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GLOBAL STABILIZATION OF FEEDFORWARD SYSTEMS UNDER PERTURBATIONS IN SAMPLING SCHEDULE*

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Abstract. For nonlinear systems that are known to be globally asymptotically stabilizable, control over networks introduces a major challenge because of the asynchrony in the transmission schedule. Maintaining global asymptotic stabilization in sampled-data implementations with zero-order hold and with perturbations in the sampling schedule is not achievable in general, but we show in this paper that it is achievable for the class of feedforward systems. We develop sampled-data feedback stabilizers which are not approximations of continuous-time designs but are discontinuous feedback laws that are specifically developed for maintaining global asymptotic stabilizability under any sequence of sampling periods that is uniformly bounded by a certain "maximum allowable sampling period."

Key words. nonlinear systems, feedforward systems, sampled-data control

AMS subject classifications. 93C10, 93C57, 93D15, 93D21

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1. Introduction.

Achieving stabilization by sampled-data feedback and ensuring robustness to perturbations in the sampling schedule are the central challenges in nonlinear control over networks, where asynchrony is ubiquitous. In this paper we achieve these goals for the class of uncertain feedforward systems, for which these goals are achievable due to the absence of exponential and finite escape time instabilities, despite the presence of nonlinearities of superlinear growth. We propose a saturation-based forwarding feedback, which we construct specifically for the sampled-data problem (namely, not as an approximation of a continuous design) and in such a way that it guarantees robustness of global asymptotic stability to perturbations in the sampling schedule.

Feedforward systems. Research on feedforward systems has played an important role in the development of nonlinear control theory, starting with the introduction of this class and the first feedback laws by Teel [54], followed by the key advances by Mazenc and Praly [31] and Jankovic, Sepulchre, and Kokotovic [15], and continuing with various extensions and generalizations by many authors [1, 2, 3, 4, 7, 8, 12, 13, 14, 16, 24, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40, 46, 47, 48, 49, 50, 51, 52, 53, 55, 56, 57] . More recently, feedforward systems with input delays and/or measurement delays have been studied [5, 6, 21, 25].

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Results of this paper. In this work we focus on the problem of robust global stabilization of uncertain feedforward systems of the form

$$\begin{aligned}
\dot{x}_1 &= u, \\
\dot{x}_2 &= x_1 + g_2(d, x_1, u) \\
&\vdots \\
\dot{x}_{n-1} &= x_{n-2} + g_{n-1}(d, x_1, \ldots, x_{n-2}, u), \\
\dot{x}_n &= x_{n-1} + g_n(d, x_1, \ldots, x_{n-1}, u), \\
x &= (x_1, \ldots, x_n)' \in \Re^n,\ u \in \Re,\ d \in D,
\end{aligned} \tag{1.1}$$

where $D \subset \Re^l$ is a nonempty compact set and all mappings $g_i : D \times \Re^{i-1} \times \Re \to \Re$ $(i = 2, \ldots, n)$ are locally Lipschitz and such that there exists a smooth nondecreasing function $L \in C^0(\Re_+; \Re_+)$ with the following property:

$$\begin{aligned}
\left|g_i(d, \underline{x}_{i-1}, u)\right| &\leq L\left(\left|(\underline{x}_{i-1}, u)\right|\right)\left|\underline{x}_{i-1}\right|^2 + L\left(\left|(\underline{x}_{i-1}, u)\right|\right)\left|\underline{x}_{i-1}\right||u|, \\
\underline{x}_{i-1} &:= (x_1, \ldots, x_{i-1})' \in \Re^{i-1}\ \forall (d, x, u) \\
&\in D \times \Re^n \times \Re \text{ and } i = 2, \ldots, n.
\end{aligned} \tag{1.2}$$

More specifically, we solve the problem of robust global stabilization of (1.1) by means of bounded sampled-data feedback control applied with zero-order hold, i.e., with a controller of the form

$$\begin{aligned}
&u(t) = k(x(\tau_i)),\ t \in [\tau_i, \tau_{i+1}), \\
&\tau_{i+1} = \tau_i + r\exp(-w(\tau_i)),\ \tau_0 = 0, \\
&w(t) \in \Re^+,
\end{aligned} \tag{1.3}$$

where $r > 0$ is a constant (the maximum allowable sampling period (MASP); see [45]) and $k : \Re^n \to \Re$ is a bounded function with $k(0) = 0$. The input $w : \Re_+ \to \Re_+$ represents possible perturbations in the sampling schedule. More specifically, the input $w : \Re_+ \to \Re_+$ allows the consideration of all sampling schedules which are partitions of $\Re_+$ with upper diameter $r > 0$, i.e., allows the consideration of all increasing sequences $\{\tau_i\}_{i=0}^{\infty}$ with $\lim \tau_i = +\infty$ and $\sup_{i=0,1,\ldots}(\tau_{i+1} - \tau_i) \leq r$.

Global stabilization under sampling. The problem of global stabilization under sampling is important for real-time implementations of control of feedforward systems, especially over networks, and to the best of our knowledge has not been addressed so far. The literature on sampled-data control provides control design methodologies that guarantee global stability for the following cases:

1. Linear stabilizable systems, $\dot{x} = Ax + Bu$, where $x \in \Re^n$, $u \in \Re^m$, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.
2. Nonlinear systems of the form $\dot{x} = f(x) + g(x)u$, $x \in \Re^n, u \in \Re$, where the vector field $f : \Re^n \to \Re^n$ is globally Lipschitz and the vector field $g : \Re^n \to \Re^n$ is locally Lipschitz and bounded, which can be stabilized by a globally Lipschitz feedback law $u = k(x)$ (see [11]).
3. Nonlinear systems of the form $\dot{x}_i = f_i(x, u) + g_i(x, u)x_{i+1}$ for $i = 1, \ldots, n-1$ and $\dot{x}_n = f_n(x, u) + g_n(x, u)u$, where the drift terms $f_i(x, u)$ $(i = 1, \ldots, n)$ satisfy the linear growth conditions $|f_i(x, u)| \leq L|x_1| + \cdots + L|x_i|$ $(i = 1, \ldots, n)$ for a certain constant $L \geq 0$ and there exist constants $b \geq a > 0$ such that $a \leq g_i(x, u) \leq b$ for all $i = 1, \ldots, n$, $x \in \Re^n, u \in \Re$ (see [19]).
4. Asymptotically controllable homogeneous systems with positive minimal power and zero degree (see [9]).

5. Systems satisfying the reachability hypotheses of Theorem 3.1 in [20] or hypotheses (A1), (A2), (A3) in section 4 of [18].
6. Nonlinear systems $\dot{x} = f(x, u)$, for which there exists a global diffeomorphism $\Theta : \Re^n \to \Re^n$ such that the change of coordinates $z = \Theta(x)$ transforms the system into one of the above cases.

However, nonlinear feedforward systems of the form (1.1) under hypothesis (1.2) rarely belong to one of the above classes (an exception is the class of linearizable feedforward systems; see [24, 49, 50, 51, 52]). On the other hand, there are well-established standard control design methodologies that guarantee stabilization of system (1.1) under sampled-data control with zero-order hold [10, 22, 23, 26, 27, 41, 42, 43, 44], but only in the practical and semiglobal sense. Therefore, the problem of robust global stabilization of (1.1) by means of bounded sampled-data feedback control applied with zero-order hold is open. Moreover, it was recently shown that the combination of a robust global sampled-data stabilizer with predictor schemes achieves global stabilization for systems with input and measurement delays [21]. Consequently, the solution of the problem of robust global stabilization of (1.1) by means of bounded sampled-data feedback control applied with zero-order hold automatically yields the solution of the same problem even in the presence of arbitrary measurement and input delays.

Relation with other approaches. The emulation-based feedback laws, or the feedback laws based on approximate discrete-time model design [10, 22, 23, 26, 27, 41, 42, 43, 44] for sampled-data stabilization of nonlinear systems guarantee semiglobal practical stability and exploit the wealth of feedback designs available in the continuous-time and the discrete-time framework. However, if global and nonpractical stabilization is the goal, then these approaches face the following theoretical obstructions when combined with the two basic design strategies for control of feedforward systems:

1. The *integrator forwarding* approach [15, 24, 31, 48] yields stabilizing feedback laws $u = k(x)$ that have (in general) superlinear growth, i.e., $\sup_{x \neq 0} \frac{|k(x)|}{|x|} = +\infty$. Moreover, the control Lyapunov functions constructed by using the integrator forwarding approach demand (in general) unbounded control inputs. Feedback laws with superlinear growth would not yield (in general) global stabilization in an emulation-based sampled-data implementation (even the scalar system $\dot{x} = u$ cannot be globally stabilized using any feedback law $u = k(x)$ of superlinear growth if the sampling rate is positive).
2. The *nested saturation* approach [31, 53, 54] results in bounded stabilizing feedback laws $u = k(x)$, but in many cases it does not yield a Lyapunov function for the closed-loop system.

It may appear from the emulation approach or the approximate discrete-time model design, which both guarantee semiglobal practical stability, that the loss of globality and attractivity of the origin is an unavoidable consequence of sampling. In fact, this is a consequence of not taking sampling into account in the design process.

With the approach developed in this paper, in which we take sampling into account in the design process, we not only prevent the loss of globality and of attractivity of the origin, but as a bonus also guarantee robustness to perturbations in the sampling schedule. Our approach results in the design of sampled-data feedback laws that are similar to the feedback laws designed by means of the nested saturation approach. Therefore, the results of the paper can be viewed as the extension of the nested saturation approach to the sampled-data case with zero-order hold.

Approach developed in the paper. The key result of the present work is the "sampled-data forwarding lemma" (Lemma 3.1 below). The sampled-data forwarding lemma deals with a composite system that consists of two subsystems, the x-subsystem

$$\dot{x} = F(d, x, u), \quad x \in \Re^n,\ d \in D,\ u \in \Re, \tag{1.4}$$

and the scalar y-subsystem

$$\dot{y} = G(d, x, u),\ y \in \Re. \tag{1.5}$$

Assuming that the x-subsystem can be robustly globally stabilized by means of sampled-data control applied with zero-order hold, under appropriate conditions on the mappings F, G we are in a position to construct a stabilizing feedback for the interconnected system (1.4), (1.5). The intuition behind the sampled-data forwarding lemma is as follows: we would like to bring the state (x, y) of system (1.4), (1.5) to a neighborhood of the origin, where the linearization of (1.4), (1.5) prevails, and keep it there using a linear feedback strategy. In order to achieve this objective, we first apply the sampled-data feedback stabilizer for (1.4), which brings the x-component of the state (x, y) close to zero. Once we have brought x close to zero, we keep x close to zero while simultaneously driving y close to zero. Having brought (x, y) to an appropriate neighborhood of zero, we apply linear feedback to drive (x, y) to zero. A set of technical conditions is assumed in order to guarantee that this control strategy is feasible.

The sampled-data forwarding lemma provides an explicit formula for the robust feedback stabilizer and can be applied recursively for the robust global sampled-data stabilization of feedforward systems (Theorem 3.7 below). Moreover, if the assumed feedback stabilizer for (1.4) is bounded, then the constructed feedback stabilizer for (1.4), (1.5) is bounded too. Robustness to perturbations in the sampling schedule is guaranteed by treating $x(\tau_i)$, where τ_i is a sampling time, as a perturbation of the current value of the state $x(t)$: by restricting the MASP $r > 0$, we are in a position to guarantee that $|x(t) - x(\tau_i)|$ is sufficiently small. The same methodology was introduced in the first author's papers [19, 20], where robustness to perturbations in the sampling schedule and global stabilization were achieved for certain classes of nonlinear systems.

Organization of the paper. The structure of the paper is as follows. Section 2 provides the stability notions used in the paper and some technical results. Section 3 contains the sampled-data forwarding lemma (Lemma 3.1 below), which is applied recursively for the stabilization of (1.1). The main result (Theorem 3.7) guarantees the solvability of the problem of robust global stabilization of (1.1) by means of bounded sampled-data feedback control applied with zero-order hold. The formulae for the feedback stabilizers for feedforward systems contain parameters which can be tuned in order to guarantee good performance. A three-dimensional feedforward example is presented in section 4 of the paper, which shows the importance of proper selection of the values of the parameters. Moreover, an additional example in section 4 indicates that the sampled-data forwarding lemma is not restricted to feedforward systems. Section 5 contains the concluding remarks of the paper. Finally, the appendix contains the proofs of all technical lemmas appearing in section 3.

Notation. Throughout this paper we adopt the following notation:

(i) Let $I \subseteq \Re_+ := [0, +\infty)$ be an interval. By $\mathrm{L}^\infty(I; U)$ ($\mathrm{L}^\infty_{loc}(I; U)$) we denote the space of measurable and (locally) essentially bounded functions $u(\cdot)$ defined on I and taking values in $U \subseteq \Re^m$. Notice that we do not identify functions in $\mathrm{L}^\infty(I; U)$ or $\mathrm{L}^\infty_{loc}(I; U)$ which differ on a measure zero set.

