Maximizing regions of attraction via backstepping and CLFs with singularities

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Abstract

When a nonlinear control law contains a singularity (that is, becomes unbounded on a set of points in the state space), the region of attraction is often only a subset of the region of feasibility of this control law. A method which is well known to suffer from this difficulty is feedback linearization. We show that the backstepping design (in its standard form) has an inherent ability to make the regions of feasibility and attraction coincide, thus maximizing the latter. The key observation that this paper provides is that a standard backstepping-style control Lyapunov function, which grows unbounded on the set where the control law becomes unbounded, has level sets that always remain in the feasibility region, which makes the feasibility region positively invariant. A simulation comparison with a feedback linearization design shows a dramatic improvement of the region of attraction with backstepping. Since our theorem imposes a strong assumption that the feasibility region for the first subsystem in the backstepping problem is positively invariant, we present examples (ranging from simple to fairly difficult) which demonstrate how this condition can be satisfied. © 1997 Elsevier Science B.V.

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1. Introduction

While the problem of global stabilization has dominated research activity in nonlinear control theory over the last several years, there are many systems which cannot be stabilized globally. Therefore, the problem of achieving maximal regions of attraction in nonlinear stabilization remains a problem of major interest.

In this paper, we study systems in which the presence of control singularities results in the loss of globality. The simplest example would be a feedback linearizable nonlinear system transformable into the Brunovsky form

\begin{equation}
\begin{aligned}
\dot{x}_1 &= x_{i+1}, & i = 1,\ldots,n-1, \\
\dot{x}_n &= \alpha(\chi) + \beta(\chi)u,
\end{aligned}
\end{equation}

where \( \beta(\chi) \) can be zero on a set of points, which may prevent global stabilizability of this system. An equivalent class of systems which has attracted attention in the context of integrator backstepping are the
pure-feedback systems [5] which are, in general, not globally stabilizable (but only regionally). The only successful effort so far to systematically influence the size of the regions of attraction of pure-feedback systems was made by Teel [8], however, only in the context of adaptive stabilization.

In the presence of control singularities, the control law is, in general, feasible only on a subset of the state space. Unfortunately, the region of attraction is usually only a (positively invariant) subset of the region of feasibility of the control law. We show that the backstepping design (in its standard form) has an inherent ability to make the regions of feasibility and attraction coincide, thus maximizing the latter. A strong assumption that we impose is that the feasibility region for the first subsystem in the backstepping problem is positively invariant. We present examples which illustrate how this assumption can be satisfied constructively. A simulation comparison with a feedback linearization design shows a dramatic improvement of the region of attraction with backstepping.

The key observation provided by this paper is that a standard backstepping-style control Lyapunov function, which grows unbounded on the set where the control law becomes unbounded, has level sets that always remain in the feasibility region, which makes the feasibility region positively invariant. In simple words, the singularity of the clf prevents the trajectories from crossing the boundary of the feasibility region.

We point out that, despite occasional terminological similarities, the control singularities treated in this paper are different from those in [1, 2, 4, 6, 7] where a control singularity usually appears at the origin, so the difficulty is not (only) the region of attraction, but the local stabilizability.

2. Main result

Consider the nonlinear system

\[ \dot{x} = f(x) + g(x)\sigma(x)u, \]  \hspace{1cm} (2.1)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R} \) are the state variable and the input respectively; \( f(\cdot), f(0) = 0 \), and \( g(\cdot) \) are smooth vector fields, and \( \sigma(\cdot), \sigma(0) > 0 \) is a smooth scalar function.

**Definition 2.1.** Suppose that there exists a feedback control law \( v = \sigma(x) \) that globally asymptotically stabilizes the equilibrium \( x = 0 \) of the system

\[ \dot{y} = f(y) + g(y)v. \]  \hspace{1cm} (2.2)

The open, path connected, subset around the origin, of the set \( \{ x \in \mathbb{R}^n \mid \sigma(x) > 0 \} \) is called the feasibility region \( \mathcal{F} \) of the control law \( u = \sigma(x)/\sigma(x) \). The boundary of \( \mathcal{F}, \partial\mathcal{F} \), is referred to as the control singularity set. The largest subset \( \mathcal{P} \) of \( \mathcal{F} \) which is positively invariant for the system

\[ \dot{x} = f(x) + g(x)\sigma(x), \]  \hspace{1cm} (2.3)

is referred to as the feasible stability region.

