Systems & Control Letters 59 (2010) 713–719

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Compensating the distributed effect of a wave PDE in the actuation or sensing path of MIMO LTI systems

Nikolaos Bekiaris-Liberis*, Miroslav Krstic

Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093-0411, USA

ARTICLE INFO

Article history:
Received 9 April 2010
Received in revised form
6 August 2010
Accepted 21 August 2010
Available online 22 September 2010

Keywords:
Distributed parameter systems
Linear systems

ABSTRACT

The problem of compensation of infinite-dimensional actuator or sensor dynamics of more complex type than pure delay was solved recently using the backstepping method for PDEs. In this paper we construct an explicit feedback law for a multi-input LTI system which compensates the wave PDE dynamics in its input and stabilizes the overall system. Our design is based on a novel infinite-dimensional backstepping–forwarding transformation. We illustrate the effectiveness of our design with a simulation example of a single-input second order system, in which the wave input enters the system through two different channels, each one located at a different point in the domain of the wave PDE. Finally, we consider a dual problem where we design an exponentially convergent observer that compensates the distributed effect of the wave sensor dynamics.

1. Introduction

Compensation of input and sensor delays in linear time-invariant systems is achieved using predictor-based techniques [1–19]. For nonlinear systems several extensions of these methods exist [20–30], while adaptive controllers are beginning to emerge [31–36].

The problem of compensating more complex input and sensor dynamics than pure delays was solved recently in [37–39] using the backstepping method for PDEs. In [37] an explicit control law is constructed for a single-input system with a heat equation as its input, whereas in [38] a string PDE is compensated in the input path of an ODE. Finally, in [39] the results from [37] are extended to incorporate a counter-convection effect. In this paper we extend the results from [38] in two different directions: (1) Firstly, although the results in [38] can be almost trivially extended in the multi-input case, when the wave propagation speeds are the same in each individual input channel, the backstepping method is not applicable in the case where the propagation speeds are different in each input channel. (2) Secondly, in this paper we consider the case where the wave PDE is entering the ODE system in a distributed way. This is in contrast to the case considered in [38] where the wave PDE in the input and the ODE system are in cascade form, i.e. the wave enters the system through a single point of its spatial domain. The backstepping method is not applicable in this case. As pointed out also in [40,41] for the cases of distributed delays and diffusion respectively, the key difficulty is that the system that is comprised of the finite-dimensional state $X(t)$ and the infinite-dimensional actuator states $u(x,t)$, $x \in [0, D]$, is not in the strict-feedback form.

The challenges of considering the present problem in comparison with the one considered in [41] are analogous to those for the problem considered in [37] in comparison with the one in [38]. These include the fact that all of the (infinitely many) eigenvalues of the wave PDE are on the imaginary axis, and due to the fact that it has a finite (limited) speed of propagation (large control does not help). Moreover the PDE system is second order in time and hence one has to deal with the coupling of two infinite-dimensional states.

As in [40] for the case of distributed input or sensor delays, and in [41] for the case of diffusion with counter-convection, we design feedback laws that are given by explicit formulae. In Section 2 we design an explicit controller. In Section 3 we develop a dual of our actuator dynamics compensator and design an infinite-dimensional observer which compensates the wave PDE dynamics of the sensor. Section 4 presents a simulation example of controller design for a single-input system, in which the wave enters the system through two different channels, each one located at a different point in the domain of the PDE.

2. Controller design

We consider the system

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{2}\left(\int_0^{D_i} B_i(y)u_i(y,t)\mathrm{d}y + \int_0^{D_i} B_{it}(y)\partial_t u_i(y,t)\mathrm{d}y\right) \quad (1)$$

* Corresponding author.
E-mail addresses: nbekiari@ucsd.edu, nikos.bekiaris@gmail.com (N. Bekiaris-Liberis).

0167-6911/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2010.08.007

$$\partial_{tt}u_1(x,t) = \partial_{xx}u_1(x,t) \tag{2}$$
$$\partial_x u_1(0,t) = 0 \tag{3}$$
$$\partial_x u_1(D_1,t) = U_1(t) \tag{4}$$
$$\partial_{tt}u_2(z,t) = \partial_{zz}u_2(z,t) \tag{5}$$
$$\partial_z u_2(0,t) = 0 \tag{6}$$
$$\partial_z u_2(D_2,t) = U_2(t), \tag{7}$$

where $x \in [0, D_1]$, $z \in [0, D_2]$, $D_1, D_2 > 0$, $X(t) \in \mathbb{R}^n$ and $U_1(t)$, $U_2(t) \in \mathbb{R}$. For notational simplicity we consider a two-input case. The same analysis can be carried out for an arbitrary number of inputs with different wave propagation speeds in each individual input channel. For this system we state next an explicit feedback controller that compensates the wave dynamics and stabilizes the overall system.

Theorem 1. *Consider the closed-loop system consisting of the plant (1)–(7) and the control law*

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} \tag{8}$$

$$\begin{aligned} U_1(t) \;=\; & K_1 Z(t) - c_{01}\left(c_{11}\int_0^{D_1} \partial_t u_1(y,t)\mathrm{d}y + u_1(D_1,t)\right) \\ & - c_{11}\partial_t u_1(D_1,t) \end{aligned} \tag{9}$$

$$\begin{aligned} U_2(t) \;=\; & K_2 Z(t) - c_{02}\left(c_{12}\int_0^{D_2} \partial_t u_2(y,t)\mathrm{d}y + u_2(D_2,t)\right) \\ & - c_{12}\partial_t u_2(D_2,t) \end{aligned} \tag{10}$$

$$\begin{aligned} Z(t) \;=\; & X(t) + \int_0^{D_1} (Ag_1(y) - B_{1t}(y) \\ & + g_1(D_1)c_{01}c_{11})u_1(y,t)\,\mathrm{d}y \\ & + \int_0^{D_1} g_1(y)\partial_t u_1(y,t)\mathrm{d}y + c_{11}g_1(D_1)u_1(D_1,t) \\ & + \int_0^{D_2} (Ag_2(y) - B_{2t}(y) + g_2(D_2)c_{02}c_{12})u_2(y,t)\,\mathrm{d}y \\ & + \int_0^{D_2} g_2(y)\partial_t u_2(y,t)\mathrm{d}y + c_{12}g_2(D_2)u_2(D_2,t), \end{aligned} \tag{11}$$

where c_{0i}, c_{1i}, $i = 1, 2$ are positive constants and

$$\begin{aligned} g_i(y) \;=\; & \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix} y} \Bigg(I + \int_0^y \mathrm{e}^{-\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix} r} \mathrm{d}r \\ & \times \begin{bmatrix} 0 \\ I \end{bmatrix} A c_{1i} c_{0i} G_i^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix} D_i} \Bigg) \begin{bmatrix} I \\ 0 \end{bmatrix} g_i(0) \\ & - \begin{bmatrix} I & 0 \end{bmatrix} \int_0^y \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(y-r)} \begin{bmatrix} 0 \\ I \end{bmatrix} (B_i(r) + AB_{it}(r))\,\mathrm{d}r \\ & - \begin{bmatrix} I & 0 \end{bmatrix} \int_0^y \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(y-r)} \mathrm{d}r \begin{bmatrix} 0 \\ I \end{bmatrix} A c_{1i} c_{0i} G_i^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \\ & \times \int_0^{D_i} \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(D_i - y)} \begin{bmatrix} 0 \\ I \end{bmatrix} (B_i(y) + AB_{it}(y))\,\mathrm{d}y \end{aligned} \tag{12}$$