(ii) By $C^0(A\ ;\ \Omega)$, we denote the class of continuous functions on $A \subseteq \Re^n$, which take values in $\Omega \subseteq \Re^m$. By $C^k(A\ ;\ \Omega)$, where $k \geq 1$, we denote the class of continuous functions on $A \subseteq \Re^n$, which have continuous derivatives of order $k \geq 1$ and take values in $\Omega \subseteq \Re^m$.

(iii) For a vector $x \in \Re^n$ we denote by x' its transpose and by $|x|$ its Euclidean norm. $A' \in \Re^{n\times m}$ denotes the transpose of the matrix $A \in \Re^{m\times n}$.

(iv) By $\mathrm{sat} : \Re \to [-1,1]$, we denote the continuous function $\mathrm{sat}(x) = \frac{x}{\max(1,|x|)}$ for all $x \in \Re$. $B(x,\rho) \subseteq \Re^n$ denotes the closed ball in $\Re^n$ of radius $\rho \geq 0$ centered at $x \in \Re^n$, i.e., $B(x,\rho) := \{\, y \in \Re^n\ :\ |y-x| \leq \rho \,\}$.

(v) We say that an increasing continuous function $\gamma : \Re_+ \to \Re_+$ is of class K_∞ if $\gamma(0) = 0$ and $\lim_{s\to+\infty}\gamma(s) = +\infty$. By KL we denote the set of all continuous functions $\sigma = \sigma(s,t) : \Re_+ \times \Re_+ \ \to\ \Re_+$ with the following properties: (i) for each $t \geq 0$ the mapping $\sigma(\,\cdot\,,t)$ is nondecreasing; (ii) for each $s \geq 0$, the mapping $\sigma(s,\,\cdot\,)$ is nonincreasing with $\lim_{t\to+\infty}\sigma(s,t) = 0$.

2. Background material and preliminary results. The stability notions used in the present work are applied to sampled-data systems of the form

$$
\begin{aligned}
&\dot{x}(t) = f(d(t),x(t),x(\tau_i)), \quad t \in [\tau_i,\tau_{i+1}),\\
&\tau_0 = 0,\ \tau_{i+1} = \tau_i + r\exp(-w(\tau_i)),\ i = 0,1,\ldots,\\
&x(t) \in \Re^n,\ d(t) \in D,\ w(t) \in \Re^+,
\end{aligned}
\tag{2.1}
$$

where $D \subset \Re^l$ is a nonempty set and $r > 0$ is a constant, under the following hypothesis:

(H) $f(d,x,x_0)$ is continuous with respect to $(d,x) \in D \times \Re^n$, and there exist a symmetric positive definite matrix $P \in \Re^{n\times n}$ and a function $a \in K_\infty$ such that the following inequalities hold:

$$
\sup\left\{ \frac{(x-y)'\,P\,(f(d,x,x_0) - f(d,y,x_0))}{|x-y|^2}\ :\ x,y,x_0 \in B(0,s),\ x \neq y,\ d \in D \right\} < +\infty\,, \forall s > 0, \tag{2.2}
$$

$$
|f(d,x,x_0)| \leq\ a\,(|x| + |x_0|) \quad \forall (d,x,x_0) \in D \times \Re^n \times \Re^n. \tag{2.3}
$$

Hypothesis (H) guarantees that $0 \in \Re^n$ is an equilibrium point for (2.1) and is automatically satisfied if $D \subset \Re^l$ is compact, $f(d,x,x_0)$ is locally bounded and locally Lipschitz with respect to $x \in \Re^n$, and $f(d,0,0) = 0 = \lim_{(x,x_0)\to(0,0)} f(d,x,x_0)$ for all $d \in D$. Moreover, by virtue of Proposition 2.5 in [17], hypothesis (H) guarantees that for every $(x_0,d,w) \in \Re^n \times \mathrm{L}^\infty_{loc}(\Re_+;D) \times \mathrm{L}^\infty_{loc}(\Re_+;\Re_+)$, system (2.1) admits a unique solution $x : [0,t_{\max}) \to \Re^n$ with $x(0) = x_0$, where $t_{\max} \in (0,+\infty]$ is the maximal existence time of the solution. Furthermore, if $t_{\max} < +\infty$, then $\limsup_{t\to t_{\max}^-}|x(t)| = +\infty$. The unique solution of (2.1) with $x(0) = x_0$ corresponding to inputs $(d,w) \in \mathrm{L}^\infty_{loc}(\Re_+;D) \times \mathrm{L}^\infty_{loc}(\Re_+;\Re_+)$ will be denoted by $x(t,x_0,d,w)$. The set of times $\{\tau_i\}_{i=0}^\infty$ is called the set of sampling times.

We next provide the definition of robust global asymptotic stability of (2.1).

Definition 2.1. *Consider system* (2.1) *under hypothesis* (H)*. We say that* $0 \in \Re^n$ *is* robustly globally asymptotically stable (RGAS) *for system* (2.1) *if the following properties hold:*

P1. *System* (2.1) *is* robustly Lagrange stable*; i.e., for every* $\varepsilon > 0$*, it holds that*

$$
\sup\{\, |x(t,x_0,d,w)|\,;\ t \geq 0,\ \ |x_0| \leq \varepsilon,\ (d,w) \in \mathrm{L}^\infty_{loc}(\Re_+;D\times\Re_+)\,\} < +\infty.
$$

P2. *System* (2.1) *is* robustly Lyapunov stable*; i.e., for every* $\varepsilon > 0$ *there exists* $\delta := \delta(\varepsilon) > 0$ *such that*

$$|x_0| \leq \delta \;\Rightarrow\; |x(t,x_0,d,w)| \leq \varepsilon \quad \forall t \geq 0, \quad \forall (d,w) \in \mathrm{L}^{\infty}_{loc}(\Re_+;D) \times \mathrm{L}^{\infty}_{loc}(\Re_+;\Re_+).$$

P3. *System* (2.1) *satisfies the* robust attractivity property*; i.e.for every* $\varepsilon > 0$ *and* $R \geq 0$*, there exists a* $T := T(\varepsilon, R) \geq 0$*, such that*

$$|x_0| \leq R \;\Rightarrow\; |x(t,x_0,d,w)| \leq \varepsilon \quad \forall t \geq T, \quad \forall (d,w) \in \mathrm{L}^{\infty}_{loc}(\Re_+;D) \times \mathrm{L}^{\infty}_{loc}(\Re_+;\Re_+).$$

Remark 2.2. Using Lemma 2.17 in [18] (with zero gain function) we can guarantee that $0 \in \Re^n$ is RGAS for system (2.1) if and only if there exists a function $\sigma \in KL$ such that the following estimate holds for all $(x_0,d,w) \in \Re^n \times \mathrm{L}^{\infty}_{loc}(\Re_+;D) \times \mathrm{L}^{\infty}_{loc}(\Re_+;\Re_+)$ and $t \geq 0$:

$$|x(t)| \leq \sigma\left(|x_0|,\ t\right). \tag{2.4}$$

The reader should also notice that the sampling period is allowed to be time-varying. The factor $\exp(-w(\tau_i)) \leq 1$, with $w(t) \geq 0$ some nonnegative function of time, is an uncertainty factor in the end-point of the sampling interval. Proving RGAS for (2.1) guarantees stability for all sampling schedules with $\tau_{i+1} - \tau_i \leq r$ (robustness to perturbations of the sampling schedule). Therefore, it is justified to call the constant $r > 0$ the maximum allowable sampling period (MASP).

We finish this section by providing a technical result which will be used in the following sections.

LEMMA 2.3. *Let* $b > a$ *be constants, and let* $x : [a,b] \to \Re^n$ *be absolutely continuous. Suppose that there exist constants* $Q, G \geq 0$ *such that*

$$|\dot{x}(t)| \leq Q\,|x(t)| + G\,|x(a)| \;\text{ for } t \in [a,b) \text{ a.e.} \tag{2.5}$$

Suppose, furthermore, that $(G+Q)(b-a)\exp(Q(b-a)) < 1$*. Then the following inequality holds for all* $t \in [a,b]$*:*

$$|x(t) - x(a)| \leq \frac{(G+Q)(b-a)\exp(Q(b-a))}{1-(G+Q)(b-a)\exp(Q(b-a))}\,|x(t)|. \tag{2.6}$$

Proof. Since $x : [a,b] \to \Re^n$ is absolutely continuous, it holds that $|x(t)-x(a)| \leq \int_a^t |\dot{x}(s)|ds$ for all $t \in [a,b]$. Inequality (2.5) implies $|x(t)-x(a)| \leq Q\int_a^t |x(s)|ds + G(b-a)|x(a)|$ for all $t \in [a,b]$, and consequently we obtain $|x(t)-x(a)| \leq Q\int_a^t |x(s) - x(a)|ds + (G+Q)(b-a)|x(a)|$ for all $t \in [a,b]$. Applying the Gronwall–Bellman lemma to the previous inequality gives

$$|x(t)-x(a)| \leq (G+Q)(b-a)\exp(Q(b-a))\,|x(a)| \quad \forall t \in [a,b].$$

The above inequality in conjunction with the triangle inequality implies that

$$\begin{aligned} |x(t)-x(a)| \leq\; & (G+Q)(b-a)\exp(Q(b-a))\,|x(a)-x(t)| \\ & + (G+Q)(b-a)\exp(Q(b-a))\,|x(t)| \quad \forall t \in [a,b]. \end{aligned}$$

Since $(G+Q)(b-a)\exp(Q(b-a)) < 1$, the above inequality directly implies (2.6). The proof is complete. $\square$

3. Main results. All the results of the present work are proved by using Lemma 3.1, which is stated next. We call it the sampled-data forwarding lemma because it provides sufficient conditions for robust global stabilization by means of sampled-data control with positive sampling rate of a system with "added integration".

Assuming that a system (whose state is denoted by x) is stabilizable by sampled-data feedback, the sampled-data forwarding lemma guarantees the existence of a sampled-data feedback stabilizer when the system is augmented by an additional state (y) in a cascade/feedforward configuration. Outside of the set $x'Px < R^2$ in the (x, y) state space (where P is an appropriate positive definite matrix and $R > 0$ is an appropriate constant) the feedback law uses the stabilizer for the x-subsystem in order to make $|x|$ small, whereas inside of the set $x'Px < R^2$ the feedback law uses a suitably saturated linear feedback law whose task is to drive $|y + c'x|$ below a prescribed small value ω^{-1} (where c is an appropriate vector), while keeping $|x|$ small. Once both $|x|$ and $|y + c'x|$ are rendered small, the linear feedback law prevails in achieving exponential regulation to the origin. Robustness to perturbations in the sampling schedule is proved by quantifying the error between the current value of the state x and its most recent sampled value, and by showing the smallness of this error for the closed-loop solutions provided all sampling periods fall uniformly below a sufficiently small MASP.

The main result of the paper, Theorem 3.7, is established by recursively applying this lemma to system (1.1) and by constructively satisfying the lemma's key assumptions (inequalities (3.3), (3.4), (3.5) below) with the help of Lemma 3.6.

