geq T$ we denote $N = \lfloor (t-T)/\Delta \rfloor$, where $\lfloor\cdot\rfloor$ is the floor function. We then proceed to show that $|\bar{x}(t)| \leq M_0 e^{-\gamma_r t}|\bar{x}(0)|$, for all $t \geq 0$, for some $M_0 > 0$, and then, with the help of Corollary 1, complete the proof of the proposition. ■

Next we consider the general n-dimensional case, which is complicated by the possibility of cross talk between different components of vectors, a difficulty only possible in higher dimensions.

Theorem 3: Consider the system

$$\dot{x} = A(t)x + B(t)u \tag{38}$$

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)|x|^2 \tag{39}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $u \in \mathbb{R}$, and let there exist $\Delta > 0$, $b_\star \geq \beta_0 > 0$, $a_\star \geq 0$, and $T \geq 0$ such that $A(t)$ and $B(t)$ satisfy

$$\frac{1}{\Delta}\int_{s}^{s+\Delta} B(\tau)B^T(\tau)d\tau \geq \beta_0 I, \quad \forall s \geq T \tag{40}$$

$$\left\langle |B|^2\right\rangle_{\Delta}(s) \leq b_\star, \quad \forall s \geq 0 \tag{41}$$

$$\left\langle |A|^2\right\rangle_{\Delta}(s) \leq a_\star, \quad \forall s \geq 0. \tag{42}$$

The origin of system (38), (39) is $1/\omega$-SPUAS with a lower bound on the average decay rate given by

$$R = \frac{1}{2\Delta}\left[\ln\left(\frac{1}{\gamma}\right) - \gamma_2\right] - \sqrt{a_\star} > 0 \tag{43}$$

where

$$\gamma = 1 - \frac{k\alpha\Delta\beta_0}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} > 0 \tag{44}$$

$$\gamma_2 = \frac{4k\alpha\Delta^3 b_\star}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} \tag{45}$$

under either of the two conditions:

(i) Given $k\alpha > 0$ and $\Delta > 0$, $a_\star$ is in the interval $(0, \bar{a}_\star)$, where

$$\bar{a}_\star = \left(\frac{\ln\left(\frac{1}{\gamma}\right)}{\Delta + \sqrt{\Delta^2 + \gamma_2\ln\left(\frac{1}{\gamma}\right)}}\right)^2. \tag{46}$$

(ii) For a given $a_\star$, the window Δ satisfies $\Delta \in (0, \bar{\Delta})$, where

$$\bar{\Delta} = \frac{1}{\sqrt{a_\star}}\min\{\bar{\Delta}_1, \bar{\Delta}_2\} \tag{47}$$

where

$$\bar{\Delta}_1 = \frac{\frac{1}{2}\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - \beta_0}\right)}{1 + \frac{\sqrt{a_\star}}{b_\star}} \tag{48}$$

$$\bar{\Delta}_2 = \frac{-1 + \sqrt{1 + \sqrt{2}\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - \beta_0}\right)}}{\sqrt{2}} \tag{49}$$

and $k\alpha > 1$ is selected such that

$$k\alpha \in \left(\frac{1}{2\sqrt{2}\Delta b_\star}, \frac{1}{2\sqrt{2}\Delta b_\star} + M(a_\star, b_\star, \beta_0, \Delta)\right) \tag{50}$$

where

$$M = \frac{\sqrt{\beta_0^2 - 8b_\star^2\left[1-e^{-2\Delta\left(\sqrt{a_\star}+\frac{a_\star}{b_\star}\right)}\right]^2}}{8\Delta b_\star^2\left[1-e^{-2\Delta\left(\sqrt{a_\star}+\frac{a_\star}{b_\star}\right)}\right]} > 0. \quad (51)$$

Remark 1: Theorem 3(i) is a robustness result. For any $k\alpha > 0$, the controller (39) allows some perturbation $A(t)x$ in the system (38), as long as the mean of $A(t)$ is sufficiently small, as quantified by (46). Theorem 3(ii) is a design result. If the window Δ is small enough, as quantified by (47) and known (it is reasonable to assume that Δ is known because otherwise the *a priori* bounds (40)–(42) would have no meaning for the user), then $k\alpha$ can be chosen in the interval (50) to guarantee stability. In summary, the controller (39) cannot dominate an arbitrarily large $A(t)$, but if $B(t)$ is persistently exciting (PE) over Δ that is sufficiently small in relation to the size of $A(t)$, then the controller (39) can stabilize the system (38). Furthermore the allowable $k\alpha$ is not arbitrarily large but is within an interval. Overly large $k\alpha$ results in instability despite $B(t)$ being PE because, for a given Δ, an overly large $k\alpha$ forces $x(t)$ to evolve within the time-varying null space of $B^T(t)$, rather than forcing $x(t)$ to converge to zero.

Proof: The closed-loop system (38), (39) is given by

$$\dot{x} = A(t)x + B(t)\alpha\sqrt{\omega}\cos(\omega t) - B(t)k\sqrt{\omega}\sin(\omega t)|x|^2 \quad (52)$$

which we decompose as

$$\dot{x} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{a,i,j}(x)\bar{u}_{a,i,j}(t) + \sum_{j=1}^{n} b_{c,j}(x)\sqrt{\omega}\hat{u}_{c,j}(t,\theta) + \sum_{j=1}^{n} b_{s,j}(x)\sqrt{\omega}\hat{u}_{s,j}(t,\theta), \quad (53)$$

where

$$\bar{u}_{a,i,j}(t) = a_{ji}(t), \quad \hat{u}_{c,j}(t,\theta) = b_j(t)\cos(\omega t)$$
$$\hat{u}_{s,j}(t,\theta) = b_j(t)\sin(\omega t)$$

and

$$b_{a,i,j}(x) = x_i e_j, \quad b_{c,j}(x) = \alpha e_j, \quad b_{s,j}(x) = -k|x|^2 e_j$$

where e_j is the standard j-th basis vector of $\mathbb{R}^n$. Applying Lie bracket averaging, we obtain the averaged system

$$\dot{\bar{x}} = A(t)\bar{x} - k\alpha B(t)B^T(t)\bar{x}. \quad (54)$$

Parts of this proof use steps developed in the proof of Theorem 4.3.2 (iii) in the second half of Section 4.8.3 in [29]. With the following Lyapunov function candidate:

$$V(\bar{x}) = \frac{|\bar{x}|^2}{2} \quad (55)$$

we get

$$\dot{V}(\bar{x}) = \bar{x}^T\dot{\bar{x}} = \bar{x}^T A(t)\bar{x} - k\alpha\bar{x}^T B(t)B^T(t)\bar{x}. \quad (56)$$

Therefore, for any $s \geq T$ we have

$$V(s+\Delta) = V(s) - k\alpha\underbrace{\int_s^{s+\Delta}\left|\bar{x}^T(\tau)B(\tau)\right|^2 d\tau}_{I_1} + \underbrace{\int_s^{s+\Delta}\bar{x}^T(\tau)A(\tau)\bar{x}(\tau)d\tau}_{I_2}. \quad (57)$$

