Technical Notes and Correspondence

Stabilization and Gevrey Regularity of a Schrödinger Equation in Boundary Feedback With a Heat Equation

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***Abstract*—We study stability of a Schrödinger equation with a collocated boundary feedback compensator in the form of a heat equation with a collocated input/output pair. Remarkably, exponential stability is achieved for both positive and negative gains, namely, as long as the gain is non-zero. We show that the spectrum of the closed-loop system consists only of two branches along two parabolas which are asymptotically symmetric relative to the line $\mathrm{Re}\lambda = -\mathrm{Im}\lambda$ (the 135° line in the second quadrant). The asymptotic expressions of both eigenvalues and eigenfunctions are obtained. The Riesz basis property and exponential stability of the system are then proved. Finally we show that the semigroup, generated by the system operator, is of Gevrey class $\delta > 2$. A numerical computation is presented for the distributions of the spectrum of the closed-loop system.**

***Index Terms*—Gevrey class.**

I. Introduction

Extensive literature exists on control of the Schrödinger equation [4]–[7], [11]. In this technical note we consider an interconnected system of Schrödinger and heat equations with boundary coupling (see Fig. 1). This configuration arises as a generalization of a static collocated output feedback for the Schrödinger equation proposed in [5], whose asymptotic spectrum is given by vertical lines. In this technical note we replace the static feedback by dynamic feedback governed by a heat equation with a collocated input/output pair and show the exponential stability and Gevrey regularity for the closed-loop system.

The spectrum of the closed loop system that we reveal in the technical note consists of two parabolas that are asymptotically symmetric relative to the line $\mathrm{Re}\lambda = -\mathrm{Im}\lambda$ (the 135° line in the second quadrant), where the spectrum generated by the heat equation is moved into the second quadrant by the influence of the Schrödinger equation.

Although, in some instances the solution of the Euler-Bernoulli beam can be obtained from the Schrödinger equation [7], significant differences arise due to boundary conditions [13] and such differences also lead to the distinct results between [15] and the present technical note.

For a single Schrödinger equation, in [5], the collocated boundary control is designed to exponentially stabilize the system

$$\begin{cases} w_t(x,t) + iw_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ w_x(1,t) = 0, & t \geq 0, \\ w(0,t) = U(t), & t \geq 0 \\ Y(t) = w_x(0,t), & t \geq 0 \end{cases} \tag{1}$$

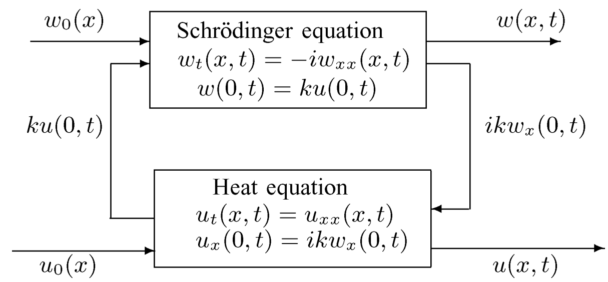

Fig. 1. Block diagram for the coupled Heat-Schrödinger system.

where $U(t)$ is the control input and $Y(t)$ is the output observation. When $U(t) = -icY(t)$, where $c > 0$ is a positive constant, the authors in [5] showed that 1) the system operator of the closed-loop system generates an exponentially stable semigroup in the energy space; and 2) the eigenvalues approach a vertical line parallel to the imaginary axis.

In this technical note, we present an alternative design method to system (1) and feed the output $Y(t)$ of the Schrödinger equation into the boundary heat flux of the heat equation while the boundary temperature of the heat equation is fed into the Schrödinger equation. Such a design improves the regularity of the closed-loop system that generates a Gevrey semigroup in the energy space, and moves the eigenvalues of the Schrödinger and heat equations into the second quadrant, which are approaching two asymptotically symmetric parabolas relative to the line $\mathrm{Re}\lambda = -\mathrm{Im}\lambda$ (the 135° line in the second quadrant).

An interconnected system of the Schrödinger and heat equations shown in Fig. 1 is written as the following closed-loop system:

$$\begin{cases} w_t(x,t) + iw_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ u_t(x,t) - u_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ w(1,t) = u_x(1,t) = 0, & t \geq 0, \\ w(0,t) = ku(0,t), & t \geq 0, \\ u_x(0,t) = ikw_x(0,t), & t \geq 0 \end{cases} \tag{2}$$

where $k \neq 0$. The energy function for (2) is given by

$$E(t) = \frac{1}{2} \int_0^1 \left[|w(x,t)|^2 + u^2(x,t) \right] dx. \tag{3}$$

We provide a detailed spectral analysis for the system (2). We show that there are two parabolas which are asymptotically symmetric relative to the line $\mathrm{Re}\lambda = -\mathrm{Im}\lambda$. The asymptotic expressions of the eigenvalues and eigenfunctions, the Riesz basis property and exponential stability of (2) are studied. Moreover, we show that the C_0-semigroup, generated by the system operator, is of Gevrey class $\delta > 2$. (Gevrey regularity is described in terms of the bounds on all derivatives of the semigroups. The differentiability of the Gevrey semigroup is slightly weaker than that of an analytic semigroup [9], [14].)

The rest of this note is organized as follows. We present the well-posedness of the system (2) and state the results in Section II. The proofs for the results are presented in Section III. A numerical computation of the eigenvalues is given in Section IV.

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II. Well-Posedness and Main Results

We consider the system (2) in the energy space $\mathcal{H} = L^2(0,1) \times L^2(0,1)$. The norm in $\mathcal{H}$ is induced by the following inner product:

$$\langle X_1, X_2\rangle = \int_0^1 \left[f_1(x)\overline{f_2(x)} + g_1(x)\overline{g_2(x,t)}\right] dx \tag{4}$$

where $X_s = (f_s, g_s) \in \mathcal{H}$, $s = 1, 2$. Define the system operator by

$$\begin{cases} \mathcal{A}(f,g) = (-if'', g''), \quad \forall (f,g) \in D(\mathcal{A}), \\ D(\mathcal{A}) = \left\{ (f,g) \in H^2(0,1) \times H^2(0,1) \,\middle|\, \begin{array}{l} f(1) = g'(1) = 0, \\ f(0) = kg(0), \\ g'(0) = ikf'(0). \end{array} \right. \end{cases} \tag{5}$$

Then (2) can be written as an evolution equation in $\mathcal{H}$

$$\begin{cases} \frac{dX(t)}{dt} = \mathcal{A}X(t), \quad t > 0, \\ X(0) = X_0. \end{cases} \tag{6}$$

where $X(t) = (w(\cdot,t), u(\cdot,t))$. Then we have the well-posedness of the system (6) as the following theorem.