$$G_i = I - \begin{bmatrix} I & 0 \end{bmatrix} \int_0^{D_i} \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(D_i - r)} \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}r A c_{1i} c_{0i} \tag{13}$$

$$g_i(0) = E_i^{-1}\Delta_i \int_0^{D_i} \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(D_i - r)} \begin{bmatrix} 0 \\ I \end{bmatrix} (B_i(r) + AB_{it}(r))\,\mathrm{d}r \tag{14}$$

$$\begin{aligned} \Delta_i \;=\; & \begin{bmatrix} 0 & I \end{bmatrix} \Bigg(I + \int_0^{D_i} \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(D_i - r)} \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}y A c_{1i} c_{0i} G_i^{-1} \\ & \times \begin{bmatrix} I & 0 \end{bmatrix} \Bigg) + (c_{0i}I + c_{1i}A)G_i^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \end{aligned} \tag{15}$$

$$E_i = \Delta_i \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix} D_i} \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad i = 1, 2. \tag{16}$$

Let the pair $(A, [g_1(D_1) \quad g_2(D_2)])$ be completely controllable and choose the positive constants c_{0i}, c_{1i}, $i = 1, 2$ such that the matrices G_i, E_i, $i = 1, 2$ are invertible. Furthermore, choose K_1, K_2 such that the matrix

$$A_{cl} = A + g_1(D_1)K_1 + g_2(D_2)K_2, \tag{17}$$

is Hurwitz, and such that the matrices

$$R_i = \begin{bmatrix} I & c_{0i}I \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A_{cl}^2 \\ I & 0 \end{bmatrix} D_i} \begin{bmatrix} c_{1i}A_{cl} \\ I \end{bmatrix}, \quad i = 1, 2, \tag{18}$$

are invertible. If $u_i(\cdot, 0) \in H^1(0, D_i)$ and $\partial_t u_i(\cdot, 0) \in L^2(0, D_i)$, $i = 1, 2$, then the closed-loop system has a unique solution $(X(t), u_1(\cdot,t), \partial_t u_1(\cdot,t), u_2(\cdot,t), \partial_t u_2(\cdot,t)) \in C([0,\infty], \mathbb{R}^n \times H^1(0, D_1) \times L^2(0, D_1) \times H^1(0, D_2) \times L^2(0, D_2))$ which is exponentially stable in the sense that there exist positive constants κ and λ such that

$$\Omega(t) \le \kappa\Omega(0)\mathrm{e}^{-\lambda t} \tag{19}$$

$$\begin{aligned} \Omega(t) \;=\; & |X(t)|^2 + \sum_{i=1}^2 \left(\int_0^{D_i} \partial_y u_i(y,t)^2 \mathrm{d}y \right. \\ & \left. + \int_0^{D_i} \partial_t u_i(y,t)^2 \mathrm{d}y + u_i(0,t)^2 \right). \end{aligned} \tag{20}$$

Moreover, if the initial condition $(u_i(\cdot,0), \partial_t u_i(\cdot,0))$, $i = 1, 2$ is compatible with controller (9) and (10) and belongs to $H^2(0, D_i) \times H^1(0, D_i)$, $i = 1, 2$, then $(X(t), u_1(\cdot,t), \partial_t u_1(\cdot,t), u_2(\cdot,t), \partial_t u_2(\cdot,t)) \in C([0,\infty], \mathbb{R}^n \times H^1(0, D_1) \times L^2(0, D_1) \times H^1(0, D_2) \times L^2(0, D_2))$ is the classical solution of the closed-loop system.

Proof. We introduce three invertible transformations, one of the finite-dimensional state $X(t)$ given in (11) and the other two for the infinite-dimensional actuator states $u_1(x,t)$ and $u_2(z,t)$ given by

$$\begin{aligned} w_1(x,t) \;=\; & u_1(x,t) - \gamma_1(x) \Bigg(X(t) + \sum_{i=1}^2 \bigg(\int_0^{D_i} (Ag_i(y) \\ & - B_{it}(y) + g_i(D_i)c_{0i}c_{1i})u_i(y,t)\,\mathrm{d}y \\ & + \int_0^{D_i} g_i(y)\partial_t u_i(y,t)\mathrm{d}y + c_{1i}g_i(D_i)u_i(D_i,t) \bigg) \Bigg) \\ & + c_{01}\int_0^x u_1(y,t)\mathrm{d}y \end{aligned} \tag{21}$$

$$\begin{aligned} w_2(z,t) \;=\; & u_2(z,t) - \gamma_2(z) \Bigg(X(t) + \sum_{i=1}^2 \bigg(\int_0^{D_i} (Ag_i(y) \\ & - B_{it}(y) + g_i(D_i)c_{0i}c_{1i})u_i(y,t)\,\mathrm{d}y \\ & + \int_0^{D_i} g_i(y)\partial_t u_i(y,t)\mathrm{d}y \\ & + c_{1i}g_i(D_i)u_i(D_i,t) \bigg) \Bigg) + c_{02}\int_0^z u_2(y,t)\mathrm{d}y, \end{aligned} \tag{22}$$

where the kernels $\gamma_1(x)$ and $\gamma_2(z)$ are to be derived to transform the plant (1)–(7), along with the control law (8)–(10), into the target system

$$\dot{Z}(t) = A_{cl}Z(t) \tag{23}$$
$$\partial_{tt}w_1(x,t) = \partial_{xx}w_1(x,t) \tag{24}$$
$$\partial_x w_1(0,t) = c_{01}w_1(0,t) \tag{25}$$
$$\partial_x w_1(D_1,t) = -c_{11}\partial_t w_1(D_1,t) \tag{26}$$
$$\partial_{tt}w_2(z,t) = \partial_{zz}w_2(z,t) \tag{27}$$
$$\partial_z w_2(0,t) = c_{02}w_2(0,t) \tag{28}$$
$$\partial_z w_2(D_2,t) = -c_{12}\partial_t w_2(D_2,t). \tag{29}$$

It is well known that system (23)–(29) associates with an exponential stable C_0-semigroup solution in the state space $Y = \mathbb{R}^n \times H^1(0,D_1) \times L^2(0,D_1) \times H^1(0,D_2) \times L^2(0,D_2)$ with the state variable $(Z(t), w_1(\cdot,t), \partial_t w_1(\cdot,t), w_2(\cdot,t), \partial_t w_2(\cdot,t))$. Denote the system operator of system (23)–(29) to be $\mathbb{B}$ and the system operator of system (1)–(11) as $\mathbb{A}$. Let the invertible transformation from $(X(t), u_1(\cdot,t), \partial_t u_1(\cdot,t), u_2(\cdot,t), \partial_t u_2(\cdot,t))$ to $(Z(t), w_1(\cdot,t), \partial_t w_1(\cdot,t), w_2(\cdot,t), \partial_t w_2(\cdot,t))$ be $\mathbb{P}$. Since, $\mathbb{A} = \mathbb{P}^{-1}\mathbb{B}\mathbb{P}$ the theorem is proved if we can show that $\mathbb{P}$ is bounded and invertible. This is given next. We first differentiate (11) and using relations (2) and (5) we get

$$\begin{aligned}\dot{Z}(t) = {} & AX(t) + \sum_{i=1}^{2}\left(\int_0^{D_i} B_i(y)u_i(y,t)\mathrm{d}y\right.\\ & + \int_0^{D_i} B_{it}(y)\partial_t u_i(y,t)\mathrm{d}y\\ & + \int_0^{D_i} (Ag_i(y) - B_{it}(y) + g_i(D_i)c_{0i}c_{1i})\partial_t u_i(y,t)\mathrm{d}y\\ & \left. + \int_0^{D_i} g_i(y)\partial_{yy}u_i(y,t)\mathrm{d}y + c_{1i}g_i(D_i)\partial_t u_i(D_i,t)\right).\end{aligned} \tag{30}$$