We first consider the term I_1 and rewrite

$$\bar{x}^T(\tau)B(\tau) = \bar{x}^T(s)B(\tau) + \left[\bar{x}(\tau)-\bar{x}(s)\right]^T B(\tau). \quad (58)$$

We can apply the inequality $(x+y)^2 \geq (1/2)x^2 - y^2$, obtaining

$$\left[\bar{x}^T(\tau)B(\tau)\right]^2 \geq \frac{1}{2}\left[\bar{x}^T(s)B(\tau)\right]^2 - \left[\left[\bar{x}(\tau)-\bar{x}(s)\right]^T B(\tau)\right]^2. \quad (59)$$

Thus, with (57) and (59) we get

$$I_1 \leq \underbrace{-k\alpha\frac{\bar{x}(s)^T}{2}\int_s^{s+\Delta}B(\tau)B^T(\tau)d\tau\bar{x}(s)}_{I_{11}} + \underbrace{k\alpha\int_s^{s+\Delta}\left[\left[\bar{x}(\tau)-\bar{x}(s)\right]^T B(\tau)\right]^2 d\tau}_{I_{12}}. \quad (60)$$

With (40) and (55) it readily follows that:

$$I_{11} \leq -\frac{\bar{x}(s)^T}{2}k\alpha\Delta I\beta_0\bar{x}(s) = -k\alpha\Delta\beta_0 V(\bar{x}). \quad (61)$$

Next we address I_{12}. Using (52) we get

$$\bar{x}(\tau)-\bar{x}(s) = \int_s^\tau \dot{\bar{x}}(\sigma)d\sigma = \int_s^\tau A(\sigma)\bar{x}(\sigma)d\sigma - k\alpha\int_s^\tau B(\sigma)B^T(\sigma)\bar{x}(\sigma)d\sigma. \quad (62)$$

Transposing (62) and multiplying by $B(\tau)$ we get

$$\left[\bar{x}(\tau)-\bar{x}(s)\right]^T B(\tau) = \int_s^\tau \bar{x}^T(\sigma)A^T(\sigma)d\sigma B(\tau) - k\alpha\int_s^\tau \bar{x}^T(\sigma)B(\sigma)B^T(\sigma)B(\tau)d\sigma. \quad (63)$$

By using the representation in (63) together with the inequality $(x-y)^2 \leq 2x^2 + 2y^2$ we get

$$I_{12} \leq \underbrace{2k\alpha \int_s^{s+\Delta} \left[\int_s^{\tau} \bar{x}^T(\sigma) A^T(\sigma) d\sigma B(\tau) \right]^2 d\tau}_{I_{13}} \underbrace{+2k\alpha \int_s^{s+\Delta} \left[k\alpha \int_s^{\tau} \bar{x}^T(\sigma) B(\sigma) B^T(\sigma) B(\tau) d\sigma \right]^2 d\tau}_{I_{14}}.$$

Next we consider the term I_{14}, to which we apply the Cauchy-Schwartz inequality followed by a change in the order of integration and obtain:

$$I_{14} \leq 2k^3\alpha^3\Delta^2 \left\langle |B|^2 \right\rangle_\Delta^2 \int_s^{s+\Delta} \left(\bar{x}^T(\sigma) B(\sigma) \right)^2 d\sigma. \quad (64)$$

Now we consider the term I_{13}, whose bound is given by

$$\begin{aligned} I_{13} &\leq 2k\alpha \int_s^{s+\Delta} |B(\tau)|^2 \left[\int_s^{\tau} |A(\sigma)| \, |x(\sigma)| \, d\sigma \right]^2 d\tau \\ &\leq 4k\alpha \int_s^{s+\Delta} |B(\tau)|^2 \int_s^{\tau} |A(\zeta)|^2 d\zeta \int_s^{\tau} V(\sigma) d\sigma d\tau \quad (65) \end{aligned}$$

and, changing the order of integration, we get

$$I_{13} \leq 4k\alpha\Delta^2 \left\langle |B|^2 \right\rangle_\Delta \left\langle |A|^2 \right\rangle_\Delta \int_s^{s+\Delta} V(\sigma) d\sigma. \quad (66)$$

Combining results (60), (61), and the bounds on I_{13} and I_{14}

$$\begin{aligned} I_1 \leq -k\alpha\Delta\beta_0 V(\bar{x}) &+ 2k^3\alpha^3\Delta^2 b_\star^2 \int_s^{s+\Delta} \left[\bar{x}^T(\sigma) B(\sigma) \right]^2 d\sigma \\ &+ 4k\alpha\Delta^2 b_\star a_\star \int_s^{s+\Delta} V(\sigma) d\sigma. \quad (67) \end{aligned}$$

Moving the second term on the right hand side of (67) to the left, we obtain

$$I_1 \leq \frac{-k\alpha\Delta\beta_0 V(\bar{x}) + 4k\alpha\Delta^2 b_\star a_\star \int_s^{s+\Delta} V(\sigma) d\sigma}{1 + 2k^2\alpha^2\Delta^2 b_\star^2}. \quad (68)$$

Now we turn our attention to the term I_2 in (57). Noting that

$$\bar{x}^T(\tau) A(\tau) \bar{x}(\tau) \leq |A(\tau)| \, \bar{x}^T \bar{x} = 2 \, |A(\tau)| \, V(\tau) \quad (69)$$

we get

$$I_2 \leq 2 \int_s^{s+\Delta} |A(\tau)| \, V(\tau) d\tau. \quad (70)$$

Combining (57), (68) and (70) we obtain

$$V(s+\Delta) \leq \gamma V(s) + 2 \int_s^{s+\Delta} |A(\tau)| \, V(\tau) d\tau + \frac{4k\alpha\Delta^2 b_\star a_\star \int_s^{s+\Delta} V(\sigma) d\sigma}{1 + 2k^2\alpha^2\Delta^2 b_\star^2}$$

which can be rewritten as

$$\gamma V(s) + \int_s^{s+\Delta} \left(2 \, |A(\tau)| + \frac{4k\alpha\Delta^2 b_\star a_\star}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} \right) V(\tau) d\tau \quad (71)$$

where γ is defined in (44). Noting that

$$\frac{k\alpha\Delta\beta_0}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} \leq \frac{\beta_0}{2\sqrt{2} b_\star} \quad (72)$$

and that $\beta_0 \leq b_\star$, we get that $\gamma \in ((2\sqrt{2} - 1/2\sqrt{2}), 1)$, which implies that γ is positive. We now apply the Bellman-Gronwall lemma, and get that for all $s \geq T$

$$V(s+\Delta) \leq \gamma e^{\left(2 \int_s^{s+\Delta} |A(\tau)| d\tau + \frac{4k\alpha\Delta^3 b_\star a_\star}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} \right)} V(s). \quad (73)$$

