Pergamon

PII: S0005-1098(97)00072-1

Automatica, Vol. 33, No. 8, pp. 1459–1473, 1997

Printed in Great Britain
0005-1098/97 $17.00 + 0.00

Optimal Design of Adaptive Tracking Controllers for Non-linear Systems*

ZHONG-HUA LI† and MIROSLAV KRSTIĆ†

An 'inverse optimal' adaptive tracking problem is solved for general non-linear systems affine in unknown parameters and control input. The cost functional incorporates penalty on the tracking error, control, and the terminal parameter error.

Key Words—Inverse optimality; adaptive tracking control; Lyapunov functions; backstepping; Sontag's formula; transient performance.

Abstract—We pose and solve an 'inverse optimal' adaptive tracking problem for non-linear systems with unknown parameters. A controller is said to be inverse optimal when it minimizes a meaningful cost functional that incorporates integral penalty on the tracking error state and the control, as well as a terminal penalty on the parameter estimation error. The basis of our method is an *adaptive tracking control Lyapunov function* (*atclf*) the existence of which guarantees the solvability of the inverse optimal problem. The controllers designed in this paper are not of certainty-equivalence type. Even in the linear case they would not be a result of solving a Riccati equation for a given value of the parameter estimate. Our abandoning of the certainty-equivalence approach is motivated by the fact that, in general, this approach does not lead to optimality of the controller with respect to the overall plant-estimator system, even though both the estimator and the controller may be optimal as separate entities. Our controllers, instead, compensate for the effect of parameter adaptation transients in order to achieve optimality of the overall system. We combine inverse optimality with backstepping to design a new class of adaptive controllers for strict-feedback systems. These controllers solve a problem left open in the previous adaptive backstepping designs, i.e. obtaining transient performance bounds that include an estimate of control effort, which is the first such result in the adaptive control literature. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Because of the burden that the Hamilton–Jacobi–Bellman (HJB) PDEs impose on the problem of optimal control of non-linear systems, the efforts made over the last few years in the control of non-linear systems with uncertainties (adaptive and robust) (see e.g. Krstić *et al.* (1995) and Marino and Tomei (1995) and the references therein) have been focused on achieving *stability* rather than optimality. Recently, Freeman and Kokotović (1996a, b) revived the interest in the optimal control problem by showing that the solvability of the (robust) *stabilization* problem implies the solvability of the (robust) *inverse optimal* control problem. Further extensive results on inverse optimal non-linear stabilization were presented by Sepulchre *et al.* (1997).

The difference between the *direct* and the *inverse* optimal control problems is that the former seeks a controller that minimizes a *given* cost, while the latter is concerned with finding a controller that minimizes *some* 'meaningful' cost. In the inverse optimal approach, a controller is designed by using a control Lyapunov function (CLF) obtained by solving the stabilization problem. The CLF employed in the inverse optimal design is, in fact, a solution to the HJB PDE with a meaningful cost.

In this paper we formulate and solve the inverse optimal *adaptive tracking* problem for non-linear systems. We focus on the tracking rather than the (set-point) regulation problem because, even when a bound on the parametric uncertainty is known, tracking cannot (in general) be achieved using robust techniques—adaptation is necessary to achieve tracking. The cost functional in our inverse optimal problem includes integral penalty on both the tracking error state and control, as well as a penalty on the terminal value of the parameter estimation error. To solve the inverse optimal adaptive tracking problem we expand upon the concept of *adaptive control Lyapunov functions* (ACLFs) introduced in our earlier paper (Krstić and Kokotović, 1995) and used it to solve the adaptive stabilization problem.

* Received 19 August 1996; received in final form 17 February 1997. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Henk Nijmeijer under the direction of Editor Tamer Başar. Corresponding author Professor Miroslav Krstic. Tel. +1 301 405-5206; Fax +1 301 314-4977; E-mail krstic@eng.umd.edu.

† Department of Mechanical Engineering, University of Maryland, College Park, MD 20742, U.S.A.

Previous efforts to design adaptive 'linear-quadratic' controllers (see e.g. Ioannou and Sun, 1995) were based on the certainty equivalence principle: a parameter estimate computed on the basis of a gradient or least-squares update law is substituted into a control law based on a Riccati equation solved for that value of the parameter estimate. Even though both the estimator and the controller independently possess optimality properties, when combined they fail to exhibit optimality (and even stability becomes difficult to prove) because the controller 'ignores' the time-varying effect of adaptation. In contrast, the Lyapunov-based approach presented in this paper results in controllers that compensate for the effect of adaptation.

A special class of systems for which we constructively solve the inverse optimal adaptive tracking problem in this paper are the *parametric strict-feedback* systems, a representative member of a broader class of systems dealt with in Krstić *et al.* (1995), which includes feedback linearizable systems and, in particular, linear systems. A number of adaptive designs for parametric strict-feedback systems are available; however, none of them is optimal. In this paper we present a new design which is optimal with respect to a meaningful cost. We also improve upon the existing transient performance results. The transient performance results achieved with the *tuning functions* design in Krstić *et al.* (1995), even though the strongest such results in the adaptive control literature, still provide only performance estimates on the tracking error but not on control effort (the control is allowed to be large to achieve good tracking performance). The inverse optimal design in this paper solves the open problem of incorporating control effort in the performance bounds.

The optimal adaptive control problem posed in the present paper is not entirely dissimilar from the problem posed in the award-winning paper by Didinsky and Başar (1994) and solved using their cost-to-come method. The difference is two-fold: (a) our approach does not require the inclusion of a noise term in the plant model in order to be able to design a parameter estimator; and (b), while Didinsky and Başar only go as far as to derive a Hamilton–Jacobi–Isaacs equation the solution of which would yield an optimal controller, we actually solve our HJB equation and obtain inverse optimal controllers for strict-feedback systems. A nice marriage of the work by Didinsky and Başar (1994) and the backstepping design in Krstić *et al.* (1995) was brought out in the paper by Pan and Başar (1996), who solved an adaptive disturbance attenuation problem for strict-feedback systems. Their cost, however, does not impose a penalty on control effort.

This paper is organized as follows. In Section 2, we pose the *adaptive* tracking problem (without optimality). The solution to this problem is given in Sections 3 and 4, which generalize the results of Krstić and Kokotović (1995). Then, in Section 5 we pose and solve the *inverse optimal* problem for general non-linear systems, assuming the existence of an *adaptive tracking control Lyapunov function* (ATCLF). A constructive method for designing ATCLFs based on backstepping is presented in Section 6, and then used to solve the inverse optimal adaptive tracking control problem for strict-feedback systems in Section 7. A summary of the transient performance analysis is given in Section 8.

2. PROBLEM STATEMENT: ADAPTIVE TRACKING

We consider the problem of global tracking for systems of the form

$$\begin{aligned} \dot{x} &= f(x) + F(x)\,\theta + g(x)\,u, \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, the mappings $f(x)$, $F(x)$, $g(x)$ and $h(x)$ are smooth, and θ is a constant *unknown* parameter vector which can take *any* value in $\mathbb{R}^p$. To make tracking possible in the presence of an unknown parameter, we make the following key assumption:

Assumption 1. For a given smooth function $y_r(t)$, there exist functions $\rho(t, \theta)$ and $\alpha_r(t, \theta)$ such that

$$\begin{aligned} \frac{\mathrm{d}\rho(t,\theta)}{\mathrm{d}t} &= f(\rho(t,\theta)) + F(\rho(t,\theta))\theta \\ &\quad + g(\rho(t,\theta))\alpha_r(t,\theta), \\ y_r(t) &= h(\rho(t,\theta)), \quad \forall t \geq 0, \forall \theta \in \mathbb{R}^p. \end{aligned} \tag{2}$$ □

Note that this implies that

$$\frac{\partial}{\partial \theta} h \circ \rho(t,\theta) = 0, \quad \forall t \geq 0, \forall \theta \in \mathbb{R}^p. \tag{3}$$

For this reason, we can replace the objective of tracking the signal $y_r(t) = h \circ \rho(t, \theta)$ by the objective of tracking $y_r(t) = h \circ \rho(t, \hat{\theta}(t))$, where $\hat{\theta}(t)$ is a time function - an estimate of θ customary in adaptive control.

Consider the signal $x_r(t) = \rho(t, \hat{\theta}(t))$, which is governed by

$$\begin{aligned} \dot{x}_r &= \frac{\partial \rho(t,\hat{\theta})}{\partial t} + \frac{\partial \rho(t,\hat{\theta})}{\partial \hat{\theta}}\dot{\hat{\theta}} \\ &= f(x_r) + F(x_r)\,\hat{\theta} + g(x_r)\,\alpha_r(t,\hat{\theta}) + \frac{\partial \rho(t,\hat{\theta})}{\partial \hat{\theta}}\dot{\hat{\theta}}. \end{aligned} \tag{4}$$

We define the tracking error $e = x - x_r = x - \rho(t, \hat{\theta})$ and compute its derivative:

$$\dot{e} = f(x) - f(x_r) + [g(x) - g(x_r)]\alpha_r(t, \hat{\theta}) + F(x)\,\theta - F(x_r)\,\hat{\theta} - \frac{\partial\rho(t, \hat{\theta})}{\partial\hat{\theta}}\dot{\hat{\theta}} + g(x)[u - \alpha_r(t, \hat{\theta})]$$

$$= \tilde{f} + \tilde{F}\theta + F_r\tilde{\theta} - \frac{\partial\rho}{\partial\hat{\theta}}\dot{\hat{\theta}} + g\tilde{u}, \quad (5)$$

where $\tilde{\theta} = \theta - \hat{\theta}$ and

$$\begin{aligned} \tilde{f}(t, e, \hat{\theta}) &:= f(x) - f(x_r) + [g(x) - g(x_r)]\alpha_r(t, \hat{\theta}), \\ \tilde{F}(t, e, \hat{\theta}) &:= F(x) - F(x_r), \\ F_r(t, \hat{\theta}) &:= F(x_r), \\ \tilde{u} &:= u - \alpha_r(t, \hat{\theta}). \end{aligned} \quad (6)$$

(With a slight abuse of notation, we will write $g(x)$ also as $g(t, e, \hat{\theta})$.) The global tracking problem is then transformed into the problem of global stabilization of the error system (5). This problem is, in general, not solvable with *static* feedback. This is obvious in the scalar case $n = p = 1$ where, even in the case $y_r(t) = x_r(t) \equiv 0$, a control law $u = \alpha(x)$ independent of θ would have the impossible task to satisfy $x[f(x) + F(x)\theta + g(x)\alpha(x)] < 0$ for all $x \neq 0$ and all $\theta \in \mathbb{R}$. Therefore, we seek *dynamic* feedback controllers to stabilize system (5) for all θ.

Definition 1. The *adaptive tracking* problem for system (1) is solvable if Assumption 1 is satisfied and there exist a function $\tilde{\alpha}(t, e, \hat{\theta})$ smooth on $\mathbb{R}_+ \times (\mathbb{R}^n\backslash\{0\}) \times \mathbb{R}^p$ with $\tilde{\alpha}(t, 0, \hat{\theta}) \equiv 0$, a smooth function $\tau(t, e, \hat{\theta})$, and a positive definite symmetric $p \times p$ matrix Γ, such that the dynamic controller

$$\tilde{u} = \tilde{\alpha}(t, e, \hat{\theta}), \quad (7)$$

$$\dot{\hat{\theta}} = \Gamma\tau(t, e, \hat{\theta}), \quad (8)$$

guarantees that the equilibrium $e = 0$, $\tilde{\theta} = 0$ of the system (5) is globally stable and $e(t) \to 0$ as $t \to \infty$ for any value of the unknown parameter $\theta \in \mathbb{R}^p$. □

3. ADAPTIVE TRACKING AND ATCLFs

Our approach is to replace the problem of adaptive stabilization of (5) by a problem of non-adaptive stabilization of a modified system. This allows us to study adaptive stabilization in the Sontag–Artstein framework of CLFs (Artstein, 1983; Sontag, 1983, 1989).