The objective of this paper is to study and design controllers which achieve \( \mathcal{P} = \mathcal{F} \). Obviously, \( \mathcal{P} = \mathcal{F} \) is achieved provided

\[ \frac{\partial \sigma(x)}{\partial x} [f(x) + g(x)\sigma(x)] > 0, \quad \forall x \in \partial \mathcal{F}. \]  \hspace{1cm} (2.4)

**Theorem 2.1.** Suppose there exist a control law \( \sigma(x) \) and a Lyapunov function \( V(x) \) such that

\[ \frac{\partial V}{\partial x} [f(x) + g(x)\sigma(x)] < -W(x), \]  \hspace{1cm} (2.5)
where $W(x)$ is a positive-definite function. Assume that $\mathcal{F} = \mathcal{P}$, that is, the feasibility region associated with $\sigma(x)$ is a positively invariant set for the system

$$
\dot{x} = f(x) + g(x)\sigma(x).
$$

(2.6)

Consider the system

$$
\dot{x} = f(x) + g(x)\sigma(x)\xi,
$$

(2.7a)

$$
\dot{\xi} = u.
$$

(2.7b)

Then the control law

$$
u = u_a(x, \xi) = -c \left( \xi - \frac{\sigma(x)}{\sigma(x)} \right) + \frac{1}{\sigma(x)} \left( \frac{\partial \sigma}{\partial x} \sigma(x) - \frac{\partial \sigma}{\partial x} \sigma(x) \right) \left( f(x) + g(x)\sigma(x)\xi \right) - \frac{\partial V}{\partial x} (x) g(x)\sigma(x),
$$

(2.8)

c > 0,

 guarantees that the equilibrium $(x, \xi) = 0$ is asymptotically stable and the set $\mathcal{F}_a = \{(x, \xi) \in \mathcal{F} \times \mathbb{R}\}$ is a feasible stability region for the closed-loop system (2.7), (2.8). The control law $u_a(x, \xi)$ is smooth on $\mathcal{F}_a$.

**Proof.** Introducing the error variable

$$
z = \xi - \frac{\sigma(x)}{\sigma(x)},
$$

(2.9)

we get

$$
\dot{x} = f(x) + g(x)\sigma(x) \left( z + \frac{\sigma(x)}{\sigma(x)} \right),
$$

(2.10a)

$$
\dot{z} = u - \frac{1}{\sigma(x)} \left( \frac{\partial \sigma}{\partial x} \sigma(x) - \frac{\partial \sigma}{\partial x} \sigma(x) \right) \left( f(x) + g(x)\sigma(x)z + g(x)\sigma(x) \right).
$$

(2.10b)

Consider the Lyapunov candidate

$$
V_a(x, \xi) = V(x) + \frac{1}{2} z^2 = V(x) + \frac{1}{2} \left( \xi - \frac{\sigma(x)}{\sigma(x)} \right)^2.
$$

(2.11)

Its derivative is

$$
\dot{V}_a = \frac{\partial V}{\partial x} \left( f + g\sigma \right) + z \left[ u - \frac{1}{\sigma^2} \left( \frac{\partial \sigma}{\partial x} \sigma - \frac{\partial \sigma}{\partial x} \sigma \right) \left( f + g\sigma \xi + g\sigma \right) + \frac{\partial V}{\partial \sigma} g\sigma \right].
$$

(2.12)

With the choice of $u$ as in (2.8), in light of (2.5), (2.12) becomes

$$
\dot{V}_a \leq - W(x) - cz^2,
$$

(2.13)

which means that $x(t)$ and $z(t)$ are bounded for $(x(0), \xi(0)) \in \mathcal{F}_a = \mathcal{F} \times \mathbb{R}$. The boundary of $\mathcal{F}_a$ is

$$
\partial \mathcal{F}_a = \partial \mathcal{F} \times \mathbb{R}.
$$

(2.14)

For any $(x, \xi) \in \partial \mathcal{F}_a$, we have

$$
\frac{\partial \sigma(x)}{\partial (x, \xi)} \left| _\sigma = \frac{\partial \sigma}{\partial x} \right| = \frac{\partial \sigma}{\partial x} (f + g\sigma) + \frac{\partial \sigma}{\partial x} g\sigma.
$$