We note that the Cauchy-Schwartz inequality yields $\int_s^{s+\Delta} |A(\tau)| d\tau \leq \Delta\sqrt{a_\star}$, so we get, for all $s \geq T$

$$V(s+\Delta) \leq \gamma e^{\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} \right)} V(s). \quad (74)$$

Evidently for convergence we require that

$$\begin{aligned} &\gamma e^{\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} \right)} \\ &= \left(1 - \frac{k\alpha\Delta\beta_0}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} \right) e^{\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} \right)} \\ &< 1 \quad (75) \end{aligned}$$

or equivalently

$$1 - \frac{k\alpha\Delta\beta_0}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} < e^{-\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} \right)}. \quad (76)$$

To prove the theorem under condition (i), we now calculate the requirement on $a_\star$ for (76) to hold. We take ln of both sides of (76) which gives us

$$a_\star \frac{4k\alpha\Delta^3 b_\star}{1 + 2k^2\alpha^2\Delta^2 b_\star^2 + 2\Delta\sqrt{a_\star}} + \ln\left(1 - \frac{k\alpha\Delta\beta_0}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} \right) < 0. \quad (77)$$

We define γ_2 as (45) and set the left side of (77) equal to zero, obtaining

$$a_\star\gamma_2 + 2\Delta\sqrt{a_\star} + \ln(\gamma) = 0. \quad (78)$$

Since $\sqrt{a_\star}$ must be positive, we only consider the positive root

$$\sqrt{a_\star} = \frac{-\Delta + \sqrt{\Delta^2 + \gamma_2 \ln\left(\frac{1}{\gamma}\right)}}{\gamma_2} \quad (79)$$

which is positive because $\gamma_2 > 0$ and $\gamma \in ((2\sqrt{2}-1)/2\sqrt{2})$ implies that $\gamma_2 \ln(1/\gamma) > 0$. So we have

$$\bar{a}_\star = \left(\frac{\ln\left(\frac{1}{\gamma}\right)}{\sqrt{\Delta^2 + \gamma_2 \ln\left(\frac{1}{\gamma}\right)} + \Delta} \right)^2. \quad (80)$$

Since the left side of (77) is increasing as a function of $a_\star > 0$, for all $a_\star \in (0, \bar{a}_\star)$ we satisfy (75). To study the convergence rate of our system we denote (75) as

$$\gamma_r = \gamma e^{\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right)} < 1. \quad (81)$$

For any $t \geq T$ we denote $N = \lfloor (t-T)/\Delta \rfloor$, where $\lfloor\cdot\rfloor$ is the floor function. Then for $t \geq T$ we have

$$t = T + \Delta\left(\frac{t-T}{\Delta} - \left\lfloor \frac{t-T}{\Delta} \right\rfloor\right) + N\Delta \quad (82)$$

and from (74) we have the bound

$$V(t) \leq \gamma_r^N V\left(T + \Delta\left(\frac{t-T}{\Delta} - \left\lfloor \frac{t-T}{\Delta} \right\rfloor\right)\right). \quad (83)$$

This bound is obtained by noting from (74) and (81) that $V(s + N\Delta) \leq \gamma_r^N V(s)$ and by substituting $s = T + \Delta(((t-T)/\Delta) - \lfloor (t-T)/\Delta \rfloor)$. Recalling that

$$\begin{aligned} \dot{V} &= \bar{x}^T \left[A(t) - k\alpha B(t) B^T(t)\right] \bar{x} \\ &\leq 2\left|A(t) - k\alpha B(t) B^T(t)\right| V \end{aligned} \quad (84)$$

for $\Delta(((t-T)/\Delta) - \lfloor (t-T)/\Delta \rfloor) \leq \Delta$ we get the bound

$$\begin{aligned} &V\left(T + \Delta\left(\frac{t-T}{\Delta} - \left\lfloor \frac{t-T}{\Delta} \right\rfloor\right)\right) \\ &\qquad \leq e^{2\int_0^{T+\Delta} \left|A(\tau) - k\alpha B(\tau) B^T(\tau)\right| d\tau} V(0) \end{aligned} \quad (85)$$

and therefore

$$V(t) \leq e^{2\int_0^{T+\Delta} \left|A(\tau) - k\alpha B(\tau) B^T(\tau)\right| d\tau} \gamma_r^N V(0). \quad (86)$$

We now consider the term γ_r^N. Since

$$N = \frac{t - T - \Delta\left(\frac{t-T}{\Delta} - \left\lfloor \frac{t-T}{\Delta} \right\rfloor\right)}{\Delta} \geq \frac{t - T - \Delta}{\Delta} \quad (87)$$

and $\gamma_r \in (0,1)$ it follows that:

$$\gamma_r^N \leq \gamma_r^{\frac{t-T-\Delta}{\Delta}}. \quad (88)$$

With (86) and (88) we obtain

$$V(t) \leq e^{2\int_0^{T+\Delta} \left|A(\tau) - k\alpha B(\tau) B^T(\tau)\right| d\tau} \gamma_r^{-\frac{T+\Delta}{\Delta}} \gamma_r^{\frac{t}{\Delta}} V(0). \quad (89)$$

We now define

$$M_0 = \sqrt{e^{2\int_0^{T+\Delta} |A(\tau) - k\alpha B(\tau) B^T(\tau)| d\tau} \gamma_r^{-\frac{T+\Delta}{\Delta}}} \quad (90)$$

and rewrite

$$\gamma_r^{\frac{t}{\Delta}} = \left(\frac{1}{\gamma_r}\right)^{\frac{-t}{\Delta}} = e^{-\frac{\ln\left(\frac{1}{\gamma_r}\right)}{\Delta} t}. \quad (91)$$

Recalling that $\gamma_r \in (0,1)$ we define

$$R(k\alpha, \Delta, \beta_0, b_\star, a_\star) = \frac{\ln\left(\frac{1}{\gamma_r}\right)}{2\Delta} > 0 \quad (92)$$

and write the exponential decay of V as

$$V(t) \leq M_0^2 e^{-2Rt} V(0). \quad (93)$$

Substituting (81) into (92), we obtain (43). Finally recalling the definition of $V(t)$ we write the exponential decay of $|\bar{x}(t)|$ as

$$|\bar{x}(t)| \leq M_0 e^{-Rt} |\bar{x}(0)|. \quad (94)$$

Therefore, by Corollary 1, the origin of system (38), (39) is $1/\omega$-SPUAS, which proves the result under condition (i). Proceeding to the proof of the theorem under condition (ii), for any given $a_\star$ we want to find a range of stabilizing values of $k\alpha$ as a function of Δ. For a given $\beta_0, b_\star, a_\star$ we first consider over what range of $\Delta \in (0, \infty)$ it is possible to satisfy the convergence condition (76). We define the function

$$F(k\alpha, \Delta) = \frac{k\alpha\Delta\beta_0}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} + e^{-\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right)} \quad (95)$$

which must achieve a value larger than 1 for (76) to be satisfied. In order to consider the maximum possible value of (95) we first fix Δ and set the derivative, with respect to $k\alpha$, of $F(k\alpha, \Delta)$ equal to zero, to find that $F(k\alpha, \Delta)$ has its maximum value at

$$(k\alpha)_{\mathrm{m}} = \frac{1}{\sqrt{2}\Delta b_\star} \quad (96)$$

and the maximum value is

$$F((k\alpha)_{\mathrm{m}}, \Delta) = \frac{\beta_0}{2\sqrt{2} b_\star} + e^{-\left(2\Delta\sqrt{a_\star} + \sqrt{2}\Delta^2 a_\star\right)}. \quad (97)$$