LEMMA 3.1 (the sampled-data forwarding lemma). *Consider the system*

$$\begin{aligned} &\dot{x} = Ax + bu + f(d, x, u), \\ &\dot{y} = x_n + g(d, x, u), \\ &x = (x_1, \dots, x_n)' \in \Re^n,\ y \in \Re,\ u \in \Re,\ d \in D \subset \Re^l, \end{aligned} \tag{3.1}$$

where $b = (1, 0, \dots, 0)' \in \Re^n$, $D \subset \Re^l$ is a nonempty compact set, $f : D \times \Re^n \times \Re \to \Re^n$, $g : D \times \Re^n \times \Re \to \Re$ are locally Lipschitz mappings with $f(d, 0, 0) = 0$, $g(d, 0, 0) = 0$ for all $d \in D$, $A = \{a_{i,j} : i, j = 1, \dots, n\}$ with $a_{i,j} = 1$ if $j = i - 1$, $i = 2, \dots, n$, and $a_{i,j} = 0$ if otherwise. We assume that the following hypothesis holds:

(H) *There exist a constant $r > 0$ and a locally bounded function $k : \Re^n \to \Re$, with $k(0) = 0$ being continuous at $0 \in \Re^n$ such that $0 \in \Re^n$ is RGAS for the following sampled-data system:*

$$\begin{aligned} &\dot{x}(t) = Ax(t) + bu(t) + f(d(t), x(t), u(t)), \\ &u(t) = k(x(\tau_i)),\ t \in [\tau_i, \tau_{i+1}), \\ &\tau_{i+1} = \tau_i + r\exp(-w(\tau_i)),\ \tau_0 = 0, \\ &d(t) \in D,\ w(t) \in \Re_+. \end{aligned} \tag{3.2}$$

Let $P \in \Re^{n\times n}$ be a symmetric positive definite matrix, and let $p \in \Re^n$ be a constant vector such that the matrix $P(A+bp')+(A'+pb')P$ is negative definite. Define $c = -(A'+pb')^{-1}(0, \dots, 0, 1)'$ and assume the existence of constants $M, R, K, \omega, \delta > 0$

such that

$$\max\left\{ x'P\,(Ax + f(d,x,u) + bu) \;:\; (d,x) \in D \times \Re^n,\ x'Px = R^2,\ |u - p'x| \le K\,|c'b| \right\} < 0, \tag{3.3}$$

$$\max\left\{ |g(d,x,u) + c'f(d,x,u)| \;:\; x'Px \le R^2,\ d \in D,\ |u - p'x| \le K\,|c'b| \right\} < K(c'b)^2, \tag{3.4}$$

$$\begin{aligned} & z\,(Mg(d,x,u) + Mc'f(d,x,u) - K(c'b)\omega b'Px) \\ & \quad \le \left(MK(c'b)^2\omega - \delta\right)|z|^2 - x'P((A + bp' + \delta I)x + f(d,x,u)) \\ & \quad \forall (x,z,d) \in \Re^n \times \Re \times D \text{ with } x'Px \le R^2, \\ & \quad \omega\,|z| \le 1 \text{ and } u = p'x - Kc'b\omega z. \end{aligned} \tag{3.5}$$

Define $\tilde{k} : \Re^n \times \Re \to \Re$ *by*

$$\tilde{k}(x,y) := \begin{cases} k(x) & \text{if } x'Px \ge R^2, \\ p'x - Kc'b\,\mathrm{sat}(\omega(y + c'x)) & \text{if } x'Px < R^2. \end{cases} \tag{3.6}$$

Then for sufficiently small $\tilde{r} > 0$, $0 \in \Re \times \Re^n$ *is RGAS for the sampled-data system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(d(t), x(t), u(t)) + bu(t), \\ \dot{y}(t) &= x_n(t) + g(d(t), x(t), u(t)), \\ u(t) &= \tilde{k}(x(\tau_i), y(\tau_i)),\ t \in [\tau_i, \tau_{i+1}), \\ \tau_{i+1} &= \tau_i + \tilde{r}\exp(-w(\tau_i)),\ \tau_0 = 0, \\ d(t) &\in D,\ w(t) \in \Re_+. \end{aligned} \tag{3.7}$$

Discussion of the assumptions of the sampled-data forwarding lemma. Hypothesis (H) is an important assumption that guarantees that the x-subsystem can be stabilized by sampled-data feedback. Inequalities (3.3) and (3.4) guarantee that the control strategy of driving y close to zero, while keeping the state component x in a neighborhood of zero, is feasible. Indeed,

- inequality (3.3) guarantees that the state cannot leave a neighborhood of the set $x = 0$ (this neighborhood is the set $\left\{(x,y) \in \Re^n \times \Re \,:\, x'Px \le R^2\right\}$) when we use controls that are "perturbations" of a nominal feedback law, where the nominal feedback law is the linear feedback $u = p'x$ and the magnitude of the "perturbation" is determined by $|u - p'x| \le K\,|c'b|$;
- inequality (3.5) guarantees that we can drive y close to zero when we use controls which are "perturbations" of a nominal feedback law.

Finally, inequality (3.5) is a standard assumption that guarantees robust local exponential *stabilizability* of the origin by means of a linear feedback. This becomes clear when inequality (3.5) is written in the following way:

$$\frac{d}{dt}\left(x'Px + M\,(y + c'x)^2\right) \le -2\delta\,|y + c'x|^2 - 2\delta x'Px$$

for all $(x,y,d) \in \Re^n \times \Re \times D$ with $x'Px \le R^2$, $\omega\,|y + c'x| \le 1$, and $u = p'x - Kc'b\omega(y + c'x)$, where d/dt indicates differentiation along the trajectories of (3.1). Therefore, it can be argued that inequalities (3.3), (3.4), and (3.5) are fundamental to having a feedback strategy for driving y close to zero while keeping the state in a neighborhood of the set $x = 0$.

The proof of the sampled-data forwarding lemma is technical and is based on the following four technical results. Their proofs are provided in the appendix.

LEMMA 3.2. *Let $P \in \Re^{n\times n}$ be a symmetric positive definite matrix, and let $p \in \Re^n$ be a constant vector such that the matrix $P(A+bp')+(A'+pb')P$ is negative definite. Define $c = -(A'+pb')^{-1}(0,\ldots,0,1)'$ and assume the existence of constants $M,R,K,\omega,\delta > 0$ such that* (3.3), (3.4), *and* (3.5) *hold. Consider the solution $(x(t),y(t)) \in \Re^n \times \Re$ of* (3.7) *under hypothesis* (H), *where $\tilde{k} : \Re^n \times \Re \to \Re$ is defined by* (3.6) *and $\tilde{r} > 0$, with arbitrary initial condition $(x(0),y(0)) \in \Re^n \times \Re$ satisfying $x'(0)Px(0) < R^2$ and corresponding to arbitrary $(d,w) \in \mathrm{L}^\infty_{loc}(\Re_+; D \times \Re_+)$. If $\tilde{r} > 0$ is sufficiently small, then $x'(t)Px(t) < R^2$ for all $t \in [0,\tau_1]$.*

Using induction and Lemma 3.2, we obtain the following result.

LEMMA 3.3. *Let $P \in \Re^{n\times n}$ be a symmetric positive definite matrix and $p \in \Re^n$ be a constant vector such that the matrix $P(A+bp')+(A'+pb')P$ is negative definite. Define $c = -(A'+pb')^{-1}(0,\ldots,0,1)'$ and assume the existence of constants $M,R,K,\omega,\delta > 0$ such that* (3.3), (3.4), *and* (3.5) *hold. Consider the solution $(x(t),y(t)) \in \Re^n \times \Re$ of* (3.7) *under hypothesis* (H), *where $\tilde{k} : \Re^n \times \Re \to \Re$ is defined by* (3.6) *and $\tilde{r} > 0$, with arbitrary initial condition $(x(0),y(0)) \in \Re^n \times \Re$ satisfying $x'(0)Px(0) < R^2$ and corresponding to arbitrary $(d,w) \in \mathrm{L}^\infty_{loc}(\Re_+; D \times \Re_+)$. If $\tilde{r} > 0$ is sufficiently small, then the solution $(x(t),y(t)) \in \Re^n \times \Re$ of* (3.7) *exists for all $t \geq 0$ and satisfies $x'(t)Px(t) < R^2$ for all $t \geq 0$.*

The following lemma uses the result of Lemma 3.3 and shows attractivity for a certain region in the state space.

LEMMA 3.4. *Let $P \in \Re^{n\times n}$ be a symmetric positive definite matrix, and let $p \in \Re^n$ be a constant vector such that the matrix $P(A+bp')+(A'+pb')P$ is negative definite. Define $c = -(A'+pb')^{-1}(0,\ldots,0,1)'$ and assume the existence of constants $M,R,K,\omega,\delta > 0$ such that* (3.3), (3.4), *and* (3.5) *hold. Consider the solution $(x(t),y(t)) \in \Re^n \times \Re$ of* (3.7) *under hypothesis* (H), *where $\tilde{k} : \Re^n \times \Re \to \Re$ is defined by* (3.6) *and $\tilde{r} > 0$, with arbitrary initial condition $(x(0),y(0)) \in \Re^n \times \Re$ satisfying $x'(0)Px(0) < R^2$ and corresponding to arbitrary $(d,w) \in \mathrm{L}^\infty_{loc}(\Re_+; D \times \Re_+)$. If $\tilde{r} > 0$ is sufficiently small, then there exists $T \in C^0(\Re;\Re_+)$ such that*

$$|z(t)| \leq \max\left\{|z(0)|, \omega^{-1}\right\} \quad \forall t \geq 0, \tag{3.8}$$

$$|z(t)| \leq \omega^{-1} \quad \forall t \geq T(z(0)), \tag{3.9}$$

where $z(t) = y(t) + c'x(t)$.

LEMMA 3.5. *Let $P \in \Re^{n\times n}$ be a symmetric positive definite matrix, and let $p \in \Re^n$ be a constant vector such that the matrix $P(A+bp')+(A'+pb')P$ is negative definite. Define $c = -(A'+pb')^{-1}(0,\ldots,0,1)'$ and assume the existence of constants $M,R,K,\omega,\delta > 0$ such that* (3.3), (3.4), *and* (3.5) *hold. Consider the solution $(x(t),y(t)) \in \Re^n \times \Re$ of* (3.7) *under hypothesis* (H), *where $\tilde{k} : \Re^n \times \Re \to \Re$ is defined by* (3.6) *and $\tilde{r} > 0$, with arbitrary initial condition $(x(0),y(0)) \in \Re^n \times \Re$ satisfying $x'(0)Px(0) < R^2$, $|y(0)+c'x(0)| \leq \omega^{-1}$ and corresponding to arbitrary $(d,w) \in \mathrm{L}^\infty_{loc}(\Re_+; D \times \Re_+)$.*

If $\tilde{r} > 0$ is sufficiently small, then there exists $\mu > 0$ such that the following differential inequality holds for $t \geq 0$ a.e.:

$$\dot{V}(t) \leq -\mu V(t), \tag{3.10}$$

where $z(t) = y(t) + c'x(t)$ and $V(t) = \frac{M}{2}z^2(t) + \frac{1}{2}x'(t)Px(t)$.

The reader should notice that by virtue of Lemmas 3.3 and 3.4 the set $S = \{(x,y) \in \Re^n \times \Re \,:\, x'Px < R^2, |y+c'x| \leq \omega^{-1}\}$ is positively invariant for system

(3.7). The differential inequality (3.10) guarantees that $V(t) \le \exp(-\mu t)V(0)$ for all $t \ge 0$, provided that $(x(0), y(0)) \in S$. Since $V(x,y) = \frac{M}{2}|y + c'x|^2(t) + \frac{1}{2}x'Px$ is a positive definite quadratic function, the previous inequality shows that local exponential stability is guaranteed for system (3.7) in the region $S \subseteq \Re^n \times \Re$. Notice that the size of the region $S \subseteq \Re^n \times \Re$ is determined by the constants R, ω.

We are now in a position to prove the sampled-data forwarding lemma.

Proof of Lemma 3.1. We will restrict $\tilde{r} > 0$ so that

$$\tilde{r} \le r. \tag{3.11}$$

Notice that Lemma 3.3 and definition (3.6) imply that the set $\{x \in \Re^n : x'Px < R^2\}$ is positively invariant. Robust Lyapunov stability for system (3.7) is a direct consequence of the differential inequality (3.10). Next we will show robust Lagrange stability and robust attractivity for system (3.7). Consider the solution $(x(t), y(t)) \in \Re^n \times \Re$ of (3.7) under hypothesis (H), with initial condition $(x(0), y(0)) \in \Re^n \times \Re$ and corresponding to arbitrary $(d, w) \in \mathrm{L}^\infty_{loc}(\Re_+; D \times \Re_+)$. By virtue of hypothesis (H), inequality (3.11) and definition (3.6), there exists $\sigma \in KL$ such that

$$|x(t)| \le \sigma(|x(0)|, t) \tag{3.12}$$

for all times $t \ge 0$ with $x'(t)Px(t) \ge R^2$. Inequality (3.12) in conjunction with Lemma 3.3 implies that there exists a constant $C > 0$ such that the following inequality holds:

$$|x(t)| \le \max(\,\sigma(|x(0)|, 0),\, C) \quad \forall t \ge 0 \tag{3.13}$$

and that there exists a nondecreasing $\tilde{T} \in C^0(\Re_+; \Re_+)$ such that

$$x'(t)Px(t) < R^2 \ \forall t \ge \tilde{T}(|x(0)|). \tag{3.14}$$

Notice that hypothesis (H) implies the existence of $\rho \in K_\infty$ such that $|k(x)| \le \rho(|x|)$ for all $x \in \Re^n$. Therefore, we can conclude that there exists $\gamma \in K_\infty$ such that

$$|y(t) + c'x(t)| \le \gamma(|x(0)| + |y(0)|) \tag{3.15}$$

for all times $t \ge 0$ with $x'(t)Px(t) \ge R^2$. In order to see why (3.15) holds, we notice that by virtue of the compactness of $D \subset \Re^l$, continuity of $g : D \times \Re^n \times \Re \to \Re$, the fact that $g(d, 0, 0) = 0$ for all $d \in D$, and the fact that $|k(x)| \le \rho(|x|)$ for all $x \in \Re^n$, there exists $\tilde{\gamma} \in K_\infty$ such that $|x_n + g(d, x, k(x_0))| \le \tilde{\gamma}(|x|) + \tilde{\gamma}(|x_0|)$ for all $(d, x, x_0) \in D \times \Re^n \times \Re^n$. Therefore, definition (3.6) and differential equations (3.7) imply that the following inequality holds for almost all times $t \ge 0$ with $x'(t)Px(t) \ge R^2$:

$$|\dot{y}(t)| \le \tilde{\gamma}(|x(t)|) + \tilde{\gamma}(|x(\tau_i)|),$$

where τ_i is an appropriate sampling time satisfying $t \in [\tau_i, \tau_{i+1})$. Using (3.12), (3.14), and the above differential inequality, we conclude that the following estimate holds for all times $t \ge 0$ with $x'(t)Px(t) \ge R^2$:

$$|y(t)| \le |y(0)| + 2\tilde{T}(|x(0)|)\,\tilde{\gamma}(\sigma(|x(0)|, 0)).$$

The above inequality shows that (3.15) holds with $\gamma(s) := s + |c|\,\sigma(s, 0) + 2\tilde{T}(s)\tilde{\gamma}(\sigma(s, 0))$. Inequality (3.15) in conjunction with Lemma 3.4 implies that

$$|y(t) + c'x(t)| \le \max\left(\gamma(|x(0)| + |y(0)|),\, |y(0) + c'x(0)|,\, \omega^{-1}\right) \quad \forall t \ge 0. \tag{3.16}$$

Estimates (3.13), (3.16) prove robust Lagrange stability. Finally, inequality (3.14) in conjunction with Lemmas 3.4 and 3.5 imply that robust attractivity holds as well. The proof is complete. □

The following result shows that the assumptions of the sampled-data forwarding lemma can be automatically satisfied for a certain class of nonlinearities.