(2.15)
Since $x$ and $z$ are bounded, $\partial \sigma / \partial x$ and $g(x)$ are bounded, and $\sigma(x) = 0$ on $\mathcal{F}_a$,

$$\frac{\partial \sigma(x)}{\partial x} \left[ \frac{f + g \sigma(x)}{z_a} \right] = \frac{\partial \sigma}{\partial x} (f + g x) > 0,$$

which means all trajectories passing through points on $\partial \mathcal{F}_a$ point inside $\mathcal{F}_a$. Inside $\mathcal{F}_a$, $\sigma(x) \neq 0$, so $\xi$ is bounded. Therefore, the equilibrium $(x, \xi) = 0$ is asymptotically stable and the set $\mathcal{F}_a = \{(x, \xi) \in \mathcal{F} \times \mathbb{R} \}$ is a feasible stability region for the closed-loop system (2.7), (2.8).

Theorem 2.1 results in the following corollary for a chain of integrators.

**Corollary 2.1.** Assume the conditions in Theorem 2.1 are satisfied with $\sigma(x) = \sigma_0(x)$. Consider the system

$$\dot{x} = f(x) + g(x) \sigma(x) \xi_1,$$

$$\dot{\xi}_i = \xi_{i+1}, \quad i = 1, \ldots, k - 1,$$

$$\dot{\xi}_k = u.$$

For this system, repeated application of Theorem 2.1 with $\xi_1, \ldots, \xi_k$ as virtual controls, results in the control law

$$u = \sigma_k(x, \xi)$$

with

$$z_1 = \xi_1 - \frac{\sigma_0(x)}{\sigma(x)},$$

$$z_i = \xi_i - \xi_{i-1}, \quad i = 2, \ldots, k,$$

$$\sigma_1(x, \xi_1) = -c_1 z_1 + \frac{1}{\sigma^2(x)} \left( \frac{\partial \sigma}{\partial x}(x) - \frac{\partial \sigma}{\partial x} \sigma(x) \right) (f(x) + g(x) \sigma(x) \xi_1) - \frac{\partial V}{\partial x} g(x) \sigma(x),$$

$$\sigma_i(x, \xi_i) = -z_{i-1} - c_i z_i + \frac{\partial \sigma_{i-1}}{\partial x} (f(x) + g(x) \sigma(x) \xi_i) + \sum_{j=1}^{i-1} \frac{\partial \sigma_{i-1}}{\partial \xi_j} \xi_{j+1},$$

$$\bar{\xi}_i = (\xi_1, \ldots, \xi_i), \quad i = 2, \ldots, k,$$

which guarantees that the equilibrium $(x, \xi) = 0$ is asymptotically stable and the set $\mathcal{F}_a = \{(x, \xi) \in \mathcal{F} \times \mathbb{R}^k \}$ is a feasible stability region for the closed-loop system (2.17), (2.18).

By examining the recursive expressions (2.19) we see that the control law (2.18) is smooth on $\mathcal{F}_a$ but becomes singular on $\partial \mathcal{F}_a$ because it includes a division by $\sigma(x)^{k+1}$.

### 3. Three examples

In view of Theorem 2.1, our main task is to select $\sigma(x)$ in order to satisfy condition (2.4). While in Example 1 this choice is obvious, in Examples 2 and (especially) 3 this choice is very intricate.

#### 3.1. Example 1: Backstepping vs. feedback linearization

Consider the following second-order system:

$$\dot{x} = -x^2 + \sigma(x) \xi = -x^2 + (1 + x) \xi,$$

$$\dot{\xi} = u.$$
with \( f(x) = -x^2 \), \( g(x) = 1 \), \( \sigma(x) = 1 + x \). The feasibility set \( \mathcal{F} \) is \( x > -1 \), and the singularity set \( \partial \mathcal{F} \) is \( x = -1 \).

First, notice that for the open-loop system

\[
\frac{\partial \sigma}{\partial x} f(x) \bigg|_{\partial \mathcal{F}} = -x^2 \big|_{x=-1} = -1 < 0,
\]

which means the set \( \mathcal{F} \) is repelling in the presence of any control law. Therefore, the singularity in the system (3.1a), (3.1b) is non-trivial.

Our objective now is to design a controller such that for the closed-loop system the feasibility region becomes positively invariant. In light of Theorem 2.1, it suffices to design an \( \sigma_1(x) \) at the first step which satisfies (2.4).