Using integration by parts together with relations (3)–(4), (6)–(7) and (11) we get

$$\begin{aligned}\dot{Z}(t) = {} & AZ(t) + \sum_{i=1}^{2}\left(\int_0^{D_i} -(A^2g_i(y) - AB_{it}(y)\right.\\ & + Ag_i(D_i)c_{0i}c_{1i})u_i(y,t)\,\mathrm{d}y\\ & - \int_0^{D_i} Ag_i(y)\partial_t u_i(y,t)\mathrm{d}y - c_{1i}Ag_i(D_i)u_i(D_i,t)\\ & + \int_0^{D_i} B_i(y)u_i(y,t)\mathrm{d}y + \int_0^{D_i} B_{it}(y)\partial_t u_i(y,t)\,\mathrm{d}y\\ & + \int_0^{D_i} (Ag_i(y) - B_{it}(y) + g_i(D_i)c_{0i}c_{1i})\partial_t u_i(y,t)\,\mathrm{d}y\\ & + g_i(D_i)U_i(t) - g'(D_i)u_i(D_i,t) + g_i'(0)u_i(0,t)\\ & \left. + \int_0^{D_i} g_i''(y)u_i(y,t)\mathrm{d}y + c_{1i}g_i(D_i)\partial_t u_i(D_i,t)\right).\end{aligned} \tag{31}$$

Consequently the following holds

$$\begin{aligned}\dot{Z}(t) = {} & AZ(t) - \sum_{i=1}^{2}\left(\int_0^{D_i} (-g_i''(y) + A^2g_i(y) - AB_{it}(y)\right.\\ & - B_i(y) + Ag_i(D_i)c_{0i}c_{1i})u_i(y,t)\,\mathrm{d}y\\ & + g_i'(0)u_i(0,t) - ((c_{1i}A + c_{0i}I)g_i(D_i) + g_i'(D_i))u_i(D_i,t)\\ & + g_i(D_i)\left(U_i(t) + c_{0i}\left(u_i(D_i,t) + c_{1i}\int_0^{D_i}\partial_t u_i(y,t)\,\mathrm{d}y\right)\right.\\ & \left.\left. + c_{1i}\partial_t u_i(D_i,t)\right)\right).\end{aligned} \tag{32}$$

Observing now that the $g_i(\cdot)$, $i = 1, 2$ in (12) are the solutions of the following boundary value problems

$$g_i''(y) = A^2g_i(y) - B_i(y) - AB_{it}(y) + c_{0i}c_{1i}Ag_i(D_i) \tag{33}$$
$$g_i'(0) = 0 \tag{34}$$
$$g_i'(D_i) = -(c_{0i}I + Ac_{1i})g_i(D_i), \quad i = 1, 2, \tag{35}$$

we arrive at

$$\begin{aligned}\dot{Z}(t) = {} & AZ(t) + \sum_{i=1}^{2} g_i(D_i)\left(U_i(t) + c_{1i}\partial_t u_i(D_i,t)\right.\\ & \left. + c_{0i}\left(c_{1i}\int_0^{D_i}\partial_t u_i(y,t)\mathrm{d}y + u_i(D_i,t)\right)\right).\end{aligned} \tag{36}$$

With controller (8)–(10) we get (23). We derive now relations (24)–(26). The derivation of (27)–(29) follows exactly the same pattern. Taking two time derivatives in (21), using (2), integration by parts and relations (11), (23) we have

$$\partial_{tt}w_1(x,t) = \partial_{tt}u_1(x,t) - \gamma_1(x)A_{cl}^2Z(t) + c_{01}\partial_x u_1(x,t). \tag{37}$$

Taking two spatial derivatives of (21) we get

$$\partial_{xx}w_1(x,t) = \partial_{xx}u_1(x,t) - \gamma_1''(x)Z(t) + c_{01}\partial_x u_1(x,t). \tag{38}$$

Let us now examine the expressions

$$w_1(0,t) = u_1(0,t) - \gamma_1(0)Z(t) \tag{39}$$
$$\partial_x w_1(0,t) = \partial_x u_1(0,t) - \gamma_1'(0)Z(t) + c_{01}u_1(0,t) \tag{40}$$
$$\partial_x w_1(D_1,t) = \partial_x u_1(D_1,t) - \gamma_1'(D_1)Z(t) + c_{01}u_1(D_1,t) \tag{41}$$
$$\begin{aligned}\partial_t w_1(D_1,t) = {} & \partial_t u_1(D_1,t) - \gamma_1(D_1)A_{cl}Z(t)\\ & + c_{01}\int_0^{D_1}\partial_t u_1(x,t)\mathrm{d}x.\end{aligned} \tag{42}$$

Note here that, although $Z(t)$ appears in relations (37)–(42), $w_1(x,t)$ and $w_2(z,t)$ are transformations of the states $X(t)$, $u_1(x,t)$ and $u_2(z,t)$ since $Z(t)$ is given in terms of $X(t)$, $u_1(x,t)$ and $u_2(z,t)$ through relation (11). Using expressions (37)–(42) together with (2)–(4) and (10)–(11) we conclude that (24)–(26) hold if $\gamma_1(x)$ satisfies the following boundary value problems

$$\gamma_1''(x) = \gamma_1(x)A_{cl}^2 \tag{43}$$
$$\gamma_1'(0) = c_{01}\gamma_1(0) \tag{44}$$
$$\gamma_1'(D_1) = K_1 - c_{11}\gamma_1(D_1)A_{cl}, \tag{45}$$

which can be solved explicitly to give

$$\gamma_1(x) = C_1\begin{bmatrix} I & c_{01}I\end{bmatrix}\mathrm{e}^{\begin{bmatrix} 0 & A_{cl}^2\\ I & 0\end{bmatrix}x}\begin{bmatrix} I\\ 0\end{bmatrix} \tag{46}$$
$$C_1 = K_1R_1^{-1}. \tag{47}$$

Similarly relations (27)–(29) hold if $\gamma_2(z)$ satisfies

$$\gamma_2(z) = C_2\begin{bmatrix} I & c_{02}I\end{bmatrix}\mathrm{e}^{\begin{bmatrix} 0 & A_{cl}^2\\ I & 0\end{bmatrix}z}\begin{bmatrix} I\\ 0\end{bmatrix} \tag{48}$$
$$C_2 = K_2R_2^{-1}. \tag{49}$$

We find next the inverse transformation of (21). We postulate an inverse transformation in the form

$$\begin{aligned}u_1(x,t) = {} & w_1(x,t) + \delta_1(x)Z(t)\\ & - c_{01}\int_0^x \mathrm{e}^{-c_{01}(x-y)}w_1(y,t)\mathrm{d}y.\end{aligned} \tag{50}$$