LEMMA 3.6. *Suppose that there exists a nondecreasing function $L \in C^0(\Re_+;\Re_+)$ such that the following inequality holds for the mappings $f : D \times \Re^n \times \Re \to \Re^n$, $g : D \times \Re^n \times \Re \to \Re$:*

$$\begin{aligned} |f(d,x,u)| + |g(d,x,u)| \le L(|(x,u)|)\, |x|^2 & \\ + L(|(x,u)|)\, |x|\, |u| \ \ \forall (d,x,u) \in D \times \Re^n \times \Re. & \end{aligned} \tag{3.17}$$

Then there exist constants $R^*, C > 0$ with $C \le 1$ such that for every $\omega > 0$, $R \in (0, R^*)$ there exist constants $M, \delta > 0$ such that (3.3), (3.4), (3.5) *hold with $K = CR$.*

THEOREM 3.7. *Consider system* (1.1) *where all mappings $g_i : D \times \Re^{i-1} \times \Re \to \Re$ are locally Lipschitz, and assume that there exists a smooth nondecreasing function $L \in C^0(\Re_+;\Re_+)$ such that* (1.2) *holds. Then there exist a bounded $k : \Re^n \to \Re$ with $k(0) = 0$ being continuous at $0 \in \Re^n$ and a constant $r > 0$ such that $0 \in \Re^n$ is RGAS for the closed-loop sampled-data system* (1.1) *with*

$$\begin{aligned} & u(t) = k(x(\tau_i)),\ t \in [\tau_i, \tau_{i+1}), \\ & \tau_{i+1} = \tau_i + r\exp(-w(\tau_i)),\ \tau_0 = 0, \\ & w(t) \in \Re_+. \end{aligned} \tag{3.18}$$

Define $Q_i \in \Re^{i\times n}$ with $Q_i x = (x_1,\dots,x_i)'$ for $i = 1,\dots,n$, $b_1 = [1] \in \Re$, $b_i = (1,0,\dots,0)' \in \Re^i$ for $i = 2,\dots,n$, $A_1 = [0] \in \Re^{1\times 1}$, $A_i = \{a_{k,j} : k,j = 1,\dots,i\} \in \Re^{i\times i}$ for $i = 2,\dots,n$ with $a_{k,j} = 1$ if $j = k-1$, $k = 2,\dots,i$, and $a_{k,j} = 0$ if otherwise. Let arbitrary constants $K_0 > 0$, $\omega_i > 0$ $(i = 0,\dots,n-1)$, arbitrary matrices $P_i \in \Re^{i\times i}(i = 1,\dots,n-1)$ being symmetric and positive definite, and arbitrary vectors $p_i \in \Re^i(i = 1,\dots,n-1)$ be such that the matrices $P_i(A_i + b_i p_i') + (A_i' + p_i b_i')P_i(i = 1,\dots,n-1)$ are negative definite. Define $c_i = -(A_i' + p_i b_i')^{-1}(0,\dots,0,1)' \in \Re^i$ for $i = 1,\dots,n-1$. Then there exist constants $r > 0$, $K_i > 0$, $R_i > 0$ $(i = 1,\dots,n-1)$ such that $0 \in \Re^n$ is RGAS for the closed-loop sampled-data system (1.1) *with* (3.18), *where $k : \Re^n \to \Re$ is defined by*

$$k(x) := p_i' Q_i x - K_i c_i' b_i \operatorname{sat}(\omega_i(x_{i+1} + c_i' Q_i x)), \tag{3.19}$$

where $i = i(x) \in \{1,\dots,n-1\}$ is the largest integer such that

$$x' Q_i' P_i Q_i x < R_i^2 \tag{3.20}$$

and

$$k(x) := -K_0 \operatorname{sat}(\omega_0 x_1) \ \text{if} \min_{l=1,\dots,n-1} \left(x' Q_l' P_l Q_l x - R_l^2 \right) \ge 0. \tag{3.21}$$

Theorem 3.7 contains two statements. The first statement (above (3.18)) is an existence-type result, which guarantees the existence of a robust global sampled-data stabilizer. The second statement (below (3.18)) is a design result which provides a family of bounded robust global sampled-data stabilizers for (1.1). The reader should notice that since the pairs of matrices (A_i, b_i) are controllable pairs for $i = 1,\dots,n-1$, it is straightforward to obtain symmetric, positive definite matrices $P_i \in \Re^{i\times i}$

$(i = 1, \dots, n-1)$ and vectors $p_i \in \Re^i$ $(i = 1, \dots, n-1)$ such that the matrices $P_i(A_i + b_i p_i') + (A_i' + p_i b_i')P_i$ $(i = 1, \dots, n-1)$ are negative definite.

Proof of Theorem 3.7. The proof is based on a repeated application of the sampled-data forwarding lemma and Lemma 3.6. Notice that the subsystem $\dot{x}_1 = u$ can be stabilized by the bounded sampled-data feedback

$$\begin{aligned}&u(t) = -K_0 \operatorname{sat}(x_1(\tau_i)),\ t \in [\tau_i, \tau_{i+1}),\\&\tau_{i+1} = \tau_i + K_0^{-1} \exp(-w(\tau_i)),\ \tau_0 = 0,\\&w(t) \in \Re_+,\end{aligned}$$

where $K_0 > 0$ is an arbitrary positive constant. The sampled-data forwarding lemma is applied for $j = 1, \dots, n-1$ with $x \in \Re^n$ replaced by $(x_1, \dots, x_j)' \in \Re^j$; $y \in \Re$ replaced by $x_{j+1} \in \Re$; $A \in \Re^{n\times n}$ replaced by $A_j \in \Re^{j\times j}$; $b \in \Re^n$ replaced by $b_j \in \Re^j$; $g(d,x,u) \in \Re$ replaced by $g_{j+1}(d, x_1, \dots, x_j, u) \in \Re$; $f(d,x,u) \in \Re^n$ replaced by $f(d,x,u) = (0, g_2(d,x_1,u), \dots, g_j(d, x_1, \dots, x_{j-1}, u))' \in \Re^j$ for $j \geq 2$ and $f(d,x,u) = 0 \in \Re$ for $j = 1$; $P \in \Re^{n\times n}$, $p \in \Re^n$; $c = -(A' + pb')^{-1}(0, \dots, 0, 1)'$ replaced by $P_j \in \Re^{j\times j}$, $p_j \in \Re^j$, $c_j = -(A_j' + p_j b_j')^{-1}(0, \dots, 0, 1)' \in \Re^j$, respectively; and $k : \Re^n \to \Re$ replaced by $k_j : \Re^j \to \Re$ which is defined by the following equalities:

- for $j \geq 2$

$$k_j(x_1, \dots, x_j) := p_i' Q_i x - K_i c_i' b_i \operatorname{sat}(\omega_i(x_{i+1} + c_i' Q_i x)), \tag{3.22}$$

  where $i = i(x_1, \dots, x_{j-1}) \in \{1, \dots, j-1\}$ is the largest integer such that (3.20) holds and

$$k_j(x_1, \dots, x_j) := -K_0 \operatorname{sat}(\omega_0 x_1) \text{ if } \min_{l=1,\dots,j-1} \left(x' Q_l' P_l Q_l x - R_l^2\right) \geq 0; \tag{3.23}$$

- for $j = 1$

$$k_1(x_1) := -K_0 \operatorname{sat}(\omega_0 x_1). \tag{3.24}$$

By virtue of (1.2), it follows that (3.17) holds with $x \in \Re^n$ replaced by $(x_1, \dots, x_j)' \in \Re^j$, $g(d,x,u) \in \Re$ replaced by $g_{j+1}(d, x_1, \dots, x_j, u) \in \Re$, $f(d,x,u) \in \Re^n$ replaced by $f(d,x,u) = (0, g_2(d,x_1,u), \dots, g_j(d, x_1, \dots, x_{j-1}, u))' \in \Re^j$ for $j \geq 2$ and $f(d,x,u) = 0 \in \Re$ for $j = 1$, and $L \in C^0(\Re_+;\Re_+)$ replaced by $L_j \in C^0(\Re_+;\Re_+)$, where $L_j(s) = jL(s)$ and L is the function involved in (1.2). Therefore, by virtue of Lemma 3.6, it follows that there exist constants $R_j^*, C_j > 0$ with $C_j \leq 1$ such that for every $\omega_j > 0$, $R_j \in (0, R_j^*)$ there exist constants $M_j, \delta_j > 0$ such that (3.3), (3.4), (3.5) hold with $K = K_j = C_j R_j$, $R = R_j$, $\omega = \omega_j$, $M = M_j$, and $\delta = \delta_j$. The proof is complete. ∎

Remark 3.8. Notice that the proof of Theorem 3.7 guarantees that for every $G > 0$, the sampled-data feedback stabilizer $k : \Re^n \to \Re$ can be selected in such a way that $|k(x)| \leq G$ for all $x \in \Re^n$. To see this, first select arbitrary constants $\omega_i > 0$ $(i = 0, \dots, n-1)$, arbitrary matrices $P_i \in \Re^{i\times i}$ $(i = 1, \dots, n-1)$ being symmetric and positive definite, and arbitrary vectors $p_i \in \Re^i$ $(i = 1, \dots, n-1)$ such that the matrices $P_i(A_i + b_i p_i') + (A_i' + p_i b_i')P_i$ $(i = 1, \dots, n-1)$ are negative definite. The selection of $K_i > 0$, $R_i > 0$ $(i = 1, \dots, n-1)$ made in the proof of Theorem 3.7 guarantees that the constants $R_i \in (0, R_i^*)$ $(i = 1, \dots, n-1)$ can be selected in an arbitrary way, where $R_i^* > 0$ $(i = 1, \dots, n-1)$ are appropriate constants. Moreover, the inequalities $K_i \leq R_i$ hold for $i = 1, \dots, n-1$. It follows from (3.19), (3.20), (3.21) that

$$|k(x)| \leq \max\left\{K_0, \max_{i=1,\dots,n-1} R_i\left(|p_i|\, a_i^{-1} + |c_i' b_i|\right)\right\} \quad \forall x \in \Re^n, \tag{3.25}$$

where $a_i > 0$ ($i = 1, \ldots, n-1$) are constants satisfying $x'Q_i'P_iQ_ix \geq a_i^2 |Q_ix|^2$ for all $x \in \Re^n$. It follows from (3.25) that if $K_0 \leq G$ and $R_i \leq \frac{G}{|p_i|a_i^{-1} + |c_i'b_i|}$ for $i = 1, \ldots, n-1$, then $|k(x)| \leq G$ for all $x \in \Re^n$. Notice that we can always select $K_0 \leq G$ and $R_i \leq \frac{G}{|p_i|a_i^{-1} + |c_i'b_i|}$ for $i = 1, \ldots, n-1$ ($K_0 > 0$ and $R_i \in (0, R_i^*)$ are free parameters).

4. Illustrative examples. In this section we present two examples that illustrate the results of the previous section. The first example shows the application of Theorem 3.7 to a feedforward system.