**Step 1:** Letting

\[
z_1 = x,
\]

\[
\sigma_1(x) = x^2 - c_1 x,
\]

the closed-loop \( x \)-subsystem with \( \sigma(x) = \sigma_1(x) \) becomes

\[
\dot{x} = f(x) + g(x)\sigma_1(x) = -c_1 x.
\]

The positive invariance condition (2.4) is satisfied:

\[
\frac{\partial \sigma_1(x)}{\partial x} (f(x) + g(x)\sigma_1(x)) \big|_{\partial \mathcal{F}} = -c_1 x \big|_{x=-1} = c_1 > 0.
\]

The stability condition (2.5) is satisfied with \( V(x) = \frac{1}{2} x^2 \).

**Step 2:** Theorem 2.1 gives

\[
z_2 = \xi - \frac{\sigma_1(x)}{1 + x} = \xi + \frac{1}{x+1} (-x^2 + c_1 x),
\]

\[
u = -c_2 z_2 + (x^2 + 2x - c_1) \frac{-x^2 + (x+1) \xi}{(x+1)^2} - (x+1)x.
\]

The Lyapunov function is

\[
V_s = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 = \frac{1}{2} x^2 + \frac{1}{2} \frac{(-x^2 + c_1 x + (x+1) \xi)^2}{(x+1)^2}.
\]

The closed-loop system is

\[
\dot{z} = \begin{bmatrix} -c_1 & 1 + x \\ -(1 + x) & -c_2 \end{bmatrix} z,
\]

which is also asymptotically stable because \( \dot{V}_s = -c_1 z_1^2 - c_2 z_2^2 \).

By Theorem 2.1, the set \( \mathcal{F}_a = \{(x, \xi) \in \mathcal{F} \times \mathbb{R} \} = \{ x > -1 \} \) is a feasible stability region for the closed-loop system (3.10). The level sets for \( V_s \) are plotted in Fig. 1. The singularity in \( V_s(x, \xi) \) which occurs along \( \partial \mathcal{F}_a \) prevents the level sets from "spilling" beyond \( \partial \mathcal{F}_a \). Simulations with design parameters \( c_1 = c_2 = 1 \) for various initial conditions are superimposed and show that \( \mathcal{P}_a = \mathcal{F}_a \). Fig. 2 shows the control effort needed to stabilize the extreme case: a trajectory starting from \( E \) and getting closest to the singularity. It is seen that the control effort spent to keep the trajectory away from the control singularity (the peak of \( u \) at \( \approx 0.4 \text{s} \)) is not too large.
Remark 3.1. As a comparison, we apply feedback linearization [3] to this system. Starting from the transformation

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = T(x, \zeta) = \begin{bmatrix} x \\ -x^2 + (x + 1)\zeta \end{bmatrix},$$

we select control

$$u = -\frac{1}{x + 1}((-2x + \zeta)\eta_2 - k_1 \eta_1 - k_2 \eta_2),$$

so the closed-loop system becomes

$$\dot{\eta} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \eta = : A\eta,$$
which is asymptotically stable because the derivative of Lyapunov function

\[ V_l = \eta^T P \eta, \]  

where \( P = P^T > 0 \) satisfies

\[ PA + A^T P = -I, \]  

is \( \dot{V}_l = -\|\eta\|^2 \).

We are interested now whether the feasible stability region \( \mathcal{P}_a \) is the same as the feasibility set \( \mathcal{F}_a \). To see this, we plot the level sets for \( V_l \) in Fig. 3 with the parameter values \( k_1 = k_2 = 1 \). It is seen from Fig. 3 that the Lyapunov level sets "bifurcate" around the point \((-1, -4)\). As a result, some trajectories cross the singularity line \( x = -1 \) instead of converging to the origin \((0, 0)\). Simulation results superimposed on the level sets show that the stability region is only the "boomerang-shape" area around the origin whose boundary is the separatrix \( B \). Thus, the linearization design has a stability region much smaller than \( \mathcal{F}_a \).

