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Compensation of state-dependent state delay for nonlinear systems

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ABSTRACT

We extend the technique for compensating state-dependent delays from systems with delayed inputs to systems with delayed states. We focus on predictor-feedback design for nonlinear systems in the strict-feedback form, having a state-dependent state delay on the virtual input. The two key challenges are the definition of the predictor state and the fact that the predictor design does not follow immediately from the delay-free design. We resolve these challenges and we establish asymptotic stability of the resulting infinite-dimensional nonlinear system for general nonnegative-valued delay functions of the state. Due to an inherent limitation on the delay rate, and since the delay rate depends on the state, we obtain only regional stability results. However, for forward-complete systems, we provide an estimate of the region of attraction in the state space of the infinite-dimensional system. We finally provide two examples, including an example of stabilization of a cooling system.

1. Introduction

State-dependent state delays appear in many engineering applications. Examples include milling processes [1], engine cooling systems [2], irrigation channels [3], network congestion control [4], population dynamics [5], supply networks [6,7] and automatic landing systems [8].

Predictor-based techniques are an indispensable part of the control design toolbox [9], for unstable linear plants with constant delays affecting the input [10–13] or simultaneously affecting inputs and states [14–17]. Various control schemes also exist for nonlinear systems with constant delays affecting the input [18–20] or state [21–23]. Yet, extensions of the predictor-feedback design to nonlinear systems with constant input delay had not been developed until recently [24,25]. Although in [26] (see also [27]), a predictor-based controller for unstable linear plants with time-varying input delay is developed, only recently a Lyapunov function was provided [28]. Finally, although nonlinear systems with simultaneous time-varying input and state delays are considered in [29], predictor-like designs for nonlinear systems with time-varying input delays [30] or simultaneous input and state delays [31] were developed recently. In [32], we introduced a technique for compensating state-dependent delays on the input of a nonlinear system. In this paper, we generalize this technique to systems that include state-dependent delays on the states of the system.

We consider forward-complete systems that are globally stabilizable in the absence of the delay. We then "backstep" one state-dependent integrator and design a predictor-based control law for the overall system, using prediction intervals that depend on the current value of the state (Section 2). Using an invertible infinite-dimensional backstepping transformation we derive explicit bounds for the norm of the closed-loop system. Due to the fundamental limitation of the allowable magnitude of the delay function's gradient (the control signal never reaches the plant if the delay rate is larger than one) we use these bounds to estimate the region of attraction of the proposed controller (Section 3). Two simulation examples illustrate the application of the control design (Sections 4 and 5).

Notation: we use the common definition of class $\mathscr{K}$, $\mathscr{K}_\infty$ and $\mathscr{K}\mathscr{L}$ functions from [33]. For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. We say that a function $\xi : \mathbb{R}_+ \times (0, 1) \mapsto \mathbb{R}_+$ belongs to class $\mathscr{K}\mathscr{C}$ if it is of class $\mathscr{K}$ with respect to its first argument for each value of its second argument and continuous with respect to its second argument. It belongs to class $\mathscr{K}\mathscr{C}_\infty$ if it is in $\mathscr{K}\mathscr{C}$ and also in $\mathscr{K}_\infty$ with respect to its first argument.

2. Problem formulation and controller design

We consider the following system

$$\dot{X}_1(t) = f_1\left(t, X_1(t), X_2\left(t - D\left(X_1(t)\right)\right)\right) \tag{1}$$

$$\dot{X}_2(t) = f_2\left(t, X_1(t), X_2(t)\right) + U(t), \tag{2}$$

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where $X_1 \in \mathbb{R}^n$, $X_2, U \in \mathbb{R}$ and $t \geq t_0 \geq 0$. We assume that $f_1 : [t_0, \infty) \times \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$ is locally Lipschitz with $f_1(t, 0, 0) = 0$ for all $t \geq t_0$ and that there exists a class $\mathcal{K}_\infty$ function $\hat{\alpha}$ such that

$$|f_1(t, X_1, X_2)| \leq \hat{\alpha}(|X_1| + |X_2|), \quad \text{for all } t \geq t_0. \tag{3}$$

We further assume that $f_2 : [t_0, \infty) \times \mathbb{R}^{n+1} \mapsto \mathbb{R}$ is locally Lipschitz with respect to $(X_1, X_2) \in \mathbb{R}^{n+1}$ with $f_2(t, 0, 0) = 0$ for all $t \geq t_0$. The goal of the paper is to show that for (1), (2) there exist functions $P(\theta)$ and $\sigma(\theta)$, where $t - D(X_1(t)) \leq \theta \leq t$, such that the controller

$$U(t) = -f_2(t, X_1(t), X_2(t)) - c_2(X_2(t) - \kappa(\sigma(t), P_1(t))) + \frac{\frac{\partial \kappa(\sigma, P_1)}{\partial \sigma} + \frac{\partial \kappa(\sigma, P_1)}{\partial P_1} f_1(\sigma(t), P_1(t), X_2(t))}{1 - \nabla D(P_1(t)) f_1(\sigma(t), P_1(t), X_2(t))}, \tag{4}$$

where c_2 is an arbitrary positive constant and

$$P_1(\theta) = X_1(t) + \int_{t-D(X_1(t))}^{\theta} \frac{f_1(\sigma(s), P_1(s), X_2(s))\, ds}{1 - \nabla D(P_1(s)) f_1(\sigma(s), P_1(s), X_2(s))}, \quad t - D(X_1(t)) \leq \theta \leq t \tag{5}$$

$$\sigma(\theta) = \theta + D(P_1(\theta)), \quad t - D(X_1(t)) \leq \theta \leq t, \tag{6}$$

for $t \geq t_0$, compensates the state-dependent state delay and achieves asymptotic stability of the resulting closed-loop system. We refer to the quantity $P_1(\theta)$ given in (5) as "predictor" since $P_1(t)$ is the $D(P_1(t))$ time units ahead predictor of $X_1(t)$, i.e., $P_1(t) = X_1(t + D(P_1(t)))$. This fact can be seen as follows. Differentiating relation (5) with respect to θ and setting $\theta = t$ we get

$$\frac{dP_1(t)}{dt} = \frac{f_1(\sigma(t), P_1(t), X_2(t))}{1 - \nabla D(P_1(t)) f_1(\sigma(t), P_1(t), X_2(t))}. \tag{7}$$

Performing a change of variables $\tau = \sigma(t)$ in the ODE for $X_1(\tau)$ given by $\frac{dX_1(\tau)}{d\tau} = f_1(\tau, X_1(\tau), X_2(\tau - D(X_1(\tau))))$, we have that

$$\frac{dX_1(\sigma(t))}{dt} = \frac{d\sigma(t)}{dt} f_1(\sigma(t), X_1(\sigma(t)), X_2(t)). \tag{8}$$