With similar derivation one can show that (50) is indeed the inverse transformation of (21) if $\delta_1(x)$ satisfy

$$\delta_1'(x) = \delta_1(x)A_{cl}^2 \quad (51)$$

$$\delta_1'(0) = 0 \quad (52)$$

$$\delta_1(0) = C_1, \quad (53)$$

which can be solved explicitly to give

$$\delta_1(x) = \begin{bmatrix} C_1 & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & A_{cl}^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (54)$$

Analogously,

$$u_2(z,t) = w_2(z,t) + \delta_2(z)Z(t) - c_{02}\int_0^z e^{-c_{02}(z-y)} w_2(y,t)dy \quad (55)$$

$$\delta_2(z) = \begin{bmatrix} C_2 & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & A_{cl}^2 \\ I & 0 \end{bmatrix} z} \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (56)$$

Finally, $X(t)$ can be expressed in terms of $w_1(x,t)$, $w_2(z,t)$ and $Z(t)$ as

$$\begin{aligned} X(t) = \Bigg(I - \sum_{i=1}^{2}\bigg(\int_0^{D_i} (Ag_i(y) - B_{it}(y) + g_i(D_i)c_{0i}c_{1i})\delta_i(y)\,dy \\ - \int_0^{D_i} g_i(y)\delta_i(y)A_{cl}dy - c_{1i}g_i(D_i)\delta_i(D_i)\bigg)\Bigg) Z(t) \\ - \sum_{i=1}^{2}\Bigg(\int_0^{D_i} (Ag_i(y) - B_{it}(y) + g_i(D_i)c_{0i}c_{1i}) \\ \times \left(w_i(y,t) - c_{0i}\int_0^y e^{-c_{0i}(y-r)} w_i(r,t)dr \right) dy \\ - \int_0^{D_i} g_i(y) \times \bigg(\partial_t w_i(y,t) \\ - c_{0i}\int_0^y e^{-c_{0i}(y-r)} \partial_t w_i(r,t)dr \bigg) dy \\ - c_{1i}g_i(D_i)\,(w_i(D_i,t) \\ - c_{0i}\int_0^{D_i} e^{-c_{0i}(D_i-y)} w_i(y,t)\,dy)\Bigg). \end{aligned} \quad (57)$$

Consider now the Lyapunov function

$$V(t) = Z(t)^T PZ(t) + E(t), \quad (58)$$

where $P = P^T > 0$ and $Q = Q^T > 0$ satisfy

$$A_{cl}^T P + PA_{cl} = -Q, \quad (59)$$

and

$$\begin{aligned} E(t) = \sum_{i=1}^{2}\Bigg(\frac{1}{2}(c_{0i}w_i(0,t)^2 + \|\partial_y w_i(t)\|^2 + \|\partial_t w_i(t)\|^2) \\ + \epsilon_i \int_0^{D_i} (1+y)\partial_y w_i(y,t)\partial_t w_i(y,t)\,dy\Bigg). \end{aligned} \quad (60)$$

Note that $\|\partial_y w_i(t)\|^2$ is a compact notation for $\int_0^{D_i} \partial_y w_i(y,t)^2 dy$ and that for sufficiently small ϵ_i, $i = 1, 2$, $E(t)$ is positive definite [42]. Using (23) and applying the same calculations as in [43], Chapter 7.2, is readily shown that there exists a positive constant M such that

$$V(t) \le MV(0)e^{-Mt}. \quad (61)$$

To show (19)–(20), it is sufficient to show that

$$\underline{M}\Omega(t) \le V(t) \le \overline{M}\Omega(t), \quad (62)$$

for some positive $\overline{M}$ and $\underline{M}$. From relations (21)–(22) and (11) we get

$$\partial_y w_i(y,t) = \partial_y u_i(y,t) - \gamma_i'(y)Z(t) + c_{0i}u_i(y,t) \quad (63)$$

$$\partial_t w_i(y,t) = \partial_t u_i(y,t) - \gamma_i(y)A_{cl}Z(t) + c_{0i}\int_0^y \partial_t u_i(r,t)dr \quad (64)$$

$$w_i(0,t) = u_i(0,t) - \gamma_i(0)Z(t), \quad i = 1, 2. \quad (65)$$

Using (11)–(12) and the fact that $2\int_0^{D_i} u_i(y,t)\partial_y u_i(y,t)\,dy + u_i(0,t)^2 = u_i(D_i,t)^2$, $i = 1, 2$ together with Poincare, Young and Cauchy–Schwarz's inequalities we conclude that there exists a positive constant m such that

$$|Z(t)|^2 \le m\left(|X(t)|^2 + \sum_{i=1}^{2}(u_i(0,t)^2 + \|\partial_y u_i(t)\|^2 + \|\partial_t u_i(t)\|^2) \right). \quad (66)$$

Hence, using relations (58), (60) and (63)–(66) together with Young and Cauchy–Schwarz's inequalities we get the upper bound in (62). The lower bound is obtained similarly using the inverse transformations (50)–(55), (57). The rest of the arguments are almost identical to [42]. This completes the proof of the theorem. □

3. Observer design

We consider the system

$$\dot{X}(t) = AX(t) + BU(t) \quad (67)$$

$$\partial_{tt}\xi_1(x,t) = \partial_{xx}\xi_1(x,t) + C_1(x)X(t) \quad (68)$$

$$\partial_x\xi_1(0,t) = 0 \quad (69)$$

$$\xi_1(D_1,t) = 0 \quad (70)$$

$$\partial_{tt}\xi_2(z,t) = \partial_{zz}\xi_2(z,t) + C_2(z)X(t) \quad (71)$$

$$\partial_z\xi_2(0,t) = 0 \quad (72)$$

$$\xi_2(D_2,t) = 0 \quad (73)$$

$$Y_1(t) = \xi_1(0,t) \quad (74)$$

$$Y_2(t) = \xi_2(0,t). \quad (75)$$

The distributed effect of the wave PDEs (68)–(73) in the sensor path of the ODE (67) is reflected from the non-homogeneous term that appear in Eqs. (68) and (71). To see this, one can write down the solution of the wave PDEs (68)–(73). We state next a new observer that compensates the sensor dynamics and achieves exponential convergence of the estimation error.

Theorem 2. *Define the observer*

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + L_1(Y_1(t) - \hat{Y}_1(t)) + L_2(Y_2(t) - \hat{Y}_2(t)) \quad (76)$$

$$\begin{aligned} \partial_{tt}\hat{\xi}_1(x,t) = \partial_{xx}\hat{\xi}_1(x,t) + C_1(x)\hat{X}(t) + \gamma_1(x)AL_1(Y_1(t) \\ - \hat{Y}_1(t)) + \gamma_1(x)L_1(\dot{Y}_1(t) - \dot{\hat{Y}}_1(t)) \\ + \gamma_1(x)AL_2(Y_2(t) - \hat{Y}_2(t)) + \gamma_1(x)L_2(\dot{Y}_2(t) - \dot{\hat{Y}}_2(t)) \end{aligned} \quad (77)$$

$$\begin{aligned} \partial_x\hat{\xi}_1(0,t) = -c_{01}\gamma_1(0)L_1(Y_1(t) - \hat{Y}_1(t)) - c_{01}(\dot{Y}_1(t) - \dot{\hat{Y}}_1(t)) \\ - c_{01}\gamma_1(0)L_2(Y_2(t) - \hat{Y}_2(t)) \end{aligned} \quad (78)$$