Definition 2. A smooth function $V_a : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$, positive definite, decrescent, and proper (radially unbounded) in e (uniformly in t) for each θ, is called an *adaptive tracking control Lyapunov function* (*ATCLF*) for (1) (or, alternatively, an adaptive control Lyapunov function (ACLF) for (5)), if Assumption 1 is satisfied and there exists a positive definite symmetric matrix $\Gamma \in \mathbb{R}^{p\times p}$ such that for each $\theta \in \mathbb{R}^p$, $V_a(t, e, \theta)$ is a CLF for the modified non-adaptive system:

$$\dot{e} = \tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^{\mathrm{T}} - \frac{\partial\rho}{\partial\theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + g\tilde{u}, \quad (9)$$

that is, V_a satisfies

$$\inf_{\tilde{u}\in\mathbb{R}}\left\{\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left[\tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^{\mathrm{T}} - \frac{\partial\rho}{\partial\theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + g\tilde{u}\right]\right\} < 0. \quad (10)$$

In the sequel we will show that in order to achieve adaptive stabilization of (5) it is necessary and sufficient to achieve non-adaptive stabilization of (9). Noting that for $\tilde{\theta}(t) \equiv 0$ the system (5) reduces to the non-adaptive system

$$\dot{e} = \tilde{f} + \tilde{F}\theta + g\tilde{u}, \quad (11)$$

and we see that the modification in (9) is

$$F\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^{\mathrm{T}} - \frac{\partial\rho}{\partial\theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}. \quad (12)$$

Since these terms are present only when Γ is non-zero, the role of these terms is to account for the effect of adaptation. Since $V_a(t, e, \theta)$ has a minimum at $e = 0$ for all t and θ, the modification terms vanish at the $e = 0$, so $e = 0$ is an equilibrium of (9).

We now show how to design an adaptive controller (7)–(8) when an ATCLF is known.

Theorem 1. The following two statements are equivalent:

(i) There exists a triplet $(\tilde{\alpha}, V_a, \Gamma)$ such that $\tilde{\alpha}(t, e, \theta)$ globally uniformly asymptotically stabilizes (9) at $e = 0$ for each $\theta \in \mathbb{R}^p$ with respect to the Lyapunov function $V_a(t, e, \theta)$.

(ii) There exists an ATCLF $V_a(t, e, \theta)$ for (1).

Moreover, if an ATCLF $V_a(t, e, \theta)$ exists, then the adaptive tracking problem for (1) is solvable. □

Proof. That (i) ⇒ (ii) is obvious because (i) implies that there exists a continuous function $W:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}_+$, positive definite in e (uniformly in t) for each θ, such that

$$\frac{\partial V_a}{\partial t}+\frac{\partial V_a}{\partial e}\left[\tilde{f}+\tilde{F}\theta+F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}}-\frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}+g\tilde{\alpha}\right]\le -W(t,e,\theta). \quad (13)$$

Thus $V_a(t,e,\theta)$ is a CLF for (9) for each $\theta\in\mathbb{R}^p$, and therefore it is an ATCLF for (1).

The proof of (ii)$\Rightarrow$(i) is based on Sontag's formula (Sontag, 1989). We assume that V_a is an ATCLF for (1), that is, a CLF for (9). Sontag's formula applied to (9) gives a control law smooth on $\mathbb{R}_+\times(\mathbb{R}^n\backslash\{0\})\times\mathbb{R}^p$:

$$\tilde{\alpha}(t,e,\theta)=\begin{cases}\dfrac{\dfrac{\partial V_a}{\partial t}+\dfrac{\partial V_a}{\partial e}\bar{f}+\sqrt{\left(\dfrac{\partial V_a}{\partial t}+\dfrac{\partial V_a}{\partial e}\bar{f}\right)^2+\left(\dfrac{\partial V_a}{\partial e}g\right)^4}}{\dfrac{\partial V_a}{\partial e}g}, & \dfrac{\partial V_a}{\partial e}g(t,e,\theta)\neq 0,\\ 0, & \dfrac{\partial V_a}{\partial e}g(t,e,\theta)=0,\end{cases} \quad (14)$$

where

$$\bar{f}=\tilde{f}+\tilde{F}\theta+F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}}-\frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}. \quad (15)$$

With the choice (14), inequality (13) is satisfied with the continuous function

$$W(t,e,\theta)=\sqrt{\left(\frac{\partial V_a}{\partial t}(t,e,\theta)+\frac{\partial V_a}{\partial e}\bar{f}(t,e,\theta)\right)^2+\left(\frac{\partial V_a}{\partial e}g(t,e,\theta)\right)^4} \quad (16)$$

which is positive definite in e (uniformly in t) for each θ, because (10) implies that

$$\frac{\partial V_a}{\partial e}g(t,e,\theta)=0\Rightarrow\frac{\partial V_a}{\partial t}(t,e,\theta)+\frac{\partial V_a}{\partial e}\bar{f}(t,e,\theta)<0,\quad \forall e\neq 0,\ t\ge 0. \quad (17)$$

We note that the control law $\tilde{\alpha}(t,e,\theta)$, which is smooth away from $e=0$, will be also continuous at $e=0$ if and only if the ATCLF V_a satisfies the following property, called the *small control property* (Sontag, 1989): for each $\theta\in\mathbb{R}^p$ and for any $\varepsilon>0$ there is a $\delta>0$ such that, if $e\neq 0$ satisfies $|e|\le\delta$, then there is some $\tilde{u}$ with $|\tilde{u}|\le\varepsilon$ such that

$$\frac{\partial V_a}{\partial t}+\frac{\partial V_a}{\partial e}\left[\tilde{f}+\tilde{F}\theta+F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}}-\frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}+g\tilde{u}\right]<0 \quad (18)$$

for all $t\ge 0$.

Assuming the existence of an ATCLF, we now show that the adaptive tracking problem for (1) is solvable. Since (ii)$\Rightarrow$(i), there exists a triple $(\tilde{\alpha},V_a,\Gamma)$ and a function W such that (13) is satisfied. Consider the Lyapunov function candidate

$$V(t,e,\hat{\theta})=V_a(t,e,\hat{\theta})+\tfrac{1}{2}(\theta-\hat{\theta})^{\mathrm{T}}\Gamma^{-1}(\theta-\hat{\theta}). \quad (19)$$

With the help of (13), the derivative of V along the solutions of (5), (7) and (8), is

$$\begin{aligned}\dot{V}&=\frac{\partial V_a}{\partial t}+\frac{\partial V_a}{\partial e}\left[\tilde{f}+\tilde{F}\theta+F_r\tilde{\theta}-\frac{\partial \rho}{\partial \hat{\theta}}\Gamma\tau(t,e,\hat{\theta})+g\tilde{\alpha}(t,e,\hat{\theta})\right]\\&\quad+\frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau(t,e,\hat{\theta})-\tilde{\theta}^{\mathrm{T}}\tau(t,e,\hat{\theta})\\&=\frac{\partial V_a}{\partial t}+\frac{\partial V_a}{\partial e}[\tilde{f}+\tilde{F}\hat{\theta}+g\tilde{\alpha}(t,e,\hat{\theta})]\\&\quad+\frac{\partial V_a}{\partial e}F\tilde{\theta}-\frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \hat{\theta}}\Gamma\tau+\frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau-\tilde{\theta}^{\mathrm{T}}\tau\\&\le -W(t,e,\hat{\theta})-\frac{\partial V_a}{\partial \hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}\\&\quad+\frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau+\frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}-\frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \hat{\theta}}\Gamma\tau\\&\quad+\tilde{\theta}^{\mathrm{T}}\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}-\tilde{\theta}^{\mathrm{T}}\tau.\end{aligned} \quad (20)$$

Choosing

$$\tau(t,e,\hat{\theta})=\left(\frac{\partial V_a}{\partial e}F(t,e,\hat{\theta})\right)^{\mathrm{T}}, \quad (21)$$

we get

$$\dot{V}\le -W(t,e,\hat{\theta}). \quad (22)$$

Thus the equilibrium $e=0$, $\tilde{\theta}=0$ of (5), (7) and (8) is globally stable and, by LaSalle's theorem, $e(t)\to 0$ as $t\to\infty$. By Definition 1, the adaptive tracking problem for (1) is solvable. □

The adaptive controller constructed in the proof of Theorem 1 consists of a control law $\bar{u}=\tilde{\alpha}(t,e,\hat{\theta})$ given by (14), and an update law $\dot{\hat{\theta}}=\Gamma\tau(t,e,\hat{\theta})$ with (21). The control law $\tilde{\alpha}(t,e,\theta)$ is stabilizing for the modified system (9), but may not be stabilizing for the original system (5). However, as the proof of Theorem 1 shows, its certainty equivalence form $\tilde{\alpha}(t,e,\hat{\theta})$ is an adaptive globally stabilizing control law for the original system (5). The modified system 'anticipates' parameter estimation transients, which results in incorporating the *tuning function* τ in the control law $\tilde{\alpha}$. Indeed, the formula (14) for $\tilde{\alpha}$ depends on τ via

$$\frac{\partial V_a}{\partial e}\bar{f}(t,e,\theta)=\frac{\partial V_a}{\partial e}(\tilde{f}+\tilde{F}\theta)+\tau^{\mathrm{T}}\Gamma\left(\frac{\partial V_a}{\partial \theta}-\frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta}\right)^{\mathrm{T}}, \quad (23)$$

which is obtained by combining (15) and (21). Using (21) to rewrite the inequality (13) as

$$\frac{\partial V_a}{\partial t}+\frac{\partial V_a}{\partial e}[\tilde{f}+\tilde{F}\theta+g\tilde{\alpha}(t,e,\theta)]+\left(\frac{\partial V_a}{\partial \theta}-\frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\tau(t,e,\theta)\le -W(t,e,\theta), \quad (24)$$

it is not difficult to see that the control law (14) containing (23) prevents τ from destroying the non-positivity of the Lyapunov derivative.

Example 1. Consider the problem of designing an adaptive tracking controller for the system:

$$\begin{aligned}\dot{x}_1&=x_2+\varphi(x_1)^{\mathrm{T}}\theta,\\ \dot{x}_2&=u,\\ y&=x_1.\end{aligned} \quad (25)$$

In light of (1), $f(x)=[x_2,0]^{\mathrm{T}}$, $F(x)=[\varphi(x_1),0]^{\mathrm{T}}$ and $g(x)=[0,1]^{\mathrm{T}}$. For any given C^2 function $y_r(t)$, the function $\rho(t,\theta)=[\rho_1(t),\rho_2(t,\theta)]^{\mathrm{T}}$ is given by $\rho_1(t)=y_r(t)$ and $\rho_2(t,\theta)=\dot{y}_r(t)-\varphi(y_r)^{\mathrm{T}}\theta$, and the reference input is $\alpha_r(t,\theta)=\ddot{y}_r(t)-[\partial\varphi(y_r)^{\mathrm{T}}/\partial y_r]\,\theta\dot{y}_r(t)$. Hence Assumption 1 is satisfied.