The convergence condition requires this maximum value (97) to be greater than 1. We note that $F((k\alpha)_{\mathrm{m}}, \Delta)$ is strictly decreasing as a function of $\Delta \in (0, \infty)$. Therefore if $F((k\alpha)_{\mathrm{m}}, \Delta^\star) = 1$, it follows that $F((k\alpha)_{\mathrm{m}}, \Delta) > 1$ for all $\Delta \in (0, \Delta^\star)$. The condition $F((k\alpha)_{\mathrm{m}}, \Delta^\star) = 1$ implies that

$$2\Delta\sqrt{a_\star} + \sqrt{2}\Delta^2 a_\star - \ln\left(\frac{2\sqrt{2} b_\star}{2\sqrt{2} b_\star - \beta_0}\right) = 0 \quad (98)$$

from which we obtain the positive root

$$\Delta^\star = \frac{-1 + \sqrt{1 + \sqrt{2}\ln\left(\frac{2\sqrt{2} b_\star}{2\sqrt{2} b_\star - \beta_0}\right)}}{\sqrt{2 a_\star}}. \quad (99)$$

Therefore it is possible to stabilize the system when $0 < \Delta < \Delta^\star$ by choosing $k\alpha = (k\alpha)_{\mathrm{m}}$ as in (96). By continuity, for any $0 < \Delta < \Delta^\star$ there must be an interval containing $(k\alpha)_{\mathrm{m}}$ such that all values of $k\alpha$ within that interval satisfy condition

(76). For $\Delta \in (0, \Delta^\star)$ we consider all values of $k\alpha$ that achieve $F(k\alpha, \Delta) > 1$. Recalling the definition of $F(k\alpha, \Delta)$

$$F(k\alpha, \Delta) = \frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2 b_\star^2} + e^{-\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right)} \quad (100)$$

to remove the $k\alpha$ dependence from the exponential in (100) we restrict our attention to $k\alpha > 1$, in which case

$$e^{-\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right)} > e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}. \quad (101)$$

We satisfy (100) by restricting $k\alpha$ to satisfy

$$\frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2 b_\star^2} + e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)} > 1. \quad (102)$$

Setting (102) equal to 1, we solve for $k\alpha$ as

$$k\alpha = \frac{\beta_0 \pm \sqrt{\beta_0^2 - 8b_\star^2\left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]^2}}{4\Delta b_\star^2\left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]}. \quad (103)$$

To ensure $k\alpha$ is real valued we impose the condition

$$\beta_0^2 \geq 8b_\star^2\left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]^2 \quad (104)$$

which implies

$$e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)} \geq 1 - \frac{\beta_0}{2\sqrt{2}b_\star}. \quad (105)$$

Taking ln of each side of (105) we obtain the condition

$$-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right) > \ln\left(1 - \frac{\beta_0}{2\sqrt{2}b_\star}\right) \quad (106)$$

which implies that the new requirement on the possible values of Δ is

$$0 < \Delta < \bar{\Delta} = \min\left\{\frac{\ln\left(\frac{1}{\sqrt{1-\frac{\beta_0}{2\sqrt{2}b_\star}}}\right)}{\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}, \Delta^\star\right\}. \quad (107)$$

With (99) and (107) we obtain (47). Returning to (103) and recalling the value $(k\alpha)_{\mathrm{m}} = 1/\sqrt{2}\Delta b_\star$ we have the roots

$$k\alpha = \frac{(k\alpha)_{\mathrm{m}}}{\eta}, \quad k\alpha = (k\alpha)_{\mathrm{m}}\eta \quad (108)$$

where

$$\eta = \frac{\beta_0 + \sqrt{\beta_0^2 - 8b_\star^2\left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]^2}}{2\sqrt{2}b_\star\left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]}. \quad (109)$$

Therefore the system is stable for

$$k\alpha \in \left(\frac{(k\alpha)_{\mathrm{m}}}{\eta}, (k\alpha)_{\mathrm{m}}\eta\right). \quad (110)$$

We have thus derived sufficient conditions on Δ and $k\alpha$ to guarantee stability of our system. For each window Δ we have given an interval of stabilizing values of $k\alpha$, (110). However we now proceed to restrict our conditions on $k\alpha$ in order to give a more intuitive condition (50). We show that the interval (110) contains $(k\alpha)_{\mathrm{m}}$ by recalling (104) and calculating

$$\eta \geq 1 + \frac{\sqrt{\beta_0^2 - 8b_\star^2\left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]^2}}{2\sqrt{2}b_\star\left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]} \quad (111)$$

and

$$\frac{1}{\eta} \leq \frac{2\sqrt{2}b_\star\left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]}{\beta_0} < 1. \quad (112)$$

Therefore the interval (110) contains the more restrictive, but more illustrative interval (50), where we have explicitly written out the value $(k\alpha)_{\mathrm{m}} = 1/\sqrt{2}\Delta b_\star$. From the presence of Δ in the denominator we see that this interval of stability grows unbounded in length as the window Δ decreases. ■

Remark 2: We recall from (47) that Δ must not exceed $\bar{\Delta}_1$. By recalling that $b_\star \geq \beta_0$, by using the fact that $\ln(2\sqrt{2}/(2\sqrt{2}-1)) < 1/\sqrt{2}$ and by noting that

$$\frac{\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - \beta_0}\right)}{2\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)} < \frac{\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - b_\star}\right)}{2\sqrt{a_\star}} = \frac{\ln\left(\frac{2\sqrt{2}}{2\sqrt{2}-1}\right)}{2\sqrt{a_\star}} < \frac{1}{2\sqrt{2a_\star}} \quad (113)$$

we get that $\bar{\Delta}_1 < 1/2\sqrt{2a_\star}$. Hence, the stabilizing values of $k\alpha$ in the interval (50) must satisfy

$$k\alpha > \frac{1}{2\sqrt{2}\Delta b_\star} > \frac{1}{2\sqrt{2}\bar{\Delta}_1 b_\star} > \frac{\sqrt{a_\star}}{b_\star}. \quad (114)$$

The condition (114) is very similar to the stability requirement that is established in the one-dimensional case, in Proposition 1. As $a_\star$ increases, stability is ensured by increasing $k\alpha$.