From (8) one observes that $P_1(t)$ satisfies the same ODE in t as $X_1(\sigma(t))$ because from (6) to (8) it follows that

$$\frac{d\sigma(\theta)}{d\theta} = \frac{1}{1 - \nabla D(X_1(\sigma(\theta))) f_1(\sigma(\theta), X_1(\sigma(\theta)), X_2(\theta))}, \quad t - D(X_1(t)) \leq \theta \leq t, \tag{9}$$

provided that $P_1(t) = X_1(\sigma(t))$. Since from (5) for $t = t_0$ and $\theta = t_0 - D(X_1(t_0))$ it follows that $P_1(t_0 - D(X_1(t_0))) = X_1(t_0)$, by defining

$$\phi(t) = t - D(X_1(t)), \quad t \geq t_0, \tag{10}$$

$$\sigma(\theta) = \phi^{-1}(\theta), \quad t - D(X_1(t)) \leq \theta \leq t, \tag{11}$$

we have that $P_1(t_0) = X_1(\sigma(t_0))$. Noting from (10) and (11) that $D(X_1(\sigma(t))) = \sigma(t) - t$, differentiating this relation, we get (9). Comparing (7) with (8) we conclude with the help of (9) that indeed $P_1(t) = X_1(\sigma(t))$ for all $t \geq t_0$.

Motivated by the need to keep the denominator in (5) and (9) positive, throughout the paper we consider the condition on the solutions which is given by

$$\mathcal{G}_c : \nabla D(P_1(\theta)) f_1(\sigma(\theta), P_1(\theta), X_2(\theta)) < c, \quad \text{for all } \theta \geq t_0 - D(X_1(t_0)), \tag{12}$$

for $c \in (0, 1]$. We refer to $\mathcal{G}_1$ as the *feasibility condition* of the controller (4)–(5).

3. Stability analysis for forward-complete systems

Throughout the section, we make the following assumptions concerning the plant (1)–(2):

Assumption 1. $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ and ∇D is locally Lipschitz.[1]

Assumption 2. There exist a smooth positive definite function R and class $\mathcal{K}_\infty$ functions α_1, α_2 and α_3 such that for the plant $\dot{X} = f_1(t, X, \omega)$, the following hold

$$\alpha_1(|X|) \leq R(t, X) \leq \alpha_2(|X|) \tag{13}$$

$$\frac{\partial R(t, X)}{\partial t} + \frac{\partial R(t, X)}{\partial X} f_1(t, X, \omega) \leq R(t, X) + \alpha_3(|\omega|), \tag{14}$$

for all $X, \omega \in \mathbb{R}^{n+1}$ and $t \geq t_0$.

Assumption 2 guarantees that the plant $\dot{X} = f_1(t, X, \omega)$ with ω as input is forward-complete.

Assumption 3. There exist functions $\kappa \in C^1([t_0, \infty) \times \mathbb{R}^n; \mathbb{R})$ and $\hat{\rho} \in \mathcal{K}_\infty$, such that the plant $\dot{X}(t) = f_1\big(t, X(t), \kappa\big(t, X(t)\big) + \omega(t)\big)$ is input-to-state stable with respect to ω and κ is uniformly bounded with respect to its first argument, that is,

$$|\kappa(t, X)| \leq \hat{\rho}(|X|), \quad \text{for all } t \geq t_0. \tag{15}$$

Theorem 1. *Consider the plant (1)–(2) together with the control law (4)–(6). Under Assumptions 1–3, there exist a class $\mathcal{K}$ function ξ_{RoA}, a class $\mathcal{KL}$ function β and a class $\mathcal{KC}_\infty$ function ξ_1 such that for all initial conditions for which X_2 is locally Lipschitz on the interval $[t_0 - D(X_1(t_0)), t_0]$ and which satisfy*

$$|X_1(t_0)| + \sup_{t_0 - D(X_1(t_0)) \leq \theta \leq t_0} |X_2(\theta)| < \xi_{\mathrm{RoA}}(c), \tag{16}$$

for some $0 < c < 1$, there exists a unique solution to the closed-loop system with $X_1 \in C^1[t_0, \infty)$, $X_2 \in C^1(t_0, \infty)$, and

$$|X_1(t)| + \sup_{t - D(X_1(t)) \leq \theta \leq t} |X_2(\theta)| \leq \beta\left(\xi_1\left(|X_1(t_0)| + \sup_{t_0 - D(X_1(t_0)) \leq \theta \leq t_0} |X_2(\theta)|, c\right), t - t_0\right), \tag{17}$$

for all $t \geq t_0$. Furthermore, there exists a class $\mathcal{K}$ function δ^*, such that for all $t \geq t_0$ the following holds

$$D(X_1(t)) \leq D(0) + \delta^*(c) \tag{18}$$

$$\left|\dot{D}(X_1(t))\right| \leq c. \tag{19}$$

The proof of Theorem 1 is based on Lemmas 1–8 which are presented next.

Lemma 1 (*Backstepping Transform of the Delayed State*)**.** *The infinite-dimensional backstepping transformation of the state X_2 defined by*

$$Z_2(\theta) = X_2(\theta) - \kappa(\sigma(\theta), P_1(\theta)), \quad t - D(X_1(t)) \leq \theta \leq t, \tag{20}$$

together with the predictor-based control law given in relations (4)–(5) transform system (1)–(2) to the "target system" given by

$$\dot{X}_1(t) = f_1(t, X_1(t), \kappa(t, X_1(t)) + Z_2(t - D(X_1(t)))) \tag{21}$$

$$\dot{Z}_2(t) = -c_2 Z_2(t). \tag{22}$$

[1] To ensure uniqueness of solutions.

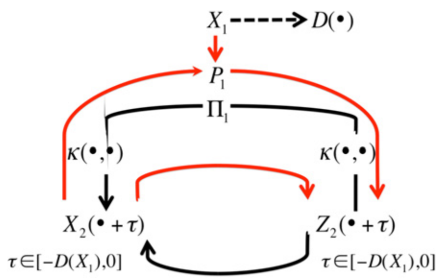

Fig. 1. Interconnections between the predictor states P_1 and Π_1 with the transformations Z_2 and X_2 in (20) and (23). The direct backstepping transformation is defined as $(X_1(t), X_2(\theta)) \mapsto (X_1(t), Z_2(\theta))$ and is given in (20), where $P_1(\theta)$ is given as a function of $X_1(t)$ and $X_2(\theta)$ through relation (5). Analogously, the inverse transformation is defined as $(X_1(t), Z_2(\theta)) \mapsto (X_1(t), X_2(\theta))$ and is given in (23) where $\Pi_1(\theta)$ is given as a function of $X_1(t)$ and $Z_2(\theta)$ through relation (23).

Proof. Using (1) and the facts that $P_1(t - D(X_1(t))) = X_1(t)$ and $\sigma(t - D(X_1(t))) = t$, which are immediate consequences of (5) and (6), we get (21). Setting $\theta = t$ in (20) and taking the derivative with respect to t of the resulting equation we get (22) using (4), (7) and (9). □

Lemma 2 (*Inverse Backstepping Transform*). *The inverse of the infinite-dimensional backstepping transformation defined in* (20) *is given by*

$$X_2(\theta) = Z_2(\theta) + \kappa(\sigma(\theta), \Pi_1(\theta)), \quad t - D(X_1(t)) \le \theta \le t, \tag{23}$$

where $\Pi_1(\theta)$ is defined in Box I.