### 3.2. Example 2: Backstepping with a special choice of design parameters

In the previous example, the condition (2.4) was easy to satisfy because the \( x \)-system was scalar. Now consider a third-order system

\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_1 + (1 + x_1 + x_2)\dot{\xi}, \\
\dot{\xi} &= u,
\end{align*}

with

\[ f(x) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \sigma(x) = 1 + x_1 + x_2. \]
The feasibility set is $1 + x_1 + x_2 > 0$, and the singularity set is $x_1 + x_2 = -1$. Again, we note that $(\partial \sigma / \partial x)(f(x)|_{\partial \sigma} = (x_1 + x_2)|_{x_1 + x_2 = -1} = -1 < 0$, which means that the feasibility region is repelling. The backstepping design proceeds as follows:

**Step 1:**

\[ z_1 = x_1, \]
\[ x_1(x) = -c_1 x_1. \]

**Step 2:** This is a key step in the design procedure. With

\[ z_2 = x_2 - x_1(x) = x_2 + c_1 x_1, \]
\[ x_2(x) = -(c_2 z_2 + c_1 x_2 + 2 x_1), \]

the closed-loop $x$-subsystem with $x(x) = z_2(x)$ becomes

\[ \dot{x} = f(x) + g(x)x_2(x) = \begin{bmatrix} x_2 \\ -x_1 - c_1 x_2 - c_2 x_2 - c_1 c_2 x_1 \end{bmatrix}. \]

The positive invariance condition (2.4) is

\[ \dot{V}(x) = (1 + c_1 c_2) x_1 + (1 - c_1 - c_2) x_2 |_{x_1 + x_2 = -1} = -(1 + c_2)(x_1 + x_2) |_{x_1 + x_2 = -1} + (2 + c_1 c_2 - c_1 - c_2) x_2 |_{x_1 + x_2 = -1}. \]

Now we are in the position to select the design parameters $c_1$ and $c_2$ to make this expression positive. Picking

\[ 2 + c_1 c_2 = c_1 + c_2, \]

(3.21) becomes

\[ \dot{V}(x) = -(1 + c_1 c_2)(x_1 + x_2) |_{x_1 + x_2 = -1} = 1 + c_1 c_2 > 0. \]

The stability condition (2.5) is satisfied with

\[ V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} (x_2 + c_1 x_1)^2. \]

**Step 3:** Theorem 2.1 now applies and gives

\[ z_3 = \xi - \frac{x_2(x)}{1 + x_1 + x_2} = \xi + \frac{1}{1 + x_1 + x_2} (c_2 z_2 + c_1 x_2 + 2 x_1), \]

\[ u = -c_3 z_3 - \frac{c_1 + c_2}{(1 + x_1 + x_2)^2} (x_1 + x_2 + (1 + x_1 + x_2) \xi) - (1 + x_1 + x_2) z_2. \]

The closed-loop system is

\[ \dot{z} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 + x_1 + x_2 \\ 0 & -(1 + x_1 + x_2) & -c_3 \end{bmatrix} z. \]

The complete Lyapunov function is

\[ V_r = \frac{1}{2} x_1^2 + \frac{1}{2} (x_2 + c_1 x_1)^2 + \frac{1}{2} \left( \xi + \frac{2 + c_1 c_2}{1 + x_1 + x_2} x_1 + (c_1 + c_2) x_2 \right)^2, \]
Fig. 4. A $\xi = 0$ section of level sets for $V_r$ with state trajectories projected. Note that a coordinate change is used.

and its derivative is

$$
\dot{V}_r = -c_1 x_1^2 - c_2 x_2^2 - c_3 x_3^2,
$$

(3.29)

which, by Theorem 2.1, guarantees that the set $\mathcal{F}_a = \{ (x, \xi) \in \mathcal{F} \times \mathbb{R} \}$ is a feasible stability region for the closed-loop system (3.27). Fig. 4 shows a $\xi = 0$ section of level sets of $V_r$ for specifically chosen design parameters $c_1 = 0.5$, $c_2 = 3$. Simulation results for various initial conditions are superimposed. It may appear that the trajectories intersect, however this is only their projection from $\mathbb{R}^3$ to the plane $\xi = 0$. For all the initial conditions within the feasibility set $\mathcal{F}_a$, the solution converges to the equilibrium $(x, \xi) = 0$, which shows that $\mathcal{F}_a = \mathcal{F}_r$.

