Further Results on Stabilization of Shock-Like Equilibria of the Viscous Burgers PDE

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***Abstract*—In this note we show that a symmetric shock profile of the linearized viscous Burgers equation under high-gain "radiation" boundary feedback is exponentially stable, though the previously reported numerical eigenvalue calculations have reported instability. We also show limitations of the radiation feedback by deriving an analytical bound on the closed-loop decay rate for a given shock profile. We prove that the decay rate goes to zero exponentially as the shock becomes sharper. This limitation in the decay rate achievable by radiation feedback highlights the importance of backstepping designs for the Burgers equation, which achieve arbitrarily fast local convergence to arbitrarily sharp shock profiles.**

***Index Terms*—Boundary control, Burgers equation, radiation feedback.**

I. Introduction

A recent paper [4] considers a problem of nonlinear stabilization of the viscous Burgers equation and, for a family of unstable symmetric "shock-like" stationary profiles (see Fig. 1), it designs stabilizing nonlinear full-state feedbacks with arbitrarily fast decay rates, using the method of infinite-dimensional backstepping.

The paper [4] highlights the inability of a simple "radiation boundary feedback" to achieve the same goals (of arbitrary decay rates for arbitrary shock profiles). Radiation boundary feedback is a form of static proportional feedback based on a collocated input-output pair, where the temperature at the boundary is measured and the heat flux at the same boundary is actuated.

The emphasis in [4] and in the present note is on *sharp* shock-like profiles. The sharpness of a shock-like profile is measured in terms of the maximum of the spatial derivative of the equilibrium profile. The sharpness is quantified in Section II in terms of a scalar parameter σ.

The evidence presented in [4, Sec. IV, Fig. 3] for the inability of radiation feedback to *exponentially stabilize* sharp shock profiles is numerical. Numerical calculations of closed-loop eigenvalues in [4, Sec. IV, Fig. 3] display the first eigenvalue which appears to remain positive for any value of the gains in the radiation boundary conditions, when the shock coefficient is sufficiently large. This numerical result happens to be incorrect and the error occurs due to high numerical sensitivity at high parameter values (high shock coefficient and high feedback gain). Rather than remaining slightly positive, as displayed in [4, Fig. 3], the first eigenvalue is slightly negative, which we show analytically in this note.

It is important to clarify that the numerical error in [4, Sec. IV, Fig. 3] does not affect any of the theoretical results in [4]. The numerical results in question are related to the linear radiation feedback, whereas the theoretical result in [4] are related to the nonlinear full-state backstepping feedback.

Manuscript received June 18, 2009, revised April 16, 2010; accepted May 03, 2010. Date of publication May 10, 2010; date of current version July 30, 2010. Recommended by Associate Editor J. J. Winkin.

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Digital Object Identifier 10.1109/TAC.2010.2050018

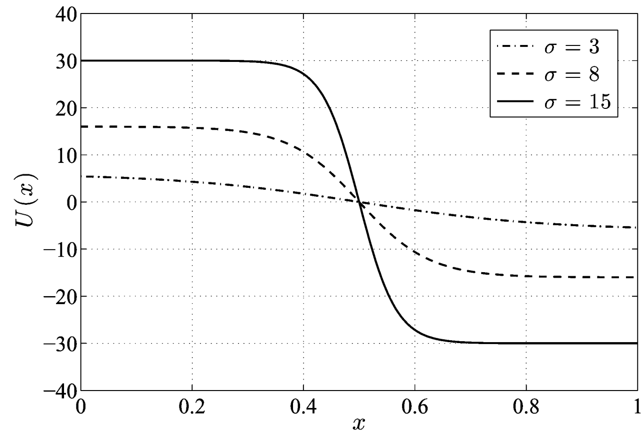

Fig. 1. Equilibrium profiles for the Burgers (1)–(3), parameterized in terms of the shock coefficient σ.

The results of this note are the following. We show that arbitrarily sharp shock profiles are stabilizable using sufficiently high-gain radiation feedback. However, we also show that, as the shock coefficient goes to infinity, the first closed-loop eigenvalue goes to zero irrespective of the radiation gain. These two analytical results are formulated as Theorems 1 and 2.

Theorem 1 is a partly redeeming result for radiation feedback as it shows that this feedback does achieve exponential stability for arbitrary shock profiles. The theorem gives a necessary and sufficient stability condition for the radiation gain. For very sharp shock profiles the gain becomes very high, making numerical calculations of the eigenvalues very sensitive, which explains the incorrect conclusions drawn based on numerical results in [4, Sec. IV, Fig. 3].