$$\hat{\xi}_1(D_1,t)=0 \tag{79}$$

$$\begin{aligned}\partial_{tt}\hat{\xi}_2(z,t)&=\partial_{zz}\hat{\xi}_2(z,t)+C_2(z)\hat{X}(t)\\&\quad+\gamma_2(z)AL_1(Y_1(t)-\hat{Y}_1(t))+\gamma_2(z)L_1(\dot{Y}_1(t)-\dot{\hat{Y}}_1(t))\\&\quad+\gamma_2(z)AL_2(Y_2(t)-\hat{Y}_2(t))+\gamma_2(z)L_2(\dot{Y}_2(t)-\dot{\hat{Y}}_2(t))\end{aligned} \tag{80}$$

$$\begin{aligned}\partial_z\hat{\xi}_2(0,t)&=-c_{02}\gamma_2(0)L_2(Y_2(t)-\hat{Y}_2(t))-c_{02}(\dot{Y}_2(t)-\dot{\hat{Y}}_2(t))\\&\quad-c_{02}\gamma_2(0)L_1(Y_1(t)-\hat{Y}_1(t))\end{aligned} \tag{81}$$

$$\hat{\xi}_2(D_2,t)=0 \tag{82}$$

$$\hat{Y}_1(t)=\hat{\xi}_1(0,t) \tag{83}$$

$$\hat{Y}_2(t)=\hat{\xi}_2(0,t), \tag{84}$$

where

$$\begin{aligned}\gamma_i(y)&=\gamma_i(0)\begin{bmatrix}I & c_{0i}A\end{bmatrix}\mathrm{e}^{\begin{bmatrix}0 & A^2\\ I & 0\end{bmatrix}y}\begin{bmatrix}I\\0\end{bmatrix}\\&\quad-\int_0^y\begin{bmatrix}0 & C_i(r)\end{bmatrix}\mathrm{e}^{\begin{bmatrix}0 & A^2\\ I & 0\end{bmatrix}(y-r)}\begin{bmatrix}I\\0\end{bmatrix}\mathrm{d}r\end{aligned} \tag{85}$$

$$\gamma_i(0)=\int_0^{D_i}\begin{bmatrix}0 & C_i(r)\end{bmatrix}\mathrm{e}^{\begin{bmatrix}0 & A^2\\ I & 0\end{bmatrix}(D_i-r)}\begin{bmatrix}I\\0\end{bmatrix}\mathrm{d}r\,\Lambda_i^{-1} \tag{86}$$

$$\Lambda_i=\begin{bmatrix}I & c_{0i}A\end{bmatrix}\mathrm{e}^{\begin{bmatrix}0 & A^2\\ I & 0\end{bmatrix}D_i}\begin{bmatrix}I\\0\end{bmatrix},\quad i=1,2. \tag{87}$$

Let the pair $\left(A,\begin{bmatrix}\gamma_1(0)\\ \gamma_2(0)\end{bmatrix}\right)$ be observable and choose the gains L_1 and L_2 such that the matrix $A-L_1\gamma_1(0)-L_2\gamma_2(0)$ is Hurwitz. Moreover, choose the positive constants c_{0i}, $i=1,2$ such that the matrices Λ_i, $i=1,2$ are invertible. Then for any $(\xi_i(\cdot,0),\hat{\xi}_i(\cdot,0))\in H^1(0,D_i)$ and $(\partial_t\xi_i(\cdot,0),\partial_t\hat{\xi}_i(\cdot,0))\in L^2(0,D_i)$, $i=1,2$ the observer error system has a unique solution $(X(t)-\hat{X}(t),\xi_1(\cdot,t)-\hat{\xi}_1(\cdot,t),\partial_t\xi_1(\cdot,t)-\partial_t\hat{\xi}_1(\cdot,t),\xi_2(\cdot,t)-\hat{\xi}_2(\cdot,t),\partial_t\xi_2(\cdot,t)-\partial_t\hat{\xi}_2(\cdot,t))\in C([0,\infty],Y)$ with $Y=\mathbb{R}^n\times H_R^1(0,D_1)\times L^2(0,D_1)\times H_R^1(0,D_2)\times L^2(0,D_2)$ and $H_R^1(0,D_i)=\left\{f\in H^1(0,D_i)|f(D_i)=0\right\}$, $i=1,2$ which is exponentially stable in the sense that there exist positive constants μ and ρ such that

$$\Xi(t)\le\mu\Xi(0)\mathrm{e}^{-\rho t} \tag{88}$$

$$\begin{aligned}\Xi(t)&=|X(t)-\hat{X}(t)|^2+\sum_{i=1}^2\left(\int_0^{D_i}(\partial_y\xi_i(y,t)-\partial_y\hat{\xi}_i(y,t))^2\mathrm{d}y\right.\\&\quad\left.+\int_0^{D_i}(\partial_t\xi_i(y,t)-\partial_t\hat{\xi}_i(y,t))^2\mathrm{d}y\right).\end{aligned} \tag{89}$$

Proof. Introducing the error variables

$$\tilde{X}(t)=X(t)-\hat{X}(t) \tag{90}$$

$$\tilde{\xi}_1(x,t)=\xi_1(x,t)-\hat{\xi}_1(x,t) \tag{91}$$

$$\tilde{\xi}_2(z,t)=\xi_2(z,t)-\hat{\xi}_2(z,t), \tag{92}$$

we obtain

$$\dot{\tilde{X}}(t)=A\tilde{X}(t)-L_1\tilde{\xi}_1(0,t)-L_2\tilde{\xi}_2(0,t) \tag{93}$$

$$\begin{aligned}\partial_{tt}\tilde{\xi}_1(x,t)&=\partial_{xx}\tilde{\xi}_1(x,t)+C_1(x)\tilde{X}(t)-\gamma_1(x)AL_1\tilde{\xi}_1(0,t)\\&\quad-\gamma_1(x)L_1\partial_t\tilde{\xi}_1(0,t)-\gamma_1(x)AL_2\tilde{\xi}_2(0,t)-\gamma_1(x)L_2\partial_t\tilde{\xi}_2(0,t)\end{aligned}$$

$$\begin{aligned}\partial_x\tilde{\xi}_1(0,t)&=c_{01}\gamma_1(0)L_1\tilde{\xi}_1(0,t)+c_{01}\partial_t\tilde{\xi}_1(0,t)\\&\quad+c_{01}\gamma_1(0)L_2\tilde{\xi}_2(0,t)\end{aligned} \tag{94}$$

$$\tilde{\xi}_1(D_1,t)=0 \tag{95}$$

$$\begin{aligned}\partial_{tt}\tilde{\xi}_2(z,t)&=\partial_{zz}\tilde{\xi}_2(z,t)+C_2(z)\tilde{X}(t)-\gamma_2(z)AL_1\tilde{\xi}_1(0,t)\\&\quad-\gamma_2(z)L_1\partial_t\tilde{\xi}_1(0,t)-\gamma_2(z)AL_2\tilde{\xi}_2(0,t)-\gamma_2(z)L_2\partial_t\tilde{\xi}_2(0,t)\end{aligned}$$