With the signal $x_r(t)=\rho(t,\hat{\theta})$ and the tracking error $e=x-x_r$, we get the error system

$$\begin{aligned}\dot{e}_1&=e_2+\tilde{\varphi}^{\mathrm{T}}\theta+\varphi_r^{\mathrm{T}}\tilde{\theta},\\ \dot{e}_2&=\tilde{u}-\frac{\partial\rho_2}{\partial\hat{\theta}}\dot{\hat{\theta}},\end{aligned} \quad (26)$$

where $\tilde{\varphi}=\varphi(x_1)-\varphi(x_{r1})$, $\varphi_r=\varphi(x_{r1})=\varphi(y_r)$ and $\tilde{u}=u-\alpha_r$. The modified non-adaptive error system is

$$\begin{aligned}\dot{e}_1&=e_2+\tilde{\varphi}^{\mathrm{T}}\theta+\varphi^{\mathrm{T}}\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^{\mathrm{T}},\\ \dot{e}_2&=\tilde{u}-\frac{\partial\rho_2}{\partial\theta}\Gamma\left(\frac{\partial V_a}{\partial e_1}\varphi^{\mathrm{T}}\right)^{\mathrm{T}}.\end{aligned} \quad (27)$$

The control law

$$\begin{aligned}\tilde{u}&=\tilde{\alpha}(t,e,\theta)\\ &=-z_1-c_2z_2-(-c_1z_1+z_2)\left(c_1+\frac{\partial\tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\theta\right)\\ &\quad-\frac{\partial\tilde{\varphi}^{\mathrm{T}}}{\partial t}\theta-\varphi^{\mathrm{T}}\Gamma\varphi\left[1,c_1+\frac{\partial\tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\theta\right]z,\end{aligned} \quad (28)$$

where $z_1=e_1$, $z_2=c_1e_1+e_2+\tilde{\varphi}^{\mathrm{T}}\theta$, and c_1, $c_2>0$, globally uniformly asymptotically stabilizes (27) at $e=0$ with respect to $V_a=\frac{1}{2}(z_1^2+z_2^2)$ with $W(t,e,\theta)=c_1z_1^2+c_2z_2^2$. By Theorem 1, the adaptive tracking problem for (25) is solved with the control law $\tilde{u}=\tilde{\alpha}(t,e,\hat{\theta})$ and the update law

$$\dot{\hat{\theta}}=\Gamma\tau(t,e,\hat{\theta})=\Gamma\varphi(t,e,\hat{\theta})\left[1,c_1+\frac{\partial\tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\hat{\theta}\right]z. \quad (29)$$

□

As is always the case in adaptive control, in the proof of Theorem 1 we used a Lyapunov function $V(t,e,\hat{\theta})$ given by (19), which is quadratic in the parameter error $\theta-\hat{\theta}$. The quadratic form is suggested by the linear dependence of (5) on θ, and the fact that θ cannot be used for feedback. We will now show that the quadratic form of (19) is both necessary and sufficient for the existence of an ATCLF.

Definition 3. The *adaptive quadratic tracking* problem for (1) is solvable if the adaptive tracking problem for (1) is solvable and, in addition, there exist a smooth function $V_a(t,e,\theta)$ positive definite, decrescent, and proper in e (uniformly in t) for each θ, and a continuous function $W(t,e,\theta)$ positive definite in e (uniformly in t) for each θ, such that the derivative of (19) along the solutions of (5), (7) and (8) is given by (22).

Corollary 1. The adaptive quadratic tracking problem for the system (1) is solvable if and only if there exists an ATCLF $V_a(t,e,\theta)$.

Proof. The 'if' part is contained in the proof of Theorem 1 where the Lyapunov function $V(t,e,\hat{\theta})$ is in the form (19). To prove the 'only if' part, we start by assuming global adaptive quadratic stabilizability of (5), and first show that $\tau(t,e,\hat{\theta})$ must be given by (21). The derivative of

V along the solutions of (5), (7) and (8), given by (20), is rewritten as

$$\dot V = \frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}[\tilde f + \tilde F\hat\theta + g\tilde\alpha(t,e,\hat\theta)] - \frac{\partial V_a}{\partial e}\frac{\partial\rho}{\partial\hat\theta}\Gamma\tau(t,e,\hat\theta) + \frac{\partial V_a}{\partial\hat\theta}\Gamma\tau(t,e,\hat\theta) - \hat\theta^{\mathrm T}\left[\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm T} - \tau\right] + \theta^{\mathrm T}\left[\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm T} - \tau\right]. \tag{30}$$

This expression has to be non-positive to satisfy (22). Since it is affine in θ, it can be non-positive for all $\theta\in\mathbb{R}^p$ only if the last term is zero, that is, only if τ is defined as in (21). Then, it is straightforward to verify that

$$\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left[\tilde f + \tilde F\hat\theta + F\Gamma\left(\frac{\partial V_a}{\partial\hat\theta}\right)^{\mathrm T} - \frac{\partial\rho}{\partial\hat\theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm T} + g\tilde\alpha\right] = \dot V + \left(\tilde\theta^{\mathrm T} - \frac{\partial V_a}{\partial\hat\theta}\Gamma\right)\left[\tau - \left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm T}\right] \le -W(t,e,\hat\theta), \tag{31}$$

for all $(t,e,\hat\theta)\in\mathbb{R}_+\times\mathbb{R}^{n+p}$. By ($1\Rightarrow 2$) in Theorem 1, $V_a(t,e,\theta)$ is an ATCLF for (1). □

4. ADAPTIVE BACKSTEPPING

With Theorem 1, the problem of adaptive stabilization is reduced to the problem of finding an ATCLF. This problem is solved recursively via backstepping.

Lemma 1. If the adaptive quadratic tracking problem for the system

$$\begin{aligned}\dot x &= f(x) + F(x)\,\theta + g(x)\,u,\\ y &= h(x)\end{aligned} \tag{32}$$

is solvable with a C^1 control law, then the adaptive quadratic tracking problem for the augmented system

$$\begin{aligned}\dot x &= f(x) + F(x)\,\theta + g(x)\,\xi\\ \dot\xi &= u,\\ y &= h(x)\end{aligned} \tag{33}$$

is also solvable.

Proof. Since the adaptive quadratic tracking problem for the system (32) is solvable, by Corollary 1 there exists an ATCLF $V_a(t,e,\theta)$ for (32), and by Theorem 1 it satisfies (13) with a control law $\tilde\alpha(t,e,\theta)$. Define $\tilde\xi = \xi - \alpha_r(t,\hat\theta)$ and consider the system

$$\begin{aligned}\dot e &= \tilde f(t,e,\hat\theta) + \tilde F(t,e,\hat\theta)\theta + F_r(t,\hat\theta)\tilde\theta - \frac{\partial\rho}{\partial\hat\theta}\dot{\hat\theta} + g(t,e,\hat\theta)\tilde\xi,\\ \dot{\tilde\xi} &= \tilde u - \frac{\partial\alpha_r}{\partial\hat\theta}\dot{\hat\theta},\end{aligned} \tag{34}$$

where $\tilde u = u - \alpha_{r1}(t,\hat\theta)$ and $\alpha_{r1}(t,\hat\theta) = \partial\alpha_r(t,\hat\theta)/\partial t$. We will now show that

$$V_1(t,e,\tilde\xi,\theta) = V_a(t,e,\theta) + \tfrac12[\tilde\xi - \tilde\alpha(t,e,\theta)]^2 \tag{35}$$

is an ATCLF for the augmented system (33) by showing that it is a CLF for the modified non-adaptive system:

$$\begin{aligned}\dot e &= \tilde f + \tilde F\theta + F\Gamma\left(\frac{\partial V_1}{\partial\theta}\right)^{\mathrm T} - \frac{\partial\rho}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm T} + g\tilde\xi,\\ \dot{\tilde\xi} &= \tilde u - \frac{\partial\alpha_r}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm T}.\end{aligned} \tag{36}$$

We present a constructive proof which shows that the control law

$$\begin{aligned}\tilde u &= \tilde\alpha_1(t,e,\tilde\xi,\theta)\\ &= -c(\tilde\xi - \tilde\alpha) + \frac{\partial\tilde\alpha}{\partial t} - \frac{\partial V_a}{\partial e}g + \frac{\partial\tilde\alpha}{\partial e}(\tilde f + \tilde F\theta + g\tilde\xi)\\ &\quad + \left(\frac{\partial\alpha_r}{\partial\theta} + \frac{\partial\tilde\alpha}{\partial\theta} - \frac{\partial\tilde\alpha}{\partial e}\frac{\partial\rho}{\partial\theta}\right)\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm T}\\ &\quad + \left(\frac{\partial V_a}{\partial\theta} - \frac{\partial V_a}{\partial e}\frac{\partial\rho}{\partial\theta}\right)\Gamma\left(\frac{\partial\tilde\alpha}{\partial e}F\right)^{\mathrm T}, \quad c>0,\end{aligned} \tag{37}$$

satisfies

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial(e,\tilde\xi)} \times \begin{bmatrix}\tilde f + \tilde F\theta + F\Gamma\left(\frac{\partial V_1}{\partial\theta}\right)^{\mathrm T} - \frac{\partial\rho}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm T} + g\tilde\xi\\ \tilde\alpha_1(t,e,\tilde\xi,\theta) - \frac{\partial\alpha_r}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm T}\end{bmatrix} \le -W - c(\tilde\xi - \tilde\alpha)^2. \tag{38}$$

Let us start by introducing, for brevity, a new error state $z = \tilde\xi - \tilde\alpha(t,e,\theta)$. With (35) we compute

$$\begin{aligned}\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial(e,\tilde\xi)}\begin{bmatrix}\tilde f + \tilde F\theta + g\tilde\xi\\ \tilde\alpha_1(t,e,\tilde\xi,\theta)\end{bmatrix} &= \frac{\partial V_a}{\partial t} - z\frac{\partial\tilde\alpha}{\partial t} + \left(\frac{\partial V_a}{\partial e} - z\frac{\partial\tilde\alpha}{\partial e}\right)(\tilde f + \tilde F\theta + g\tilde\xi) + \frac{\partial V_1}{\partial\tilde\xi}\tilde\alpha_1\\ &= \frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}(\tilde f + \tilde F\theta + g\tilde\alpha)\\ &\quad + z\left[\tilde\alpha_1 - \frac{\partial\tilde\alpha}{\partial t} + \frac{\partial V_a}{\partial e}g - \frac{\partial\tilde\alpha}{\partial e}(\tilde f + \tilde F\theta + g\tilde\xi)\right].\end{aligned} \tag{39}$$

On the other hand, in view of (35), we have

$$\frac{\partial V_1}{\partial(e,\tilde{\xi})}\begin{bmatrix} F\Gamma\left(\frac{\partial V_1}{\partial\theta}\right)^{\mathrm{T}} - \frac{\partial\rho}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} \\ -\frac{\partial\alpha_r}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} \end{bmatrix}$$

$$= \left(\frac{\partial V_a}{\partial e} - z\frac{\partial\tilde{\alpha}}{\partial e}\right)\left[F\Gamma\left(\frac{\partial V_a}{\partial\theta} - z\frac{\partial\tilde{\alpha}}{\partial\theta}\right)^{\mathrm{T}} - \frac{\partial\rho}{\partial\theta}\Gamma\left(\frac{\partial V_a}{\partial e}F - z\frac{\partial\tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}}\right] - z\frac{\partial\alpha_r}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}}$$

$$= \frac{\partial V_a}{\partial e}F\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^{\mathrm{T}} - \frac{\partial V_a}{\partial e}\frac{\partial\rho}{\partial\theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} - z\left[\left(\frac{\partial\alpha_r}{\partial\theta} + \frac{\partial\tilde{\alpha}}{\partial\theta} - \frac{\partial\tilde{\alpha}}{\partial e}\frac{\partial\rho}{\partial\theta}\right)\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} + \left(\frac{\partial V_a}{\partial\theta} - \frac{\partial V_a}{\partial e}\frac{\partial\rho}{\partial\theta}\right)\Gamma\left(\frac{\partial\tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}}\right]. \tag{40}$$

Adding (39) and (40), with (13) and (37) we get (38). This proves by Theorem 1 that $V_1(t,e,\tilde{\xi},\theta)$ is an ATCLF for system (33), or, an ACLF for (34) and, by Corollary 1, the adaptive quadratic tracking problem for this system is solvable.