3.3. Example 3: Backstepping with a special choice of $\alpha(x)$

Finally, we consider the following third-order system which is slightly different but considerably more difficult than (3.15a)-(3.15c):

$$
\begin{align*}
\dot{x}_1 &= x_1 + x_2, \\
\dot{x}_2 &= \sigma(x) \xi = (1 + x_1 + x_2) \xi, \\
\dot{\xi} &= u,
\end{align*}
$$

(3.30)

with

$$
\begin{align*}
f(x) &= \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix}, & g(x) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \sigma(x) = 1 + x_1 + x_2.
\end{align*}
$$

The feasibility set $\mathcal{F}$ and the singularity set $\partial \mathcal{F}$ are the same as in Example 2. The set $\mathcal{F}$ is again repelling because $(\partial \sigma/\partial x)f(u)(x_1 + x_2) = (x_1 + x_2)|_{x_1+x_2=-1} = -1 < 0$. Different from Examples 1 and 2, however, applying
the standard backstepping design as before cannot make $\mathcal{P} = \mathcal{F}$. In fact, even though the system (3.30a), (3.30b) is linear with $a(x)\xi$ as a control, a stabilizing linear feedback $\alpha(x)$ cannot satisfy the condition (2.4). Thus, we are forced to depart from the standard backstepping design.

Step 1:

\[ z_1 = x_1, \quad (3.31) \]
\[ \dot{z}_1(x) = -x_1 - \frac{x_1}{2\sqrt{1 + x_1^2}}, \quad (3.32) \]

Note that a bounded nonlinear term $x_1/2\sqrt{1 + x_1^2}$ in (3.32) replaces a linear term $c_1 x_1$ in (3.17). Instead of $\frac{1}{2} x_1^2$, we use a modified Lyapunov function

\[ V_1(x_1) = \frac{1}{2} \int_0^{x_1} \left( 1 + \frac{1}{2(1 + x_1^2)^{3/2}} \right) \frac{x_1}{\sqrt{1 + x_1^2}} \, dx_1, \quad (3.33) \]

which is positive-definite and radially unbounded.

Step 2:

\[ z_2 = x_2 - \dot{z}_1(x) = x_2 + x_1 + \frac{x_1}{2\sqrt{1 + x_1^2}}, \quad (3.34) \]
\[ \dot{z}_2(x) = -\left( 2 + \frac{1}{2(1 + x_1^2)^{3/2}} \right) z_2. \quad (3.35) \]

The closed-loop $x$-subsystem with $\alpha(x) = \dot{z}_2(x)$ becomes

\[ \dot{x} = f(x) + g(x) \dot{z}_2(x) = \begin{bmatrix} x_1 + x_2 \\ - \left( 2 + \frac{1}{2(1 + x_1^2)^{3/2}} \right) z_2 \end{bmatrix}. \quad (3.36) \]

Due to our choices made in (3.32) and (3.35), the positive invariance condition (2.4) is satisfied:

\[ \frac{\partial \sigma(x)}{\partial x} (f(x) + g(x) \dot{z}_2(x))|_{\mathcal{P}} = \left( x_1 + x_2 - \left( 2 + \frac{1}{2(1 + x_1^2)^{3/2}} \right) z_2 \right)_{x_1 + x_2 = -1} \]
\[ = 1 - \frac{x_1}{\sqrt{1 + x_1^2}} + \frac{1}{2(1 + x_1^2)^{3/2}} \left( 1 - \frac{x_1}{2\sqrt{1 + x_1^2}} \right) > 0. \quad (3.37) \]

The stability condition (2.5) is satisfied with

\[ V(x) = V_1(x_1) + \frac{1}{2} z_2^2 = \frac{1}{2} \left( \sqrt{1 + x_1^2} - \frac{1}{4(1 + x_1^2)} - \frac{3}{4} \right) + \frac{1}{2} \left( x_1 + x_2 + \frac{x_1}{2\sqrt{1 + x_1^2}} \right)^2. \quad (3.38) \]

Step 3: Theorem 2.1 gives

\[ z_3 = \xi - \frac{\dot{z}_2(x)}{1 + x_1 + x_2} = \xi + \frac{1}{1 + x_1 + x_2} \left( 2 + \frac{1}{2(1 + x_1^2)^{3/2}} \right) z_2, \quad (3.39) \]
\[ u = -c_3 z_3 + \frac{A(x, \xi)}{1 + x_1 + x_2} + \frac{B(x, \xi)}{(1 + x_1 + x_2)^2} - (1 + x_1 + x_2) z_2, \quad (3.40) \]
where