V. Linear Systems in Strict-Feedback Form

In this section we consider linear systems in strict-feedback form and design a controller based on the backstepping approach [34], [58].

Theorem 4: Consider the plant

$$\dot{x}_i = \sum_{j=1}^{i} a_{ij}(t)x_j + x_{i+1}, \quad 1 \leq i \leq n-1 \quad (115)$$

$$\dot{x}_n = \sum_{j=1}^{n} a_{nj}(t)x_j + b(t)u \quad (116)$$

with the control law

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)\left(\sum_{i=1}^{n-1}\left(\prod_{j=i}^{n-1} c_j\right)x_i + x_n\right)^2 \quad (117)$$

and let $\beta_0 > 0$ and $a_{\max} > 0$ be known such that for some $T > 0$ and $\Delta > 0$, for all $s \geq T$

$$\frac{1}{\Delta}\int_s^{s+\Delta} b^2(\tau)d\tau > \beta_0 \quad (118)$$

$$\frac{1}{\Delta}\int_s^{s+\Delta} |a_{ij}(\tau)|\, d\tau \leq a_{\max}, \quad \forall i, j. \quad (119)$$

If $c_1, c_2, \ldots, c_n$ are chosen recursively so that

$$c_i > a_{\max} + \left\{ C_{1i}, \max_{2\le j\le i-2} C_{2ij}, c_{i-1} \right\}, \quad 1 \le i \le n-1 \tag{120}$$

where $c_0 = 0$ and

$$C_{1i} = \frac{(n-1)^2(1+\bar{d}_{i,i-1})^2}{4\bar{d}_{i-1,i-1}} \tag{121}$$

$$C_{2ij} = \frac{(n-1)^2\bar{d}_{ij}^2}{4\bar{d}_{jj}} \tag{122}$$

and

$$\bar{d}_{ij}=a_{\max} + a_{\max}c_j + c_{i-1}\bar{d}_{i-1,j}, \quad \begin{matrix} 1 \le i \le n \\ 1 \le j \le i-2 \end{matrix} \tag{123}$$

$$\bar{d}_{ii}=c_i - c_{i-1} + a_{\max}, \quad 1 \le i \le n-1 \tag{124}$$

$$\bar{d}_{nn}=b^2k\alpha - c_{n-1} + a_{\max} \tag{125}$$

then if

$$k\alpha > \frac{c_{n-1} + a_{\max}}{\beta_0} \tag{126}$$

the origin of system (115)–(117) is $1/\omega$-SPUAS.

Proof: We define

$$z_i = x_i + \sum_{k=1}^{i-1}\left(\prod_{j=k}^{i-1} c_j\right) x_k, \quad 1 \le i \le n \tag{127}$$

and rewrite the controller (117) as

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)z_n^2. \tag{128}$$

We get the Lie bracket averaged system (115)–(117) as

$$\dot{\bar{z}} = D\bar{z} \tag{129}$$

where

$$D=\begin{pmatrix} -d_{11} & 1 & \cdots & 0 & 0 \\ d_{21} & -d_{22} & \cdots & 0 & 0 \\ d_{31} & d_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & -d_{n-1,n-1} & 1 \\ d_{n1} & d_{n2} & \cdots & d_{n,n-1} & -d_{nn} \end{pmatrix} \tag{130}$$

with the diagonal terms of (130) satisfying

$$d_{ii} = c_i - c_{i-1} - a_{ii}, \quad 1 \le i \le n-1$$
$$d_{nn} = b^2k\alpha - c_{n-1} - a_{nn}.$$

The off-diagonal terms are defined as

$$d_{ij} = a_{ij} - a_{i,j+1}c_j + c_{i-1}d_{i-1,j}, \quad \begin{matrix} 1 \le i \le n, \\ 1 \le j \le i-2. \end{matrix}$$

Considering the Lyapunov function

$$V = \frac{1}{2}\sum_{i=1}^{n}\bar{z}_i^2 \tag{131}$$

we get

$$\dot{V} = -\sum_{i=1}^{n} d_{ii}\bar{z}_i^2 + \sum_{i=2}^{n}(1+d_{i,i-1})\bar{z}_i\bar{z}_{i-1} + \sum_{i=3}^{n}\sum_{j=1}^{i-2} d_{ij}\bar{z}_i\bar{z}_j$$

which we rewrite as

$$\dot{V} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[-\frac{d_{ii}}{n-1}\bar{z}_i^2 + (1+d_{ji})\bar{z}_i\bar{z}_j - \frac{d_{jj}}{n-1}\bar{z}_j^2\right] \tag{132}$$

Note that $d_{ii} > 0\ \forall i$ for c_i and $k\alpha$ that satisfy

$$c_i > c_{i-1} + a_{\max}, \quad 1 \le i \le n-1, \quad c_0 = 0. \tag{133}$$

We now rewrite (132) as

$$\dot{V} = \frac{-2}{n-1}\sum_{i=1}^{n-1} d_{ii}\sum_{j=i+1}^{n}\begin{bmatrix} z_i \\ z_j \end{bmatrix}^T \hat{D}_{ij}\begin{bmatrix} z_i \\ z_j \end{bmatrix} \tag{134}$$

where

$$\hat{D}_{ij} = \frac{1}{2}\begin{bmatrix} 1 & \frac{(n-1)(1+d_{ij})}{d_{ii}} \\ \frac{(n-1)(1+d_{ij})}{d_{ii}} & \frac{d_{jj}}{d_{ii}} \end{bmatrix}. \tag{135}$$

To ensure that $\dot{V} < 0$, the matrices $\hat{D}_{ij} \ne D_{nn}$ are made positive definite by choosing

$$\sqrt{\frac{d_{ii}}{d_{i-1,i-1}}} > \frac{(n-1)(1+d_{i,i-1})}{2d_{i-1,i-1}}, \quad 2 \le i \le n \tag{136}$$

and

$$\sqrt{\frac{d_{ii}}{d_{jj}}} > \frac{(n-1)d_{ij}}{2d_{jj}}, \quad 3 \le i \le n, \quad 2 \le j \le i-2 \tag{137}$$

which is accomplished by choosing c_i such that

$$c_i = a_{ii} + d_{ii} > a_{ii} + \frac{(n-1)^2(1+d_{i,i-1})^2}{4d_{i-1,i-1}}, \quad 2 \le i \le n$$

$$c_i = a_{ii} + d_{ii} > a_{ii} + \frac{(n-1)^2d_{ij}^2}{4d_{jj}}, \quad \begin{matrix} 3 \le i \le n, \\ 2 \le j \le i-2. \end{matrix}$$

Finally, by choosing

$$k\alpha > \frac{c_{n-1} + a_{\max}}{\beta_0} \tag{138}$$

we ensure that $\int_s^{s+\Delta} D_{nn}(\tau)d\tau < 0$, and proceeding as in the proof of Proposition 1, we ensure that $V(s+\Delta) < V(s)$ for all $s \ge T$, and as in Theorem 1 we guarantee that the origin is an exponentially stable equilibrium point of system (129). Therefore by Corollary 1, the origin of system (115)–(117) is $1/\omega$-SPUAS. ■

A closer look at the control law (117) and the CLF (131), along with (127), shows that the control law is not exactly in the forms (14) and (18). The terms $z_1^2, \ldots, z_{n-1}^2$ are omitted because $L_gV = L_g(z_n^2/2) = z_n$.