The new tuning function is determined by the new ATCLF V_1 and given by

$$\tau_1(t,e,\tilde{\xi},\theta) = \left(\frac{\partial V_1}{\partial(e,\tilde{\xi})}\begin{bmatrix}F\\0\end{bmatrix}\right)^{\mathrm{T}} = \left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} = \left[\left(\frac{\partial V_a}{\partial e} - (\tilde{\xi}-\tilde{\alpha})\frac{\partial\tilde{\alpha}}{\partial e}\right)F\right]^{\mathrm{T}} = \tau(t,e,\theta) - \left(\frac{\partial\tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}}(\tilde{\xi}-\tilde{\alpha}). \tag{41}$$

The control law $\tilde{\alpha}_1(t,e,\tilde{\xi},\theta)$ in (37) is only one out of many possible control laws. Once we have shown that V_1 given by (35) is an ATCLF for (33), or, an ACLF for (34), we can use, for example, the C^0 control law $\tilde{\alpha}_1$ given by Sontag's formula (14).

Example 2 (Example 1 continued). Let us consider the system:

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi(x_1)^{\mathrm{T}}\theta,\\ \dot{x}_2 &= x_3,\\ \dot{x}_3 &= u,\\ y &= x_1.\end{aligned} \tag{42}$$

We treat the state x_3 as an integrator added to the (x_1,x_2) subsystem for Example 1, so Lemma 1 is applicable. Defining $z_3 = \tilde{x}_3 - \tilde{\alpha}(t,e,\theta)$, where $\tilde{x}_3 = x_3 - \alpha_r$, by Lemma 1 the function $V_1(t,e,\tilde{x}_3,\theta) = \frac{1}{2}(z_1^2+z_2^2+z_3^2)$ is an ATCLF for the system (42). With (37) and (41) we obtain

$$\begin{aligned}\tilde{u} &= \tilde{\alpha}_1(t,e,\tilde{\xi},\theta)\\ &= -z_2 - cz_3 + \frac{\partial\tilde{\alpha}}{\partial t} + \frac{\partial\tilde{\alpha}}{\partial e}\begin{bmatrix}e_2+\tilde{\varphi}^{\mathrm{T}}\theta\\ \tilde{x}_3\end{bmatrix}\\ &\quad + \left(\frac{\partial\alpha_r}{\partial\theta} + \frac{\partial\tilde{\alpha}}{\partial\theta} + \frac{\partial\tilde{\alpha}}{\partial e_2}\varphi(y_r)^{\mathrm{T}}\right)\Gamma\tau_1\\ &\quad + \varphi^{\mathrm{T}}\Gamma\varphi\frac{\partial\tilde{\alpha}}{\partial e_1}z_2,\end{aligned} \tag{43}$$

$$\tau_1 := \tau_1(t,e,\tilde{x}_3,\theta) = \tau - \frac{\partial\tilde{\alpha}}{\partial e_1}\varphi z_3. \tag{44}$$

The actual control is $u = \tilde{u} + \alpha_{r1}$, where

$$\alpha_{r1} = \frac{\partial\alpha_r}{\partial t} = \dddot{y}_r(t) - \frac{\partial\varphi(y_r)^{\mathrm{T}}}{\partial y_r}\theta\ddot{y}_r(t) - \frac{\partial^2\varphi(y_r)^{\mathrm{T}}}{\partial y_r^2}\theta(\dot{y}_r(t))^2.$$

□

A repeated application of Lemma 1 (generalized as in Krstić *et al.* (1995, page 138) to the case where $\dot{\xi} = u + \phi(x,\xi)^{\mathrm{T}}\theta)$ recovers our earlier result (Krstić *et al.*, 1992).

Corollary 2 (Krstić *et al.*, 1992). The adaptive quadratic tracking problem for the following system is solvable:

$$\begin{aligned}\dot{x}_n &= x_{i+1} + \varphi_i(x_1,\ldots,x_i)^{\mathrm{T}}\theta, \quad i = 1,\ldots,n-1\\ \dot{x}_n &= u + \varphi_n(x_1,\ldots,x_n)^{\mathrm{T}}\theta\\ y &= x_1.\end{aligned} \tag{45}$$

□

5. INVERSE OPTIMAL ADAPTIVE TRACKING

While in the previous sections our objective was only to achieve adaptive tracking, in this section our objective is to achieve its optimality in a certain sense.

Definition 4. The *inverse optimal adaptive tracking* problem for system (1) is solvable if there exist a positive constant β, a positive real-valued function $r(t,e,\theta)$, a real-valued function $l(t,e,\theta)$ positive definite in e for each θ, and a dynamic feedback law ((7) and (8)) which solves the adaptive quadratic tracking problem and also minimizes the cost functional

$$J = \beta\lim_{t\to\infty}|\theta - \hat{\theta}(t)|^2_{\Gamma^{-1}} + \int_0^\infty [l(t,e,\hat{\theta}) + r(t,e,\hat{\theta})\tilde{u}^2]\,\mathrm{d}t, \tag{46}$$

for any $\theta \in \mathbb{R}^p$. □

This definition of optimality puts penalty on e and $\tilde{u}$ as well as on the terminal value of $|\tilde{\theta}|$. Even though $\tilde{\theta}(t)$ is not guaranteed to have a limit in the general tracking case (it is guaranteed to have a limit in the case of set-point regulation (Krstić, 1996; Li and Krstić, 1996)), the existence of $\lim_{t\to\infty}|\tilde{\theta}|^2_{\Gamma^{-1}}$ is ensured by the assumption that the adaptive *quadratic* tracking problem is solvable. This can be seen by noting that, since $V(t)\geq 0$, and from (22) $V(t)$ is non-increasing, $V(t)$ has a limit. Since (22) guarantees that $V_a(t)\to 0$, it follows from (19) that $|\tilde{\theta}|^2_{\Gamma^{-1}}$ has a limit. The absence of an integral penalty on $\tilde{\theta}$ in (46) should not be surprising, because adaptive feedback systems, in general, do not guarantee parameter convergence to a true value.

Theorem 2. Suppose there exists an ATCLF $V_a(t, e, \theta)$ for (1) and a control law $\tilde{u} = \tilde{\alpha}(t, e, \theta)$ that stabilizes the system

$$\dot{e} = \underbrace{\tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}}_{\bar{f}} + g\tilde{u} \tag{47}$$

has the form

$$\tilde{\alpha}(t, e, \theta) = -r^{-1}(t, e, \theta)\frac{\partial V_a}{\partial e}g, \tag{48}$$

where $r(t, e, \theta) > 0$ for all t, e and θ. Then:

(i) The non-adaptive control law

$$\tilde{u} = \tilde{\alpha}^*(t, e, \theta) = \beta\tilde{\alpha}(t, e, \theta), \quad \beta \geq 2, \tag{49}$$

minimizes the cost functional

$$J_a = \int_0^\infty [l(t, e, \theta) + r(t, e, \theta)\tilde{u}^2]\,\mathrm{d}t, \quad \forall\theta \in \mathbb{R}^p \tag{50}$$

along the solutions of the non-adaptive system (47), where

$$l(t, e, \theta) = -2\beta\left[\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}(\bar{f} + g\tilde{\alpha})\right] + \beta(\beta-2)r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2. \tag{51}$$

(ii) The inverse optimal adaptive tracking problem is solvable. □

Proof. (*Part* 1). In light of (13), we have

$$\begin{aligned} l(t, e, \theta) &= -2\beta\left[\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}(\bar{f} + g\tilde{\alpha})\right] + \beta(\beta-2)r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2 \\ &\geq 2\beta W(t, e, \theta) + \beta(\beta-2)r^{-2}\left(\frac{\partial V_a}{\partial e}g\right)^2. \end{aligned} \tag{52}$$

Since $\beta \geq 2$, $r(t, e, \theta) > 0$, and $W(t, e, \theta)$ is positive definite, $l(t, e, \theta)$ is also positive definite. Therefore J_a defined in (50) is a meaningful cost functional, which puts a penalty on both e and $\tilde{u}$. Substituting $l(t, e, \theta)$ and

$$v = \tilde{u} - \tilde{\alpha}^* = \tilde{u} + \beta r^{-1}\frac{\partial V_a}{\partial e}g \tag{53}$$

into J_a, we get

$$\begin{aligned} J_a &= \int_0^\infty \left[-2\beta\frac{\partial V_a}{\partial t} - 2\beta\frac{\partial V_a}{\partial e}\bar{f} + \beta^2 r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2 + rv^2 - 2\beta v\frac{\partial V_a}{\partial e}g + \beta^2 r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2\right]\mathrm{d}t \\ &= -2\beta\int_0^\infty\left[\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}(\bar{f} + g\tilde{u})\right]\mathrm{d}t + \int_0^\infty rv^2\,\mathrm{d}t \\ &= -2\beta\int_0^\infty \mathrm{d}V_a + \int_0^\infty rv^2\,\mathrm{d}t \\ &= 2\beta V_a(0, e(0), \hat{\theta}(0)) - 2\beta\lim_{t\to\infty}V_a(t, e(t), \hat{\theta}(t)) + \int_0^\infty rv^2\,\mathrm{d}t. \end{aligned} \tag{54}$$

Since the control input $\tilde{u}(t)$ solves the adaptive quadratic tracking problem, $\lim_{t\to\infty}e(t) = 0$, and we have that $\lim_{t\to\infty}V_a(t, e(t), \hat{\theta}(t)) = 0$. Therefore, the minimum of (54) is reached only if $v = 0$, and hence the control $\tilde{u} = \tilde{\alpha}^*(t, e, \theta)$ is an optimal control.

(*Part* 2). Since there exists an ATCLF V_a for (1), the adaptive quadratic tracking problem is solvable. Next, we show that the dynamic control law

$$\tilde{u} = \tilde{\alpha}^*(t, e, \hat{\theta}), \tag{55}$$

$$\dot{\hat{\theta}} = \Gamma\tau(t, e, \hat{\theta}) = \Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}, \tag{56}$$

minimizes the cost functional (46). The choice of the update law (56) is due to the requirement that (55) and (56) solve the adaptive *quadratic* tracking problem (see the proof of Corollary 1). Substituting $l(t, e, \hat\theta)$ and

$$v = \bar u + \beta r^{-1}\frac{\partial V_a}{\partial e}g \tag{57}$$

into J, along the solutions of (5) and (56) we get

$$\begin{aligned}
J &= \beta \lim_{t\to\infty}|\tilde\theta|^2_{\Gamma^{-1}} + \int_0^\infty \Big\{-2\beta\frac{\partial V_a}{\partial t} - 2\beta\frac{\partial V_a}{\partial e}\Big[\bar f + \bar F\theta \\
&\quad + F_r\tilde\theta - \frac{\partial\rho}{\partial\hat\theta}\Gamma\Big(\frac{\partial V_a}{\partial e}F\Big)^{\mathrm T}\Big] \\
&\quad - 2\beta\Big(\frac{\partial V_a}{\partial\hat\theta} - \tilde\theta^{\mathrm T}\Gamma^{-1}\Big)\Gamma\Big(\frac{\partial V_a}{\partial e}F\Big)^{\mathrm T} \\
&\quad + \beta^2 r^{-1}\Big(\frac{\partial V_a}{\partial e}g\Big)^2 + rv^2 - 2\beta v\frac{\partial V_a}{\partial e}g \\
&\quad + \beta^2 r^{-1}\Big(\frac{\partial V_a}{\partial e}g\Big)^2\Big\}\,\mathrm{d}t \\
&= \beta \lim_{t\to\infty}|\tilde\theta|^2_{\Gamma^{-1}} - 2\beta\int_0^\infty \Big\{\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\Big[\bar f + \bar F\theta \\
&\quad + F_r\tilde\theta - \frac{\partial\rho}{\partial\hat\theta}\Gamma\Big(\frac{\partial V_a}{\partial e}F\Big)^{\mathrm T} + g\bar u\Big] \\
&\quad + \Big(\frac{\partial V_a}{\partial\hat\theta} - \tilde\theta^{\mathrm T}\Gamma^{-1}\Big)\Gamma\Big(\frac{\partial V_a}{\partial e}F\Big)^{\mathrm T}\Big\}\,\mathrm{d}t + \int_0^\infty rv^2\,\mathrm{d}t \\
&= \beta \lim_{t\to\infty}|\tilde\theta|^2_{\Gamma^{-1}} - 2\beta\int_0^\infty \mathrm{d}\big(V_a + \tfrac12\tilde\theta^{\mathrm T}\Gamma^{-1}\tilde\theta\big) \\
&\quad + \int_0^\infty rv^2\,\mathrm{d}t \\
&= 2\beta V_a(0, e(0), \hat\theta(0)) + \beta\,|\tilde\theta(0)|^2_{\Gamma^{-1}} \\
&\quad - 2\beta\lim_{t\to\infty} V_a(t, e(t), \hat\theta(t)) + \int_0^\infty rv^2\,\mathrm{d}t.
\end{aligned} \tag{58}$$