Example 4.1. We consider the three-dimensional feedforward system

$$\begin{aligned} \dot{x}_1 &= u, \\ \dot{x}_2 &= x_1 + x_1 u, \\ \dot{x}_3 &= x_2 + x_1^2, \\ x &= (x_1, x_2, x_3)' \in \Re^3, \; u \in \Re. \end{aligned} \tag{4.1}$$

The solution map of system (4.1) can be explicitly found: the resulting discrete-time system that corresponds to a constant sampling period $r > 0$ and input $u \in \Re$ applied with zero-order hold is given by the following equations:

$$\begin{aligned} x_1^+ &= x_1 + ur, \\ x_2^+ &= x_2 + (x_1 + ux_1)r + (u + u^2)\frac{r^2}{2}, \\ x_3^+ &= x_3 + (x_2 + x_1^2)r + (x_1 + 3ux_1)\frac{r^2}{2} + (u + 3u^2)\frac{r^3}{6}. \end{aligned} \tag{4.2}$$

However, as already noted in the introduction, system (4.1) is not included in one of the classes of systems noted in the introduction for which there exists a feedback design methodology that results in the design of a globally stabilizing sampled-data feedback (notice that (4.1) is not linearizable). Other approaches for sampled-data systems can be also applied (see [10, 22, 23, 26, 27, 41, 42, 43, 44]), but the result is semiglobal and practical sampled-data stabilization of system (4.1).

Here we apply the step-by-step feedback design methodology described in Theorem 3.7. The feedback law will be given by (3.19), (3.20), (3.21). For simplicity, we select $\omega_0 = \omega_1 = \omega_2 = 1$ and $K_0 = 1$. We also select

$$P_1 = [1], \; p_1 = [-1], \; P_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \; p_2 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

Using the formula $c_i = -(A_i' + p_ib_i')^{-1}(0, \ldots, 0, 1)' \in \Re^i$ for $i = 1, 2$, we obtain $c_1 = [1]$ and $c_2 = \left[\begin{smallmatrix} 1/2 \\ 1 \end{smallmatrix}\right]$. The only constants that remain to be determined are R_1, R_2, K_1, K_2.

In order to determine $R_1, K_1 > 0$, we use the sampled-data forwarding lemma. We apply the sampled-data forwarding lemma with $n = 1$, $A = [0]$, $b = [1]$, $P = [1]$, $p = [-1]$, $c = [1]$, $f(d, x, u) \equiv 0$, and $g(d, x, u) = x_1 u$. Conditions (3.3), (3.4), (3.5) are satisfied with $M = \frac{K}{R+K}$, $\omega = 1$ for $\delta > 0$ sufficiently small, provided that

$$\frac{R^2}{1 - R} < K < R \text{ and } R + K < 1. \tag{4.3}$$

Inequalities (4.3) hold with $R = \frac{3}{8}$ and $K = \frac{1}{4}$. Therefore, we select $R_1 = \frac{3}{8}$ and $K_1 = \frac{1}{4}$.

In order to determine $R_2, K_2 > 0$, we use again the sampled-data forwarding lemma. We apply the sampled-data forwarding lemma with $n = 2$, $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $p = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$, $c = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, $f(d,x,u) = \begin{bmatrix} 0 \\ x_1 u \end{bmatrix}$, and $g(d,x,u) = x_1^2$. After some tedious calculations, we conclude that conditions (3.3), (3.4), (3.5) are satisfied with $M = K\frac{2+(3+2\sqrt{2})R}{4R}$, $\omega = 1$ for $\delta > 0$ sufficiently small, provided that

$$\frac{4R^2}{1-2\sqrt{2}R} < K < 2R\frac{1-2\left(2+\sqrt{2}\right)R}{R+1} \text{ and } (4+2\sqrt{2})R + (3-2\sqrt{2})R^2 < 1. \tag{4.4}$$

Inequalities (4.4) hold with $R = K = \frac{1}{20}$. Therefore, we select $R_2 = K_2 = \frac{1}{20}$. We conclude that the sampled-data feedback law (3.18) defined by

$$k(x) := \begin{cases} -\mathrm{sat}(x_1) \quad \text{if} \quad 8|x_1| \geq 3 \quad \text{and} \quad 20\sqrt{x_2^2 + (x_1+x_2)^2} \geq 1, \\ -x_1 - \frac{1}{4}\mathrm{sat}(x_2 + x_1) \quad \text{if} \quad 8|x_1| < 3 \quad \text{and} \quad 20\sqrt{x_2^2 + (x_1+x_2)^2} \geq 1, \\ -2(x_1+x_2) - \frac{1}{40}\,\mathrm{sat}\left(x_3 + x_2 + \frac{1}{2}x_1\right) \quad \text{if} \quad 20\sqrt{x_2^2 + (x_1+x_2)^2} < 1 \end{cases} \tag{4.5}$$

achieves global stabilization of system (4.1) provided that the MASP $r > 0$ is sufficiently small. Indeed, simulations show that global stabilization of system (4.1) is achieved with $r = 0.01$. However, under these conservative choices of design parameters, which satisfy the sufficient conditions of Theorem 3.7, the closed-loop system shows different dynamic behaviors in different time scales. The state variables x_1, x_2 converge very fast, while the state variable x_3 exhibits slow convergence, which lasts about 900 time units.

Therefore, it is crucial to determine tight bounds for the range of values for $R_2, K_2 > 0$ which guarantee global asymptotic stability. Numerical experiments show that values higher than 0.05 for $R_2, K_2 > 0$ can guarantee global asymptotic stability for $r = 0.2$. Figure 1 shows the evolution of the state variables for the closed-loop system (4.1) with (3.18), where k is defined by

$$k(x) := \begin{cases} -\mathrm{sat}(x_1) \quad \text{if} \quad 8|x_1| \geq 3 \quad \text{and} \quad \sqrt{x_2^2 + (x_1+x_2)^2} \geq 1, \\ -x_1 - \frac{1}{4}\mathrm{sat}(x_2 + x_1) \quad \text{if} \quad 8|x_1| < 3 \quad \text{and} \quad \sqrt{x_2^2 + (x_1+x_2)^2} \geq 1, \\ -2(x_1+x_2) - \frac{1}{2}\,\mathrm{sat}\left(x_3 + x_2 + \frac{1}{2}x_1\right) \quad \text{if} \quad \sqrt{x_2^2 + (x_1+x_2)^2} < 1 \end{cases} \tag{4.6}$$

with $r = 0.2$, $w(t) = \ln(\frac{2}{1+|\sin(t)|})$, and initial condition $x_1(0) = x_2(0) = x_3(0) = 1$. It is clear that the selection $R_2 = K_2 = 1$ guarantees good performance even when perturbations of the sampling schedule are present. Figure 2 shows the corresponding input behavior, and Figure 3 focuses on the evolution of the input for $t \in [4, 8]$.

The second example shows that the sampled-data forwarding Lemma (Lemma 3.1) can be also applied to some nonlinear systems outside of the class of feedforward systems.

Example 4.2. Consider the nonlinear system

$$\begin{aligned} &\dot{x} = Ax + bu + f(d,x), \\ &\dot{y} = x_n + g(d,x), \\ &x \in \Re^n,\ y \in \Re,\ d \in D,\ u \in \Re, \end{aligned} \tag{4.7}$$

where $b = (1, 0, \ldots, 0)' \in \Re^n$, $A = \{a_{i,j} : i, j = 1, \ldots, n\}$ with $a_{i,j} = 1$ if $j = i-1$, $i = 2, \ldots, n$, and $a_{i,j} = 0$ if otherwise, $D \subset \Re^l$ is a nonempty compact set,

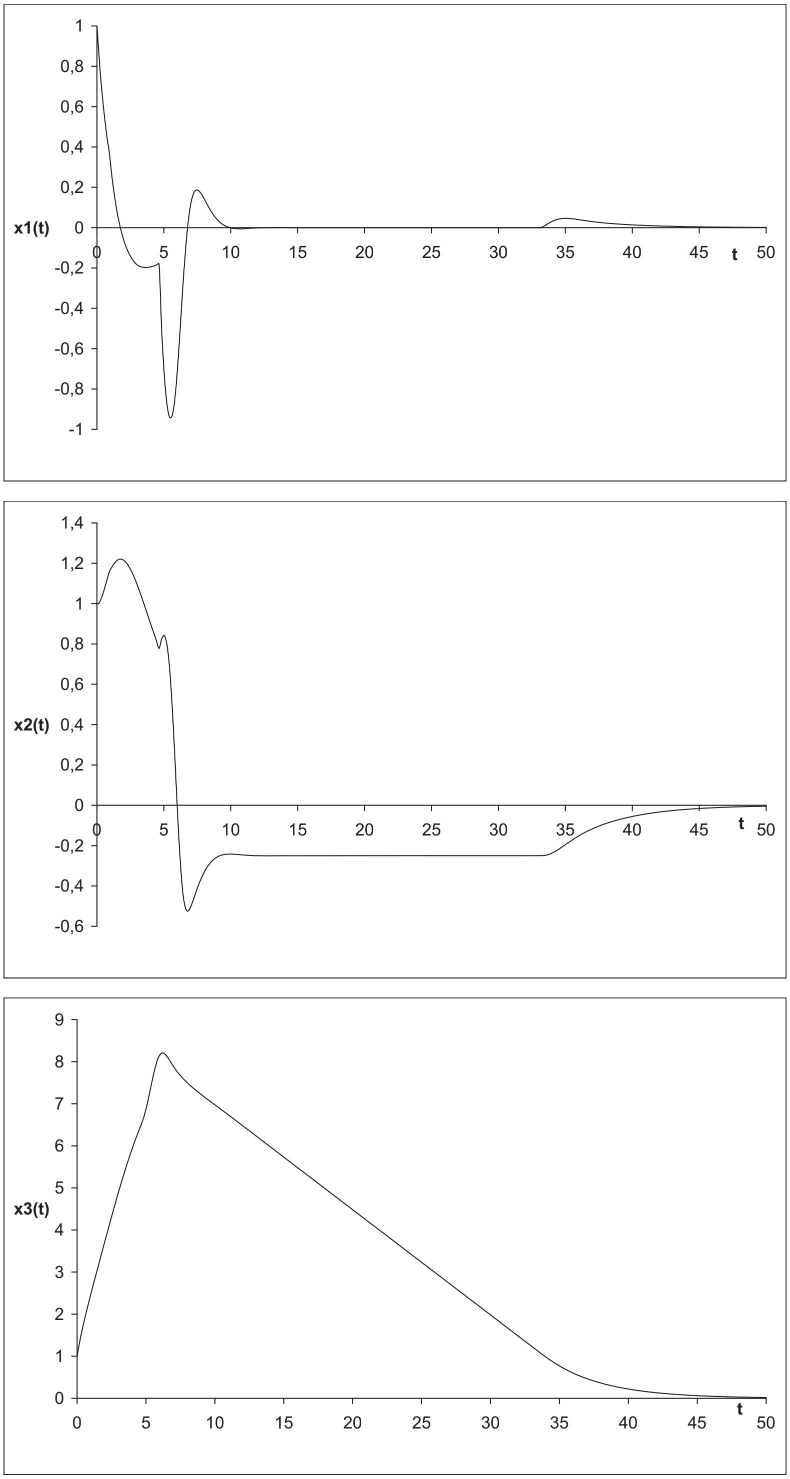

FIG. 1. *Time evolution of the state variables* $x_1(t), x_2(t), x_3(t)$ *for the closed-loop system* (4.1) *with* (3.18)*, where* k *is defined by* (4.6) *with* $r = 0.2$*,* $w(t) = \ln(\frac{2}{1+|\sin(t)|})$*, and initial condition* $x_1(0) = x_2(0) = x_3(0) = 1$.