Theorem 2.1: Let $\mathcal{A}$ be given by (5). Then $\mathcal{A}^{-1}$ exists and is compact. Hence, $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues of finite algebraic multiplicity only. Moreover $\mathcal{A}$ is dissipative in $\mathcal{H}$ and $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions in $\mathcal{H}$.

Let us now consider the eigenvalue problem of $\mathcal{A}X = \lambda X$, where $X = (f,g) \in D(\mathcal{A})$, if and only if f, g satisfy

$$\begin{cases} f''(x) - i\lambda f(x) = 0, \\ g''(x) - \lambda g(x) = 0, \\ f(1) = g'(1) = 0, \quad f(0) = kg(0), \quad g'(0) = ikf'(0). \end{cases} \tag{7}$$

Lemma 2.1: Let $\mathcal{A}$ be defined by (5). Then for each $\lambda \in \sigma(\mathcal{A})$, we have $\mathrm{Re}\lambda < 0$.

Due to Lemma 2.1 that all the eigenvalues are located in the left half complex plane, we consider

$$\lambda := i\rho^2, \quad \rho \in \mathcal{S} := \left\{\rho \in \mathbb{C} \middle| 0 \le \arg\rho \le \frac{\pi}{2}\right\}. \tag{8}$$

Note that if we denote $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ with

$$\begin{cases} \mathcal{S}_1 := \left\{\rho \in \mathbb{C} \middle| \frac{\pi}{8} < \arg\rho \le \frac{3\pi}{8}\right\}, \\ \mathcal{S}_2 := \left\{\rho \in \mathbb{C} \middle| 0 \le \arg\rho \le \frac{\pi}{8}\right\}, \\ \mathcal{S}_3 := \left\{\rho \in \mathbb{C} \middle| \frac{3\pi}{8} < \arg\rho \le \frac{\pi}{2}\right\} \end{cases} \tag{9}$$

then we have

$$\begin{cases} \forall \rho \in \mathcal{S}_1 \cup \mathcal{S}_3, \\ \mathrm{Re}(i\rho) = -|\rho|\sin(\arg\rho) \le -|\rho|\sin\left(\frac{1}{8}\pi\right) < 0; \\ \forall \rho \in \mathcal{S}_2, \\ \mathrm{Re}(-\sqrt{i}\rho) = -|\rho|\cos\left(\frac{\pi}{4} + \arg\rho\right) \le -|\rho|\cos\left(\frac{3}{8}\pi\right) < 0; \\ \forall \rho \in \mathcal{S}_3, \\ \mathrm{Re}(\sqrt{i}\rho) = |\rho|\cos\left(\frac{\pi}{4} + \arg\rho\right) \le -|\rho|\cos\left(\frac{3}{8}\pi\right) < 0. \end{cases} \tag{10}$$

Now substituting $\lambda = i\rho^2$ with $\rho \ne 0$ into (7), we have the eigenvalue system of (2) in ρ

$$\begin{cases} f''(x) + \rho^2 f(x) = 0, \\ g''(x) - i\rho^2 g(x) = 0, \\ f(1) = g'(1) = 0, \\ f(0) = kg(0), \quad g'(0) = ikf'(0). \end{cases} \tag{11}$$

Let

$$f(x) = ae^{i\rho x} + be^{-i\rho x}, \quad g(x) = ce^{\sqrt{i}\rho x} + de^{-\sqrt{i}\rho x} \tag{12}$$

where a, b and c, d are constants. Substituting these into the boundary conditions of (11), we have

$$\begin{cases} ae^{i\rho} + be^{-i\rho} = 0, \\ \sqrt{i}\rho[ce^{\sqrt{i}\rho} - de^{-\sqrt{i}\rho}] = 0, \\ a + b - ck - dk = 0, \\ \rho[ak - bk + c\sqrt{i} - d\sqrt{i}] = 0. \end{cases} \tag{13}$$

Then (11) has the nontrivial solution if and only if the characteristic equation $\det\Delta(\rho) = 0$, where

$$\Delta(\rho) = \begin{bmatrix} e^{i\rho} & e^{-i\rho} & 0 & 0 \\ 0 & 0 & e^{\sqrt{i}\rho} & -e^{-\sqrt{i}\rho} \\ 1 & 1 & -k & -k \\ k & -k & \sqrt{i} & -\sqrt{i} \end{bmatrix}. \tag{14}$$

Lemma 2.2: Let $\lambda = i\rho^2$ with $\rho \in \mathcal{S}$ and let $\Delta(\rho)$ be given by (11). Then the following expansion holds:

$$2\det\Delta(\rho) = a_1 e^{i\rho}e^{\sqrt{i}\rho} + a_2 e^{i\rho}e^{-\sqrt{i}\rho} + a_2 e^{-i\rho}e^{\sqrt{i}\rho} + a_1 e^{-i\rho}e^{-\sqrt{i}\rho} \tag{15}$$

where

$$a_1 = 2k^2 + \sqrt{2} + i\sqrt{2}, \quad a_2 = 2k^2 - \sqrt{2} - i\sqrt{2}. \tag{16}$$

Moreover, when $\rho \in \mathcal{S}_i$, $i = 1, 2, 3$, $\det\Delta(\rho)$ has more accurate asymptotic expansions, respectively

$$2e^{i\rho}\det\Delta(\rho) = a_2 e^{\sqrt{i}\rho} + a_1 e^{-\sqrt{i}\rho} + \mathcal{O}\left(e^{-c_1|\rho|}\right), \quad \rho \in \mathcal{S}_1, \tag{17}$$

$$2e^{-\sqrt{i}\rho}\det\Delta(\rho) = a_1 e^{i\rho} + a_2 e^{-i\rho} + \mathcal{O}\left(e^{-c_2|\rho|}\right), \quad \rho \in \mathcal{S}_2 \tag{18}$$

and

$$2e^{\sqrt{i}\rho}e^{i\rho}\det\Delta(\rho) = a_1 + \mathcal{O}\left(e^{-c_3|\rho|}\right), \quad \rho \in \mathcal{S}_3 \tag{19}$$

where c_1, c_2 and c_3 are three positive constants, and $\mathcal{O}(e^{-c_j|\rho|})$, $j = 1, 2, 3$ denote that there is a constant M_0 such that $\mathcal{O}(e^{-c_j|\rho|}) \le M_0 e^{-c_j|\rho|}$.

Remark 2.1: From (15), (16), it is found that the sign of the feedback gain k does not affect zero distributions of $\det\Delta(\rho)$, that is, the sign of the feedback gain k does not affect the eigenvalues of $\mathcal{A}$.