Theorem 2 shows that, as the shock coefficient grows (shock becomes sharper), the best achievable decay rate under radiation feedback goes to zero. Moreover, we prove that even for infinite gain this convergence is exponential in the shock coefficient, so even for mild profiles (like the middle profile in Fig. 1) radiation feedback results in extremely sluggish closed-loop response.

The results presented in this note amplify the importance of the backstepping design in [4] which, unlike the radiation boundary feedback, is capable of assigning an arbitrarily fast decay, with an explicit control law, using either full-state feedback, or using feedback based only on boundary measurement [5], as in the case of radiation feedback.

II. Burgers Equation Under "Radiation Feedback"

Consider the viscous Burgers equation

$$u_t(x,t) = u_{xx}(x,t) - u_x(x,t)u(x,t), \quad x \in (0,1) \tag{1}$$

with boundary conditions

$$u_x(0,t) = \omega_0(t) \tag{2}$$

$$u_x(1,t) = \omega_1(t) \tag{3}$$

where ω_0 and ω_1 are the control inputs.

The following family of "shock-like" stationary profiles exists for (1)–(3):

$$U(x) = -2\sigma \tanh\left(\sigma\left(x - \frac{1}{2}\right)\right) \tag{4}$$

where σ is a nonnegative constant parameter (see Fig. 1), which we refer to as the "shock coefficient." Introducing $\tilde{u}(x,t) = u(x,t) - U(x)$ one gets

$$\tilde{u}_t = \tilde{u}_{xx} - U(x)\tilde{u}_x - U'(x)\tilde{u} - \tilde{u}_x\tilde{u} \quad (5)$$
$$\tilde{u}_x(0,t) = \tilde{\omega}_0(t) \quad (6)$$
$$\tilde{u}_x(1,t) = \tilde{\omega}_1(t) \quad (7)$$

where $\tilde{\omega}_0(t) = \tilde{\omega}_0(t) - U'(0)$ and $\tilde{\omega}_1(t) = \tilde{\omega}_1(t) - U'(1)$. Using the "radiation feedback" $\tilde{\omega}_0(t) = k\tilde{u}(0,t)$, $\tilde{\omega}_1(t) = -k\tilde{u}(1,t)$ and linearizing the closed-loop system (with the new state θ), we get

$$\begin{aligned}\theta_t(x,t) &= \theta_{xx}(x,t)\\ &\quad + 2\sigma\left(\tanh\left(\sigma\left(x-\frac{1}{2}\right)\right)\theta(x,t)\right)_x \quad (8)\\ \theta_x(0,t) &= k\theta(0,t) \quad (9)\\ \theta_x(1,t) &= -k\theta(1,t). \quad (10)\end{aligned}$$

In Sections III–IV we derive a condition on k that ensures stability of (8)–(10). In Sections V–VI we prove that even for $k = +\infty$, the decay rate decreases exponentially w.r.t. the parameter σ.

III. Eigenvalue Problem

To analyze stability properties of the system (8)–(10) we look at the eigenvalues. Introducing $u(x,t) = e^{\lambda t}\phi(x)$ we obtain the following two-point boundary value (Sturm-Liouville) problem for ϕ:

$$\phi''(x) + 2\sigma\left(\tanh\left(\sigma\left(x-\frac{1}{2}\right)\right)\phi(x)\right)' = \lambda\phi(x) \quad (11)$$
$$\phi'(0) = k\phi(0) \quad (12)$$
$$\phi'(1) = -k\phi(1). \quad (13)$$

Let us make a change of variables

$$\xi = \tanh\left(\sigma\left(x-\frac{1}{2}\right)\right), \quad \phi(x) = \sqrt{\xi^2-1}\hat{\phi}(\xi). \quad (14)$$

Equation (11) becomes

$$(1-\xi^2)\hat{\phi}''(\xi) - 2\xi\hat{\phi}'(\xi) + \left(2 - \frac{\lambda+\sigma^2}{\sigma^2}\frac{1}{1-\xi^2}\right)\hat{\phi}(\xi) = 0 \quad (15)$$

where $\xi \in (-\tanh(\sigma/2), \tanh(\sigma/2))$.

The general solution of (15) is given by [1]

$$\hat{\phi}(\xi) = C_1 P_1^{\sqrt{1+\lambda/\sigma^2}}(\xi) + C_2 Q_1^{\sqrt{1+\lambda/\sigma^2}}(\xi) \quad (16)$$

where $P_1^{\nu}(\cdot)$, $Q_1^{\nu}(\cdot)$ are the associated Legendre functions of 1st and 2nd kind, respectively.