\[
A(x, \xi) = \frac{3x_1}{2(1 + x_1^2)^{3/2}} \left( x_1 + x_2 + \frac{x_1}{2\sqrt{1 + x_1^2}} \right) - \left( 2 + \frac{1}{2(1 + x_1^2)^{3/2}} \right) \left( 1 + x_1 + x_2 \right) \xi + x_1 + x_2 + \frac{1}{2(1 + x_1^2)^{3/2}},
\]

\[
B(x, \xi) = \left( 2 + \frac{1}{2(1 + x_1^2)^{3/2}} \right) \left( x_1 + x_2 + \frac{x_1}{2\sqrt{1 + x_1^2}} \right) \left( x_1 + x_2 + (1 + x_1 + x_2) \xi \right).
\]

The closed-loop system is

\[
\begin{align*}
\dot{z}_1 &= -\frac{x_1}{2\sqrt{1 + x_1^2}} + z_2, \\
\dot{z}_2 &= -\frac{1}{2} \left( 1 + \frac{1}{2(1 + x_1^2)^{3/2}} \right) \frac{x_1}{\sqrt{1 + x_1^2}} - z_2 + (1 + x_1 + x_2)z_3, \tag{3.41} \\
\dot{z}_3 &= (1 + x_1 + x_2)z_2 - z_3.
\end{align*}
\]

The Lyapunov function is

\[
V_m(x, \xi) = V(x) + \frac{1}{2} x_3^2 = \frac{1}{2} \left( \sqrt{1 + x_1^2} - \frac{1}{4(1 + x_1^2)} - \frac{3}{4} \right) + \frac{1}{2} \left( x_1 + x_2 + \frac{x_1}{2\sqrt{1 + x_1^2}} \right)^2 + \frac{1}{2} \left( \xi + \frac{2 + (1/2(1 + x_1^2)^{3/2})}{1 + x_1 + x_2} \left( x_1 + x_2 + (x_1/2\sqrt{1 + x_1^2}) \right) \right)^2. \tag{3.42}
\]

Its derivative is

\[
\dot{V}_m = -\frac{1}{4} \left( 1 + \frac{1}{2(1 + x_1^2)^{3/2}} \right) \frac{x_1}{1 + x_1^2} - z_2^2 - z_3^2. \tag{3.43}
\]

The origin is asymptotically stable with a feasible stability region \( \mathcal{F}_0 = \{(x, \xi) \in \mathcal{F} \times \mathbb{R}\} \). The \( \xi = 0 \) section of the level sets of \( V_m \) are plotted in Fig. 5. Simulation results (the projection to \( \xi = 0 \)) are also superimposed on Fig. 5 for various initial conditions. For all the initial conditions within the feasibility set \( \mathcal{F}_0 \), the solution converges to the equilibrium \( (x, \xi) = 0 \). It is seen that \( \mathcal{P}_0 = \mathcal{F}_0 \).

4. Conclusions

We studied a class of systems with control singularities for which feedback linearization results in regions of stability which can be only small subsets of regions of feasibility. We showed that backstepping leads to controllers which make the regions of feasibility and stability coincident, thus maximizing the latter. Our key tool is to employ control Lyapunov functions which are singular on the control singularity set, and whose level sets follow the shape of the feasibility region.

So far, our idea is applicable to a restricted class of systems \( (2.17) \). For instance, it is not clear at present how it would be extended to strict feedback systems with virtual control singularities:

\[
\begin{align*}
\dot{x}_i &= g_i(\bar{x}_i)x_{i+1} + \varphi_i(\bar{x}_i), \quad i = 1, \ldots, n - 1, \\
\dot{x}_n &= g_n(x)u + \varphi_n(x)
\end{align*} \tag{4.1}
\]

with \( g_i(\bar{x}_i) \) which can be zero away from the origin, where \( \bar{x}_i = [x_1, \ldots, x_i]^T \).
It would also be of interest to extend Theorem 2.1 to the system not affine in virtual control
\[
\dot{x} = f(x, \xi),
\]
\[
\dot{\xi} = u,
\]
(4.2)
as well as to pure-feedback systems
\[
\dot{x}_i = \varphi_i(\bar{x}_i, x_{i+1}), \quad i = 1, \ldots, n - 1,
\]
\[
\dot{x}_n = \varphi_n(x, u).
\]
(4.3)
This would allow us to tackle interesting non-affine practical problems such as, for example, magnetic suspension, where the current enters quadratically.

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References