Theorem 2.2: Let $\mathcal{A}$ be defined by (5). The spectrum $\sigma(\mathcal{A})$ has two families

$$\sigma(\mathcal{A}) = \{\lambda_{1n}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, n \in \mathbb{N}\} \tag{20}$$

where λ_{1n} and λ_{2n} are asymptotically symmetric with respect to the straight line $\mathrm{Im}\lambda = -\mathrm{Re}\lambda$ on the spectrum λ-plane, and have the following asymptotic expansions:

$$\begin{cases} \lambda_{1n} = \frac{1}{4}(\ln r)^2 - \left[n\pi + \frac{1}{2}\theta_1\right]^2 \\ \qquad + \ln r\left[n\pi + \frac{1}{2}\theta_1\right] i + \mathcal{O}(e^{-c_1 n}), \\ \lambda_{2n} = -\left[n\pi + \frac{1}{2}\theta_2\right]\ln r \\ \qquad + \left[\left[n\pi + \frac{1}{2}\theta_2\right]^2 - \frac{1}{4}(\ln r)^2\right] i + \mathcal{O}(e^{-c_2 n}). \end{cases} \tag{21}$$

Here r, θ_1 and θ_2 are three constants given by

$$r=\frac{\sqrt{k^8+1}}{k^4-\sqrt{2}k^2+1}>1,\ \ln r>0,\ \ln r^{-1}=-\ln r<0 \tag{22}$$

$$\theta_1=\begin{cases}\pi+\arctan\frac{\sqrt{2}k^2}{k^4-1}, & |k|>1,\\ \frac{3\pi}{2}, & k=\pm 1,\\ -\arctan\frac{\sqrt{2}k^2}{1-k^4}, & 0<|k|<1\end{cases} \tag{23}$$

and

$$\theta_2=\begin{cases}-\pi-\arctan\frac{\sqrt{2}k^2}{k^4-1}, & |k|>1,\\ \frac{\pi}{2,} & k=\pm 1,\\ \arctan\frac{\sqrt{2}k^2}{1-k^4}, & 0<|k|<1.\end{cases} \tag{24}$$

Therefore

$$\text{Re}\lambda_{1n},\quad \text{Re}\lambda_{2n}\to-\infty,\quad \text{as}\quad n\to\infty. \tag{25}$$

We now investigate the asymptotic behavior of the eigenfunctions.

Theorem 2.3: Let $\mathcal{A}$ be defined by (5) and let $\sigma(\mathcal{A})=\{\lambda_{1n}, n\in\mathbb{N}\}\cup\{\lambda_{2n}, n\in\mathbb{N}\}$ be the spectrum of $\mathcal{A}$. Then there are two families of approximate normalized eigenfunctions of $\mathcal{A}$:

(i) one family $\{\Phi_{1n}, n\in\mathbb{N}\}$, where $\Phi_{1n}=(f_{1n},g_{1n})$ is the eigenfunction of $\mathcal{A}$ with respect to the eigenvalue λ_{1n}, has the following asymptotic expression:

$$\Phi_{1n}(x)=\begin{pmatrix}f_{1n}(x)\\ g_{1n}(x)\end{pmatrix}=\begin{pmatrix}0\\ \cos\left[\left[n\pi+\frac{1}{2}\theta_1\right](1-x)-\frac{1}{2}i\ln r(1-x)\right]\end{pmatrix}+\mathcal{O}(e^{-c_1 n}) \tag{26}$$

(ii) the other family $\{\Phi_{2n}, n\in\mathbb{N}\}$, where $\Phi_{2n}=(f_{2n},g_{2n})$ is the eigenfunction of $\mathcal{A}$ with respect to the eigenvalue pairs λ_{2n}, has the following asymptotic expression:

$$\Phi_{2n}(x)=\begin{pmatrix}f_{2n}(x)\\ g_{2n}(x)\end{pmatrix}=\begin{pmatrix}\sin\left[\left[n\pi+\frac{1}{2}\theta_2\right](1-x)+\frac{1}{2}i\ln r(1-x)\right]\\ 0\end{pmatrix}+\mathcal{O}(e^{-c_2 n}). \tag{27}$$

Now, we get the Riesz basis property of the system (6) and then establish its exponential stability.

Before going on, let us recall some notation. For a closed operator $\mathbf{A}$ in a Hilbert space $\mathbf{H}$, a nonzero element $\phi\in\mathbf{H}$ is called a generalized eigenvector of $\mathbf{A}$, corresponding to an eigenvalue λ of $\mathbf{A}$, if there is an integer $\nu\geq 1$ such that $(\lambda I-\mathbf{A})^\nu\phi=0$. If $\nu=1$, then ϕ is an eigenvector. A sequence $\{\phi_n\}_{n=1}^\infty$ in $\mathbf{H}$ is called a Riesz basis for $\mathbf{H}$ if there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ in $\mathbf{H}$ and a linear bounded invertible operator T such that

$$T\phi_n=e_n,\quad n=1,2,\ldots.$$

Let $\{\lambda_n\}_{n=1}^\infty=\sigma(\mathbf{A})$, the spectrum of $\mathbf{A}$. Suppose each λ_n has finite algebraic multiplicity m_n, and let $\{\psi_{n_i}\}_1^{m_n}$ be the set of generalized eigenvectors of $\mathbf{A}$ corresponding to λ_n. If $\{\psi_{n_i}|1\leq i\leq m_n, n=1,2,\ldots\}$ form a Riesz basis for $\mathbf{H}$, then the C_0-semigroup generated by $\mathbf{A}$ can be represented as

$$e^{\mathbf{A}t}x=\sum_{n=1}^\infty e^{\lambda_n t}\sum_{j=1}^{m_n}a_{nj}f_{nj}(t)\psi_{nj},\ \forall x=\sum_{n=1}^\infty\sum_{j=1}^{m_n}a_{nj}\psi_{nj}\in\mathbf{H} \tag{28}$$

where $f_{nj}(t)$ are the polynomials of t with order not greater than m_n. In particular, if m_n has the uniformly upper bound and $\{\psi_{n_i}\}_1^{m_n}$ is the eigenvector (not generalized eigenvector) set of $\mathcal{A}$ with respect to λ_n for all sufficiently large n, then the spectrum determined growth condition holds, i.e., $\omega(\mathbf{A})=s(\mathbf{A})$, where $\omega(\mathbf{A})$ is the growth bound of $e^{\mathbf{A}t}$, and $s(\mathbf{A})$ is the spectral bound of $\mathbf{A}$ (see [3]). Now we establish the Riesz basis property of the system (6).

Theorem 2.4: Let $\mathcal{A}$ be defined by (5) and let $k\neq\pm 1$. Then there is a sequence of generalized eigenfunctions of $\mathcal{A}$, which forms a Riesz basis for $\mathcal{H}$. Moreover, all eigenvalues with sufficient large modulus are algebraically simple.

Now we establish the exponential stability of the system (6).