Going back to the original variables, we get

$$\phi(x) = \frac{C_1 P_1^{\sqrt{1+\lambda/\sigma^2}}\left(\tanh\left(\sigma\left(\frac{x-1}{2}\right)\right)\right)}{\cosh\left(\sigma\left(\frac{x-1}{2}\right)\right)} + \frac{C_2 Q_1^{\sqrt{1+\lambda/\sigma^2}}\left(\tanh\left(\sigma\left(\frac{x-1}{2}\right)\right)\right)}{\cosh\left(\sigma\left(\frac{x-1}{2}\right)\right)}. \quad (17)$$

To obtain characteristic equation for λ, it remains to substitute (17) into the boundary conditions (12) and (13). However, it is clear that the resulting equation is impossible to solve analytically. Therefore, instead of deriving this equation, we are going to use (17) to find a condition on k that ensures exponential stability of the zero equilibrium of (8)–(10).

IV. Stability Condition

Suppose that for some σ and k there is an eigenvalue at zero. Setting $\lambda = 0$ in (17), and using the definitions [1]

$$Q_1^1(z) = -\frac{z}{\sqrt{1-z^2}} - \frac{\sqrt{1-z^2}}{2}\ln\frac{1+z}{1-z} \quad (18)$$

and $P_1^1(z) = -\sqrt{1-z^2}$, after simplifications we get

$$\phi_0(x) = C_2\left[\tanh\left(\sigma\left(\frac{x-1}{2}\right)\right) + \frac{\sigma x}{\cosh^2\left(\sigma\left(\frac{x-1}{2}\right)\right)}\right] - \frac{C_1}{\cosh^2\left(\sigma\left(\frac{x-1}{2}\right)\right)}. \quad (19)$$

Substituting (19) into the boundary conditions (12), (13), we get

$$\left(2\sigma + \left(\frac{k}{2}\right)\sinh(\sigma)\right)C_2 - \left(2\sigma\tanh\left(\frac{\sigma}{2}\right) - k\right)C_1 = 0 \quad (20)$$

and

$$\left(2\sigma - 2\sigma^2\tanh\left(\frac{\sigma}{2}\right) + k\sigma + \left(\frac{k}{2}\right)\sinh(\sigma)\right)C_2 + \left(2\sigma\tanh\left(\frac{\sigma}{2}\right) - k\right)C_1 = 0 \quad (21)$$

respectively. The system (20)–(21) has a non-trivial solution for C_1 and C_2 only when its determinant is zero. We get the following condition:

$$\left(2\sigma\tanh\left(\frac{\sigma}{2}\right) - k\right) \times \left(k(\sigma + \sinh(\sigma)) + 4\sigma - 2\sigma^2\tanh\left(\frac{\sigma}{2}\right)\right) = 0. \quad (22)$$

This equation has two solutions for k

$$k_1^* = 2\sigma\tanh\left(\frac{\sigma}{2}\right) \quad (23)$$
$$k_2^* = \frac{2\sigma^2\tanh\left(\frac{\sigma}{2}\right) - 4\sigma}{\sinh(\sigma) + \sigma}. \quad (24)$$

Note that $k_1^* > k_2^*$ for all $\sigma \geq 0$. We have the following result.

Theorem 1: The system (8)–(10) is exponentially stable in $L^2(0,1)$ norm if and only if $k > k_1^*$. For each $k > k_1^*$, there exist $M(k,\sigma) > 0$ and $m(k,\sigma) > 0$ such that

$$\|\theta(t)\| \leq Me^{-mt}\|\theta_0\|. \quad (25)$$

Proof: It is easy to check that the differential operator that corresponds to (8)–(10) is self-adjoint. In particular, this implies that all the eigenvalues are real. From Lemma A.1 it follows that for $k = +\infty$ all eigenvalues are negative. From Theorem A.2 it follows that as k decreases from $k = +\infty$, all the eigenvalues continuously and monotonically increase until at $k = k_1^*$ the first (largest) eigenvalue becomes zero. Therefore, for all $k_1^* < k < \infty$ all the eigenvalues are negative. The corresponding eigenfunctions form an orthogonal basis that spans