Again, since $\bar u(t)$ solves the adaptive quadratic tracking problem, $\lim_{t\to\infty} e(t) = 0$, and we have that $\lim_{t\to\infty} V_a(t, e(t), \hat\theta(t)) = 0$. Therefore, the minimum of (58) is reached only if $v = 0$ and thus the control $\bar u = \bar\alpha^*(t, e, \hat\theta)$ minimizes the cost functional (46). □

Remark 1. Even though not explicit in the proof of the above theorem, the ATCLF $V_a(t, e, \theta)$ solves the following family of HJB equations parametrized in $\beta \ge 2$:

$$\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\Big[\bar f + \bar F\theta + \underbrace{F\Gamma\Big(\frac{\partial V_a}{\partial\theta}\Big)^{\mathrm T} - \frac{\partial\rho}{\partial\theta}\Gamma\Big(\frac{\partial V_a}{\partial e}F\Big)^{\mathrm T}}\Big]$$
$$-\frac{\beta}{2r(t, e, \theta)}\Big(\frac{\partial V_a}{\partial e}g\Big)^2 + \frac{l(t, e, \theta)}{2\beta} = 0. \tag{59}$$

The underbraced terms represent the 'non-certainty-equivalence' part of this HJB equation. Their role is to take into account the time-varying effect of parameter adaptation and make the control law optimal *in the presence of an update law.* □

Remark 2. The freedom in selecting the parameter $\beta \ge 2$ in the control law (49) means that the inverse optimal adaptive controller has an infinite gain margin. □

Example 3. Consider the scalar *linear* system

$$\begin{aligned}\dot x &= u + \theta x,\\ y &= x.\end{aligned} \tag{60}$$

For simplicity, we focus on the regulation case, $y_r(t) \equiv 0$. Since the system is scalar, $V_a = \frac12 x^2$, $L_g V_a = x$ and $L_{\bar f} V_a = x^2\theta$. We choose the control law based on Sontag's formula

$$u_s = -x(\theta + \sqrt{\theta^2 + 1}) = -2r^{-1}(\theta)x, \tag{61}$$

where

$$r(\theta) = \frac{2}{\theta + \sqrt{\theta^2 + 1}} > 0, \qquad \forall\theta. \tag{62}$$

The control $u_s/2$ is stabilizing for the system (60) because

$$\dot V_a|_{u_s/2} = -\tfrac12(-\theta + \sqrt{\theta^2 + 1})x^2. \tag{63}$$

By Theorem 2, the control u_s is optimal with respect to the cost functional

$$J_u = \int_0^\infty [l(x, \theta) + ru^2]\,\mathrm{d}t = 2\int_0^\infty \frac{x^2 + u^2}{\theta + \sqrt{\theta^2 + 1}}\,\mathrm{d}t, \tag{64}$$

with a value function $J_a^*(x) = 2x^2$. Meanwhile, the dynamic control

$$u_s = -x(\hat\theta + \sqrt{\hat\theta^2 + 1}), \tag{65}$$
$$\dot{\hat\theta} = x^2, \tag{66}$$

is optimal with respect to the cost functional

$$J = 2[\theta - \hat\theta(\infty)]^2 + 2\int_0^\infty \frac{x^2 + u^2}{\hat\theta + \sqrt{\hat\theta^2 + 1}}\,\mathrm{d}t, \tag{67}$$

with a value function $J^*(x, \hat\theta) = 2[x^2 + (\theta - \hat\theta)^2]$.

We point out that $\hat\theta(\infty)$ exists both due to the scalar (in parameter θ) nature of the problem and because it is a problem of regulation (Krstić, 1996). Note that, even though the penalty coefficient on x and u in (67) varies with $\hat\theta(t)$, the penalty coefficient is always positive and finite. □

Remark 3. The control law (61) is, in fact, a linear quadratic regulator (LQR) for the system (60) when the parameter θ is known. The control law can be also written as $u_s = -p(\theta)x$, where $p(\theta)$ is the solution of the Riccati equation

$$p^2 - 2\theta p - 1 = 0. \qquad (68)$$

It is of interest to compare the approach in our paper with the 'adaptive LQR scheme' for linear systems given in Ioannou and Sun (1995, Section 7.4.4):

- Even though both methodologies result in the same control law (61) for the *scalar linear* system in Example 3, they employ different update laws. The *gradient* update law in Ioannou and Sun (1995, Section 7.4.4) is optimal with respect to an (instantaneous) cost on an *estimation* error; however, when its estimates $\hat{\theta}(t)$ are substituted into the control law (65), this control law is not (guaranteed to be) optimal for the overall system. (Even its proof of stability is a non-trivial matter!) In contrast, the update law (66) guarantees optimality of the control law (65) for the overall system with respect to the meaningful cost (67).
- The true difference between the approach in our paper and the adaptive LQR scheme in Ioannou and Sun (1995, Section 7.4.4) arises for systems of higher order. Then the underbraced non-certainty-equivalence terms in (59) start to play a significant role. The certainty-equivalence approach in Ioannou and Sun (1995) would be to set $\Gamma = 0$ in the HJB (Riccati - for linear systems) equation (59) and combine the resulting control law with a gradient or least-squares update law. The optimality of the non-adaptive controller would be lost in the presence of adaptation due to the time-varying $\hat{\theta}(t)$. In contrast, a solution to (59) with $\Gamma > 0$ would lead to optimality with respect to (46). □

Corollary 3. If there exists an ATCLF $V_a(t, e, \theta)$ for (1), then the inverse optimal adaptive tracking problem is solvable.

Proof. Consider the Sontag-type control law $u_s = \tilde{\alpha}(t, e, \theta)$, where $\tilde{\alpha}(t, e, \theta)$ is defined by (14). The control law $\tilde{u}_s/2 = \frac{1}{2}\tilde{\alpha}(t, e, \theta)$ is an asymptotic stabilizing controller for system (9) because inequality (13) is satisfied with

$$W(t, e, \theta) = \frac{1}{2}\left[-\left(\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\bar{f}\right) + \sqrt{\left(\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\bar{f}\right)^2 + \left(\frac{\partial V_a}{\partial e}g\right)^4}\,\right], \qquad (69)$$

which is positive definite in e (uniformly in t) for each θ, since $(\partial V_a/\partial t) + (\partial V_a/\partial e)\bar{f} < 0$ whenever $(\partial V/\partial e)g = 0$, $e \neq 0$ and $t \geq 0$. Since $\frac{1}{2}\tilde{\alpha}(t, e, \theta)$ is of the form $\frac{1}{2}\tilde{\alpha}(t, e, \theta) = -r^{-1}(\partial V_a/\partial e)g$ with $r(t, e, \theta) > 0$ given by

$$r(t, e, \theta) = \begin{cases} \dfrac{2\left(\dfrac{\partial V_a}{\partial e}g\right)^2}{\dfrac{\partial V_a}{\partial t} + \dfrac{\partial V_a}{\partial e}\bar{f} + \sqrt{\left(\dfrac{\partial V_a}{\partial t} + \dfrac{\partial V_a}{\partial e}\bar{f}\right)^2 + \left(\dfrac{\partial V_a}{\partial e}g\right)^4}}, & \dfrac{\partial V_a}{\partial e}g \neq 0, \\ \text{any positive real number}, & \dfrac{\partial V_a}{\partial e}g = 0, \end{cases} \qquad (70)$$

by Theorem 2 the inverse optimal adaptive tracking problem is solvable. The optimal control is the formula (14) itself. □

Corollary 4. The inverse optimal adaptive tracking problem for the following system is solvable:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_i(x_1, \ldots, x_i)^{\mathrm{T}}\theta, \quad i = 1, \ldots, n-1, \\ \dot{x}_n &= u + \varphi_n(x_1, \ldots, x_n)^{\mathrm{T}}\theta, \qquad (71) \\ y &= x_1. \end{aligned}$$

Proof. By Corollary 2 and Corollary 1, there exists an ATCLF $V_a(t, e, \theta)$ for (71). It then follows from Corollary 3 that the inverse optimal adaptive tracking problem for system (71) is solvable. □

6. INVERSE OPTIMALITY VIA BACKSTEPPING

With Theorem 2, the problem of inverse optimal adaptive tracking is reduced to the problem of finding an ATCLF. However, the control law (14) based on Sontag's formula is not guaranteed to be smooth at the origin. In this section we develop controllers based on backstepping which are *smooth everywhere*, and hence can be employed in a recursive design.

Lemma 2. If the adaptive quadratic tracking problem for the system

$$\begin{aligned} \dot{x} &= f(x) + F(x)\,\theta + g(x)\,u \qquad (72) \\ y &= h(x) \end{aligned}$$

is solvable with a smooth control law $\tilde{\alpha}(t, e, \theta)$, and (13) is satisfied with $W(t, e, \theta) = e^{\mathrm{T}}\Omega(t, e, \theta)e$, where $\Omega(t, e, \theta)$ is positive definite and symmetric for all t, e and θ; then the inverse optimal adaptive tracking problem for the augmented system

$$\begin{aligned} \dot{x} &= f(x) + F(x)\,\theta + g(x)\,\xi, \\ \dot{\xi} &= u \qquad (73) \\ y &= h(x) \end{aligned}$$

is also solvable with a smooth control law. □

Proof. Since the adaptive quadratic tracking problem for the system (72) is solvable, by Lemma 1 and Corollary 1, $V_1(t, e, \xi, \theta) = V_a(t, e, \theta) + \frac{1}{2}[\xi - \tilde{\alpha}(t, e, \theta)]^2$ is an ATCLF for the augmented system (73), i.e. a CLF for the modified non-adaptive error system (36). Adding (39) and (40), with (13), we get

$$\begin{aligned}\dot{V}_1 &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial(e, \xi)} \\ &\times \begin{bmatrix} \tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_1}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} + g\xi \\ \tilde{u} - \frac{\partial \alpha_r}{\partial \theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} \end{bmatrix} \\ &\le -W + z\Big[\tilde{u} - \frac{\partial \tilde{\alpha}}{\partial t} + \frac{\partial V_a}{\partial e}g \\ &\quad - \frac{\partial \tilde{\alpha}}{\partial e}[\tilde{f} + \tilde{F}\theta + g(\tilde{\alpha} + z)] \\ &\quad - \left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} - \frac{\partial \tilde{\alpha}}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} \\ &\quad + \left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} - \frac{\partial \tilde{\alpha}}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial \tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}} z \\ &\quad - \left(\frac{\partial V_a}{\partial \theta} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial \tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}}\Big], \end{aligned} \tag{74}$$

where $z = \xi - \tilde{\alpha}(t, e, \theta)$. To render $\dot{V}_1$ negative definite, one choice is (37), which cancels all the non-linear terms inside the bracket in (74). However, the cancellation controller (37) is not (guaranteed to be) optimal. Therefore, we have to use other techniques in the design of our control law. One such technique we will use here is 'non-linear damping' (Krstić *et al.*, 1995).