Theorem 2.5: Let $\mathcal{A}$ be defined by (5) and let $k\neq\pm 1$. Then the spectrum-determined growth condition $\omega(\mathcal{A})=s(\mathcal{A})$ holds true for the C_0-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$. Moreover, the system (6) is exponentially stable, that is, there exist two positive constants M and ω such that the C_0-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ satisfies

$$\|e^{\mathcal{A}t}\|\leq Me^{-\omega t}. \tag{29}$$

In what follows, we show that the C_0-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is of a Gevrey class δ with any $\delta>2$. We recall the definition.

Definition 2.1: ([14]) A C_0-semigroup $T(t)$ is of a Gevrey class $\delta>1$ for $t>t_0$ if $T(t)$ is infinitely differentiable for $t>t_0$ and for every compact subset $K\subset(t_0,\infty)$ and each $\theta>0$, there is a constant $C=C(K,\theta)$ such that

$$\left\|T^{(n)}(t)\right\|\leq C\theta^n(n!)^\delta,\quad \forall t\in K,\quad n=0,1,2,\ldots.$$

Now we establish the Gevrey regularity of the system (6).

Theorem 2.6: Let $\mathcal{A}$ be defined by (5) and let $k\neq\pm 1$. Then the semigroup $e^{\mathcal{A}t}$, generated by $\mathcal{A}$, is of a Gevrey class $\delta>2$ with $t_0=0$.

III. Proofs of the Results

In this section, we present the proofs of the results in the previous section.

Proof of Theorem 2.1: For any given $(f_1,g_1)\in\mathcal{H}$, solve

$$\mathcal{A}(f,g)=(-if'',g'')=(f_1,g_1)$$

to get

$$\begin{cases}f''(x)=if_1(x), & g''=g_1(x),\\ f(1)=g'(1)=0, & f(0)=kg(0),\quad g'(0)=ikf'(0)\end{cases}$$

which gives

$$\begin{cases}f'(x)=f'(0)+i\int_0^x f_1(\xi)d\xi,\\ g'(x)=-\int_x^1 g_1(\xi)d\xi,\quad f'(0)=-\frac{1}{ik}\int_0^1 g_1(\xi)d\xi.\end{cases} \tag{30}$$

By the boundaries $f(1)=0$ and $f(0)=kg(0)$, we further have

$$\begin{cases}f(x)=f'(0)(x-1)-i\int_0^x(1-x)f_1(\xi)d\xi\\ \qquad -i\int_x^1(1-\xi)f_1(\xi)d\xi,\\ g(x)=g(0)-\int_0^x\xi g_1(\xi)d\xi-\int_x^1 xg_1(\xi)d\xi,\\ g(0)=\frac{1}{k}\left[-f'(0)-i\int_0^1(1-\xi)f_1(\xi)d\xi\right].\end{cases} \tag{31}$$

Hence, we get the unique $(f,g) \in D(\mathcal{A})$. Hence, $\mathcal{A}^{-1}$ exists and is compact on $\mathcal{H}$ by the Sobolev embedding theorem. Therefore, $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues of finite algebraic multiplicity only. Now we show that $\mathcal{A}$ is dissipative in $\mathcal{H}$. Let $X = (f,g,h) \in D(\mathcal{A})$. Then we have

$$\begin{aligned}\langle \mathcal{A}X, X\rangle &= \left\langle (-if'', g''), (f,g)\right\rangle = -i\int_0^1 f''\overline{f}dx + \int_0^1 g''\overline{g}dx\\ &= -if'\overline{f}\,\big|_0^1 + g'\overline{g}\big|_0^1 + i\int_0^1 |f'|^2 dx - \int_0^1 |g'|^2 dx\\ &= i\int_0^1 |f'|^2dx - \int_0^1 |g'|^2 dx\end{aligned}$$

and

$$\mathrm{Re}\langle \mathcal{A}X, X\rangle = -\int_0^1 |g'|^2 dx \le 0. \tag{32}$$

Hence $\mathcal{A}$ is dissipative and $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions in $\mathcal{H}$ by the Lumer-Philips theorem [10]. The proof is complete. □

Proof of Lemma 2.1: By Theorem 2.1, since $\mathcal{A}$ is dissipative, we have for each $\lambda \in \sigma(\mathcal{A})$, $\mathrm{Re}\lambda \le 0$. So we only need to show there is no any eigenvalues on the imaginary axis. Let $\lambda = \pm i\mu^2 \in \sigma(\mathcal{A})$ with $\mu \in \mathbb{R}^+$ and $X = (f,g) \in D(\mathcal{A})$ be its associated eigenfunction of $\mathcal{A}$. Then by (32), we have

$$\mathrm{Re}\langle \mathcal{A}X, X\rangle = -\int_0^1 |g'|^2 dx = 0$$

and hence $g'(x) = 0$. By the second equation of (7), we have $g(x) = 0$. So $f'(0) = f(0) = 0$. Moreover $\mathcal{A}X = i\mu^2 X$ further gives that f satisfies the following:

$$\begin{cases} f''(x) \pm \mu^2 f(x) = 0,\\ f(0) = f(1) = f'(0) = 0.\end{cases}$$

A direct computation yields that the above equation only has the trivial solution. Hence $f = g = 0$ and $X = 0$. Therefore, there are no eigenvalues on the imaginary axis. The proof is complete. □

Proof of Lemma 2.2: From (14), a direct computation gives

$$\begin{aligned}\det\Delta(\rho) &= -\begin{vmatrix} e^{i\rho} & e^{-i\rho}\\ 1 & 1\end{vmatrix}\begin{vmatrix} e^{\sqrt{i}\rho} & -e^{-\sqrt{i}\rho}\\ \sqrt{i} & -\sqrt{i}\end{vmatrix}\\ &\quad+\begin{vmatrix} e^{i\rho} & e^{-i\rho}\\ k & -k\end{vmatrix}\begin{vmatrix} e^{\sqrt{i}\rho} & -e^{-\sqrt{i}\rho}\\ -k & -k\end{vmatrix}\\ &= \frac{1}{2}[a_1e^{i\rho}e^{\sqrt{i}\rho} + a_2e^{i\rho}e^{-\sqrt{i}\rho} + a_2e^{-i\rho}e^{\sqrt{i}\rho}\\ &\qquad + a_1e^{-i\rho}e^{-\sqrt{i}\rho}]\end{aligned}$$

where a_1 and a_2 are given by (16). So (15) is obtained. Moreover, when $\rho \in \mathcal{S}_1$ and $\rho \in \mathcal{S}_2$, from (10), we have

$$\begin{cases} e^{-i\rho} \to \infty, & \text{as } |\rho| \to \infty, \quad \rho \in \mathcal{S}_1,\\ e^{\sqrt{i}\rho} \to \infty, & \text{as } |\rho| \to \infty, \quad \rho \in \mathcal{S}_2\end{cases}$$

and hence, $\det\Delta(\lambda)$ has the more accurate asymptotic expressions given by (17) and (18) in $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Finally, when $\rho \in \mathcal{S}_3$, from (10), we have

$$e^{-i\rho} \to \infty, \quad e^{-\sqrt{i}\rho} \to \infty, \quad \text{as} \quad |\rho| \to \infty, \quad \rho \in \mathcal{S}_3$$

which yields (19). The proof is complete. □

Proof of Theorem 2.2: From (19), it is found that there is no solution of $\det\Delta(\rho)$ in $\mathcal{S}_3$ when the modulus of ρ is large enough, so we only need to find the solutions of $\det\Delta(\rho)$ in $\mathcal{S}_1 \cup \mathcal{S}_2$. Let $\det\Delta(\rho) = 0$. By (17), $\rho \in \mathcal{S}_1$ satisfies

$$a_2e^{\sqrt{i}\rho} + a_1e^{-\sqrt{i}\rho} + \mathcal{O}\left(e^{-c_1|\rho|}\right) = 0. \tag{33}$$