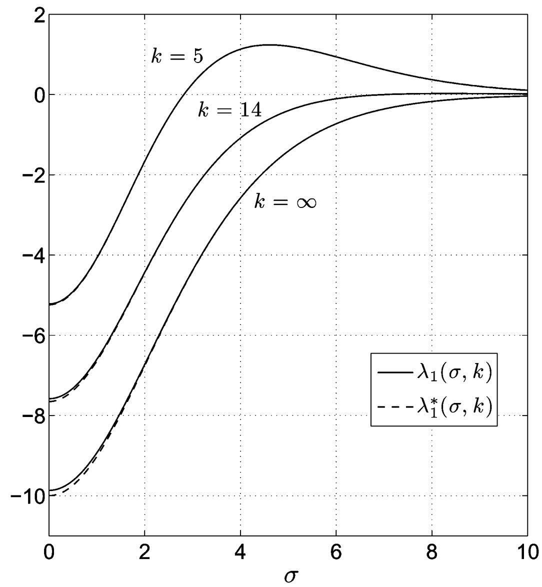

Fig. 2. Largest eigenvalue $\lambda_1(\sigma,k)$ of the system (8)–(10) and the bound (27) for different values of k.

$L^2(0,1)$ (this follows from the corresponding operator being self-adjoint and standard properties of the Sturm-Liouville problem, see [7], [8]). Therefore, the system (8)–(10) is exponentially stable in $L^2(0,1)$. ■

V. Maximum Decay Rate Achievable With Radiation Feedback

In the previous section we established that for sufficiently high k, the system (8)–(10) is exponentially stable. In this section we show that as σ increases, the system's decay rate exponentially goes to zero for any $k > k_1^*$.

Theorem 2: The largest eigenvalue $\lambda_1(\sigma,k)$ of the Sturm-Liouville problem (11)–(13) satisfies

$$\lambda_1(\sigma,k) \geq \lambda_1^*(\sigma,k) \geq \lambda_1^{**}(\sigma,k) \tag{26}$$

for all $\sigma \geq 0$ and all $k > 2\sigma\tanh(\sigma/2)$, where

$$\lambda_1^* = \frac{-4\sigma^2(K\sinh(\sigma)-\sigma)}{\sinh(\sigma)+\sigma+K(\cosh(\sigma)+1)(K\sinh(\sigma)-2\sigma)} \tag{27}$$

$$\lambda_1^{**} = -\frac{32}{K}e^{-\sigma/2} \tag{28}$$

and

$$K = \frac{k}{k-2\sigma\tanh\left(\frac{\sigma}{2}\right)}. \tag{29}$$

Corollary 3: For all $\sigma \geq 0$ and all $k > k_1^*$, the decay rate in (25) satisfies

$$m(k,\sigma) < 32e^{-\sigma/2} \tag{30}$$

and hence

$$\lim_{k\to+\infty} m(k,\sigma) = 0, \quad \forall\, k > k_1^*. \tag{31}$$

This corollary follows from (28), the fact that $K \geq 1$ for $k > k_1^*$, and the arguments in the proof of Theorem 1.

In Fig. 2 we show the bound (27) and the numerically computed first eigenvalue as functions of σ for different values of k. Note that the bound (27) is very accurate for all $\sigma \geq 0$ and all $k > k_1^*$. The error between the bound (27) and the actual eigenvalue is largest at $k=+\infty$, $\sigma=0$, when it is equal to $(10-\pi^2)/10 \approx 1.3\%$. It is clear that for σ greater than ≈ 8, the time response of the system (8)–(10) would be too sluggish since the largest eigenvalue is very close to zero.

Proof of Theorem 2

A. Rayleigh-Ritz Method

To bound the first eigenvalue of (11)–(13), we are going to use the well known Rayleigh-Ritz method (see, e.g., [6]). A version of this method tailored to our problem is given below.

Theorem 4: For the Sturm-Liouville problem

$$\varphi''(x) - \alpha(x)\varphi(x) = \mu\varphi(x) \tag{32}$$

$$\varphi_x(0) = \beta\varphi(0) \tag{33}$$

$$\varphi_x(1) = -\beta\varphi(0) \tag{34}$$

where $\alpha(x)$ is a smooth function, the lower bound on the first (largest) eigenvalue μ_1 is $\mu_1 \geq \mu_1^*$

$$\mu_1^* = \frac{-\int_0^1(\psi'^2(x)+\alpha(x)\psi^2(x))dx - \beta(\psi^2(0)+\psi^2(1))}{\int_0^1\psi^2(x)dx} \tag{35}$$

where $\psi(x)$ is an arbitrary function that satisfies $\psi_x(0) = \beta\psi(0)$, $\psi_x(1) = -\beta\psi(1)$. Equality in (35) is achieved only when $\psi(x)$ is the eigenfunction corresponding to the first eigenvalue.