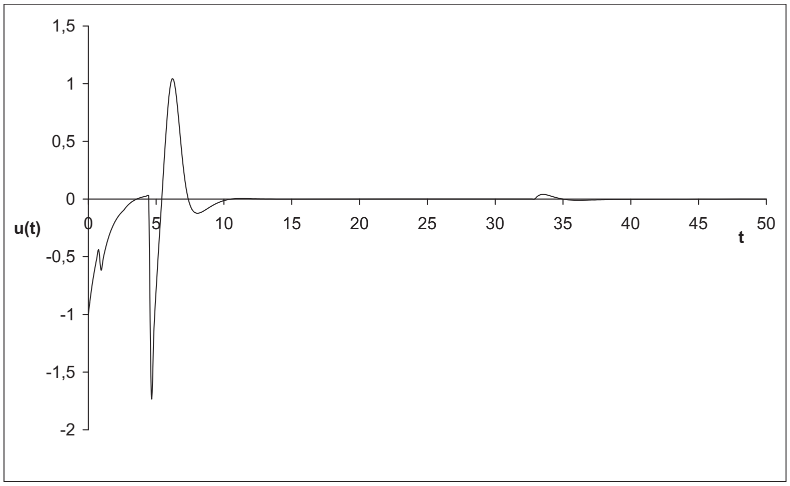

FIG. 2. *Time evolution of the input $u(t)$ for the closed-loop system* (4.1) *with* (3.18)*, where k is defined by* (4.6) *with $r = 0.2$, $w(t) = \ln(\frac{2}{1+|\sin(t)|})$, and initial condition $x_1(0) = x_2(0) = x_3(0) = 1$.*

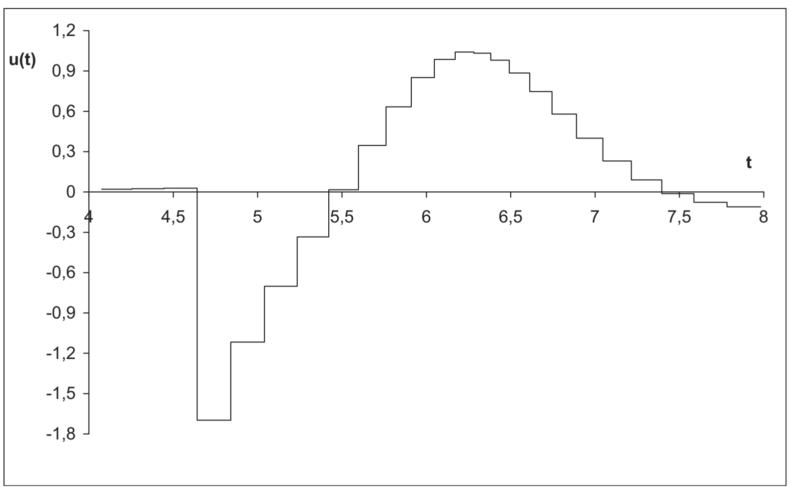

FIG. 3. *Time evolution of the input $u(t)$, $t \in [4, 8]$ for the closed-loop system* (4.1) *with* (3.18)*, where k is defined by* (4.6) *with $r = 0.2$, $w(t) = \ln(\frac{2}{1+|\sin(t)|})$, and initial condition $x_1(0) = x_2(0) = x_3(0) = 1$.*

$f : D \times \Re^n \to \Re^n$, $g : D \times \Re^n \to \Re$ are locally Lipschitz mappings that satisfy the following inequalities:

$$\max\{\,|f(d,x)|\,,\,|g(d,x)|\,\} \le L_1\,|x| \;\; \forall (d,x) \in D \times B(0,\rho), \tag{4.8}$$

$$|f(d,x)| \le L_2\,|x| \;\; \forall (d,x) \in D \times \Re^n \tag{4.9}$$

for certain constants $L_2 \ge L_1 > 0$ and $\rho > 0$. At this point we should note the crucial difference between (4.8), (4.9), and (3.17). While in (3.17) the nonlinearities f and g are restricted to be locally quadratic, and in (4.8), (4.9) the nonlinearities are allowed to have linear growth at the origin.

We also assume the existence of a symmetric positive definite matrix $P \in \Re^{n\times n}$, a constant vector $p \in \Re^n$, and a constant $q > 0$ such that the matrix $P(A + bp') + (A' + pb')P$ is negative definite and such that

$$x'P(A + bp')x + x'Pf(d, x) \leq -q\,|x|^2 \quad \forall\ (d, x) \in D \times \Re^n. \tag{4.10}$$

Finally, we assume that

$$L_1 < \frac{qa_1\,|c'b|}{(1 + |c|)\,a_2\,|Pb|}, \tag{4.11}$$

where $c = -(A' + pb')^{-1}(0, \ldots, 0, 1)'$ and $a_2 \geq a_1 > 0$ are constants satisfying

$$a_1^2 x'Px \leq |x|^2 \leq a_2^2 x'Px \quad \forall x \in \Re^n. \tag{4.12}$$

Inequalities (4.8), (4.9), (4.10), and (4.11) are the "translation" of inequalities (3.3), (3.4), (3.5) for the case (4.7). Therefore, inequalities (4.8), (4.9), (4.10), and (4.11) guarantee that the control strategy of driving y close to zero, while keeping the state in a neighborhood of the set $x = 0$, is feasible.

We show next that for every $\omega, R > 0$ with $a_2 R \leq \rho$ there exist constants $K, \tilde{r} > 0$ such that $0 \in \Re \times \Re^n$ is RGAS for the closed-loop system (4.7) with

$$\begin{aligned} &u(t) = \tilde{k}(x(\tau_i), y(\tau_i)),\ t \in [\tau_i, \tau_{i+1}),\\ &\tau_{i+1} = \tau_i + \tilde{r}\exp(-w(\tau_i)),\ \tau_0 = 0,\\ &d(t) \in D,\ w(t) \in \Re_+, \end{aligned} \tag{4.13}$$

where $\tilde{k} : \Re^n \times \Re \to \Re$ is defined by

$$\tilde{k}(x, y) := \begin{cases} p'x & \text{if } x'Px \geq R^2,\\ p'x - Kc'b\,\mathrm{sat}(\omega(y + c'x)) & \text{if } x'Px < R^2. \end{cases} \tag{4.14}$$

Indeed, this can be shown by a direct application of the sampled-data forwarding lemma with $k(x) := p'x$, arbitrary $\omega, R > 0$ with $a_2 R \leq \rho$, constants $M, K > 0$ satisfying

$$\frac{(1 + |c|)\,L_1 a_2 R}{|c'b|^2} < K < \frac{qa_1 R}{|Pb|\,|c'b|} \text{ and } M = \frac{K\,|c'b|\,\omega\,|Pb|}{(1 + |c|)\,L_1}, \tag{4.15}$$

and sufficiently small constant $\delta > 0$. Notice that by virtue of (4.10) and (4.12), inequality (3.3) is satisfied provided that $K\,|Pb|\,|c'b| < qa_1 R$ (a direct consequence of (4.15)). Moreover, since $a_2 R \leq \rho$, it follows from (4.8) and (4.12) that inequality (3.4) holds provided that $(1 + |c|)\,L_1 a_2 R < K(c'b)^2$ (a direct consequence of (4.15)). Finally, using the fact that $a_2 R \leq \rho$ in conjunction with (4.8), (4.10), (4.12), we conclude that inequality (3.5) with M as defined in (4.15) holds for sufficiently small $\delta > 0$ provided $(1 + |c|)\,L_1\,|Pb| < q\,|c'b|$ (a direct consequence of (4.11)).

The only thing that remains to be shown is that hypothesis (H) of Lemma 3.1 holds with $k(x) := p'x$ and sufficiently small $r > 0$. By virtue of (4.10), we notice that for every $(x_0, d, w) \in \Re^n \times \mathrm{L}^\infty_{loc}(\Re_+; D) \times \mathrm{L}^\infty_{loc}(\Re_+; \Re_+)$, the solution of $\dot{x} = Ax + bu + f(d, x)$ with $x(0) = x_0$, $u(t) = p'x(\tau_i)$ satisfies the following differential inequality for almost all $t \in [\tau_i, \tau_{i+1})$:

$$\dot{V}(t) \leq -2q\,|x(t)|^2 + 2\,|x(t)|\,|Pb|\,|p|\,|x(\tau_i) - x(t)|\,, \tag{4.16}$$

where $V(t) = x'(t)Px(t)$, $\tau_{i+1} = \tau_i + r\exp(-w(\tau_i))$. By virtue of (4.11), it follows that the hypotheses of Lemma 2.3 hold for the absolutely continuous mapping $x : [\tau_i, \tau_{i+1}] \to \Re^n$. Using Lemma 2.3 and inequality (4.16), we conclude that for sufficiently small $r > 0$ there exists $\tilde{q} > 0$ such that the differential inequality $\dot{V}(t) \le -\tilde{q}\,|x(t)|^2$ holds for almost all $t \ge 0$. Therefore, hypothesis (H) of Lemma 3.1 holds as well.

Notice that the case (4.7) includes systems which are not necessarily feedforward systems. For example, the three-dimensional system

$$\begin{aligned} \dot{x}_1 &= k_1 d_1 x_1 + u, \\ \dot{x}_2 &= k_2 d_2 x_2 + x_1, \\ \dot{y} &= x_2 + d_3 g(x), \\ x &= (x_1, x_2)' \in \Re^2,\ y \in \Re, d = (d_1, d_2, d_3)' \in [-1,1]^3,\ u \in \Re, \end{aligned}$$

where $k_1, k_2 > 0$ and $g \in C^1(\Re^2; \Re)$ with $g(0) = 0$, is not a feedforward system of the form (1.1). Inequalities (4.8), (4.9) hold for every $\rho > 0$ with $L_2 = L_1 = \max\{k_1, k_2, \max_{x \in B(0,\rho)} |\nabla g(x)|\}$. Moreover, inequality (4.10) holds with $P = \begin{bmatrix} 1 & 1+k_2 \\ 1+k_2 & (1+k_2)^2+1 \end{bmatrix}$, $p' = -[\ 1 + S + k_2 \quad 1 + S(1+k_2)\ ]$, $S = \frac{1}{2} + k_1 + \frac{1}{2}(1 + k_2)^2(k_2 + k_1)^2$, $q = \frac{\sqrt{(1+k_2)^2+4} - 1 - k_2}{2 + 2k_2 + 2\sqrt{(1+k_2)^2+4}}$.

5. Concluding remarks. To construct a globally asymptotically stabilizing sampled-data feedback for feedforward systems subject to perturbations in the sampling schedule, we have developed the recursive sampled-data feedback synthesis tool—the sampled-data forwarding lemma, which is used to construct our main result in Theorem 3.7.

Useful examples have shown that formulae (3.19), (3.20), (3.21) can be used in a straightforward way in order to design a globally stabilizing sampled-data feedback for an uncertain feedforward system of the form (1.1) under hypothesis (A2) in section 4 of [18]. However, the selection of the parameters $K_i > 0$, $R_i > 0$ $(i = 1, \ldots, n-1)$ involved in formulae (3.19), (3.20), (3.21) is crucial for performance: low values for $K_i > 0$ will result in slow convergence of some state variables and high overshoot.

The results of the present paper in combination with the approach we introduced in [21] allow us to compensate any amount of actuation or sensing delay when controlling systems within the feedforward class using the sampled-data controllers introduced in the present paper.

Appendix.

Proof of Lemma 3.2. Define

$$\text{(A.1)} \qquad \begin{aligned} \delta := -\max\big\{ & x'P\,(Ax + f(d,x,u) + bu) \ :\ (d,x) \in D \times \Re^n,\ x'Px = R^2, \\ & |u - p'x| \le K\,|c'b| \,\big\} > 0. \end{aligned}$$

The fact that δ as defined by (A.1) is positive is a consequence of (3.3). Clearly, by virtue of continuity of the solution $x(t)$, there exists $\tau \in (0, \tau_1]$ such that $\max_{t\in[0,\tau]} x'(t)Px(t) < R^2$. The structure of system (3.7) guarantees that the solution of (3.7) exists for all times $\tau \in (0, \tau_1]$ with $\max_{t\in[0,\tau]} x'(t)Px(t) \le R^2$.

We prove by contradiction that $\max_{t\in[0,\tau_1]} x'(t)Px(t) < R^2$. We therefore assume that there exists $\tau \in (0, \tau_1]$ with $x'(\tau)Px(\tau) \ge R^2$. We define

$$\text{(A.2)} \qquad T := \inf\left\{ t \in [0, \tau_1] \ :\ x'(t)Px(t) \ge R^2 \right\}$$

and notice that $T \in (0, \tau_1]$. Definition (A.2) and continuity of the solution $x(t)$ imply that $\max_{t\in[0,T]} x'(t)Px(t) = x'(T)Px(T) = R^2$. Define $V(t) = x'(t)Px(t)$ and $z(t) = y(t) + c'x(t)$. Notice that inequalities (3.2), (3.4) and the fact that $u(t) := p'x(0) - Kc'b\,\mathrm{sat}(\omega\, z(0))$ for all $t \in [0,T)$ imply that the following inequality holds for almost all $t \in [0,T)$:

$$\begin{aligned}\dot{V}(t) \le & -2x'(t)P\left(Ax(t) + b(p'x(t)+v) + f(d(t),x(t),p'x(t)+v)\right) \\ & + 2x'(t)P\left(f(d(t),x(t),p'x(0)+v) - f(d(t),x(t),p'x(t)+v)\right) \\ & + 2x'(t)Pbp'(x(0)-x(t)),\end{aligned} \tag{A.3}$$

where $v := -Kc'b\,\mathrm{sat}(\omega\, z(0))$. The differential equation (3.7), in conjunction with $\max_{t\in[0,T]} x'(t)Px(t) \le R^2$ and $u(t) := p'x(0) - Kc'b\,\mathrm{sat}(\omega\, z(0))$, implies that there exists a constant $S > 0$ such that $|\dot{x}(t)| \le S$ for almost all $t \in [0,T)$. Consequently, the following inequality holds for all $t \in [0,T]$:

$$|x(t) - x(0)| \le S\, t. \tag{A.4}$$

Using the facts that $f(d,x,u)$ is locally Lipschitz, $t \le T \le \tau_1 \le \tilde{r}$, $|v| \le K|c'b|$, and $\max_{t\in[0,T]} x'(t)Px(t) = x'(T)Px(T) = R^2$ in conjunction with definition (A.1) and inequalities (A.3), (A.4), we guarantee the existence of a constant $L > 0$ such that the following inequality holds for almost all $t \in [0,T)$ sufficiently close to T:

$$\dot{V}(t) \le -\delta + L\,\tilde{r}. \tag{A.5}$$

If $\tilde{r} > 0$ is sufficiently small, then inequality (A.5) implies that $\dot{V}(t) \le -\delta/2$ for almost all $t \in [0,T)$ sufficiently close to T. This contradicts the assumption $\max_{t\in[0,T]} x'(t)$ $Px(t) = x'(T)Px(T) = R^2$. The proof is complete. ◻

Proof of Lemma 3.4. Using the fact that $c = -(A'+pb')^{-1}(0,\dots,0,1)'$, we obtain $c'b \ne 0$. To see why $c'b \ne 0$, notice that the definition $c = -(A'+pb')^{-1}(0,\dots,0,1)'$ implies $(A'+pb')c = -(0,\dots,0,1)'$. Consequently, if $c'b = 0$, then we obtain $A'c = -(0,\dots,0,1)'$ and $(0,\dots,0,1)A'c = -1$, which is a contradiction, since $(0,\dots,0,1)A'x = 0$ for all $x \in \Re^n$ (notice that $a_{i,n} = 0$ for all $i = 1,\dots,n$).