By (16), $a_2e^{\sqrt{i}\rho} + a_1e^{-\sqrt{i}\rho} = 0$ yields

$$e^{2\sqrt{i}\rho} = -\frac{a_1}{a_2} = -\frac{2k^2+\sqrt{2}+i\sqrt{2}}{2k^2-\sqrt{2}-i\sqrt{2}} = -\frac{k^4-1+\sqrt{2}k^2 i}{k^4-\sqrt{2}k^2+1} = re^{i\theta_1} \tag{34}$$

where r and θ_1 are given by (22) and (23) respectively. Note that for any $k \in \mathbb{R}$ with $k \neq 0$, we have $k^4 - \sqrt{2}k^2 + 1 > 0$, so

$$r^2 = \frac{k^8+1}{k^8+1-2\sqrt{2}k^2[k^4-\sqrt{2}k^2+1]} > 1. \tag{35}$$

Hence, the roots of $a_2e^{\sqrt{i}\rho} + a_1e^{-\sqrt{i}\rho} = 0$ are

$$\tilde{\rho}_{1n} = \frac{1}{2\sqrt{i}}\ln r + \left[n\pi + \frac{1}{2}\theta_1\right]\sqrt{i}, \quad n = 1, 2, \ldots.$$

By Rouché's theorem, the roots of (33) have the following asymptotic expression:

$$\rho_{1n} = \frac{1}{2\sqrt{i}}\ln r + \left[n\pi + \frac{1}{2}\theta_1\right]\sqrt{i} + \mathcal{O}(e^{-c_1 n}), \quad n > N_1 \tag{36}$$

where N_1 is a sufficiently large positive integer. Similarly, from (18), it follows that $\rho \in \mathcal{S}_2$ satisfies

$$a_1e^{i\rho} + a_2e^{-i\rho} + \mathcal{O}\left(e^{-c_2|\rho|}\right) = 0. \tag{37}$$

By (16), $a_1e^{i\rho} + a_2e^{-i\rho} = 0$ yields

$$e^{2i\rho} = -\frac{a_2}{a_1} = -\frac{2k^2-\sqrt{2}-i\sqrt{2}}{2k^2+\sqrt{2}+i\sqrt{2}} = -\frac{k^4-1-\sqrt{2}k^2 i}{k^4+\sqrt{2}k^2+1} = \frac{1}{r}e^{i\theta_2} \tag{38}$$

where r and θ_2 are given by (22) and (24), respectively. Hence, the roots of $a_1e^{i\rho} + a_2e^{-i\rho} = 0$ are

$$\tilde{\rho}_{2n} = -\frac{1}{2i}\ln r + \left[n\pi + \frac{1}{2}\theta_2\right], \quad n = 1, 2, \ldots.$$

By Rouché's theorem, the roots of (37) are given by the following asymptotic expression:

$$\rho_{2n} = -\frac{1}{2i}\ln r + \left[n\pi + \frac{1}{2}\theta_2\right] + \mathcal{O}(e^{-c_2 n}),\ n > N_2 \tag{39}$$

where N_2 is a sufficiently large positive integer. Finally, by using $\lambda = i\rho^2$, we eventually get λ_{in}, $i = 1, 2$, given by (21). The proof is complete. □

Proof of Theorem 2.3: From (10), (12), (14), and with some linear algebra calculations (see [8, pp. 74–75]), for each $\lambda \in \sigma(\mathcal{A})$ with $\lambda = i\rho^2$, the corresponding eigenfunction $g(x)$ and $f(x)$ are given, respectively, by

$$\begin{aligned} g(x) &= \begin{vmatrix} e^{i\rho} & e^{-i\rho} & 0 & 0 \\ 0 & 0 & e^{\sqrt{i}\rho} & -e^{-\sqrt{i}\rho} \\ 1 & 1 & -k & -k \\ 0 & 0 & e^{\sqrt{i}\rho x} & e^{-\sqrt{i}\rho x} \end{vmatrix} \\ &= -\left[e^{i\rho} - e^{-i\rho}\right]\left[e^{\sqrt{i}\rho(1-x)} + e^{-\sqrt{i}\rho(1-x)}\right] \\ &= -4i\sin\rho\cosh\left[\sqrt{i}\rho(1-x)\right] \end{aligned}$$

and

$$\begin{aligned} f(x) &= \begin{vmatrix} e^{i\rho} & e^{-i\rho} & 0 & 0 \\ 0 & 0 & e^{\sqrt{i}\rho} & -e^{-\sqrt{i}\rho} \\ 1 & 1 & -k & -k \\ e^{i\rho x} & e^{-i\rho x} & 0 & 0 \end{vmatrix} \\ &= -k\left[e^{i\rho(1-x)} - e^{-i\rho(1-x)}\right]\left[e^{\sqrt{i}\rho} + e^{-\sqrt{i}\rho}\right] \\ &= -4ki\cosh[\sqrt{i}\rho]\sin\left[\rho(1-x)\right]. \end{aligned}$$

Hence, for eigenvalue λ_{1n}, the corresponding normalized eigenfunction Φ_{1n} has the form

$$\Phi_{1n}(x) = \begin{pmatrix} f_{1n}(x) \\ g_{1n}(x) \end{pmatrix} = -\frac{1}{4i\sin\rho_{1n}}\begin{pmatrix} f(x,\rho_{1n}) \\ g(x,\rho_{1n}) \end{pmatrix} \tag{40}$$

and for λ_{2n}, the corresponding normalized eigenfunction Φ_{2n} has the form

$$\Phi_{2n}(x) = \begin{pmatrix} f_{2n}(x) \\ g_{2n}(x) \end{pmatrix} = \frac{-1/(4ki)}{\cosh[\sqrt{i}\rho_{2n}]}\begin{pmatrix} f(x,\rho_{2n}) \\ g(x,\rho_{2n}) \end{pmatrix}. \tag{41}$$