Since $\tilde{\alpha}$, $\partial\tilde{\alpha}/\partial t$, $\tilde{f}$, $\tilde{F}$, $\partial V_a/\partial e$ and $\partial V_a/\partial \theta$ are smooth and vanish for $e = 0$, then we can write

$$\begin{aligned} &-\frac{\partial \tilde{\alpha}}{\partial t} + \frac{\partial V_a}{\partial e}g - \frac{\partial \tilde{\alpha}}{\partial e}(\tilde{f} + \tilde{F}\theta + g\tilde{\alpha}) \\ &- \left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} - \frac{\partial \tilde{\alpha}}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} \\ &- \left(\frac{\partial V_a}{\partial \theta} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial \tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}} \\ &= \Psi_1(t, e, \theta)^{\mathrm{T}}\,\Omega(t, e, \theta)^{1/2}\,e, \end{aligned} \tag{75}$$

where $\Psi_1(t, e, \theta)$ is a vector-valued smooth function and $\Omega(t, e, \theta)^{1/2}$ is invertible for all t, e and θ. In addition, let us denote

$$-\frac{\partial \tilde{\alpha}}{\partial e}g + \left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} - \frac{\partial \tilde{\alpha}}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial \tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}} = \Psi_2(t, e, \theta). \tag{76}$$

Then (74) is re-written as

$$\dot{V}_1 \le -|\Omega^{1/2}e|^2 + z\tilde{u} + z\Psi_1\Omega^{1/2}e + \Psi_2 z^2. \tag{77}$$

The choice

$$\tilde{u} = \tilde{\alpha}_1(t, e, \xi, \theta) = -\left(c + \frac{|\Psi_1|^2}{2} + \frac{\Psi_2^2}{2c}\right)z, \quad c > 0, \tag{78}$$

renders

$$\dot{V} \le -\tfrac{1}{2}|\Omega^{1/2}e|^2 - \frac{c}{2}z^2. \tag{79}$$

Since the control law $\tilde{u} = \tilde{\alpha}_1$ defined in (78) is of the form

$$\tilde{\alpha}_1(t, e, \xi, \theta) = -R^{-1}(t, e, \xi, \theta)\frac{\partial V_1}{\partial(e, \xi)}\begin{bmatrix}\mathbf{0}\\1\end{bmatrix}, \tag{80}$$

where

$$R^{-1}(t, e, \xi, \theta) = \left(c + \frac{|\Psi_1|^2}{2} + \frac{\Psi_2^2}{2c}\right) > 0, \quad \forall t, e, \xi, \theta, \tag{81}$$

by Theorem 2, the dynamic feedback control (adaptive control) $\tilde{u}^* = \beta\tilde{\alpha}_1(t, e, \xi, \hat{\theta})$, $\beta \ge 2$, with

$$\dot{\hat{\theta}} = \Gamma\tau_1(t, e, \xi, \hat{\theta}) = \Gamma\left(\frac{\partial V_1}{\partial e}F(t, e, \xi, \hat{\theta})\right)^{\mathrm{T}}, \tag{82}$$

is optimal for the closed-loop tracking error system (34) and (82). □

Example 4 (Example 2 revisited). For the system (42), we designed a controller (43) which is not optimal due to its cancellation property. With Lemma 2, we can design an optimal control as follows. First we note that $\tilde{\alpha}$ given by (28) in Example 1 is of the form

$$\begin{aligned}\tilde{\alpha}(t, e, \theta) &= -\left[1 + \varphi^{\mathrm{T}}\Gamma\varphi - \left(c_1 + \frac{\partial \tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\theta\right)c_1 + \frac{\partial \phi^{\mathrm{T}}}{\partial t}\theta\right]z_1 \\ &\quad - \left[c_2 + \left(c_1 + \frac{\partial \tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\theta\right)(1 + \varphi^{\mathrm{T}}\Gamma\varphi)\right]z_2 \\ &:= a(t, e, \theta)z_1 + b(t, e, \theta)z_2, \end{aligned} \tag{83}$$

because $\tilde{\varphi} = \varphi(x_1) - \varphi(x_{r1}) = \varphi(e_1 + x_{r1}) - \varphi(x_{r1}) = e_1\phi(e_1)$ and $\partial\tilde{\varphi}/\partial t = z_1(\partial\phi/\partial t)$. Instead of (43) we choose the 'non-linear damping' control suggested by Lemma 2:

$$\begin{aligned}\tilde{u} &= \tilde{\alpha}_1(t, e, \xi, \theta) \\ &= -\Big\{c_3 + \frac{1}{2c_1}\Big[\frac{\partial \tilde{\alpha}}{\partial e_1}c_1 - \frac{\partial a}{\partial t} - \frac{\partial \tilde{\alpha}}{\partial e_2}a \\ &\quad - \left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial e_2}\varphi_r^{\mathrm{T}}\right)\Gamma\varphi\Big]^2\end{aligned}$$

$$+\frac{1}{2c_2}\left[1+\frac{\partial\tilde{\alpha}}{\partial e_1}\varphi^{\mathrm{T}}\Gamma\varphi-\frac{\partial\tilde{\alpha}}{\partial e_1}-\frac{\partial b}{\partial t}\right.$$
$$-\frac{\partial\tilde{\alpha}}{\partial e_2}b-\left(\frac{\partial\alpha_r}{\partial\theta}+\frac{\partial\tilde{\alpha}}{\partial\theta}+\frac{\partial\tilde{\alpha}}{\partial e_2}\varphi_r^{\mathrm{T}}\right)\Gamma\varphi$$
$$\left.\times\left(c_1+\frac{\partial\tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\theta\right)\right]^2+\frac{1}{2c_3}\left(\frac{\partial\tilde{\alpha}}{\partial e_2}\right)^2\Bigg\}z_3. \quad (84)$$

The tuning function τ_1 is the same as in (44). The control law $\tilde{u}^*=\beta\tilde{\alpha}_1(t, e, \xi, \hat{\theta})$, $\beta\geq 2$, with $\dot{\hat{\theta}}=\Gamma\tau_1$ is optimal. □

7. DESIGN FOR STRICT-FEEDBACK SYSTEMS

We now consider the *parametric strict-feedback systems* (Krstić *et al.*, 1995):

$$\begin{aligned}\dot{x}_1&=x_{i+1}+\varphi_i(\bar{x}_i)^{\mathrm{T}}\theta,\quad i=1,\dots,n-1,\\ \dot{x}_n&=u+\varphi_n(x)^{\mathrm{T}}\theta\\ y&=x_1,\end{aligned}\quad (85)$$

where we use the compact notation $\bar{x}_i=(x_i,\dots,x_i)$, and develop a procedure for optimal adaptive tracking of a given signal $y_r(t)$. An inverse optimal design following from Corollary 4 (based on Sontag's formula) would be non-smooth at $e=0$. In this section we develop a design which is smooth everywhere. This design is also different from the non-optimal design in Krstić *et al.* (1992, 1995).

For the class of systems (85), assumption 1 is satisfied for any function $y_r(t)$, and there exist functions $\rho_1(t)$, $\rho_2(t,\theta),\dots,\rho_n(t,\theta)$, and $\alpha_r(t,\theta)$ such that

$$\begin{aligned}\dot{\rho}_i&=\rho_{i+1}+\varphi_i(\bar{\rho}_i)^{\mathrm{T}}\theta,\quad i=1,\dots,n-1,\\ \dot{\rho}_n&=\alpha_r(t,\theta)+\varphi_n(\rho)^{\mathrm{T}}\theta,\\ y_r(t)&=\rho_1(t).\end{aligned}\quad (86)$$

Consider the signal $x_r(t)=\rho(t,\hat{\theta})$, which is governed by

$$\begin{aligned}\dot{x}_{ri}&=x_{r,i+1}+\varphi_{ri}(\bar{x}_{ri})^{\mathrm{T}}\hat{\theta}+\frac{\partial\rho_i}{\partial\hat{\theta}}\dot{\hat{\theta}},\quad i=1,\dots,n-1,\\ \dot{x}_{rn}&=\alpha_r(t,\hat{\theta})+\varphi_{rn}(x_r)^{\mathrm{T}}\hat{\theta}+\frac{\partial\rho_n}{\partial\hat{\theta}}\dot{\hat{\theta}},\\ y_r&=x_{r1}.\end{aligned}\quad (87)$$

The tracking error $e=x-x_r$ is governed by the system

$$\begin{aligned}\dot{e}_i&=e_{i+1}+\tilde{\varphi}_i(t,\bar{e}_i,\hat{\theta})^{\mathrm{T}}\theta+\varphi_{ri}(t,\hat{\theta})^{\mathrm{T}}\tilde{\theta}-\frac{\partial\rho_i}{\partial\hat{\theta}}\dot{\hat{\theta}},\\ &\qquad i=1,\dots,n-1,\\ \dot{e}_n&=\tilde{u}+\tilde{\varphi}_n(t,e,\hat{\theta})^{\mathrm{T}}\theta+\varphi_{rn}(t,\hat{\theta})^{\mathrm{T}}\tilde{\theta}-\frac{\partial\rho_n}{\partial\hat{\theta}}\dot{\hat{\theta}},\end{aligned}\quad (88)$$

where $\tilde{u}=u-\alpha_r(t,\hat{\theta})$ and $\tilde{\varphi}_i=\varphi_i(\bar{x}_i)-\varphi_{ri}(\bar{x}_{ri})$, $i=1,\dots,n$. For an ATCLF V_a, the modified non-adaptive error system is

$$\begin{aligned}\dot{e}_i&=e_{i+1}+\tilde{\varphi}_i^{\mathrm{T}}\theta+\varphi_i^{\mathrm{T}}\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^{\mathrm{T}}-\frac{\partial\rho_i}{\partial\hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}},\\ &\qquad i=1,\dots,n-1,\\ \dot{e}_n&=\tilde{u}+\tilde{\varphi}_n(t,e,\hat{\theta})^{\mathrm{T}}\theta\\ &\quad+\varphi_n^{\mathrm{T}}\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^{\mathrm{T}}-\frac{\partial\rho_n}{\partial\hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}},\end{aligned}\quad (89)$$

where $F=[\varphi_1^{\mathrm{T}},\dots,\varphi_n^{\mathrm{T}}]^{\mathrm{T}}$.