Proof. By direct verification, noting also that $\Pi_1(\theta) = P_1(\theta)$ for all $t - D(X_1(t)) \le \theta \le t$, where $\Pi_1(\theta)$ is driven by the transformed state $Z_2(\theta)$, whereas $P_1(\theta)$ is driven by the state $X_2(\theta)$ for $t - D(X_1(t)) \le \theta \le t$. See Fig. 1. □

Lemma 3 (*Stability Estimate for Target System*). *There exists a class $\mathcal{KL}$ function β^* such that for all solutions of system* (1), (2) *satisfying* (12) *for $0 < c < 1$, the following holds for all $t \ge t_0$*

$$|X_1(t)| + \sup_{t-D(X_1(t))\le\theta\le t} |Z_2(\theta)| \le \beta^*\left(|X_1(t_0)| + \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)|, t - t_0\right). \tag{25}$$

Proof. Solving (22), and using the facts that $\sigma(t_0) = t_0 + D\big(X_1(\sigma(t_0))\big)$ and that $\phi(t)$ is increasing for all $t \ge t_0$ we get

$$\sup_{t-D(X_1(t))\le\theta\le t} |Z_2(\theta)| \le \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)| \times e^{-c_2(t-D(X_1(t))-t_0)}, \quad \text{for all } t \ge \sigma(t_0), \tag{26}$$

where we have also used the trivial inequality $|Z_2(t_0)| \le \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)|$. Similarly, for all $t_0 \le t \le \sigma(t_0)$ we get

$$\sup_{t-D(X_1(t))\le\theta\le t} |Z_2(\theta)| \le \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)| + \sup_{t_0\le\theta\le t\le\sigma(t_0)} |Z_2(\theta)|, \tag{27}$$

and hence, combining (27) with (22), we get

$$\sup_{t-D(X_1(t))\le\theta\le t} |Z_2(\theta)| \le 2 \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)|, \quad \text{for all } t_0 \le t \le \sigma(t_0). \tag{28}$$

Therefore, using (26) and (28) we get

$$\sup_{t-D(X_1(t))\le\theta\le t} |Z_2(\theta)| \le 2 \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)|\, e^{c_2(D(0)+\delta_1(|X_1(t)|))} e^{-c_2(t-t_0)}, \quad \text{for all } t \ge t_0. \tag{29}$$

Under Assumption 3 and [34], (see also [35,36]), there exist class $\mathcal{KL}$ function $\hat{\beta}$ and class $\mathcal{K}$ function $\hat{\gamma}$ such that

$$|X_1(t)| \le \hat{\beta}(|X_1(s)|, t - s) + \hat{\gamma}\left(\sup_{s\le\tau\le t} |Z_2(\tau - D(X_1(\tau)))|\right), \quad \text{for all } t \ge s \ge t_0. \tag{30}$$

Setting $s = t_0$ we have that

$$|X_1(t)| \le \hat{\beta}(|X_1(t_0)|, 0) + \hat{\gamma}\left(\sup_{\phi(t_0)\le\theta\le t_0} |Z_2(\theta)| + \sup_{t_0\le\theta\le\phi(t)} |Z_2(\theta)|\right), \quad \text{for all } t \ge t_0, \tag{31}$$

and hence,

$$|X_1(t)| \le \hat{\beta}(|X_1(t_0)|, 0) + \hat{\gamma}\left(2 \sup_{\phi(t_0)\le\theta\le t_0} |Z_2(\theta)|\right), \quad \text{for all } t \ge t_0. \tag{32}$$

Setting $s = \frac{t+t_0}{2}$ in (30) we have that

$$|X_1(t)| \le \hat{\beta}\left(\left|X_1\left(\frac{t+t_0}{2}\right)\right|, \frac{t-t_0}{2}\right) + \hat{\gamma}\left(\sup_{\phi\left(\frac{t+t_0}{2}\right)\le\theta\le\phi(t)} |Z_2(\theta)|\right). \tag{33}$$

We estimate now $\sup_{\phi\left(\frac{t+t_0}{2}\right)\le\theta\le\phi(t)} |Z_2(\theta)|$. Solving (22) we get

$$\sup_{\phi\left(\frac{t+t_0}{2}\right)\le\theta\le\phi(t)} |Z_2(\theta)| \le 2 \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)|\, e^{-c_2\left(\phi\left(\frac{t+t_0}{2}\right)-t_0\right)}, \quad \text{for all } t \ge 2\sigma(t_0) - t_0. \tag{34}$$

With the help of relations (22) and (34) we get

$$\sup_{\phi\left(\frac{t+t_0}{2}\right)\le\theta\le\phi(t)} |Z_2(\theta)| \le \sup_{\phi\left(\frac{t+t_0}{2}\right)\le\theta\le t_0} |Z_2(\theta)| + \sup_{t_0\le\theta\le\phi(t)} |Z_2(\theta)| \le 2 \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)|, \quad \text{for all } t_0 \le t \le 2\sigma(t_0) - t_0. \tag{35}$$

Hence, using the fact that $\phi\left(\frac{t+t_0}{2}\right) = \frac{t+t_0}{2} - D\left(X_1\left(\frac{t+t_0}{2}\right)\right)$ we get for all $t \ge t_0$

$$\sup_{\phi\left(\frac{t+t_0}{2}\right)\le\theta\le\phi(t)} |Z_2(\theta)| \le 2 \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)| \times e^{c_2\left(D(0)+\delta_1\left(\left|X_1\left(\frac{t+t_0}{2}\right)\right|\right)\right)} e^{-\frac{c_2}{2}(t-t_0)}. \tag{36}$$

$$\Pi_1(\theta) = X_1(t) + \int_{t-D(X_1(t))}^{\theta} \frac{f_1\left(\sigma(s), \Pi_1(s), \kappa\left(\sigma(s), \Pi_1(s)\right) + Z_2(s)\right) \mathrm{d}s}{1 - \nabla D\left(\Pi_1(s)\right) f_1\left(\sigma(s), \Pi_1(s), \kappa\left(\sigma(s), \Pi_1(s)\right) + Z_2(s)\right)}, \quad t - D\left(X_1(t)\right) \le \theta \le t. \tag{24}$$

Box I.