Noting that ρ_{1n} and ρ_{2n} are given by (36) and (39), respectively, by using their Taylor expansions, we have

$$\begin{aligned} &\cosh\left[\sqrt{i}\rho_{1n}(1-x)\right] \\ &= \cosh\left[\left[\frac{1}{2}\ln r + \left[n\pi + \frac{1}{2}\theta_1\right]i + \mathcal{O}(e^{-c_1 n})\right](1-x)\right] \\ &= \cos\left[\left[n\pi + \frac{1}{2}\theta_1\right](1-x) - \frac{1}{2}i\ln r(1-x)\right] + \mathcal{O}(e^{-c_1 n}) \end{aligned}$$

and

$$\begin{aligned} &\sin\left[\rho_{2n}(1-x)\right] \\ &= \sin\left[\left[-\frac{1}{2i}\ln r + n\pi + \frac{1}{2}\theta_2 + \mathcal{O}(e^{-c_2 n})\right](1-x)\right] \\ &= \sin\left[\left[n\pi + \frac{1}{2}\theta_2\right](1-x) + \frac{1}{2}i\ln r(1-x)\right] + \mathcal{O}(e^{-c_2 n}). \end{aligned}$$

Therefore, we finally get $\Phi_{1n}(x)$ and $\Phi_{2n}(x)$ as given by (26) and (27), respectively. The proof is complete. □

Before proving Theorem 2.4, we introduce the following lemma [1, Theorem 1, 2 and Remark 1] (or see [12, Theorem 5.23]) on the extension result of the Kadets 1/4-Theorem:

Lemma 3.1: Let $\lambda_n \in \mathbb{C}, n = 1, 2, \ldots$, be a sequence that satisfies $\sup_n |\mathrm{Im}\lambda_n| \le M$, where M is a positive constant. Then

1) the sine system $\{\sin\lambda_n x, n \ge 1\}$ is a Riesz basis in $L^2(0,1)$ provided that the sequence $\{\lambda_n\}$ satisfies one of the following conditions:
   (1.1) $\sup_n |\mathrm{Re}\lambda_n - n\pi| < (\pi/4)$;
   (1.2) $\sup_n |\mathrm{Re}\lambda_n - n\pi + (1/2)\pi| < \pi/4$;
2) the cosine system $\{1, \cos\lambda_n x, n \ge 1\}$, is a Riesz basis in $L^2(0,\pi)$ provided that the sequence $\{\lambda_n\}$ satisfies one of the following conditions:
   (2.1) $\sup_n |\mathrm{Re}\lambda_n - n\pi| < (\pi/4)$;
   (2.1) $\sup_n |\mathrm{Re}\lambda_n - n\pi - (1/2)\pi| < (\pi/4)$.

Now we prove Theorem 2.4.

Proof of Theorem 2.4: Let $\{\Psi_{1n}(x), n = 0,1,2,\ldots\}$ and $\{\Psi_{2n}(x), n = 1,2,\ldots\}$ be two subsets in $L^2(0,1) \times L^2(0,1)$ given by

$$\begin{cases} \Psi_{1n}(x) = \begin{pmatrix} 0 \\ \cos\left[\left[n\pi + \frac{1}{2}\theta_1\right](1-x) - \frac{1}{2}i\ln r(1-x)\right] \end{pmatrix}, \\ \Psi_{2n}(x) = \begin{pmatrix} \sin\left[\left[n\pi + \frac{1}{2}\theta_2\right](1-x) + \frac{1}{2}i\ln r(1-x)\right] \\ 0 \end{pmatrix}. \end{cases} \tag{42}$$

When $0 < |k| < 1$, from (23) and (24), we have

$$-\frac{\pi}{4} < \frac{1}{2}\theta_1 < 0, \quad 0 < \frac{1}{2}\theta_2 < \frac{\pi}{4}, \quad 0 < |k| < 1. \tag{43}$$

Similarly, when $|k| > 1$, from (23) and (24), we have

$$\frac{1}{2}\pi < \frac{1}{2}\theta_1 < \frac{3}{4}\pi, \quad -\frac{3}{4}\pi < \frac{1}{2}\theta_2 < -\frac{1}{2}\pi, \quad |k| > 1. \tag{44}$$

So, by using (2.1) and (2.2) of Lemma 3.1 for $0 < |k| < 1$ and $|k| > 1$, respectively, the sequence

$$\left\{1, \cos\left[\left[n\pi + \frac{1}{2}\theta_1\right](1-x) - \frac{1}{2}i\ln r(1-x)\right], n = 1, 2, \ldots\right\}$$

forms a Riesz basis in $L^2(0,1)$. Also, by using (1.1) and (1.2) of Lemma 3.1 for $0 < |k| < 1$ and $|k| > 1$, respectively, the sequence

$$\left\{\sin\left[\left[n\pi + \frac{1}{2}\theta_2\right](1-x) + \frac{1}{2}i\ln r(1-x)\right], n = 1, 2, \ldots\right\}$$

forms a Riesz basis in $L^2(0,1)$. Therefore, for $0 < |k| < 1$ and $|k| > 1$, we have that $\{\Psi_{10}(x), \Psi_{1n}(x), \Psi_{2n}(x), n = 1, 2, \ldots\}$ forms a Riesz basis on $L^2(0,1) \times L^2(0,1)$. By the expressions of Φ_{1n} and Φ_{2n} given by (26) and (27) respectively, we get that there is an $N > 0$ such that

$$\sum_{j=1}^{2}\sum_{n \ge N}^{\infty} \|\Phi_{jn} - \Psi_{jn}\|^2 = \sum_{j=1}^{2}\sum_{n \ge N}^{\infty} \mathcal{O}(e^{-2cn}) < \infty \tag{45}$$

where c is a positive constant. Hence, by Theorem 6.3 of [3], we conclude that the generalized eigenfunctions of $\mathcal{A}$ form a Riesz basis in $\mathcal{H}$ and all eigenvalues of $\mathcal{A}$ with sufficiently large modulus are algebraically simple. The proof is complete. □

Proof of Theorem 2.5: The spectrum-determined growth condition follows from Theorem 2.4. By Lemma 2.1, for each $\lambda \in \sigma(\mathcal{A})$, we have $\mathrm{Re}\lambda < 0$. This, together with (20)–(25) and the spectrum-determined growth condition, shows that $e^{\mathcal{A}t}$ is exponentially stable. The proof is complete. □

In order to prove Theorem 2.6, we need the following lemma established by Taylor in [14, Theorem 4, Chapter 5].