$$\begin{aligned}\partial_z\tilde{\xi}_2(0,t)&=c_{02}\gamma_2(0)L_2\tilde{\xi}_2(0,t)+c_{02}\partial_t\tilde{\xi}_2(0,t)\\&\quad+c_{02}\gamma_2(0)L_1\tilde{\xi}_1(0,t)\end{aligned} \tag{96}$$

$$\tilde{\xi}_2(D_2,t)=0. \tag{97}$$

Consider now the transformations

$$\tilde{\zeta}_1(x,t)=\tilde{\xi}_1(x,t)-\gamma_1(x)\tilde{X}(t) \tag{98}$$

$$\tilde{\zeta}_2(z,t)=\tilde{\xi}_2(z,t)-\gamma_2(z)\tilde{X}(t), \tag{99}$$

where $\gamma_1(x)$ and $\gamma_2(z)$ are given in (85). Transformations (98)–(99) transform system (93)–(97) to the exponentially stable system

$$\begin{aligned}\dot{\tilde{X}}(t)&=(A-L_1\gamma_1(0)-L_2\gamma_2(0))\tilde{X}(t)\\&\quad-L_1\tilde{\zeta}_1(0,t)-L_2\tilde{\zeta}_2(0,t)\end{aligned} \tag{100}$$

$$\partial_{tt}\tilde{\zeta}_1(x,t)=\partial_{xx}\tilde{\zeta}_1(x,t) \tag{101}$$

$$\partial_x\tilde{\zeta}_1(0,t)=c_{01}\partial_t\tilde{\zeta}_1(0,t) \tag{102}$$

$$\tilde{\zeta}_1(D_1,t)=0 \tag{103}$$

$$\partial_{tt}\tilde{\zeta}_2(z,t)=\partial_{zz}\tilde{\zeta}_2(z,t) \tag{104}$$

$$\partial_z\tilde{\zeta}_2(0,t)=c_{02}\partial_t\tilde{\zeta}_2(0,t) \tag{105}$$

$$\tilde{\zeta}_2(D_2,t)=0. \tag{106}$$

It is well known that system (101)–(106) associates with an exponential stable C_0-semigroup solution in the state space $Y=\mathbb{R}^n\times H_R^1(0,D_1)\times L^2(0,D_1)\times H_R^1(0,D_2)\times L^2(0,D_2)$ where $H_R^1(0,D_i)=\{f\in H^1(0,D_i)|f(D_i)=0\}$, $i=1,2$ with the state variable $(\tilde{X}(t),\tilde{\zeta}_1(\cdot,t),\partial_t\tilde{\zeta}_1(\cdot,t),\tilde{\zeta}_2(\cdot,t),\partial_t\tilde{\zeta}_2(\cdot,t))$. Denote the system operator of system (101)–(106) to be $\mathbb{B}$ and the system operator of system (93)–(97) as $\mathbb{A}$. Let the invertible transformation from $(\tilde{X}(t),\tilde{\xi}_1(\cdot,t),\partial_t\tilde{\xi}_1(\cdot,t),\tilde{\xi}_2(\cdot,t),\partial_t\tilde{\xi}_2(\cdot,t))$ to $(\tilde{X}(t),\tilde{\zeta}_1(\cdot,t),\partial_t\tilde{\zeta}_1(\cdot,t),\tilde{\zeta}_2(\cdot,t),\partial_t\tilde{\zeta}_2(\cdot,t))$ be $\mathbb{P}$. Since, $\mathbb{A}=\mathbb{P}^{-1}\mathbb{B}\mathbb{P}$ the theorem is proved if we can show that $\mathbb{P}$ is bounded and invertible. This is given next. We match systems (93)–(97) and (101)–(106). Since the $\gamma_i(\cdot)$, $i=1,2$ in (85) satisfy the following boundary value problems

$$\gamma_i''(y)=\gamma_i(y)A^2-C_i(y) \tag{107}$$

$$\gamma_i(D_i)=0 \tag{108}$$

$$\gamma_i'(0)=c_{0i}\gamma_i(0)A,\quad i=1,2, \tag{109}$$

we get (101)–(106). We choose a Lyapunov function as

$$V(t)=\tilde{X}(t)^TP\tilde{X}(t)+\alpha E(t) \tag{110}$$

$$\begin{aligned}E(t)&=\sum_{i=1}^2\left(\frac{1}{2}(\|\partial_y\tilde{\zeta}_i(t)\|^2+\|\partial_t\tilde{\zeta}_i(t)\|^2)\right.\\&\quad\left.+\epsilon_i\int_0^{D_i}(-1-D_i+y)\partial_y\tilde{\zeta}_i(y,t)\partial_t\tilde{\zeta}_i(y,t)\,\mathrm{d}y\right),\end{aligned} \tag{111}$$

where the positive constant α is to be chosen later, $P=P^T>0$ and $Q=Q^T>0$ satisfy

$$\begin{aligned}&(A-L_1\gamma_1(0)-L_2\gamma_2(0))^TP+P(A-L_1\gamma_1(0)\\&\quad-L_2\gamma_2(0))=-Q.\end{aligned} \tag{112}$$

Employing Young's inequality in (111) similarly with the calculations in [42] we show that for sufficiently small ε_i, $i=1,2$ there exist positive constants r_1 and r_2 such that

$$r_1\Phi(t) \le V(t) \le r_2\Phi(t) \tag{113}$$

$$\Phi(t) = |\tilde{X}(t)|^2 + \sum_{i=1}^{2}(\|\partial_y\tilde{\zeta}_i(t)\|^2 + \|\partial_t\tilde{\zeta}_i(t)\|^2). \tag{114}$$

Using (98)–(100) we have that

$$\partial_y\tilde{\zeta}_i(y,t) = \partial_y\tilde{\xi}_i(y,t) - \gamma_i'(y)\tilde{X}(t) \tag{115}$$

$$\partial_t\tilde{\zeta}_i(y,t) = \partial_t\tilde{\xi}_i(y,t) - \gamma_i(y)A\tilde{X}(t) + \gamma_i(y)\left(L_1\tilde{\xi}_1(0,t) + L_2\tilde{\xi}_2(0,t)\right) \tag{116}$$

$$\partial_t\tilde{\xi}_i(y,t) = \partial_t\tilde{\zeta}_i(y,t) + \gamma_i(y)(A - L_1\gamma_1(0) - L_2\gamma_2(0))\tilde{X}(t) - \gamma_i(y)(L_1\tilde{\zeta}_1(0,t) + L_2\tilde{\zeta}_2(0,t)), \quad i = 1, 2. \tag{117}$$

By employing Young, Agmon and Poincare's inequalities in (115)–(117) it is possible to show that there exist positive constants r_3 and r_4 such that

$$r_3\Xi(t) \le \Phi(t) \le r_4\Xi(t). \tag{118}$$

Taking now the time derivative of (110) and using (101)–(106), is readily shown that

$$\dot{V}(t) = -\tilde{X}(t)^TQ\tilde{X}(t) - 2\tilde{X}(t)^TPL_1\tilde{\zeta}_1(0,t) - 2\tilde{X}(t)^TPL_2\tilde{\zeta}_2(0,t) + \alpha\dot{E}(t) \tag{119}$$

$$\dot{E}(t) = \sum_{i=1}^{2}\left(\left(\frac{\epsilon_i(D_i+1)}{2}(1+c_{0i}^2) - c_{0i}\right)\partial_t\tilde{\zeta}_i(0,t)^2 - \frac{\epsilon_i}{2}\partial_y\tilde{\zeta}_i(D_i,t)^2 - \frac{\epsilon_i}{2}(\|\partial_y\tilde{\zeta}_i(t)\|^2 + \|\partial_t\tilde{\zeta}_i(t)\|^2)\right). \tag{120}$$