First, we search for an ATCLF for the system (85). Repeated application of Lemma 1 gives an ATCLF

$$\begin{aligned}V_a&=\tfrac{1}{2}\sum_{i=1}^{n}z_i^2,\\ z_i&=e_i-\tilde{\alpha}_{i-1}(t,\bar{e}_{i-1},\theta),\end{aligned}\quad (90)$$

where the $\tilde{\alpha}_i$ terms are to be determined. For notational convenience we define $z_0:=0$ and $\tilde{\alpha}_0:=0$. We then have

$$\frac{\partial V_a}{\partial\theta}=-\sum_{j=1}^{n}\frac{\partial\tilde{\alpha}_{j-1}}{\partial\theta}z_j \quad (91)$$

$$\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}=\sum_{j=1}^{n}\frac{\partial V_a}{\partial e_j}\varphi_j=\sum_{j=1}^{n}\left(z_j-\sum_{k=j+1}^{n}\frac{\partial\tilde{\alpha}_{k-1}}{\partial e_j}z_k\right)\varphi_j$$
$$=\sum_{j=1}^{n}w_jz_j, \quad (92)$$

where

$$w_j(t,\bar{e}_j,\theta)=\varphi_j-\sum_{k=1}^{j-1}\frac{\partial\tilde{\alpha}_{j-1}}{\partial e_k}\varphi_k. \quad (93)$$

Therefore, the modified non-adaptive error system (89) becomes

$$\begin{aligned}\dot{e}_i&=e_{i+1}+\tilde{\varphi}_i^{\mathrm{T}}\theta-\sum_{j=1}^{n}\frac{\partial\tilde{\alpha}_{j-1}}{\partial\theta}\Gamma\varphi_iz_j-\sum_{j=1}^{n}\frac{\partial\rho_i}{\partial\theta}\Gamma w_jz_j,\\ &\qquad i=1,\dots,n-1,\\ \dot{e}_n&=\tilde{u}+\tilde{\varphi}_n^{\mathrm{T}}\theta-\sum_{j=1}^{n}\frac{\partial\tilde{\alpha}_{j-1}}{\partial\theta}\Gamma\varphi_nz_j-\sum_{j=1}^{n}\frac{\partial\rho_n}{\partial\theta}\Gamma w_jz_j.\end{aligned}\quad (94)$$

The functions $\tilde{\alpha}_1,\dots,\tilde{\alpha}_{n-1}$ are yet to be determined to make V_a defined in (90) a CLF for system (94). To design these functions, we apply the backstepping technique as in Krstić *et al.* (1995). We perform cancellations at all the steps before step n. At the final step n, we depart from Krstić *et al.* (1995) and choose the actual control $\tilde{u}$ in a form which, according to Lemma 2, is inverse optimal.

Step $i = 1, \ldots, n-1$.

$$\tilde{\alpha}_i(t, \bar{e}_i, \theta) = -z_{i-1} - c_i z_i + \frac{\partial \tilde{\alpha}_{i-1}}{\partial t} + \sum_{k=1}^{i-1} \frac{\partial \tilde{\alpha}_{i-1}}{\partial e_k} e_{k+1} - \tilde{w}_i^{\mathrm{T}} \theta - \sum_{k=1}^{i-1} (\sigma_{ki} + \sigma_{ik}) z_k - \sigma_{ii} z_i, \quad c_i > 0, \quad (95)$$

$$\tilde{w}_i(t, \bar{e}_i, \theta) = \bar{\varphi}_i - \sum_{k=1}^{i-1} \frac{\partial \tilde{\alpha}_{i-1}}{\partial e_k} \bar{\varphi}_k, \quad (96)$$

$$\sigma_{ik} = -\left(\frac{\partial \tilde{\alpha}_{i-1}}{\partial \theta} + \frac{\partial \rho_i}{\partial \theta} - \sum_{j=2}^{i-1} \frac{\partial \tilde{\alpha}_{i-1}}{\partial e_j} \frac{\partial \rho_j}{\partial \theta} \right) \Gamma w_k. \quad (97)$$

Step n. With the help of Lemma A1 in the Appendix, the derivative of V_a is

$$\dot{V}_a = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n \left[z_{n-1} + \sum_{k=1}^{n-1} (\sigma_{kn} + \sigma_{nk}) z_k + \sigma_{nn} z_n + \tilde{u} - \frac{\partial \tilde{\alpha}_{n-1}}{\partial t} - \sum_{k=1}^{n-1} \frac{\partial \tilde{\alpha}_{n-1}}{\partial e_k} e_{k+1} + \tilde{w}_n^{\mathrm{T}} \theta \right]. \quad (98)$$

We are now at the position to choose the actual control $\tilde{u}$. We may choose $\tilde{u}$ such that all the terms inside the bracket are cancelled and the bracketed term multiplying z_n is equal to $-c_n z_n^2$, as in Krstić *et al.* (1995), but the controller designed in that way is not guaranteed to be inverse optimal. To design a controller which is inverse optimal, according to Theorem 2, we should choose a control law of the form

$$\tilde{u} = \tilde{\alpha}_n(t, e, \theta) = -r^{-1}(t, e, \theta) \frac{\partial V_a}{\partial e} g, \quad (99)$$

where $r(t, e, \theta) > 0$, $\forall t, e, \theta$. In light of (94) and (90), (99) simplifies to

$$\tilde{u} = \tilde{\alpha}_n(t, e, \theta) = -r^{-1}(t, e, \theta) z_n, \quad (100)$$

i.e. we must choose $\tilde{\alpha}_n$ with z_n as a factor.

Since $e_{k+1} = z_{k+1} + \tilde{\alpha}_k$ $(k = 1, \ldots, n-1)$, and the expression in the second line in (98) vanishes at $e = 0$, it is easy to see that it also vanishes for $z = 0$. Therefore, there exist smooth functions ϕ_k $(k = 1, \ldots, n)$ such that

$$-\frac{\partial \tilde{\alpha}_{n-1}}{\partial t} - \sum_{k=1}^{n-1} \frac{\partial \tilde{\alpha}_{n-1}}{\partial e_k} e_{k+1} + \tilde{w}_n^{\mathrm{T}} \theta = \sum_{k=1}^{n} \phi_k z_k. \quad (101)$$

Thus (98) becomes

$$\dot{V}_a = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n \tilde{u} + \sum_{k=1}^{n} z_n \Phi_k z_k, \quad (102)$$

where

$$\Phi_k = \sigma_{kn} + \sigma_{nk} + \phi_k, \quad k = 1, \ldots, n-2,$$
$$\Phi_{n-1} = 1 + \sigma_{n-1,n} + \sigma_{n,n-1} + \phi_{n-1}, \quad (103)$$
$$\Phi_n = \sigma_{nn} + \phi_n.$$

A control law of the form (100) with

$$r(t, e, \theta) = \left(c_n + \sum_{k=1}^{n} \frac{\Phi_k^2}{2c_k} \right)^{-1} > 0, \quad c_k > 0, \quad \forall t, e, \theta, \quad (104)$$

results in

$$\dot{V}_a = -\tfrac{1}{2} \sum_{k=1}^{n} c_k z_k^2 - \sum_{k=1}^{n} \frac{c_k}{2} \left(z_k - \frac{\Phi_k}{c_k} z_n \right)^2. \quad (105)$$

By Theorem 2, the inverse optimal adaptive tracking problem is solved through the dynamic feedback control (adaptive control) law:

$$\tilde{u} = \tilde{\alpha}_n^*(t, e, \hat{\theta}) = 2\tilde{\alpha}_n(t, e, \hat{\theta}),$$
$$\dot{\hat{\theta}} = \Gamma \left(\frac{\partial V_a}{\partial e} F \right)^{\mathrm{T}} = \Gamma \sum_{j=1}^{n} w_j z_j. \quad (106)$$

8. TRANSIENT PERFORMANCE

In this brief section, we give an $\mathscr{L}_2$ bound on the error state z and control $\tilde{u}$ for the inverse optimal adaptive controller designed in Section 7. According to Theorem 2, the control law (106) is optimal with respect to the cost functional

$$J = 2 \lim_{t \to \infty} |\theta - \hat{\theta}(t)|_{\Gamma^{-1}}^2 + 2 \int_0^\infty \left[\sum_{k=1}^{n} c_k z_k^2 + \sum_{k=1}^{n} c_k \left(z_k - \frac{\Phi_k}{c_k} z_n \right)^2 + \frac{\tilde{u}^2}{2\left(c_n + \sum_{k=1}^{n} \frac{\Phi_k^2}{2c_k} \right)} \right] \mathrm{d}t \quad (107)$$

with a value function

$$J^* = 2 |\theta - \hat{\theta}|_{\Gamma^{-1}}^2 + 2 |z|^2. \quad (108)$$

In particular, we have the following $\mathscr{L}_2$ performance result.

Theorem 3. In the adaptive system (88) and (106), the following inequality holds;

$$\int_0^\infty \left[\sum_{k=1}^{n} c_k z_k^2 + \frac{\tilde{u}^2}{2\left(c_n + \sum_{k=1}^{n} \frac{\Phi_k^2}{2c_k} \right)} \right] \mathrm{d}t \le |\tilde{\theta}(0)|_{\Gamma^{-1}}^2 + |z(0)|^2. \quad (109)$$

□

This theorem presents the first performance bound in the adaptive control literature that includes an estimate of *control effort*.

The bound (109) depends on $z(0)$, which is dependent on the design parameters $c_1, \dots, c_n$, and Γ. To eliminate this dependency and allow a systematic improvement of the bound on $\|z\|_2$, we employ *trajectory initialization* as in Krstić *et al.* (1995, section 4.3.2) to set $z(0) = 0$ and obtain

$$\int_0^\infty \left[\sum_{k=1}^n c_k z_k^2 + \frac{\tilde{u}^2}{2\left(c_n + \sum_{k=1}^n \frac{\Phi_k^2}{2c_k}\right)} \right] dt \le |\tilde{\theta}(0)|^2_{\Gamma^{-1}}. \tag{110}$$

9. CONCLUSIONS

In this paper we have shown that the solvability of the inverse optimal adaptive control problem for a given system is implied by the solvability of the HJB (non-adaptive) equation for a modified system. Our results can be readily extended to the multi-input case and the case where the input vector field depends on the unknown parameter.

In constructing an inverse optimal adaptive controller for strict-feedback systems, we followed the simplest path of using an ATCLF designed by the tuning functions method in Krstić *et al.* (1995). Numerous other (possibly better) choice are possible, including an inverse optimal adaptive design at each step of backstepping. The relative merit of different approaches is yet to be established.

REFERENCES

Arstein, Z. (1983). Stabilization with relaxed controls. *Nonlinear Anal.*, **TMA-7**, 1163–1173.

Didinsky, G. and Başar, T. (1994). Minimax adaptive control of uncertain plants. In *Proc. 33rd IEEE Conf. on Decision and Control*, Lake Buena Vista, FL, pp. 2839–2844.

Freeman, R. A. and Kokotović, P. V. (1996a). Inverse optimality in robust stabilization. *SIAM J. Control Optim.*, **34**, 1365–1391.

Freeman, R. A. and Kokotović, P. V. (1996b). *Robust Nonlinear Control Design*. Birkhäuser, Boston.

Ioannou, P. A. and Sun, J. (1995). *Robust Adaptive Control*. Prentice-Hall, Englewood Cliffs, NJ.

Krstić, M. (1996). Invariant manifolds and asymptotic properties of adaptive nonlinear systems. *IEEE Trans. Autom. Control*, **AC-41**, 817–829.

Krstić, M. and Kokotović, P. V. (1995). Control Lyapunov functions for adaptive nonlinear stabilization. *Syst. Control Lett.*, **26**, 17–23.

Krstić, M., Kanellakopoulos, I. and Kokotović, P. V. (1992). Adaptive nonlinear control without overparametrization. *Syst. Control Lett.*, **19**, 177–185.

Krstić, M., Kanellakopoulos, I. and Kokotović, P. V. (1995). *Nonlinear and Adaptive Control Design*. Wiley, New York.

Li, Z. H. and Krstić, M. (1996). Geometric/asymptotic properties of adaptive nonlinear systems with partial excitation. In *Proc. 35th IEEE Conf. on Decision and Control*, Kobe, Japan, pp. 4683–4688. [To appear in *IEEE Trans. Autom. Control*.]

Marino, R. and Tomei, P. (1995). *Nonlinear Control Design: Geometric, Adaptive, and Robust*. Prentice-Hall, London.

Pan, Z. and Başar, T. (1996). Adaptive controller design for tracking and disturbance attenuation in parametric-strict-feedback nonlinear systems. In *Proc. 13th IFAC Congress on Automatic Control*, San Francisco, CA, Vol. F, pp. 323–328.