Setting $s = t_0$ and replacing t by $\frac{t+t_0}{2}$ we get from (30) that

$$\left|X_1\left(\frac{t+t_0}{2}\right)\right| \le \hat{\beta}\left(\left|X_1(t_0)\right|, \frac{t-t_0}{2}\right) + \hat{\gamma}\left(\sup_{\phi(t_0)\le\theta\le\phi\left(\frac{t+t_0}{2}\right)} |Z_2(\theta)|\right). \tag{37}$$

Since,

$$\sup_{\phi(t_0)\le\theta\le\phi\left(\frac{t+t_0}{2}\right)} |Z_2(\theta)| \le \sup_{\phi(t_0)\le\theta\le t_0} |Z_2(\theta)| + \sup_{t_0\le\theta\le\phi(t)} |Z_2(\theta)| \le 2 \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)|, \tag{38}$$

we arrive at

$$\left|X_1\left(\frac{t+t_0}{2}\right)\right| \le \hat{\beta}\left(\left|X_1(t_0)\right|, 0\right) + \hat{\gamma}\left(2 \sup_{t_0-D(X_1(t_0))\le\theta\le t_0} |Z_2(\theta)|\right), \quad \text{for all } t \ge t_0. \tag{39}$$

Combining (29), (32), (33), (36) and (39), we get the statement of the lemma with

$$\beta^*(s, t-t_0) = \hat{\beta}\left(\hat{\beta}(s,0) + \hat{\gamma}(2s), \frac{t-t_0}{2}\right) + \hat{\gamma}\left(2se^{c_2\left(D(0)+\delta_1\left(\hat{\beta}(s,0)+\hat{\gamma}(2s)\right)\right)} e^{-\frac{c_2}{2}(t-t_0)}\right) + 2se^{c_2\left(D(0)+\delta_1\left(\hat{\beta}(s,0)+\hat{\gamma}(2s)\right)\right)} e^{-c_2(t-t_0)}. \quad \square \tag{40}$$

Lemma 4 (*Bound on Predictor in Terms of System's States*). *There exists a class $\mathcal{KC}_\infty$ function ξ such that for all solutions of system (1), (2) satisfying (12) for $0 < c < 1$, the following holds*

$$|P_1(\theta)| \le \xi\left(|X_1(t)| + \sup_{t-D(X_1(t))\le\tau\le t} |X_2(\tau)|, c\right), \quad t - D\left(X_1(t)\right) \le \theta \le t. \tag{41}$$

Proof. Consider the following ODE in θ which follows by differentiating (5) together with the initial condition $P_1\left(t - D\left(X_1(t)\right)\right) = X_1(t)$

$$\frac{\mathrm{d}P_1(\theta)}{\mathrm{d}\theta} = \frac{f_1\left(\sigma(\theta), P_1(\theta), X_2(\theta)\right)}{1 - \nabla D\left(P_1(\theta)\right) f_1\left(\sigma(\theta), P_1(\theta), X_2(\theta)\right)}, \quad t - D\left(X_1(t)\right) \le \theta \le t. \tag{42}$$

With the change of variables

$$y = \sigma(\theta), \tag{43}$$

we re-write (42) as

$$\frac{\mathrm{d}P_1(\phi(y))}{\mathrm{d}y} = f_1\left(y, P_1(\phi(y)), X_2\left(y - D\left(P_1(\phi(y))\right)\right)\right), \quad t \le y \le \sigma(t). \tag{44}$$

Using (14) we get

$$\frac{\mathrm{d}R\left(y, P_1(\phi(y))\right)}{\mathrm{d}\theta} \frac{\mathrm{d}\theta}{\mathrm{d}y} \le R\left(y, P_1(\phi(y))\right) + \alpha_3\left(\left|X_2\left(y - D\left(P_1(\phi(y))\right)\right)\right|\right). \tag{45}$$

With (12) we have

$$\frac{\mathrm{d}R\left(\sigma(\theta), P_1(\theta)\right)}{\mathrm{d}\theta} \le \frac{1}{1-c}\left(R\left(\sigma(\theta), P_1(\theta)\right) + \alpha_3\left(|X_2(\theta)|\right)\right), \quad t - D\left(X_1(t)\right) \le \theta \le t. \tag{46}$$

Under Assumption 1, there exists a function $\delta_1 \in \mathcal{K}_\infty \cap C^1$ such that

$$D(X_1) \le D(0) + \delta_1\left(|X_1|\right), \tag{47}$$

and hence, using the comparison principle we have from (46) for all $t - D\left(X_1(t)\right) \le \theta \le t$ that

$$R\left(\sigma(\theta), P_1(\theta)\right) \le e^{\frac{D(0)+\delta_1(|X_1(t)|)}{1-c}}\left(R\left(t, X_1(t)\right) + \sup_{t-D(X_1(t))\le\tau\le t} \alpha_3\left(|X_2(\tau)|\right)\right). \tag{48}$$

With standard properties of class $\mathcal{K}_\infty$ functions we get the statement of the lemma with $\xi \in \mathcal{KC}_\infty$ given by

$$\xi(s, c) = \alpha_1^{-1}\left(\left(\alpha_2(s) + \alpha_3(s)\right) e^{\frac{D(0)+\delta_1(s)}{1-c}}\right). \quad \square \tag{49}$$

Lemma 5 (*Bound on Predictor in Terms of Transformed System's States*). *There exists a class $\mathcal{K}$ function γ such that for all solutions of system (1), (2) satisfying (12) for $0 < c < 1$, the following holds*

$$|\Pi_1(\theta)| \le \gamma\left(|X_1(t)| + \sup_{t-D(X_1(t))\le\tau\le t} |Z_2(\tau)|\right), \quad t - D\left(X_1(t)\right) \le \theta \le t. \tag{50}$$

Proof. Let $Y(s)$ be the solution of $\frac{\mathrm{d}Y(s)}{\mathrm{d}s} = f_1\big(s, Y(s), \kappa\big(s, Y(s)\big) + \omega(s)\big)$ for $s \ge s_0 \ge 0$. Under Assumption 3 and [34], (see also [35,36]), there exist class $\mathcal{KL}$ function $\hat{\beta}$ and class $\mathcal{K}$ function $\hat{\gamma}$ such that

$$|Y(s)| \le \hat{\beta}\left(|Y(s_0)|, s - s_0\right) + \hat{\gamma}\left(\sup_{s_0\le r\le s} |\omega(r)|\right), \quad \text{for all } s \ge s_0. \tag{51}$$

Using the change of variable (43) and definition (24), we have that

$$\frac{\mathrm{d}\Pi_1(\phi(y))}{\mathrm{d}y} = f_1\left(y, \Pi_1(\phi(y)), \kappa\left(y, \Pi_1(\phi(y))\right) + Z_2(\phi(y))\right), \quad t \le y \le \sigma(t). \tag{52}$$

Since $\Pi_1(\phi(y))$ satisfies the same ODE in y as the ODE for $Y(s)$ in s we have that

$$|\Pi_1(\phi(y))| \le \hat{\beta}\left(|X_1(t)|, y-t\right) + \hat{\gamma}\left(\sup_{t\le y\le\sigma(t)} |Z_2(\phi(y))|\right),$$
$$\text{for all } t \le y \le \sigma(t). \quad (53)$$