Using again Young, Agmon and Poincare's inequalities in (119) and if we choose $\alpha \ge \frac{40\max_{i=1,2}\{|PL_i|^2\}}{\min_{i=1,2}\{\epsilon_i\}\lambda_{\min}(Q)}$ we show that there exist ρ such that $V(t) \le \rho V(0)e^{-\rho t}$. Using (113) and (118) we get (88) with $\mu = \frac{\rho r_2 r_4}{r_1 r_3}$. With similar arguments from [42] the proof is completed. □

4. Simulations

In this section we consider a special case of system (1)–(7) as

$$\dot{X}(t) = AX(t) + B_0u(0,t) + B_1u(D,t) \tag{121}$$

$$\partial_{tt}u(x,t) = \partial_{xx}u(x,t) \tag{122}$$

$$\partial_x u(0,t) = 0 \tag{123}$$

$$\partial_x u(D,t) = U(t), \tag{124}$$

where, we choose $D = 1$, $A = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$, $B_0 = \begin{bmatrix}1\\ 1\end{bmatrix}$ and $B_1 = \begin{bmatrix}0\\ 1\end{bmatrix}$. It is important here to observe that neither the pair (A, B_0) nor (A, B_1) are controllable, however, the pair $(A, g(1))$ is. To clarify this we calculate explicitly $g(1)$. Using (13) we have that

$$I = G^{-1} - \begin{bmatrix}I & 0\end{bmatrix}\int_0^D e^{\begin{bmatrix}0 & I\\ A^2 & 0\end{bmatrix}(D-r)}\begin{bmatrix}0\\ I\end{bmatrix}drAc_1c_0G^{-1}. \tag{125}$$

By taking into account the fact that $B(y) = B_1\delta(D-y) + B_0\delta(y)$, where $\delta(y)$ is the Dirac function we get that

$$g(D) = G^{-1}\begin{bmatrix}I & 0\end{bmatrix}e^{\begin{bmatrix}0 & I\\ A^2 & 0\end{bmatrix}D}\begin{bmatrix}I\\ 0\end{bmatrix}g(0) - G^{-1}\begin{bmatrix}I & 0\end{bmatrix}e^{\begin{bmatrix}0 & I\\ A^2 & 0\end{bmatrix}D}\begin{bmatrix}0\\ I\end{bmatrix}B_0, \tag{126}$$

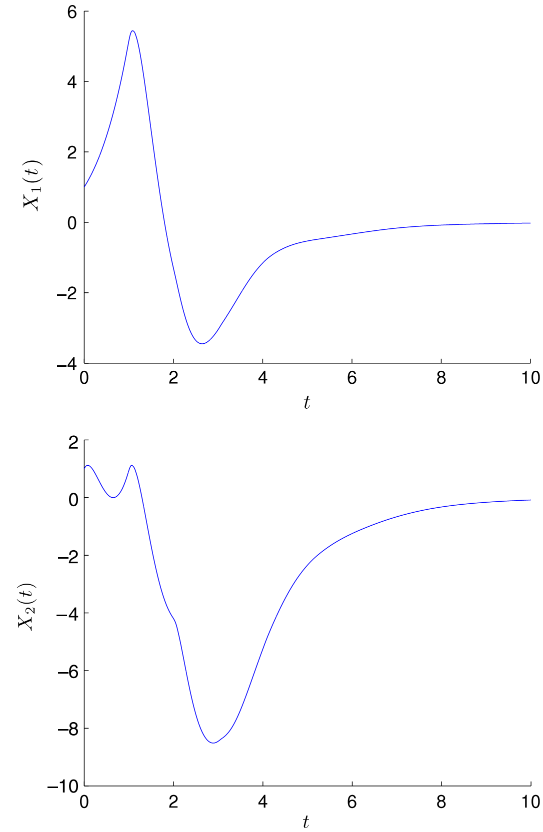

Fig. 1. The response of system (121) with initial conditions $X_1(0) = X_2(0) = 1$.

and

$$g(0) = E^{-1}\Delta\left(e^{\begin{bmatrix}0 & I\\ A^2 & 0\end{bmatrix}D}\begin{bmatrix}0\\ I\end{bmatrix}B_0 + \begin{bmatrix}0\\ I\end{bmatrix}B_1\right). \tag{127}$$

The most involved calculations are those that incorporate the matrices Δ and G, due to the integral term $\int_0^D e^{\begin{bmatrix}0 & I\\ A^2 & 0\end{bmatrix}(D-r)}dr$. This integral can be calculated either using numerical approximation, or explicitly using the Jordan representation of a matrix and then by explicitly computing its value using the formula for the matrix exponential. In the present example we have

$$\int_0^D e^{\begin{bmatrix}0 & I\\ A^2 & 0\end{bmatrix}(D-r)}dr = V\begin{bmatrix}D & \frac{D^2}{2} & 0 & 0\\ 0 & D & 0 & 0\\ 0 & 0 & -e^{-D}+1 & 0\\ 0 & 0 & 0 & e^D - 1\end{bmatrix}V^{-1}, \tag{128}$$

where V is such that $A = VJV^{-1}$ and J is the Jordan form of matrix A. Using the above relations we get that $g(1) = \begin{bmatrix}0.1226\\ 0.6226\end{bmatrix}$.

The initial conditions are chosen as $X_1(0) = X_2(0) = 1$, $u(x,0) = 1\ \forall x \in [0,1]$. Finally K is chosen such that the eigenvalues of $A + g(1)K$ are -2 and -1 and $c_0 = 2$, $c_1 = 1$. The response of the system is shown in Figs. 1 and 2. From Figs. 1 and 2 one can observe that the closed-loop system is exponentially stable, as Theorem 1 predicts.

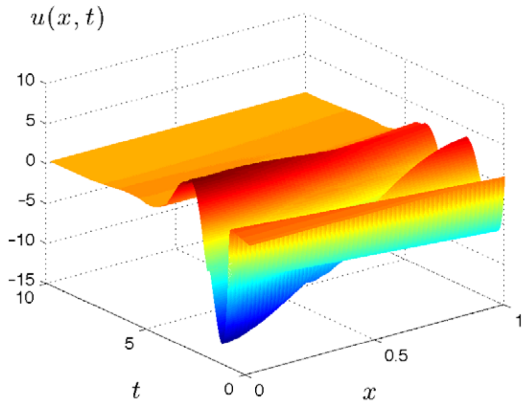

Fig. 2. The function $u(x, t)$ with initial condition $u(x, 0) = 1$, $\forall x \in [0, 1]$ for system (121)–(124).

5. Conclusions

In the present work we construct an explicit feedback law for an ODE system with distributed inputs which satisfy wave PDEs. Our design is based on novel transformations of the finite-dimensional state of the plant and of the infinite-dimensional actuator states. Using a Lyapunov functional we prove exponential stability of the transformed system. The invertibility of our transformations guarantees the exponential stability of the original system. The effectiveness of our controller is demonstrated with a numerical example. Finally, we develop an observer and prove exponential stability of the observer estimation error.

References

[1] Z. Artstein, Linear systems with delayed controls: a reduction, IEEE Trans. Automat. Control 27 (1982) 869–879.