Lemma 3.2: Let $e^{\mathcal{A}t}$ be a C_0-semigroup satisfying $\|e^{\mathcal{A}t}\| \le Me^{\omega t}$. Suppose that for some $\mu \ge \omega$ and α satisfying $0 < \alpha \le 1$

$$\lim_{|\tau| \to \infty} \sup |\tau|^\alpha \|R(\mu + i\tau, \mathcal{A})\| = C < \infty, \quad \tau \in \mathbb{R}.$$

Then $e^{\mathcal{A}t}$ is of Gevrey class δ with $\delta > 1/\alpha$ for $t > 0$.

Proof of Theorem 2.6: From Theorem 2.5, $\mathcal{A}$ generates an exponentially stable C_0-semigroup $e^{\mathcal{A}t}$ in $\mathcal{H}$. So, by Lemma 3.2, we only need to show

$$\lim_{|\tau|\to\infty} |\tau| \, \|R(i\tau,\mathcal{A})\|^2 = C < \infty, \quad \tau \in \mathbb{R}. \tag{46}$$

By Theorem 2.4, $\left\{\{\{\Phi_{s,n,j}\}_{j=1}^{m_{sn}}\}_{n<N} \cup \{\Phi_{s,n}\}_{n\geq N}\right\}_{s=1}^{2}$ forms a Riesz basis in $\mathcal{H}$. Then for each $Y \in \mathcal{H}$, we have

$$Y = \sum_{n=1}^{N-1}\sum_{s=1}^{2}\sum_{j=1}^{m_{sn}} a_{s,n,j}\Phi_{s,n,j} + \sum_{n=N}^{\infty}\sum_{s=1}^{2} a_{s,n}\Phi_{s,n} \tag{47}$$

and

$$\|Y\|^2 \asymp \sum_{n=1}^{N-1}\sum_{s=1}^{2}\sum_{j=1}^{m_{sn}} |a_{s,n,j}|^2 + \sum_{n=N}^{\infty}\sum_{s=1}^{2} |a_{s,n}|^2. \tag{48}$$

Let $\tau \in \mathbb{R}$ and $\tau > 0$. Then we have $i\tau \in \rho(\mathcal{A})$, and, in addition

$$\begin{aligned} R(i\tau,\mathcal{A})Y = &\sum_{n=1}^{N-1}\sum_{s=1}^{2}\sum_{j=1}^{m_{sn}} \frac{a_{s,n,j}\Phi_{s,n,j}}{i\tau-\lambda_{sn}} + \sum_{n=N}^{\infty}\sum_{s=1}^{2} \frac{a_{s,n}\Phi_{s,n}}{i\tau-\lambda_{sn}} \\ &+ \sum_{n=1}^{N-1}\sum_{s=1}^{2} \mathcal{O}\left(\frac{1}{|i\tau-\lambda_{sn}|^2}\right) \end{aligned} \tag{49}$$

and

$$\|R(i\tau,\mathcal{A})Y\|^2 \asymp \sum_{n=1}^{N-1}\sum_{s=1}^{2}\sum_{j=1}^{m_{sn}} \frac{|a_{s,n,j}|^2}{|i\tau-\lambda_{sn}|^2} + \sum_{n=N}^{\infty}\sum_{s=1}^{2} \frac{|a_{s,n}|^2}{|i\tau-\lambda_{sn}|^2} \tag{50}$$

where $\{\lambda_{1n}, \lambda_{2n}, n \in \mathbb{N}\}$, given by (21), are eigenvalues of $\mathcal{A}$.

Now we estimate $|i\tau - \lambda_{sn}|^2$, $s = 1, 2$. By (21), for $|\tau|$ large enough and $s = 1, 2$, we have

$$|i\tau - \lambda_{sn}|^2 = \left|i\tau - i\rho_{sn}^2\right|^2 = \left|\tau - \rho_{sn}^2\right|^2 = |\sqrt{\tau} + \rho_{sn}|^2 |\sqrt{\tau} - \rho_{sn}|^2 \tag{51}$$

where $\lambda_{sn} = i\rho_{sn}^2$, and ρ_{1n} and ρ_{2n} are given by (36) and (39) respectively. Noting that

$$\begin{aligned} &|\sqrt{\tau} + \rho_{1n}|^2 \\ &= \left|\sqrt{\tau} + \frac{1}{2\sqrt{i}}\ln r + \left[n\pi + \frac{1}{2}\theta_1\right]\sqrt{i} + \mathcal{O}(e^{-c_1 n})\right|^2 \\ &= \left|\sqrt{\tau} + \frac{\sqrt{2}}{2}\left[\frac{1}{2}\ln r + n\pi + \frac{1}{2}\theta_1\right]\right. \\ &\qquad \left. + i\frac{\sqrt{2}}{2}\left[\frac{1}{2}\theta_1 + n\pi - \frac{1}{2}\ln r\right] + \mathcal{O}(e^{-c_1 n})\right|^2 \\ &= \left[\sqrt{\tau} + \frac{\sqrt{2}}{2}\left[\frac{1}{2}\ln r + n\pi + \frac{1}{2}\theta_1\right]\right]^2 \\ &\qquad + \frac{1}{2}\left[\frac{1}{2}\theta + n\pi - \frac{1}{2}\ln r\right]^2 + \mathcal{O}(e^{-c_1 n}), \\ &|\sqrt{\tau} - \rho_{1n}|^2 \\ &= \left|\sqrt{\tau} - \frac{1}{2\sqrt{i}}\ln r - \left[n\pi + \frac{1}{2}\theta_1\right]\sqrt{i} + \mathcal{O}(e^{-c_1 n})\right|^2 \\ &= \left[\sqrt{\tau} - \frac{\sqrt{2}}{2}\left[\frac{1}{2}\ln r + n\pi + \frac{1}{2}\theta_1\right]\right]^2 \\ &\qquad + \frac{1}{2}\left[\frac{1}{2}\ln r - \frac{1}{2}\theta_1 - n\pi\right]^2 + \mathcal{O}(e^{-c_1 n}) \end{aligned}$$

and

$$\begin{aligned} &|\sqrt{\tau} + \rho_{2n}|^2 \\ &= \left|\sqrt{\tau} + \left(n\pi + \frac{1}{2}\theta_2\right) - \frac{1}{2i}\ln r + \mathcal{O}(e^{-c_2 n})\right|^2 \\ &= \left[\sqrt{\tau} + n\pi + \frac{1}{2}\theta_2\right]^2 + \frac{1}{4}(\ln r)^2 + \mathcal{O}(e^{-c_2 n}), \\ &|\sqrt{\tau} - \rho_{2n}|^2 \\ &= \left|\sqrt{\tau} - \left(n\pi + \frac{1}{2}\theta_2\right) + \frac{1}{2i}\ln r + \mathcal{O}(e^{-c_2 n})\right|^2 \\ &= \left[\sqrt{\tau} - \left(n\pi + \frac{1}{2}\theta_2\right)\right]^2 + \frac{1}{4}(\ln r)^2 + \mathcal{O}(e^{-c_2 n}) \end{aligned}$$

there are $M_1, M_2 > 0$ such that

$$\begin{aligned} |i\tau - \lambda_{2n}|^2 &= |\sqrt{\tau} + \rho_{2n}|^2 |\sqrt{\tau} - \rho_{2n}|^2 \\ &\geq M_1\left[\tau + \left(n\pi + \frac{1}{2}\theta_2\right)^2\right] \end{aligned} \tag{52}$$

and

$$\begin{aligned} |i\tau - \lambda_{1n}|^2 &= |\sqrt{\tau} + \rho_{1n}|^2 |\sqrt{\tau} - \rho_{1n}|^2 \\ &\geq M_2\left[\frac{1}{2}\ln r - \frac{1}{2}\theta_1 - n\pi\right]^2 \\ &\quad \times \left[\tau + \frac{1}{8}[\ln r + 2n\pi + \theta_1]^2\right]. \end{aligned} \tag{53}$$