Using (43) (which can be also written as $\theta = \phi(y)$) with the fact that $\hat{\beta}(s,r) \le \hat{\beta}(s,0)$ for all $r \ge 0$, we get from (53)

$$|\Pi_1(\theta)| \le \hat{\beta}\left(|X_1(t)|, 0\right) + \hat{\gamma}\left(\sup_{t-D(X_1(t))\le\tau\le t} |Z_2(\tau)|\right),$$
$$t - D\left(X_1(t)\right) \le \theta \le t. \quad (54)$$

With the properties of class $\mathcal{K}$ functions we get the statement of the lemma where $\gamma(s) = \hat{\beta}(s,0) + \hat{\gamma}(s)$. □

Lemma 6 (*Equivalence of Norms for Original and Target System*)**.** *There exist a function ξ_1 of class $\mathcal{KC}_\infty$ and a class $\mathcal{K}_\infty$ function α_4 such that for all solutions of system (1), (2) satisfying (12) for $0 < c < 1$, the following holds*

$$|X_1(t)| + \sup_{t-D(X_1(t))\le\theta\le t} |Z_2(\theta)| \le \xi_1\left(|X_1(t)| + \sup_{t-D(X_1(t))\le\theta\le t} |X_2(\theta)|, c\right) \quad (55)$$

$$|X_1(t)| + \sup_{t-D(X_1(t))\le\theta\le t} |X_2(\theta)| \le \alpha_4\left(|X_1(t)| + \sup_{t-D(X_1(t))\le\theta\le t} |Z_2(\theta)|\right), \quad (56)$$

for all $t \ge t_0$.

Proof. Using the direct backstepping transformation (20) and the bounds (15), (41) we get the bound (55) with $\xi_1(s,c) = s + \hat{\rho}\left(\xi(s,c)\right)$. Using the inverse backstepping transformation (23) and the bounds (15), (50) we get the bound (56) with $\alpha_4(s) = s + \hat{\rho}\left(\gamma(s)\right)$. □

Lemma 7 (*Ball Around the Origin within the Feasibility Region*)**.** *There exists a function $\bar{\xi}_c$ of class $\mathcal{KC}_\infty$ such that for all solutions of system (1), (2) that satisfy*

$$|X_1(t)| + \sup_{t-D(X_1(t))\le\theta\le t} |X_2(\theta)| < \bar{\xi}_c(c,c), \quad (57)$$

for $0 < c < 1$ also satisfy (12).

Proof. Since $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ there exists class $\mathcal{K}_\infty$ function δ_2 such that

$$|\nabla D(X_1)| \le |\nabla D(0)| + \delta_2(|X_1|). \quad (58)$$

If a solution satisfies

$$\left(|\nabla D(0)| + \delta_2\left(|P_1(\theta)|\right)\right)\hat{\alpha}\left(|P_1(\theta)| + \sup_{t-D(X_1(t))\le\theta\le t} |X_2(\theta)|\right) \le c, \quad t - D\left(X_1(t)\right) \le \theta \le t, \quad (59)$$

for $0 < c < 1$, then it also satisfies (12). Using Lemma 4, (59) is satisfied for $0 < c < 1$ as long as bound (57) holds, where the class $\mathcal{KC}_\infty$ function ξ_c is given by

$$\xi_c(s,c) = \left(|\nabla D_1(0)| + \delta_2\left(\xi(s,c)\right)\right)\hat{\alpha}\left(\xi(s,c) + s\right), \quad (60)$$

and with $\bar{\xi}_c$ we denote the inverse function of ξ_c with respect to ξ_c's first argument. □

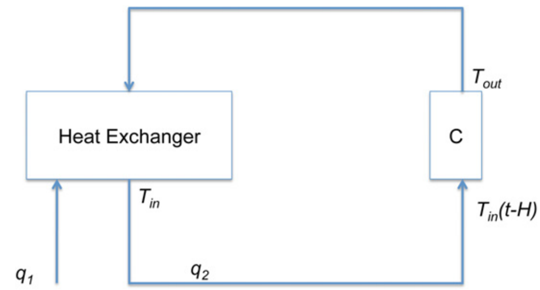

Fig. 2. A typical marine cooling system with one consumer.

Lemma 8 (*Estimate of the Region of Attraction*)**.** *There exists a class $\mathcal{K}$ function $\xi_{\rm RoA}$ such that for all initial conditions of the closed-loop system (1), (2), (4) and (5) that satisfy (16), the solutions of system (1), (2) satisfy (57) for $0 < c < 1$ and hence satisfy (12).*

Proof. Using Lemma 6, with the help of (25) we have that

$$\Omega(t) \le \alpha_4\left(\beta^*\left(\xi_1\left(\Omega(t_0), c\right), t - t_0\right)\right), \quad (61)$$

where

$$\Omega(t) = |X_1(t)| + \sup_{t-D(X_1(t))\le\theta\le t} |X_2(\theta)|. \quad (62)$$

Hence, for all initial conditions that satisfy bound (16) with any class $\mathcal{K}$ choice $\xi_{\rm RoA}(c) \le \bar{\xi}^*{}_{\rm RoA}\left(\bar{\xi}_c(c,c), c\right)$, where $\bar{\xi}^*{}_{\rm RoA}(s,c)$ is the inverse of the function $\xi^*{}_{\rm RoA}(s,c) = \alpha_4\left(\beta^*\left(\xi_1(s,c), 0\right)\right) \in \mathcal{KC}_\infty$ with respect to $\xi^*_{\rm RoA}$'s first argument, the solutions satisfy (57). Moreover, for all those initial conditions, the solutions verify (12) for all $\theta \ge t_0 - D\left(X_1(t_0)\right)$. □

Proof of 1. Using (61) we get (17) with $\beta(s,t) = \alpha_4\left(\beta^*(s,t)\right)$. From (1), Assumption 1 and the Lipschitzness of X_2 in $[t_0 - D(X_1(t_0)), t_0)$ guarantee the existence and uniqueness of $X_1 \in C^1[t_0, \sigma(t_0))$ where $\sigma(t_0) = t_0 + D(X_1(\sigma(t_0)))$. The target system (21) and (22) guarantees the existence and uniqueness of $X_1 \in C^1(\sigma(t_0), \infty)$, and (23) with the continuity of X_2 at t_0 guarantees that $X_1 \in C^1[t_0, \infty)$. Using the fact that Π_1 satisfies $\dot{\Pi}_1 = \frac{f_1(\sigma(t),\Pi_1(t),\kappa(\sigma(t),\Pi_1(t))+Z_2(t))}{1-\nabla D(\Pi_1(t))f_1(\sigma(t),\Pi_1(t),\kappa(\sigma(t),\Pi_1(t))+Z_2(t))}$ for $t \ge t_0$, the local Lipschitzness of ∇D, relation (12) and Assumption 3 ($\kappa \in C^1\left([t_0,\infty) \times \mathbb{R}^n; \mathbb{R}\right)$) guarantee that $\Pi_1 \in C^1(t_0, \infty)$. Using (23), with the help of (22) and the fact that $\kappa \in C^1\left([t_0,\infty) \times \mathbb{R}^n; \mathbb{R}\right)$ we get that $X_2 \in C^1(t_0, \infty)$. Using Lemma 8 together with (47) and (58) we get (18), (19) with any class $\mathcal{K}$ choice $\delta^*(c) \ge \delta_1\left(\bar{\xi}_c(c,c)\right)$. □