Sepulchre, R., Jankovic, M. and Kokotović, P. V. (1997). *Constructive Nonlinear Control*. Springer-Verlag, New York.

Sontag, E. D. (1983). A Lyapunov-like characterization of asymptotic controllability. *SIAM J. Control Optim.*, **21**, 462–471.

Sontag, E. D. (1989). A 'universal' construction of Artstein's theorem on nonlinear stabilization. *Syst. Control Lett.*, **13**, 117–123.

APPENDIX: A TECHNICAL LEMMA

Lemma A1. The time derivative of V_a in (90) along the solutions of system (94) with (95)–(97) is given by

$$\dot{V}_a = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n \left[z_{n-1} + \sum_{k=1}^{n-1} (\sigma_{kn} + \sigma_{nk}) z_k + \sigma_{nn} z_n + \tilde{u} - \frac{\partial \tilde{\alpha}_{n-1}}{\partial t} - \sum_{k=1}^{n-1} \frac{\partial \tilde{\alpha}_{n-1}}{\partial e_k} e_{k+1} + \tilde{w}_n^{\mathrm{T}} \theta \right]. \tag{A.1}$$

Proof. First we prove that the closed-loop system after i steps is

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} -c_1 & 1+\pi_{12} & \pi_{13} & \cdots & & \pi_{1,i-1} & \pi_{1i} \\ -1-\pi_{12} & -c_2 & 1+\pi_{23} & \cdots & & \pi_{2,i-1} & \pi_{2i} \\ -\pi_{13} & -1-\pi_{23} & \ddots & & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & & \ddots & 1+\pi_{i-2,i-1} & \pi_{i-2,i} \\ -\pi_{1,i-1} & -\pi_{2,i-1} & \cdots & -1-\pi_{i-2,i-1} & & -c_{i-1} & 1+\pi_{i-1,i} \\ -\pi_{1i} & -\pi_{2i} & \cdots & -\pi_{i-2,i} & & -1-\pi_{i-1,i} & -c_i \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix} + \begin{bmatrix} \pi_{1,i+1} & \pi_{1,i+2} & \cdots & \pi_{1n} \\ \pi_{2,i+1} & \pi_{2,i+2} & \cdots & \pi_{2n} \\ \vdots & & & \\ \pi_{i-1,i+1} & \pi_{i-1,i+2} & \cdots & \pi_{i-1,n} \\ 1+\pi_{i,i+1} & \pi_{i,i+2} & \cdots & \pi_{i,n} \end{bmatrix} \begin{bmatrix} z_{i+1} \\ \vdots \\ z_n \end{bmatrix}, \tag{A.2}$$

and the resulting $\dot{V}_i$ is

$$\dot{V}_i = -\sum_{k=1}^{i} c_k z_k^2 + z_i z_{i+1} + \sum_{j=i+1}^{n} z_j \sum_{k=1}^{i} \pi_{kj} z_k, \tag{A.3}$$

where

$$\pi_{ik} = \eta_{ik} + \xi_{ik} \tag{A.4}$$

$$\eta_{ik} = -\frac{\partial \tilde{\alpha}_{k-1}}{\partial \theta} \Gamma w_i \tag{A.5}$$

$$\xi_{ik} = -\left(\frac{\partial \rho_i}{\partial \theta} - \sum_{j=2}^{i-1} \frac{\partial \tilde{\alpha}_{k-1}}{\partial e_j} \frac{\partial \rho_j}{\partial \theta} \right) \Gamma w_k. \tag{A.6}$$

The proof is by induction.

Step 1: Substituting $\tilde{\alpha}_1 = -c_1 z_1 - \bar{\varphi}_1^{\mathrm{T}}\theta$ into (94) with $i = 1$, using (90) and noting $\tilde{\alpha}_0 = 0$ and $\partial\rho_1/\partial\theta = 0$, we get

$$\dot{z}_1 = -c_1 z_1 + z_2 - \sum_{j=1}^{n} z_j \frac{\partial\tilde{\alpha}_{j-1}}{\partial\theta}\Gamma w_1 = -c_1 z_1 + z_2 - \pi_{12}z_3 - \ldots - \pi_{1n}z_n, \tag{A.7}$$

and

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 - \sum_{j=1}^{n} z_j \frac{\partial\tilde{\alpha}_{j-1}}{\partial\theta}\Gamma w_1 z_1 = -c_1 z_1^2 + z_1 z_2 - \sum_{j=2}^{n} z_j \pi_{1j} z_1, \tag{A.8}$$

which shows that (A.2) and (A.3) are true for $i = 1$.

Assume that (A.2) and (A.3) are true for $i - 1$, that is,

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_{i-1} \end{bmatrix} = \begin{bmatrix} -c_1 & 1+\pi_{12} & \pi_{13} & \cdots & \pi_{1,i-2} & \pi_{1,i-1} \\ -1-\pi_{12} & -c_2 & 1+\pi_{23} & \cdots & \pi_{2,i-2} & \pi_{2,i-1} \\ -\pi_{13} & -1-\pi_{23} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1+\pi_{i-2,i-2} & \pi_{i-2,i-1} \\ -\pi_{1,i-2} & -\pi_{2,i-2} & \cdots & -1-\pi_{i-3,i-2} & -c_{i-2} & 1+\pi_{i-2,i-1} \\ -\pi_{1,i-1} & -\pi_{2,i-1} & \cdots & -\pi_{i-3,i-1} & -1-\pi_{i-2,i-1} & -c_{i-1} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix} + \begin{bmatrix} \pi_{1,i} & \pi_{1,i+1} & \cdots & \pi_{1n} \\ \pi_{2,i} & \pi_{2,i+1} & \cdots & \pi_{2n} \\ \vdots & & & \\ \pi_{i-2,i} & \pi_{i-2,i+1} & \cdots & \pi_{i-2,n} \\ 1+\pi_{i-1,i} & \pi_{i-1,i+1} & \cdots & \pi_{i-1,n} \end{bmatrix} \begin{bmatrix} z_i \\ \vdots \\ z_n \end{bmatrix}, \tag{A.9}$$

and

$$\dot{V}_{i-1} = -\sum_{k=1}^{i-1} c_k z_k^2 + z_{i-1}z_i + \sum_{j=i}^{n} z_j \sum_{k=1}^{i-1} \pi_{kj} z_k. \tag{A.10}$$

The z_i subsystem is given by

$$\begin{aligned} \dot{z}_i &= z_{i+1} + \tilde{\alpha}_i + \bar{\varphi}_i^{\mathrm{T}}\theta - \sum_{j=1}^{n} \frac{\partial\tilde{\alpha}_{j-1}}{\partial\theta}\Gamma\varphi_i z_j - \sum_{j=1}^{n} \frac{\partial\rho_i}{\partial\theta}\Gamma w_j z_j \\ &\quad - \frac{\partial\tilde{\alpha}_{i-1}}{\partial t} - \sum_{k=1}^{i-1} \frac{\partial\tilde{\alpha}_{i-1}}{\partial e_k}\left(e_{k+1} + \bar{\varphi}_k^{\mathrm{T}}\theta - \sum_{j=1}^{n} \frac{\partial\tilde{\alpha}_{j-1}}{\partial\theta}\Gamma\varphi_k z_j - \sum_{j=1}^{n} \frac{\partial\rho_k}{\partial\theta}\Gamma w_j z_j\right) \\ &= z_{i+1} + \tilde{\alpha}_i - \frac{\partial\tilde{\alpha}_{i-1}}{\partial t} - \sum_{k=1}^{i-1} \frac{\partial\tilde{\alpha}_{i-1}}{\partial e_k} e_{k+1} + \tilde{w}_i^{\mathrm{T}}\theta + \sum_{j=1}^{n} \pi_{ij} z_j. \end{aligned} \tag{A.11}$$

The derivative of $V_i = V_{i-1} + \frac{1}{2}z_i^2$ is calculated as

$$\begin{aligned} \dot{V}_1 &= -\sum_{k=1}^{i-1} c_k z_k^2 + z_i z_{i+1} + \sum_{j=i+1}^{n} z_j \sum_{k=1}^{i-1} \pi_{kj} z_k + \sum_{j=i+1}^{n} z_j \pi_{ij} z_i \\ &\quad + z_i\left[z_{i-1} + \sum_{k=1}^{i-1} \pi_{ki} z_k + \tilde{\alpha}_i - \frac{\partial\tilde{\alpha}_{i-1}}{\partial t} - \sum_{k=1}^{i-1} \frac{\partial\tilde{\alpha}_{i-1}}{\partial e_k} e_{k+1} + \tilde{w}_i^{\mathrm{T}}\theta + \sum_{k=1}^{i} \pi_{ik} z_k\right] \\ &= -\sum_{k=1}^{i-1} c_k z_k^2 + z_i z_{i+1} + \sum_{j=i+1}^{n} z_j \sum_{k=1}^{i} \pi_{kj} z_k \\ &\quad + z_i\left[\tilde{\alpha}_i + z_{i-1} - \frac{\partial\tilde{\alpha}_{i-1}}{\partial t} - \sum_{k=1}^{i-1} \frac{\partial\tilde{\alpha}_{i-1}}{\partial e_k} e_{k+1} + \tilde{w}_i^{\mathrm{T}}\theta + \sum_{k=1}^{i-1} (\pi_{ki} + \pi_{ik}) z_k + \pi_{ii} z_i\right]. \end{aligned} \tag{A.12}$$

From the definitions of π_{ik}, η_{ik}, δ_{ik} and σ_{ik}, it is easy to show that $\pi_{ki} + \pi_{ik} = \sigma_{ki} + \sigma_{ik}$ and that $\pi_{ii} = \sigma_{ii}$. Then the choice of $\tilde{\alpha}_i$ as in (95) results in (A.3) and

$$\dot{z}_i = -\sum_{k=1}^{i-2} \pi_{ki} z_k - (1 + \pi_{i-1,i}) z_{i-1} - c_i z_i + (1 + \pi_{i,i+1}) z_{i+1} + \sum_{k=i+2}^{n} \pi_{ik} z_k. \tag{A.13}$$

Combining (A.13) with (A.9), we get (A.2).

We now rewrite the last equation of (94) as

$$\begin{aligned} \dot{z}_n &= \tilde{u} + \bar{\varphi}_n^{\mathrm{T}}\theta - \sum_{j=1}^{n} \frac{\partial\tilde{\alpha}_{j-1}}{\partial\theta}\Gamma\varphi_n z_j - \sum_{j=1}^{n} \frac{\partial\rho_n}{\partial\theta}\Gamma w_j z_j - \frac{\partial\tilde{\alpha}_{n-1}}{\partial t} - \sum_{k=1}^{n-1} \frac{\partial\tilde{\alpha}_{n-1}}{\partial e_k}\left(e_{k+1} + \bar{\varphi}_k^{\mathrm{T}}\theta - \sum_{j=1}^{n} \frac{\partial\tilde{\alpha}_{j-1}}{\partial\theta}\Gamma\varphi_k z_j - \sum_{j=1}^{n} \frac{\partial\rho_k}{\partial\theta}\Gamma w_j z_j\right) \\ &= \tilde{u} - \frac{\partial\tilde{\alpha}_{n-1}}{\partial t} - \sum_{k=1}^{n-1} \frac{\partial\tilde{\alpha}_{n-1}}{\partial e_k} e_{k+1} + \tilde{w}_n^{\mathrm{T}}\theta + \sum_{j=1}^{n} \pi_{nj} z_j, \end{aligned} \tag{A.14}$$

where $\tilde{w}_n$ follows the same definition as in (96). Noting that $V_a = V_{n-1} + \frac{1}{2}z_n^2$ and $\pi_{kn} + \pi_{nk} = \sigma_{kn} + \sigma_{nk}$, and $\pi_{nn} = \sigma_{nn}$, (A.1) follows readily from (A.3) and (A.14). □