Hence, by (48)–(53), there is an $M > 0$ such that

$$\lim_{\tau\to\infty} |\tau \, \|R(i\tau,\mathcal{A})\|^2 = M < \infty. \tag{54}$$

On the other hand, when $\tau \in \mathbb{R}$ and $\tau < 0$, the same argument yields

$$\begin{aligned} &|-i|\tau| - \lambda_{1n}|^2 \\ &\quad \geq M_3\left[\frac{1}{2}\ln r + \frac{1}{2}\theta_1 + n\pi\right]^2 \left[\tau + \frac{1}{8}[\ln r - 2n\pi - \theta_1]^2\right] \end{aligned}$$

and

$$|-i|\tau| - \lambda_{2n}|^2 \geq M_4\left[|\tau| + \frac{1}{4}|\ln r|^2\right]$$

where $M_3, M_4 > 0$. Hence, as $\tau \to -\infty$, we have

$$\lim_{\tau\to-\infty} |\tau| \, \|R(i\tau,\mathcal{A})\|^2 = M < \infty. \tag{55}$$

Therefore, this together with (54) yields (46), and by Lemma 3.2, the semigroup $e^{\mathcal{A}t}$, generated by $\mathcal{A}$, is of a Gevrey class $\delta > 2$ with $t_0 = 0$. The proof is complete. □

IV. SIMULATIONS

The Legendre spectral method [2] is adopted to present a numerical calculation of the spectrum of $\mathcal{A}$ for the feedback gains $k = 10$. Fig. 2 displays the two branches of spectrum along two parabolas in the second quadrant, as predicted by Theorem 2.2. The spectrum is qualitatively the same for both positive and negative k.

V. CONCLUSION

In this technical note we provide a study of stabilization of the Schrödinger equation by collocated boundary control with the help of a compensator given by the heat equation. Three interesting observations arise: 1) in contrast to static feedback which assigns asymptotically a constant negative value to the real parts of the eigenvalues, the heat equation compensator results in a quadratic asymptotic growth of

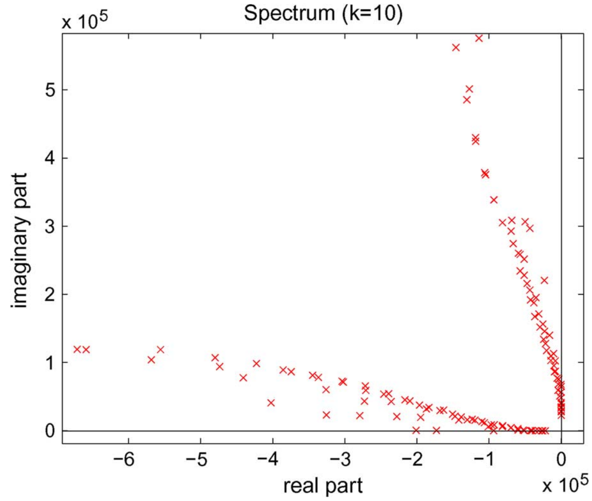

Fig. 2. Spectrum of the system for $k = 10$.

the real part of the dominant eigenvalues; 2) to accomplish this, the eigenvalues of the heat equation depart from the negative real axis and form asymptotically a symmetric image to the eigenvalues of the controlled Schrödinger equation relative to the 135° line in the second quadrant; 3) the presence of the heat equation endows the Schrödinger equation with a higher degree of regularity (of Gevrey class $\delta > 2$).

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Robust Stability Analysis of Nonlinear Discrete-Time Systems With Application to MPC

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***Abstract*—The regional Input-to-State Stability of nonlinear, possibly discontinuous, discrete-time systems is studied under the assumption that the equilibrium of the corresponding nominal model is asymptotically stable. The obtained results are used for the synthesis of a nominal Model Predictive Control law ensuring inherent robustness. A numerical example is reported that witnesses the effectiveness of the approach.**

***Index Terms*—Discontinuous systems, input-to-state stabilization, model predictive control (MPC).**

I. INTRODUCTION

Stability properties of equilibria in discrete-time nonlinear systems may be lost in the presence of disturbances. As it is discussed in [2], [5], [9], discontinuity of the model function, and of the Lyapunov functions for the nominal system, can be detrimental for robustness properties or even disrupt them. Hence, design methodologies are desirable for nominally stabilizing feedback laws capable of guaranteeing inherent robustness to the closed-loop dynamics. This is the goal of this note where, following [4], [11], robustness is analyzed in terms of regional Input-to-State Stability (ISS) and control synthesis is based on model predictive control (MPC).

While the theory on stabilizing MPC is consolidated [20], the topic of robustness still poses critical issues. As it has been clarified in [2], nominal MPC can be nonrobust even with respect to arbitrarily small disturbances. Moreover, the aforementioned discontinuity issue is crucial in MPC, where both the resulting feedback law and the available Lyapunov function can be discontinuous. For this reason, attention has been recently focused on the development of MPC algorithms ensuring desired robustness properties, see the review papers [14], [19]. This activity has led to the development of two broad classes of algorithms: one is based on a min-max formulation of the optimization problem that defines MPC, see [12], [19] and references therein; the other one relies on the *a priori* evaluation of the disturbance effect over the prediction horizon and the enforcement of tighter and tighter constraints to the predicted state trajectories, see [3], [8], [10], [15]. In any case, robust MPC methods require either a heavy on-line computational burden, or an involved off-line design phase. It is hence still of interest to analyze under which conditions nominal MPC can guarantee inherent robustness. Except for the recently published paper [16], pioneering results in this vein are only available (see, e.g., [21]) and they are all relying on regularity assumptions on the value function or on the MPC law.

In this technical note, robust stability properties in perturbed conditions are deduced by the behavior of a Lyapunov function V for the nominal system. Taking advantage of $\mathcal{K}_\infty$-functions used to bound V and its variation along trajectories [7], a robustness energy measure

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