4. Example 1: application to cooling systems

In marine transportation of materials, the design of control laws for the ship's cooling system is of paramount importance due to the significant potential of the cooling system in terms of energy optimization [2]. In Fig. 2, we show a typical marine cooling circuit with one consumer, denoted by C, and a Heat Exchanger. We denote with $T_{\rm in}$ the input temperature towards the consumer, i.e., the output temperature of the Heat Exchanger. Due to the transportation time of the coolant (typically water) from the Heat Exchanger to the consumer C, the actual input temperature $T_{\rm in}$ in the consumer is delayed by H, namely, $T_{\rm in}(t-H)$. The delay time H depends on the flow rate q_2 which can be controlled through a pump. In order to design a feedback law q_2 we take into account that the flow rate q_2 has to be proportional to the temperature at the other end of the consumer, which we denote with $T_{\rm out}$. This is because it makes sense to increase the flow rate if the outer temperature of the consumer C is increasing. A simple choice is $q_2 = k_1 T_{\rm in} + k_2$. The control objective is to regulate the temperature

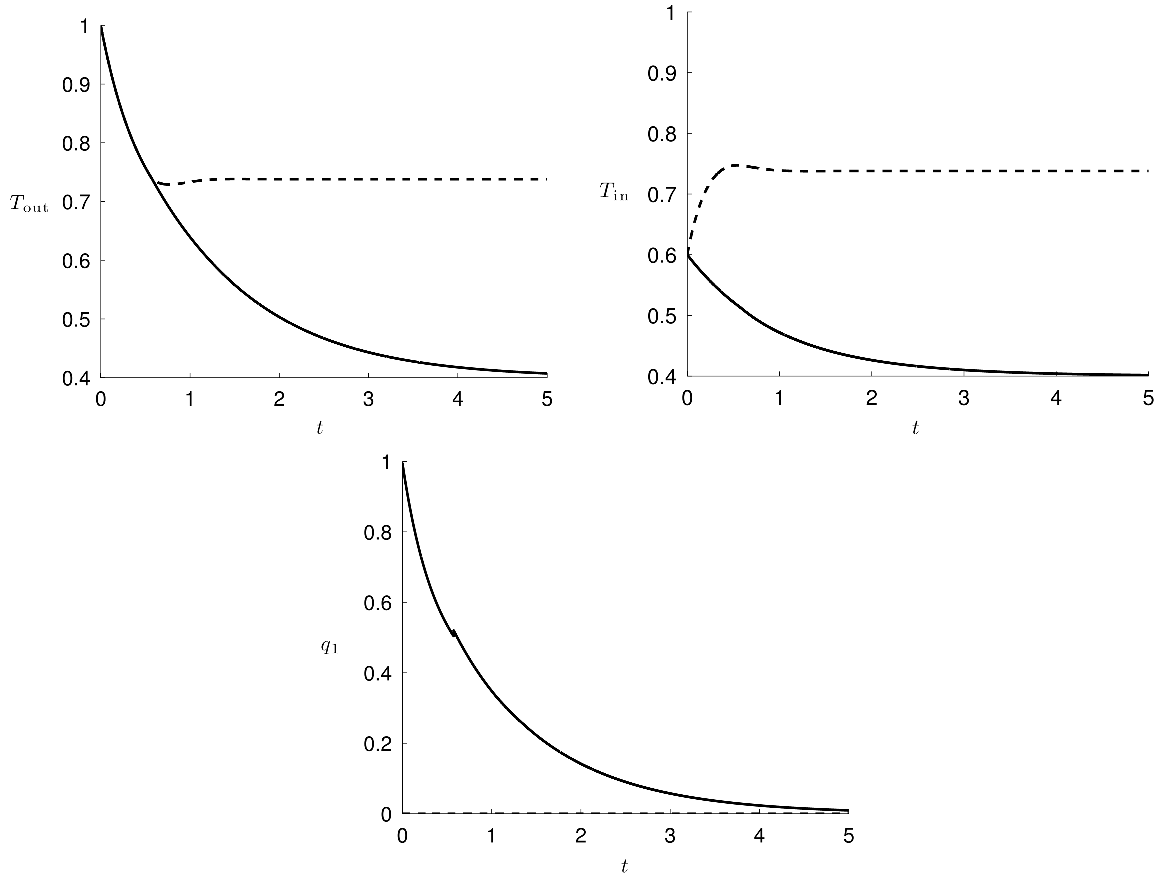

Fig. 3. Response of the cooling model (63)–(64) with the predictor-based controller (65) (solid line) and in open-loop (dashed line).

$T_{\rm out}$ to a constant set-point, say $T_{\rm eq} > 0$. This is achieved by controlling the flow rate q_1 at the input of the Heat Exchanger through a pump.

Denoting by $T_{\rm out} = X_1$, $T_{\rm in} = X_2$, $q_2 = k_1X_1 + k_2$, $H = \frac{b}{q_2} = D$ and $q_1 = U$ and by neglecting the effect of the hydraulics in the system (since the hydraulic dynamics assumed to be much faster than the heat dynamics [2]), the equations that describe the thermodynamics of the cooling circuit of Fig. 2 are

$$\dot{X}_1(t) = a\left(X_1(t) - X_2\left(t - \frac{b}{k_1X_1(t)+k_2}\right)\right) \times (k_1X_1(t)+k_2) \tag{63}$$

$$\dot{X}_2(t) = (k_1X_1(t)+k_2)(X_1(t) - X_2(t)) - U(t), \tag{64}$$

where $a < 0$, and $b, k_1, k_2 > 0$. Since the coolant flows only in one direction $q_1 > 0$, and hence $U > 0$. Since the coolant is typically water, both $T_{\rm out}$ and $T_{\rm in}$ cannot fall below zero, and hence $X_1, X_2 > 0$. In addition since the consumer always adds heat (due to its functioning), $T_{\rm out} \geq T_{\rm in}(t-H)$ and $T_{\rm out} \geq T_{\rm in}$, and hence $X_1(t) - X_2\left(t - \frac{b}{k_1X_1+k_2}\right) \geq 0$ and $X_1(t) - X_2(t) \geq 0$ for all $t \geq 0$. Assumption 2 is satisfied (in the domain of interest) since X_1 remains bounded because $q_2 = k_1X_1(t) + k_2 \geq 0$, which follows form the fact that if $q_2 = 0$ then $\dot{X}_1(t) = 0$ (so q_2 cannot cross from being positive to being negative). Assumption 1 is satisfied for all $X_1 > 0$ and Assumption 3 is satisfied since for $q_2 > 0$, (63) is input-to-state stable from the "disturbance" $w = X_2\left(t - \frac{b}{k_1X_1(t)+k_2}\right)$. We choose the control law $\kappa(X_1) = X_1 + \frac{c_1}{a}\frac{X_1}{k_1X_1+k_2}$, and hence, the predictor-based control law for this system becomes