VI. Nonlinear MIMO Systems With Matched Uncertainties

While in Section III we presented a general approach for nonlinear systems based on an assumed availability of a CLF V that satisfies the strong L_gV-stabilizability condition, in this section we turn our attention to a specific construction of such a CLF for a limited but relevant class of systems that illustrates how to overcome the challenge of dealing with unknown nonlinearities.

In this section we study multi-input systems with the same number of controls and states. Admittedly, this is a class of "glorified first-order systems." However, we use this class to illustrate clearly how to deal with nonlinearities that are not only unknown but also have arbitrary growth (super-linear, exponential, or even faster than exponential). For systems with more states than controls, such as nth order systems in the strict-feedback form with one control and with only bounds on nonlinearities known, CLFs satisfying Assumption 1 can be constructed using the approach introduced in [41, see Theorem 3.1, with (26) and (27) being the key steps], which we have actually used for linear strict-feedback systems in Section V.

We consider only time-invariant nonlinear systems in this section. Time-varying systems, albeit linear, have already been dealt with in Section IV. The nonlinear systems studied in this section can be approached similarly but, for the sake of clarity, we choose not to pursue time-varying extensions here. Since the systems we consider here have the same number of controls and states, the input matrix is square. Given that the input matrix is not time-varying and thus persistency of excitation cannot be exploited in stabilization, we make an assumption that the input matrix multiplied by its transpose is positive definite for all x, which means that the system is completely controllable, though its control directions are unknown. Furthermore, the non-zero assumption on the input matrix $G(x)$ is motivated by the possible finite escape time of general nonlinear systems.

Theorem 5: Consider the system

$$\dot{x} = f(x) + G(x)u \tag{139}$$

where $u, x \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and let there exist $\beta_0 > 0$, and $\eta \in \mathcal{K}_\infty$ such that $f(x)$ and $G(x)$ satisfy the following bounds for all $x \in \mathbb{R}^n$:

$$G(x)G^T(x) \geq \beta_0 I \tag{140}$$

$$|f(x)| \leq \eta\left(|x|\right). \tag{141}$$

If k and α are chosen such that

$$k\alpha > \frac{1}{\beta_0} \tag{142}$$

then the controller

$$u_i = \alpha\sqrt{\omega\omega_i'}\cos\left(\omega\omega_i' t\right) - k\sqrt{\omega\omega_i'}\sin\left(\omega\omega_i' t\right)V(x) \tag{143}$$

where

$$V(x) = \int_0^{|x|}\eta(r)dr \tag{144}$$

and the frequencies ω_i' are rational and distinct, renders the origin of (139), (143) $1/\omega$-SPUAS.

Proof: A common period for all of the controller components is given by $T = 2\pi\mathrm{LCM}\{1/\omega_i'\}$. Therefore

$$\begin{aligned}&\int_0^T \cos\left(\omega\omega_i' t\right)\cos\left(\omega\omega_j' t\right)dt\\ &\quad= \int_0^T \sin\left(\omega\omega_i' t\right)\sin\left(\omega\omega_j' t\right)dt\\ &\quad= \int_0^T \sin\left(\omega\omega_i' t\right)\cos\left(\omega\omega_j' t\right)dt = 0, \quad \forall\, i \neq j.\end{aligned} \tag{145}$$

Consider the closed loop system

$$\begin{aligned}\dot{x} = f(x) + \sqrt{\omega}\sum_{i=1}^{n}\Big[&\alpha G(x)e_i\sqrt{\omega_i'}\cos\left(\omega_i'\theta\right)\\ &-kG(x)e_iV(x)\sqrt{\omega_i'}\sin\left(\omega_i'\theta\right)\Big], \quad \theta = \omega t.\end{aligned} \tag{146}$$

System (146) is in the form of system (4) to which we can apply Lie bracket averaging. Considering property (145), terms of different frequency combinations integrate to zero. Therefore the Lie bracket terms we are left with are

$$\left[G(\bar{x})e_i, G(\bar{x})e_iV(\bar{x})\right] = G(\bar{x})e_ie_i^TG^T(\bar{x})\left(\frac{\partial V(\bar{x})}{\partial\bar{x}}\right)^T. \tag{147}$$

Combining all terms of the form (147) we get

$$\sum_{i=1}^{n}Ge_ie_i^TG^T\left(\frac{\partial V}{\partial\bar{x}}\right)^T = GG^T\left(\frac{\partial V}{\partial\bar{x}}\right)^T \tag{148}$$

resulting in the Lie bracket averaged system

$$\dot{\bar{x}} = f(\bar{x}) - \frac{k\alpha}{2}G(\bar{x})G^T(\bar{x})\eta\left(|\bar{x}|\right)\frac{\bar{x}}{|\bar{x}|} \tag{149}$$

where we have used the fact that

$$\frac{\partial V(\bar{x})}{\partial\bar{x}} = \eta\left(|\bar{x}|\right)\frac{\bar{x}^T}{|\bar{x}|}. \tag{150}$$

With another Lyapunov function candidate

$$W(\bar{x}) = \frac{|\bar{x}|^2}{2} \tag{151}$$

we get

$$\dot{W}(\bar{x}) = \bar{x}^T\dot{\bar{x}} = \bar{x}^Tf(\bar{x}) - k\alpha\frac{\eta\left(|\bar{x}|\right)}{|\bar{x}|}\bar{x}^TG(\bar{x})G^T(\bar{x})\bar{x}. \tag{152}$$

From (141) we have

$$|\bar{x}^Tf| \leq |\bar{x}||f| \leq |\bar{x}|\eta\left(|\bar{x}|\right) \tag{153}$$

and from (140) we have that

$$k\alpha\frac{\eta\left(|\bar{x}|\right)}{|\bar{x}|}\bar{x}^TG(\bar{x})G^T(\bar{x})\bar{x} \geq k\alpha\frac{\eta\left(|\bar{x}|\right)}{|\bar{x}|}\beta_0|\bar{x}|^2. \tag{154}$$

Plugging (153) and (154) into the equation for $\dot{W}(\bar{x})$ we get

$$\begin{aligned}\dot{W}(\bar{x}) &\leq |\bar{x}|\eta(|\bar{x}|) - k\alpha\beta_0|\bar{x}|\eta(|\bar{x}|)\\ &= (1-k\alpha\beta_0)|\bar{x}|\eta(|\bar{x}|)\end{aligned} \tag{155}$$

therefore by our choice of $k\alpha > 1/\beta_0$, we guarantee that (155) is negative definite and therefore the Lie bracket averaged system (149) is globally uniformly asymptotically stable. By Corollary 1, system (139) is $1/\omega$-SPUAS. ■