The definition $c = -(A'+pb')^{-1}(0,\dots,0,1)'$ implies that $x_n + c'Ax = -c'bp'x$ for all $x \in \Re^n$. The previous equality, in conjunction with the fact that $u(t) := p'x(0) - Kc'b\,\mathrm{sat}(\omega\, z(0))$ for all $t \in [0,\tau_1)$, implies that the following differential equation holds for almost all $t \in [0,\tau_1)$:

$$\dot{z}(t) = c'bp'(x(0)-x(t)) + g(d(t),\, x(t), u(t)) + c'f(d(t),x(t),u(t)) - K(c'b)^2\mathrm{sat}\,(\omega\, z(0))\,. \tag{A.6}$$

Define

$$\begin{aligned}J := \max\big\{&|g(d,x,u) + c'f(d,x,u)| \,:\, x'Px \le R^2,\, d \in D, \\ &|u - p'x| \le K\,|c'b|\big\} < K(c'b)^2.\end{aligned} \tag{A.7}$$

Using definition (A.7) and the fact that the mappings f, g are locally Lipschitz, and since $x'(t)Px(t) < R^2$ for all $t \ge 0$ (a consequence of Lemma 3.3), we obtain for all $t \in [0,\tau_1)$,

$$\begin{aligned}&|g(d(t),x(t),u(t)) + c'f(d(t),x(t),u(t))| \\ &\quad\le |g(d(t),x(t),p'x(t)+v) + c'f(d(t),x(t),k'x(t)+v)| \\ &\qquad + |g(d(t),x(t),p'x(0)+v) - g(d(t),x(t),p'x(t)+v)| \\ &\qquad + |c|\; |f(d(t),x(t),p'x(0)+v) - f(d(t),x(t),p'x(t)+v)| \\ &\quad\le J + L\,|x(t) - x(0)|\,,\end{aligned}$$

where $v = -Kc'b\,\mathrm{sat}(\omega\, z(0))$ and $L > 0$ is an appropriate constant. Assuming that $K(c'b)^2\omega\,\tilde{r} < 1$, integrating (A.6), and exploiting the above inequality, we conclude that the following inequality holds for all $t \in [0, \tau_1]$:

$$|z(t)| \leq \left(1 - t\frac{K\,(c'b)^2\,\omega}{\max\left(1, \omega\,\,|z(0)|\right)}\right)|z(0)| + t\,\left[Q \max_{0\leq s\leq t}|x(s) - x(0)| + J\right], \tag{A.8}$$

where $Q > 0$ is an appropriate constant. The differential equations (3.7) in conjunction with $\max_{t\in[0,\tau_1]} x'(t)Px(t) \leq R^2$ and $u(t) := p'x(0) - Kc'b\,\mathrm{sat}(\omega\, z(0))$ imply that there exists a constant $S > 0$ such that $|\dot{x}(t)| \leq S$ for almost all $t \in [0, \tau_1)$. Consequently, inequality (A.4) holds for all $t \in [0, \tau_1]$. Combining (A.4), (A.8), we can conclude that the following inequality holds for all $t \in [0, \tau_1]$:

$$|z(t)| \leq \left(1 - t\frac{K\,(c'b)^2\,\omega}{\max\left(1, \omega\,\,|z(0)|\right)}\right)|z(0)| + QS\,t^2 + J\,t. \tag{A.9}$$

The above inequality, in conjunction with inequality (3.4) (which implies that $J < K(c'b)^2$), shows that the following implications hold for sufficiently small $\tilde{r} > 0$:

1. If $\omega\,|z(0)| \geq 1$, then $|z(t)| \leq |z(0)|$ for all $t \in [0, \tau_1]$.
2. If $\omega\,|z(0)| \leq 1$, then $|z(t)| \leq \omega^{-1}$ for all $t \in [0, \tau_1]$.

It follows that $|z(t)| \leq \max(\,|z(0)|, \omega^{-1})$ for all $t \in [0, \tau_1]$. Using induction, it can be shown that

$$|z(t)| \leq \max\left(|z(\tau_i)|\,, \omega^{-1}\right) \text{ when } t \in [\tau_i, \tau_{i+1}] \quad \forall i = 0, 1, 2, \ldots. \tag{A.10}$$

Moreover, inequality (A.9) shows that the following implication holds for sufficiently small $\tilde{r} > 0$:

$$\text{If } \omega\,|z(\tau_i)| \geq 1, \text{ then } |z(\tau_{i+1})| \leq |z(\tau_i)|\, - G\,\left(\tau_{i+1} - \tau_i\right), \tag{A.11}$$

where $G > 0$ is an appropriate constant.

Implication (A.11) shows that (3.9) holds with $T(z) := \frac{\max(\,0,\,\omega|z|\,-1)}{\omega G} + \tilde{r}$. Indeed, we prove this by contradiction. Suppose that there exists $t > \frac{\max(\,0,\,\omega|z(0)|\,-1)}{\omega G} + \tilde{r}$ with $|z(t)| > \omega^{-1}$. Let $m \in Z^+$ be the largest integer with $\tau_m \leq t < \tau_{m+1}$. By virtue of (A.10) we conclude that $|z(\tau_m)| > \omega^{-1}$. Moreover, since $\tau_{m+1} \leq \tau_m + \tilde{r}$ it follows that $\tau_m > \frac{\max(\,0,\,\omega|z(0)|\,-1)}{\omega G}$. Estimate (A.10) shows that $|z(\tau_i)| > \omega^{-1}$ for all $i = 0, \ldots, m$. Implication (A.11) gives $|z(\tau_m)| \leq |z(0)| - \tau_m G$, which is a contradiction.

The proof is complete. ◻

Proof of Lemma 3.5. By virtue of Lemmas 3.3 and 3.4 the solution of (3.7) satisfies $x'(t)Px(t) < R^2$, $|z(t)| \leq \omega^{-1}$ for all $t \geq 0$. Let $t \geq 0$ be a time where $V(t) = \frac{M}{2}z^2(t) + \frac{1}{2}x'(t)Px(t)$ is differentiable. Let $m \in Z^+$ be the largest integer with $\tau_m \leq t < \tau_{m+1}$. Using (A6) we obtain

$$\dot{V}(t) = S_1(t) + S_2(t), \tag{A.12}$$

where

$$\begin{aligned}
\text{(A.13)}\quad S_1(t) := &-MK(c'b)^2\omega\, z^2(t) - Kc'b\omega z(t)x'(t)Pb \\
&+ x'(t)P(Ax(t) + bp'x(t) + f(d(t), x(t), p'x(t) - Kc'b\omega z(t))) \\
&+ Mz(t)g(d(t),\, x(t), p'x(t) - Kc'b\omega z(t)) \\
&+ Mz(t)c'f(d(t),\, x(t), p'x(t) - Kc'b\omega z(t)),
\end{aligned}$$

$$\begin{aligned}
\text{(A.14)}\quad S_2(t) := &\,[Mz(t)c'b + x'(t)Pb]\, p'(x(\tau_m) - x(t)) \\
&+ Mz(t)[g(d(t),\, x(t), p'x(\tau_m) - Kc'b\omega z(\tau_m)) \\
&\qquad - g(d(t),\, x(t), p'x(t) - Kc'b\omega z(t))] \\
&+ (x'(t)P + Mz(t)c')\,[f(d(t), x(t), p'x(\tau_m) - Kc'b\omega z(\tau_m)) \\
&\qquad\qquad - f(d(t), x(t), p'x(t) - Kc'b\omega z(t))] \\
&+ Kc'b\omega\,[Mc'b\, z(t) + x'(t)Pb]\,(z(t) - z(\tau_m)).
\end{aligned}$$

Notice that inequality (3.5) implies that

$$\text{(A.15)}\qquad S_1(t) \le -\delta\, z^2(t) - \delta\, |x(t)|^2\,.$$

Moreover, since the mappings f, g are locally Lipschitz and $x'(t)Px(t) < R^2$, $|z(t)| \le \omega^{-1}$ for all $t \ge 0$, it follows that the hypotheses of Lemma 2.3 hold on the interval $[\tau_m, \tau_{m+1}]$ for the absolutely continuous map $(z(t), x(t))$ for appropriate constants Q, G. Therefore, for sufficiently small $\tilde{r} > 0$, there exists $\Gamma > 0$ such that the following inequality holds for all $t \in [\tau_m, \tau_{m+1}]$:

$$\text{(A.16)}\qquad \max\left(\,|x(t) - x(\tau_m)|\,,\, |z(t) - z(\tau_m)|\,\right) \le \Gamma\, \tilde{r}\, |(x(t), z(t))|\,.$$

Using the facts that the mappings f, g are locally Lipschitz and $x'(t)Px(t) < R^2$, $x'(\tau_m)Px(\tau_m) < R^2$, $|z(\tau_m)| \le \omega^{-1}$, $|z(t)| \le \omega^{-1}$ in conjunction with inequality (A.16) and definition (A.14), we obtain

$$\text{(A.17)}\qquad S_2(t) \le \Gamma\, \tilde{r}q\, |z(t)|^2 + \Gamma\, \tilde{r}q\, |x(t)|^2$$

for a certain appropriate constant $q > 0$. Selecting $\tilde{r} > 0$ sufficiently small and using inequalities (A.15), (A.17), we can conclude that (3.10) holds with $\mu = \frac{\delta}{2}$. The proof is complete. $\square$

Proof of Lemma 3.6. Since $P \in \Re^{n\times n}$ is a symmetric positive definite matrix, there exist constants $a_2 \ge a_1 > 0$ satisfying

$$\text{(A.18)}\qquad a_1^2 x'Px \le |x|^2 \le a_2^2 x'Px \quad \forall x \in \Re^n.$$

Since $P(A + bp') + (A' + pb')P$ is negative definite, there exists a constant $q > 0$ such that

$$\text{(A.19)}\qquad x'P(A + bp')x \le -q\, |x|^2 \quad \forall x \in \Re^n.$$

Let $C > 0$ be an arbitrary number with $C < \frac{qa_1}{|Pb||c'b|}$, $C \le 1$, and let $R^* > 0$ be a positive number such that

$$\text{(A.20)}\; Q\,(R^*)\, a_2 R^* < \frac{q\,|c'b|}{(1 + |c|)\left(\frac{\lambda\, q}{1+|p|} + |Pb|\right)\left(1 + |p| + a_2^{-1} C\, |c'b|\right) + (1 + |p|)\, |P|\, |c'b|},$$

$$\text{(A.21)}\; Q\,(R^*)\, a_2 R^* < \frac{C(c'b)^2}{(1 + |c|)\,((1 + |p|)\, a_2 + C\, |c'b|)},$$

$$\text{(A.22)}\; Q(R^*) a_2 R^* < \frac{qa_1 - C\, |Pb|\, |c'b|}{(1 + |p|)\, |P|\, a_2 + |P|\, C\, |c'b|},$$

where $Q(R) := L((1+|p|)a_2 R + R|c'b|)$. We claim that there exists a constant $\delta > 0$ such that (3.3), (3.4), (3.5) hold with arbitrary $\omega > 0$, $R \in (0, R^*)$, $K = CR$, $M = \frac{C|c'b|\omega}{(1+|c|)Q(R)} \frac{|P|Q(R)a_2R+|Pb|}{(1+|p|)a_2+C|c'b|}$ for the case $Q(R) > 0$, and $M = \frac{CR|Pb|^2\omega}{4q} + 1$ for the case $Q(R) = 0$. Indeed, using (3.17), (A.18), (A.19), in conjunction with the fact that $C \le 1$, we conclude that conditions (3.3), (3.4) with $K = CR$ are satisfied provided that

$$(1+|p|)\,|P|\,Q(R)a_2^2 R + (|Pb| + |P|\,Q(R)a_2 R)\,C\,|c'b| < qa_1, \tag{A.23}$$

$$(1+|c|)\,(1+|p|)\,Q\,(R)\,a_2^2 R + (1+|c|)\,Q\,(R)\,a_2 CR\,|c'b| < C(c'b)^2. \tag{A.24}$$