$$\begin{aligned} U(t) &= (k_1X_1(t)+k_2)(X_1(t) - X_2(t)) \\ &+ c_2\left(X_2(t) - P_1(t) - \frac{c_1}{a}\frac{P_1(t) - T_{\rm eq}}{k_1P_1(t)+k_2}\right) \\ &- \left(1 + \frac{c_1}{ak_1}\frac{T_{\rm eq} + \frac{k_2}{k_1}}{\left(P_1(t) + \frac{k_2}{k_1}\right)^2}\right) \\ &\times \frac{a\left(P_1(t) - X_2(t)\right)\left(k_1P_1(t)+k_2\right)}{1 + \frac{bk_1}{(k_1P_1(t)+k_2)^2}a\left(P_1(t) - X_2(t)\right)\left(k_1P_1(t)+k_2\right)}, \end{aligned} \tag{65}$$

where $P_1(t)$ is defined in Box II. We choose the parameters of the plant and of the controller as $a = -1$ and $c_1 = c_2 = b = k_1 = k_2 = 1$ and the initial conditions as $X_1(0) = 1$ and $X_2(\theta) = 0.2$ for all $-\frac{b}{k_1X_1(0)+k_2} \leq \theta \leq 0$. We show in Fig. 3 the temperatures $T_{\rm out}, T_{\rm in}$ together with the input flow q_1. We compare the response of the system with the predictor-based controller (65) and with no control. In the case of the open-loop response we observe that the temperatures $T_{\rm out}, T_{\rm in}$ converge to the same value but not at the desired set-point. This is what one expects since the system has an equilibrium at $X_1 = X_2$. In contrast, the predictor based controller regulates the temperatures $T_{\rm out}, T_{\rm in}$ at the desired set point $T_{\rm eq}$.

5. Example 2

We consider the system

$$\dot{s}(t) = v(t - r_1\sin^2(\omega s(t))) \tag{67}$$

$$\dot{v}(t) = a(t), \tag{68}$$

where the state variables are denoted with s and v and the control variable is denoted with $a(t)$. This system has some resemblance with the model considered in [8] for the "soft" automatic

$$P_1(t) = X_1(t) + \int_{t-\frac{b}{k_1X_1(t)+k_2}}^{t} \frac{a\,(P_1(\theta) - X_2(\theta))\,(k_1P_1(\theta) + k_2)}{1 + \frac{bk_1}{(k_1P_1(\theta)+k_2)^2} a\,(P_1(\theta) - X_2(\theta))\,(k_1P_1(\theta) + k_2)} \mathrm{d}\theta. \tag{66}$$

Box II.

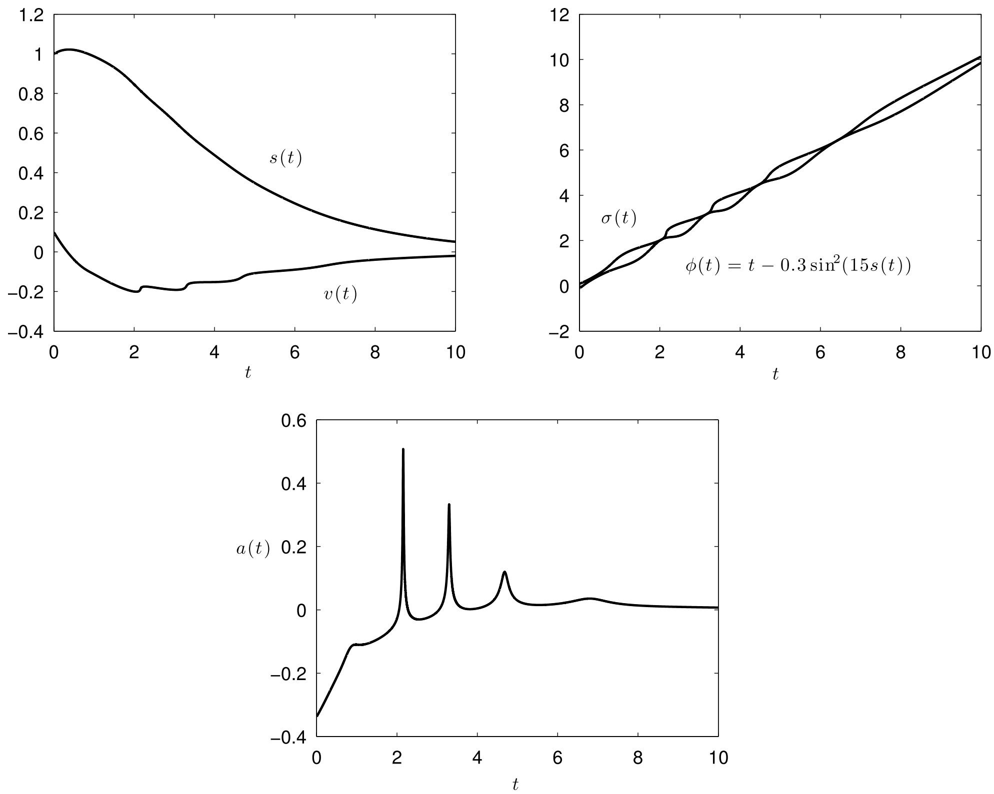

Fig. 4. Response of system (67)–(68) with the controller (69)–(70) and initial conditions as $s(0) = 1$ and $v(\theta) = 0.1$, for all $-r_1 \sin^2(\omega s(0)) \le \theta \le 0$.

landing. In view of condition (12) one should expect that global stabilization is not achievable. This fact can be viewed in terms of the system dynamics (67)–(68) and in particular from relation $v(t - r_1 \sin^2(\omega s(t)))$. Subsystem (67) is forward-complete from v since it is linear, and the delay $r_1 \sin^2(\omega s(t))$ satisfies Assumption 1. Hence, Theorem 1 applies. We choose the parameters of the plant as $r_1 = 0.3$ and $\omega = 15$, the initial conditions of the plant as $s(0) = 1$ and $v(\theta) = 0.1$ for all $-r_1 \sin^2(\omega s(0)) \le \theta \le 0$ and the parameters of the nominal controller as $c_1 = c_2 = 0.5$. The predictor-based controller is given by

$$a(t) = -c_2\,(v(t) + c_1P_1(t)) - c_1 \frac{v(t)}{1 - r_1\omega \sin(\omega P_1(t)) \cos(\omega P_1(t))\, v(t)}, \tag{69}$$

where for all $t - r_1 \sin^2(\omega s(t)) \le \theta \le t$

$$P_1(\theta) = s(t) + \int_{t-r_1 \sin^2(\omega s(t))}^{\theta} \frac{v(s)\, ds}{1 - r_1\omega \sin(\omega P_1(s)) \cos(\omega P_1(s))\, v(s)}. \tag{70}$$

The control signal reaches the state s at the time t^* which satisfies $0.3 \sin^2(15(0.1t^* + 1)) = t^* = 0.0887$. In Fig. 4, we show the response of the system. As Theorem 1 predicts $s(t)$, $v(t)$ converge to zero, whereas $\phi(t)$ and $\sigma(t)$ remain increasing for all times. From Fig. 4, we also observe that at the time instants where $0.3 \sin^2(15s(t_1)) = 0$ we have that $\phi(t) = t = \sigma(t)$. Moreover, the peaks of the control signal $a(t)$ occur at the time instants where $\sigma(t)$ increases rapidly.