Remark 3: Condition (140) can be relaxed to a functional lower bound $G(x)G^T(x) \geq \beta(|x|)I$ for some $\beta \in \mathcal{K}$. Then, for the average system, the Lyapunov inequality (155) is replaced by $\dot{W}(\bar{x}) \leq (1 - k\alpha\beta(|\bar{x}|))|\bar{x}|\eta(|\bar{x}|)$, which guarantees that, for $k\alpha > 1/\beta(\infty)$, the averaged system is globally uniformly ultimately bounded (GUUB) with an ultimate bound $\beta^{-1}(1/k\alpha)$. Though Theorem 1 only allows us to relate global asymptotic stability (GUAS) of the averaged system with $1/\omega$-SPUAS stability of the actual system, a similar relationship can be established between GUUB and what we would refer to as $1/\omega$-Semiglobal Practical Uniform Ultimate Boundedness ($1/\omega$-SPUUB) of a system. The $1/\omega$-SPUUB property and its applications in tracking for unknown systems are the subjects of ongoing research and beyond the scope of the present paper.

VII. Simulation Results

A. LTV Stabilization

To demonstrate the extremum seeking controller's ability to handle unknown, quickly time varying control direction we consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}2.1 & 4.9\\ -7.5 & 3.6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\cos(10t+.3)\\ \sin(10t+.3)\end{bmatrix}u. \tag{156}$$

A physical motivation for this example can be that $x = (x_1, x_2)$ is the planar coordinate of a mobile robot, with its angular velocity actuator failed and stuck at 10, and which has to be stabilized to the origin using the forward velocity input u only, in the presence of a position-dependent perturbation given by $\begin{bmatrix}2.1 & 4.9\\ -7.5 & 3.6\end{bmatrix}x$. The uncontrolled system is unstable with poles at $2.85 \pm 10.7i$. We apply ES control

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)\left[x_1^2(t) + x_2^2(t)\right] \tag{157}$$

with $\omega = 100$, $k = 4$, $\alpha = 2$ and starting from $x_1(0) = 1$, $x_2(0) = -1$, Fig. 2 shows the system's time response.

B. Strict-Feedback Form

Consider controlling the position and velocity of an object experiencing destabilizing forces proportional to its velocity and its distance from the origin, by applying a force u whose gain b is of unknown sign. The dynamics are governed by Newton's law, $F_{\text{total}} = ma = m\ddot{x} = k_x x + k_v\dot{x} + b\sin(10t)u$, which may be written in strict-feedback form

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{k_x}{m}x_1 + \frac{k_v}{m}x_2 + \frac{b}{m}\sin(10t)u. \tag{158}$$

We implement the feedback controller

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)(2x_1 + x_2)^2. \tag{159}$$

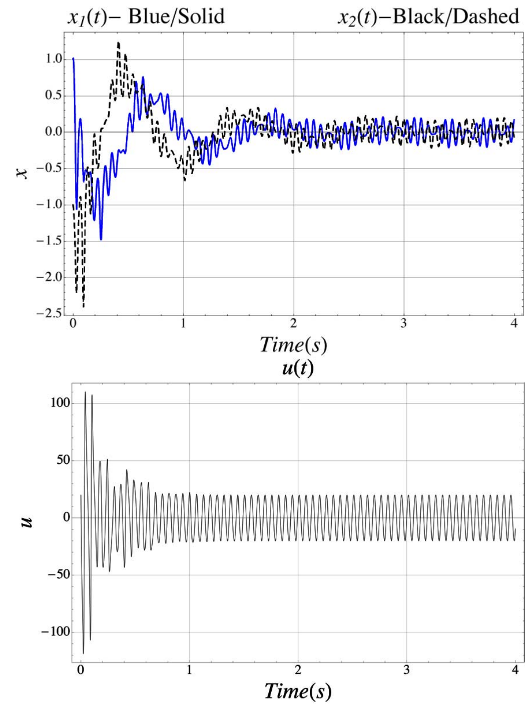

Fig. 2. After a transient which, in the average sense, is underdamped, the solution of (156), (157) settles to an $O(\alpha/\sqrt{\omega})$ neighborhood of the origin.

For the case $k_x = 1$, $k_v = 2$, $m = 1$, and $b = 1$, and with controller parameters $k = 4$, $\alpha = 2$, and $\omega = 100$, the simulation, with initial condition $x_1(0) = 1$, $x_2(0) = -1$, is shown in Fig. 3.

C. Nonlinear Scalar Example

We demonstrate the controller's ability to stabilize nonlinear systems with the following example:

$$\dot{x} = f(x) + \left(1 - \frac{1}{2}\sin(x)\right)u, \qquad f(x) = x^2. \tag{160}$$

Assuming that we know that the nonlinearity $f(x)$ is polynomial, we know that $f(x)$ satisfies a bound of the form $|f(x)| < \gamma|x|e^{|x|}$, for $f(x) = x^2$, $\gamma = 1$. Assuming γ to be known, and noting that $\int re^r dr = (r-1)e^r$, we choose the controller

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)\left[1 + (|x| - 1)e^{|x|}\right]. \tag{161}$$

With $k = 7.5$, $\alpha = 0.25$ and $\omega = 70$, simulation results starting from $x(0) = 2$ are shown in Fig. 4.

D. Comparison With Nussbaum-Type Control

We now consider the scalar example

$$\dot{x} = x + \cos(10t)u \tag{162}$$

and compare our static time-varying feedback

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)x^2 \tag{163}$$

to the dynamic feedback by Mudgett and Morse [47]

$$u = y^2\cos(y)x, \qquad \dot{y} = x^2 \tag{164}$$

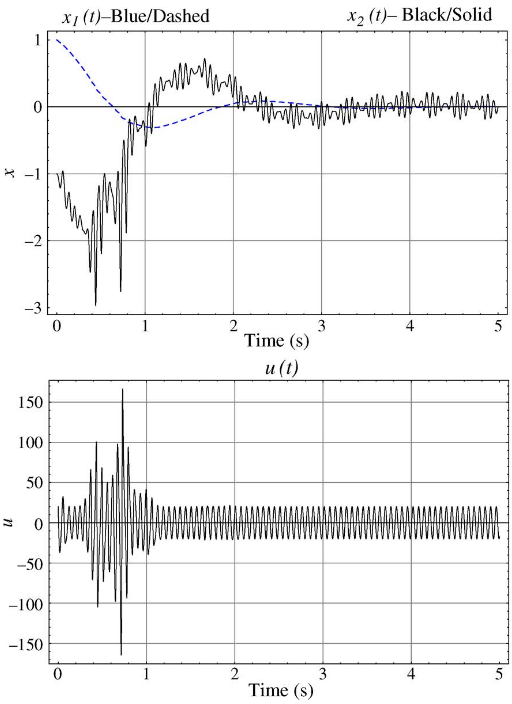

Fig. 3. Although the sign of the applied force is unknown to the controller the position x_1 and velocity x_2 of system (58), (59) quickly settle to $O(1/\sqrt{\omega})$ neighborhoods of the origin.