[2] Y.A. Fiagbedzi, A.E. Pearson, Feedback stabilization of linear autonomous time lag systems, IEEE Trans. Automat. Control 31 (1986) 847–855.

[3] K. Gu, S.-I. Niculescu, Survey on recent results in the stability and control of time-delay systems, ASME Trans. Dynam. Syst. Meas. Control 125 (2003) 158–165.

[4] M. Jankovic, Recursive predictor design for linear systems with time delay, in: 2008 American Control Conference, Seattle, WA, 2008.

[5] M. Krstic, A. Smyshlyaev, Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays, Systems Control Lett. 57 (2008) 750–758.

[6] M. Krstic, Lyapunov tools for predictor feedbacks for delay systems: inverse optimality and robustness to delay mismatch, Automatica 44 (2008) 2930–2935.

[7] W.H. Kwon, A.E. Pearson, Feedback stabilization of linear systems with delayed control, IEEE Trans. Automat. Control 25 (1980) 266–269.

[8] J.J. Loiseau, Algebraic tools for the control and stabilization of time-delay systems, Ann. Rev. Control 24 (2000) 135–149.

[9] A.Z. Manitius, A.W. Olbrot, Finite spectrum assignment for systems with delays, IEEE Trans. Automat. Control 24 (1979) 541–553.

[10] F. Mazenc, S. Mondie, S.I. Niculescu, Global asymptotic stabilization for chains of integrators with a delay in the input, IEEE Trans. Automat. Control 48 (1) (2003) 57–63.

[11] F. Mazenc, S. Mondie, R. Francisco, Global asymptotic stabilization of feedforward systems with delay at the input, IEEE Trans. Automat. Control 49 (2004) 844–850.

[12] S. Mondie, W. Michiels, Finite spectrum assignment of unstable time-delay systems with a safe implementation, IEEE Trans. Automat. Control 48 (2003) 2207–2212.

[13] A.W. Olbrot, Stabilizability, detectability, and spectrum assignment for linear autonomous systems with general time delays, IEEE Trans. Automat. Control 23 (1978) 887–890.

[14] J.-P. Richard, Time-delay systems: an overview of some recent advances and open problems, Automatica 39 (2003) 1667–1694.

[15] O.J.M. Smith, A controller to overcome dead time, ISA Trans. 6 (1959) 28–33.

[16] K. Watanabe, E. Nobuyama, T. Kitamori, M. Ito, A new algorithm for finite spectrum assignment of single-input systems with time delay, IEEE Trans. Automat. Control 37 (1992) 1377–1383.

[17] Q.-C. Zhong, L. Mirkin, Control of integral processes with dead time—part 2: quantitative analysis, IEE Proc., Control Theory Appl. 149 (2002) 291–296.

[18] Q.-C. Zhong, On distributed delay in linear control laws—part I: discrete-delay implementation, IEEE Trans. Automat. Control 49 (2006) 2074–2080.

[19] Q.-C. Zhong, Robust Control of Time-delay Systems, Springer, 2006.

[20] M. Jankovic, Control Lyapunov–Razumikhin functions and robust stabilization of time delay systems, IEEE Trans. Automat. Control 46 (2001) 1048–1060.

[21] M. Jankovic, Control of nonlinear systems with time delay, in: 2003 IEEE Conference on Decision and Control, Maui, Hawai, 2003.

[22] M. Jankovic, Forwarding, backstepping, and finite spectrum assignment for time delay systems, in: 2006 American Control Conference, Minneapolis, MN, 2006.

[23] M. Jankovic, Control of cascade systems with time delay— the integral cross-term approach, in: 2006 IEEE Conference on Decision and Control, San Diego, CA, 2006.

[24] I. Karafyllis, Finite-time global stabilization by means of time-varying distributed delay feedback, SIAM J. Control Optim. 45 (1) (2006) 320–342.

[25] I. Karafyllis, Z.P. Jiang, Control Lyapunov functionals and robust stabilization of nonlinear time-delay systems, in: 2008 IEEE Conference on Decision and Control, Cancun, Mexico, 2008.

[26] I. Karafyllis, Stabilization by means of approximate predictors for systems with delayed input, in: 2010 IFAC Symposium on Time Delay Systems, Prague, Czech Republic, 2010.

[27] M. Krstic, On compensating long actuator delays in nonlinear control, IEEE Trans. Automat. Control 53 (2008) 1684–1688.

[28] M. Krstic, Input delay compensation for forward complete and feedforward nonlinear systems, IEEE Trans. Automat. Control 55 (2010) 287–303.

[29] F. Mazenc, P.-A. Bliman, Backstepping design for timedelay nonlinear systems, IEEE Trans. Automat. Control 51 (2004) 149–154.

[30] F. Mazenc, S.-I. Niculescu, Stabilizing controllers for delay systems subject to positivity constraints, in: 2010 IFAC Symposium on Time Delay Systems, Prague, Czech Republic, 2010.

[31] D. Bresch-Pietri, Miroslav Krstic, Delay-adaptive full-state predictor feedback for systems with unknown long actuator delay, in: 2009 American Control Conference, St. Louis, Missouri, 2009.

[32] D. Bresch-Pietri, Miroslav Krstic, Adaptive trajectory tracking despite unknown input delay and plant parameters, Automatica 45 (2009) 2074–2081.

[33] S. Diop, I. Kolmanovsky, P.E. Moraal, M. van Nieuwstadt, Preserving stability/performance when facing an unknown time-delay, Control Eng. Pract. 9 (2001) 1319–1325.

[34] S. Evesque, A.M. Annaswamy, S. Niculescu, A.P. Dowling, Adaptive control of a class of time-delay systems, ASME Trans. Dynam. Syst. Meas. Control 125 (2003) 186–193.

[35] M. Krstic, A. Banaszuk, Multivariable adaptive control of instabilities arising in jet engines, Control Eng. Pract. 14 (2006) 833–842.

[36] S.-I. Niculescu, A.M. Annaswamy, An adaptive Smith- controller for time-delay systems with relative degree $n \geq 2$, Systems Control Lett. 49 (2003) 347–358.

[37] M. Krstic, Compensating actuator and sensor dynamics governed by diffusion PDEs, Systems Control Lett. 58 (2009) 372–377.

[38] M. Krstic, Compensating a string PDE in the actuation or sensing path of an unstable ODE, IEEE Trans. Automat. Control 54 (2009) 1362–1368.

[39] G.A. Susto, M. Krstic, Control of PDE-ODE cascades with Neumann interconnections, J. Franklin Inst. 347 (2010) 284–314.

[40] N. Bekiaris-Liberis, M. Krstic, Lyapunov stability of linear predictor feedback for distributed input delays, in: 2010 IEEE Conference on Decision and Control, Atlanta, Georgia, 2010 (in press).

[41] N. Bekiaris-Liberis, M. Krstic, Compensating the distributed effect of counter-convection and diffusion in multi-input and multi-output LTI systems, in: 2010 IEEE Conference on Decision and Control, Atlanta, Georgia, 2010 (in press).

[42] M. Krstic, B.-Z. Guo, A. Balogh, A. Smyshlyaev, Output-feedback stabilization of an unstable wave equation, Automatica 44 (2008) 63–74.

[43] M. Krstic, A. Smyshlyaev, Boundary Control of PDEs: A course on Backstepping Designs, SIAM, 2008.