6. Conclusions

Although we consider plants with only state-dependent state delay the results of this paper can be extended to the case of simultaneous state-dependent input and state delays. The tools that one has to use are the ones from the present paper and from [31]. However, the stability analysis will be much more involved: one has to satisfy not only one, but two (one for each delay) feasibility conditions.

Since we obtain only regional results one may wonder if the results of this paper can be applied to locally stabilizable plants. The answer to this question is positive by applying similar techniques to the ones from [32].

Acknowledgments

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References

[1] Y. Altintas, S. Engin, E. Budak, Analytical stability prediction and design of variable pitch cutters, ASME Journal of Manufacturing Science and Engineering 121 (1999) 173–178.

[2] M. Hansen, J. Stoustrup, J.-D. Bendtsen, Modeling of nonlinear marine cooling systems with closed circuit flow, in: 2011 IFAC World Congress, Milan, Italy, 2011.
[3] X. Litrico, V. Fromion, Analytical approximation of open-channel flow for controller design, Applied Mathematical Modelling 28 (2004) 677–695.
[4] S. Liu, T. Basar, R. Srikant, Pitfalls in the fluid modeling of RTT variations in window-based congestion control, in: 2005 IEEE INFOCOM, Miami, Florida, 2005.
[5] J.M. Mahaffy, J. Blair, M. Mackey, Hematopoietic model with moving boundary condition and state dependent delay: applications in erythopoiesis, Journal of Theoretical Biology 190 (1998) 135–146.
[6] R. Sipahi, S. Lammer, S.-I. Niculescu, D. Helbing, On stability analysis and parametric design of supply networks under the presence of transportation delays, in: 2006 ASME-IMECE Conference, Chicago, 2006.
[7] J.D. Sterman, Business Dynamics: Systems Thinking and Modeling for a Complex World, McGraw-Hill, 2000.
[8] H.-O. Walther, On a model for soft landing with state-dependent delay, Journal of Dynamical Systems and Differential Equations 19 (2007) 593–622.
[9] J.-P. Richard, Time-delay systems: an overview of some recent advances and open problems, Automatica 39 (2003) 1667–1694.
[10] A.Z. Manitius, A.W. Olbrot, Finite spectrum assignment for systems with delays, IEEE Transactions on Automatic Control 24 (1979) 541–553.
[11] S.-I. Niculescu, Delay Effects on Stability, Springer, 2001.
[12] Q.-C. Zhong, On distributed delay in linear control laws—part I: discrete-delay implementation, IEEE Transactions on Automatic Control 49 (2006) 2074–2080.
[13] Q.-C. Zhong, Robust Control of Time-Delay Systems, Springer, 2006.
[14] N. Bekiaris-Liberis, M. Krstic, Stabilization of strict-feedback linear systems with delayed integrators, Automatica 46 (2010) 1902–1910.
[15] Y.A. Fiagbedzi, A.E. Pearson, Feedback stabilization of linear autonomous time lag systems, IEEE Transactions on Automatic Control 31 (1986) 847–855.
[16] M. Jankovic, Forwarding, backstepping, and finite spectrum assignment for time delay systems, Automatica 45 (2009) 2–9.
[17] M. Jankovic, Recursive predictor design for state and output feedback controllers for linear time delay systems, Automatica 46 (2010) 510–517.
[18] F. Mazenc, P.-A. Bliman, Backstepping design for time-delay nonlinear systems, IEEE Transactions on Automatic Control 51 (2004) 149–154.
[19] F. Mazenc, S. Mondie, R. Francisco, Global asymptotic stabilization of feedforward systems with delay at the input, IEEE Transactions on Automatic Control 49 (2004) 844–850.
[20] F. Mazenc, S.-I. Niculescu, Generating positive and stable solutions through delayed state feedback, Automatica 47 (2011) 525–533.
[21] M. Jankovic, Control Lyapunov–Razumikhin functions and robust stabilization of time delay systems, IEEE Transactions on Automatic Control 46 (2001) 1048–1060.
[22] M. Jankovic, Cross-term forwarding for systems with time delay, IEEE Transactions on Automatic Control 54 (2009) 498–511.
[23] F. Mazenc, M. Malisoff, Z. Lin, Stabilization of a chemostat model with Haldane growth functions and a delay in the measurements, Automatica 46 (2010) 1428–1436.
[24] M. Krstic, On compensating long actuator delays in nonlinear control, IEEE Transactions on Automatic Control 53 (2008) 1684–1688.
[25] M. Krstic, Input delay compensation for forward complete and feedforward nonlinear systems, IEEE Transactions on Automatic Control 55 (2010) 287–303.
[26] Z. Artstein, Linear systems with delayed controls: a reduction, IEEE Transactions on Automatic Control 27 (1982) 869–879.
[27] M. Nihtila, Finite pole assignment for systems with time-varying input delays, in: 1991 IEEE Conf. Decision Control, 1991.
[28] M. Krstic, Lyapunov stability of linear predictor feedback for time-varying input delay, IEEE Transactions on Automatic Control 55 (2010) 554–559.
[29] I. Karafyllis, Finite-time global stabilization by means of time-varying distributed delay feedback, SIAM Journal on Control and Optimization 45 (2006) 320–342.
[30] N. Bekiaris-Liberis, M. Krstic, Compensation of time-varying input delay for nonlinear systems, in: 2011 Mediterranean Conference on Control and Automation, Corfu, Greece, 2011.
[31] N. Bekiaris-Liberis, M. Krstic, Stabilization of nonlinear strict-feedback systems with time-varying delayed integrators, in: 2011 ASME Dynamic Systems and Control Conference, Arlington, Virginia, 2011.
[32] N. Bekiaris-Liberis, M. Krstic, Compensation of state-dependent input delay for nonlinear systems, IEEE Transactions on Automatic Control (in press).
[33] H. Khalil, Nonlinear Systems, Prentice Hall, 2002.
[34] I. Karafyllis, J. Tsinias, Nonuniform in time input-to-state stability and the small gain theorem, IEEE Transactions on Automatic Control 49 (2004) 196–216.
[35] M. Malisoff, F. Mazenc, Further remarks on strict input-to-state stable Lyapunov functions for time-varying systems, Automatica 41 (2005) 1973–1978.
[36] E.D. Sontag, Smooth stabilization implies coprime factorization, IEEE Transactions on Automatic Control 34 (1989) 435–443.