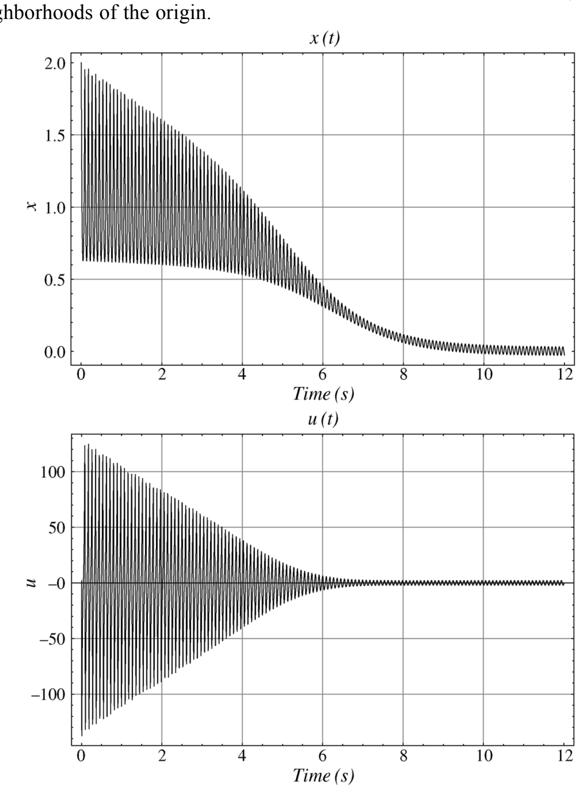

Fig. 4. As system (160) settles to within a $O(1/\sqrt{\omega})$ neighborhood of the origin, the control effort, (161), initially large, settles to a steady state magnitude of $\alpha\sqrt{\omega}$.

which admittedly was designed only for constant input coefficients. We simulate the two closed loop systems starting from $x(0) = 5$, with $\omega = 100$, $k = 5$, $\alpha = 5$ for our controller and $y(0) = 10$ for the controller of Mudgett and Morse. As shown in Fig. 5, the extremum-seeking method's performance is only slightly changed by the alternating sign of the input

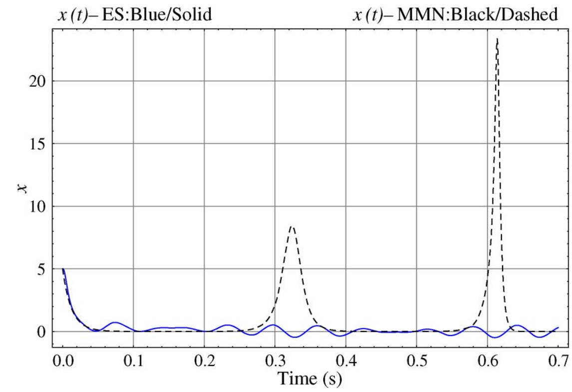

Fig. 5. Comparison of the extremum seeking (ES-Blue/Solid) and Mudgett-Morse-Nussbaum (MMN-Black/Dashed) schemes for the system $\dot{x} = x + \cos(10t)u$, with $x(0) = 5$, $y(0) = 10$, $k = 5$, $\alpha = 5$, and $\omega = 100$. The repeated overshoot caused by the MMN-controller takes place whenever $\cos(10t)\cos(y(t)) > 0$, furthermore, because, when x is non-zero, y is always growing, the overshoots grow in severity as time goes on.

coefficient, at most kicking the system $\alpha/\sqrt{\omega}$ (the size of the perturbing signal) in the wrong direction. The MMN method on the other hand suffers from overshoot each time the sign change happens as $y(t)$ cannot change fast enough to maintain $\cos(10t)\cos(y(t)) < 0$. Worse yet, the growing size of $y(t)$ causes growth of the overshoot size as well.

VIII. Conclusion

The ES algorithm creates a closed loop system that is independent of the control coefficients' signs. This is a useful property which allows us to stabilize *unknown*, unstable, control direction-varying systems using a particular form of time-varying nonlinear high-gain feedback. The only restriction to the applicability of the control law (39) is that, for a given bound on $A(t)$, the vector $B(t)$ be persistently exciting over a sufficiently short window Δ, namely, that the variations of $B(t)$ are sufficiently fast.

The results presented in the paper put more emphasis on linear problems for clarity, but their applicability to nonlinear systems is also illustrated, both in the general case, where the extremum seeking controller emulates $-L_gV$ controllers, and in the special case of MIMO nonlinear systems, where a universal CLF construction is presented that enables stabilization for matched unknown nonlinearities of arbitrary growth rate. Some minimal a priori knowledge is needed in all our theorems in order to choose the gain $k\alpha$ sufficiently high. As an alternative, one would consider employing an adaptive gain $k(t)$ that raises the gain to a sufficient level to achieve stabilization without requiring any a priori knowledge regarding the uncertainties. For example, for all systems in Section VI, if the vector field $f(x)$ is polynomial of unknown polynomial order with $f(0) = 0$ and hence satisfying $|f(x)| \leq \gamma|x|e^{|x|}$ for some unknown $\gamma \geq 0$, and if $G(x)$ satisfies the condition (140) with an unknown bound β_0, then the adaptive ES controller

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)\left[1 + (|x| - 1)\,e^{|x|}\right] \quad (165)$$

$$\dot{k} = \left[1 + (|x| - 1)\,e^{|x|}\right]|x| \quad (166)$$

achieves global stability of the equilibrium $\bar{x} = 0$, $\bar{k} = (\gamma + 1)/\alpha\beta_0$ of the average system and the convergence of the state component $\bar{x}(t)$ of the average system to zero. However, since this equilibrium is not globally asymptotically stable ($\bar{k}(t)$ is not guaranteed to converge to $(\gamma+1)/\alpha\beta_0$, though it is guaranteed to remain bounded), the perturbation theory of Moreau and Aeyels [46] does not apply and hence we cannot conclude that the original system is stable in a suitable semi-global practical sense. Worse yet, since $x(t)$ can be expected to converge only in a practical sense (near zero rather than to zero), the integrator for $k(t)$ in (166) is expected to exhibit a drift towards infinity, making this attempt towards removing all requirements for *a priori* knowledge unfruitful. New research and ideas are needed to find ways of adapting k or determining a fixed sufficiently large k so that stability is guaranteed without a priori knowledge on $f(x)$ and $g(x)$.

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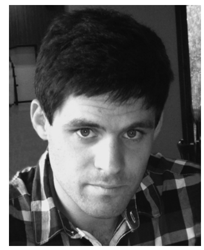

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