Inequalities (A.23), (A.24) are direct consequences of (A.21), (A.22), and the fact that $R \le R^*$. Finally, after tedious calculations and using (3.17), (A.18), (A.19) in conjunction with the fact that $C \le 1$, we conclude that condition (3.5) is satisfied provided that there exists $\delta > 0$ such that the following inequality holds for all $(x,z) \in \Re^n \times \Re$:

$$\begin{aligned}
&(M\,(1+|c|)\,(1+|p|)\,Q\,(R)\,a_2 R + K\,|P|\,|c'b|\,\omega Q\,(R)\,a_2 R + K\,|c'b|\,|Pb|\,\omega \\
&\quad +M\,(1+|c|)\,Q\,(R)\,K\,|c'b|)\,|x|\,|z| - MK(c'b)^2\omega z^2 \\
&\quad - (q - (1+|p|)\,|P|\,Q\,(R)\,a_2 R)\,|x|^2 \\
&\quad \le -\delta\,|z|^2 - \delta\,|x|^2\,.
\end{aligned}$$

The existence of sufficiently small $\delta > 0$ such that the above inequality holds is a direct consequence of (A.20), the facts that $K = CR$, $R \le R^*$, and the selections $M = \frac{C|c'b|\omega}{(1+|c|)Q(R)} \frac{|P|Q(R)a_2R+|Pb|}{(1+|p|)a_2+C|c'b|}$ for the case $Q(R) > 0$ and $M = \frac{CR|Pb|^2\omega}{4q} + 1$ for the case $Q(R) = 0$. The proof is complete. $\square$

REFERENCES

[1] X. Albouy and L. Praly, *On the use of dynamic invariants and forwarding for swinging up a spherical inverted pendulum*, in Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, NSW, Austrialia, 2000, pp. 1667–1672.

[2] M. Arcak, A. R. Teel, and P. V. Kokotovic, *Robust nonlinear control of feedforward systems with unmodeled dynamics*, Automatica, 37 (2001), pp. 265–272.

[3] C. Barbu, R. Sepulchre, W. Lin, and P. V. Kokotovic, *Global asymptotic stabilization of the ball-and-beam system*, in Proceedings of the 36th IEEE Conference on Decision and Control, San Diego, CA, 1997, pp. 2351–2355.

[4] J. M. Coron, L. Praly, and A. R. Teel, *Feedback stabilization of nonlinear systems: Sufficient conditions and Lyapunov and input-output techniques*, in Trends in Control, A. Isidori, ed., Springer-Verlag, Berlin, 1995, pp. 293–347.

[5] C. De Persis, *n-bit stabilization of n-dimensional nonlinear systems in feedforward form*, IEEE Trans. Automat. Control, 50 (2005), pp. 299–311.

[6] C. De Persis, *Minimal data rate stabilization of nonlinear systems over networks with large delays*, Internat. J. Robust Nonlinear Control, 20 (2010), pp. 1097–1111.

[7] F. Grognard, R. Sepulchre, and G. Bastin, *Global stabilization of feedforward systems with exponentially unstable Jacobian linearization*, Syst. Control Lett., 37 (1999), pp. 107–115.

[8] F. Grognard, R. Sepulchre, G. Bastin, and L. Praly, *Nested linear low-gain design for semiglobal stabilization of feedforward systems*, Journal A Soft Vision, 41 (2000), pp. 35–40.

[9] L. Grüne, *Homogeneous state feedback stabilization of homogeneous systems*, SIAM J. Control Optim., 38 (2000), pp. 1288–1308.

[10] L. Grüne and D. Nešić, *Optimization-based stabilization of sampled-data nonlinear systems via their approximate discrete-time models*, SIAM J. Control Optim., 42 (2003), pp. 98–122.

[11] G. Herrmann, S. K. Spurgeon, and C. Edwards, *Discretization of sliding mode based control schemes*, in Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, AZ, 1999, pp. 4257–4262.

[12] M. Jankovic, *Forwarding, backstepping, and finite spectrum assignment for time delay systems*, Proceedings of the American Control Conference, New York, NY, 2006, pp. 5618–5624.
[13] M. Jankovic, *Control of cascade systems with time delay-the integral cross-term approach*, Proceedings of the 45th IEEE Conference on Decision and Control, San Diego, CA, 2006, pp. 2547–2552.
[14] M. Jankovic, *Recursive predictor design for linear systems with time delay*, Proceedings of the American Control Conference, Seattle, WA, 2008, pp. 4904–4909.
[15] M. Jankovic, R. Sepulchre, and P. V. Kokotovic, *Constructive Lyapunov stabilization of nonlinear cascade systems*, IEEE Trans. Automat. Control, 41 (1996), pp. 1723–1735.
[16] M. Jankovic, R. Sepulchre, and P. V. Kokotovic, *Global adaptive stabilization of cascade nonlinear systems*, Automatica, 33 (1997), pp. 263–268.
[17] I. Karafyllis, *System-theoretic framework for a wide class of systems* I: *Applications to numerical analysis*, J. Math. Anal. Appl., 328 (2007), pp. 876–899.
[18] I. Karafyllis and Z.-P. Jiang, *A small-gain theorem for a wide class of feedback systems with control applications*, SIAM J. Control Optim., 46 (2007), pp. 1483–1517.
[19] I. Karafyllis and C. Kravaris, *Global stability results for systems under sampled-data control*, Internat. J. Robust Nonlinear Control, 19 (2009), pp. 1105–1128.
[20] I. Karafyllis and C. Kravaris, *Robust global stabilizability by means of sampled-data control with positive sampling rate*, Internat. J. Control, 82 (2009), pp. 755–772.
[21] I. Karafyllis and M. Krstic, *Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold*, IEEE Trans. Automat. Control, 57 (2012), pp. 1141–1154.
[22] C. M. Kellett, H. Shim, and A. R. Teel, *Further results on robustness of (possibly discontinuous) sample and hold feedback*, IEEE Trans. Automat. Control, 49 (2004), pp. 1081–1089.
[23] H. K. Khalil, *Performance recovery under output feedback sampled-data stabilization of a class of nonlinear systems*, IEEE Trans. Automat. Control, 49 (2004), pp. 2173–2184.
[24] M. Krstic, *Feedback linearizability and explicit integrator forwarding controllers for classes of feedforward systems*, IEEE Trans. Automat. Control, 49 (2004), pp. 1668–1682.
[25] M. Krstic, *Input delay compensation for forward complete and feedforward nonlinear systems*, IEEE Trans. Automat. Control, 55 (2010), pp. 287–303.
[26] D. S. Laila, D. Nešić, and A. R. Teel, *Open and closed-loop dissipation inequalities under sampling and controller emulation*, European J. Control, 18 (2002), pp. 109–125.
[27] D. S. Laila and A. Astolfi, *Input-to-state stability for discrete-time time-varying systems with applications to robust stabilization of systems in power form*, Automatica, 41 (2005), pp. 1891–1903.
[28] W. Lin and X. Li, *Synthesis of upper-triangular nonlinear systems with marginally unstable free dynamics using state-dependent saturation*, Internat. J. Control, 72 (1999), pp. 1078–1086.
[29] W. Lin and C. Qian, *New results on global stabilization of feedforward systems via small feedback*, in Proceedings of the 37th IEEE Conference on Decision and Control, Phoenix, AZ, 1998, pp. 873–878.
[30] L. Marconi and A. Isidori, *Stabilization of nonlinear feedforward systems: A robust approach*, in Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, FL, 2001, pp. 2778–2783.
[31] F. Mazenc and L. Praly, *Adding integrations, saturated controls, and stabilization for feedforward systems*, IEEE Trans. Automat. Control, 41 (1996), pp. 1559–1578.
[32] F. Mazenc, *Stabilization of feedforward systems approximated by a nonlinear chain of integrators*, Systems Control Lett., 32 (1997), pp. 223–229.
[33] F. Mazenc and H. Nijmeijer, *Forwarding in discrete-time nonlinear systems*, Internat. J. Control, 71 (1998), pp. 823–835.
[34] F. Mazenc, R. Sepulchre, and M. Jankovic, *Lyapunov functions for stable cascades and applications to global stabilization*, IEEE Trans. Automat. Control, 44 (1999), pp. 1795–1800.
[35] F. Mazenc and L. Praly, *Asymptotic tracking of a reference state for systems with a feedforward structure*, Automatica, 36 (2000), pp. 179–187.
[36] F. Mazenc and L. Praly, *Strict Lyapunov functions for feedforward systems and applications*, in Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, NSW, Australia, 2000, pp. 3875–3880.
[37] F. Mazenc and S. Bowong, *Tracking trajectories of feedforward systems: Application to the cart-pendulum system*, in Proceedings of the 40th IEEE Conference on Decision and Control, 2001, pp. 479–484.

[38] F. Mazenc and S. Bowong, *Tracking trajectories of feedforward systems*, IEEE Trans. Automat. Control, 47 (2002), pp. 1338–1342.
[39] F. Mazenc and J. C. Vivalda, *Global asymptotic output feedback stabilization of feedforward systems*, European J. Control, 8 (2002), pp. 519–530.
[40] F. Mazenc, S. Mondie, and R. Francisco, *Global asymptotic stabilization of feedforward systems with delay at the input*, IEEE Trans. Automat. Control, 49 (2004), pp. 844–850.
[41] D. Nešić, A. R. Teel, and P. V. Kokotovic, *Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations*, Systems Control Lett., 38 (1999), pp. 259–270.
[42] D. Nešić and A. R. Teel, *Sampled-data control of nonlinear systems: An overview of recent results*, in Perspectives on Robust Control, R.S.O. Moheimani, ed., Springer-Verlag, New York, 2001, pp. 221–239.
[43] D. Nešić and A. R. Teel, *A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models*, IEEE Trans. Automat. Control, 49 (2004), pp. 1103–1122.
[44] D. Nešić and L. Grüne, *Lyapunov-based continuous-time nonlinear controller redesign for sampled-data implementation*, Automatica, 41 (2005), pp. 1143–1156.
[45] D. Nešić, A. R. Teel, and D. Carnevale, *Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems*, IEEE Trans. Automat. Control, 54 (2009), pp. 619–624.
[46] L. Praly, *An introduction to some Lyapunov designs of global asymptotic stabilizers*, in Lecture Notes for the Banach Center Summer School on Mathematical Control Theory, Poland, 2002, pp. 2–20.
[47] R. Sepulchre, M. Jankovic, and P. Kokotovic, *Constructive Nonlinear Control*, Springer, New York, 1997.
[48] R. Sepulchre, M. Jankovic, and P. V. Kokotovic, *Integrator forwarding: A new recursive nonlinear robust design*, Automatica, 33 (1997), pp. 979–984.
[49] I. A. Tall and W. Respondek, *Transforming a single-input nonlinear system to a strict feedforward form via feedback*, in Nonlinear Control in the Year, 2000, Vol. 2, A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek, eds., Lecture Notes in Control and Inform. Sci. 259, Springer, London, 2000, pp. 527–542.
[50] I. A. Tall, *Linearizable feedforward systems: A special class*, in Proceedings of the 17th IEEE International Conference on Control Applications, San Antonio, TX, 2008, pp. 1201–1206.
[51] I. A. Tall and W. Respondek, *On linearizability of strict feedforward systems*, in Proceedings of the 2008 American Control Conference, Seattle, WA, 2008, pp. 1929–1934.
[52] I. A. Tall and W. Respondek, *Feedback linearizability of strict feedforward systems*, in Proceedings of the 47th IEEE Conference on Decision and Control, Cancun, Mexico, 2008, pp. 2499–2504.
[53] A. R. Teel, *Feedback Stabilization: Nonlinear Solutions to Inherently Nonlinear Problems*, Ph.D. dissertation, University of California, Berkeley, CA, 1992.
[54] A. R. Teel, *A nonlinear small gain theorem for the analysis of control systems with saturation*, IEEE Trans. Automat. Control, 41 (1996), pp. 1256–1270.
[55] A. R. Teel, *On $\mathcal{L}_2$ performance induced by feedbacks with multiple saturations*, ESAIM Contrôle Optim. Calc. Variations, 1 (1995/96), pp. 225–240.
[56] J. Tsinias and M. P. Tzamtzi, *An explicit formula of bounded feedback stabilizers for feedforward systems*, Systems Control Lett., 43 (2001), pp. 247–261.
[57] Y. Xudong, *Universal stabilization of feedforward nonlinear systems*, Automatica, 39 (2003), pp. 141–147.