Proof: Let us denote $F[\psi] = \mu_1^*$, where μ_1^* is given by (35). Note that $F[\varphi_1] = \mu_1$, which is easy to see if one integrates by parts in (35) and uses (32). From general Sturm-Liouville theory it follows that eigenfunctions $\{\varphi_n\}_{n=1}^\infty$ form an orthogonal basis. Therefore, it is possible to write $\psi(x)$ as a linear combination of functions φ_n, i.e., $\psi = \sum_{n=1}^\infty c_n\varphi_n$. Using the orthogonality property and integrating by parts in $F[\psi]$, one can write $F[\psi]$ as

$$F[\psi] = \frac{\int_0^1 \sum_{n=1}^\infty c_n\varphi_n(x)\sum_{m=1}^\infty c_m\mu_m\varphi_m(x)\,dx}{\int_0^1\left(\sum_{n=1}^\infty c_n\varphi_n(x)\right)^2 dx} = \frac{\sum_{n=1}^\infty c_n^2\mu_n}{\sum_{n=1}^\infty c_n^2}.$$

Therefore, $F[\psi]-\mu_1 = \sum_{n=1}^\infty c_n^2(\mu_n-\mu_1)/\sum_{n=1}^\infty c_n^2 \leq 0$ (since μ_1 is the largest eigenvalue), and we get $\mu_1 \geq \mu_1^*$. ■

The key to obtaining a useful lower bound on the first eigenvalue (i.e., a bound that, as σ increases, behaves asymptotically in the same way as the true first eigenvalue) is choosing a test function $\psi(x)$. For example, the simple function $\psi(x) = x(1-x)$, often used in Raleigh-Ritz method, leads to a bound which grows in absolute value as $\sigma \to \infty$, while the absolute value of the true eigenvalue decreases. Therefore, we have to come up with a more sophisticated choice for $\psi(x)$.

Before we start choosing $\psi(x)$, let us convert the eigenvalue problem (11)–(13) into the form (32)–(34). Using the transformation $\bar\phi(x) = \phi(x)\cosh(\sigma(x-1/2))$, we obtain the following equation for $\bar\phi$:

$$\bar\phi''(x) + \left(\frac{2\sigma^2}{\cosh^2\left(\sigma\left(\frac{x-1}{2}\right)\right)-\sigma^2}\right)\bar\phi(x) = \lambda\bar\phi(x) \tag{36}$$

$$\bar\phi'(0) = -\left(k-\sigma\tanh\left(\frac{\sigma}{2}\right)\right)\bar\phi(0) \tag{37}$$

$$\bar\phi'(1) = m\left(k-\sigma\tanh\left(\frac{\sigma}{2}\right)\right)\bar\phi(1) \tag{38}$$

so that $\alpha(x) = \sigma^2 - 2\sigma^2\cosh^{-2}(\sigma(x-1/2))$ and $\beta = k - \sigma\tanh(\sigma/2)$.

B. Choosing a Test Function in Theorem 4

Since equality in (35) is achieved when $\psi(x)$ is the eigenfunction corresponding to the first eigenvalue and we want to show that the first eigenvalue becomes closer and closer to the imaginary axis as σ increases, a good first candidate for $\psi(x)$ is a function that satisfies (36) for $\lambda = 0$. Using (19), we have

$$\psi_1(x) = C_2\left[\sinh\left(\sigma\left(\frac{x-1}{2}\right)\right) + \frac{\sigma x}{\cosh\left(\sigma\left(\frac{x-1}{2}\right)\right)}\right] - \frac{C_1}{\cosh\left(\sigma\left(\frac{x-1}{2}\right)\right)}. \quad (39)$$

Using the boundary condition $\psi_1'(0) = (k - \sigma\tanh(\sigma/2))\psi_1(0)$, we get $C_1 = -1/2kKC_2(4\sigma + k\sinh(\sigma))$ and

$$\psi_1(x) = C_2\left[\sinh\left(\sigma\left(\frac{x-1}{2}\right)\right) + \frac{K\sinh(\sigma) + 2\sigma x + \frac{4\sigma}{k}K}{2\cosh\left(\sigma\left(\frac{x-1}{2}\right)\right)}\right].$$

Let us drop the terms "$+2\sigma x + 4\sigma/kK$" in ψ_1. This is for two reasons: first, for large σ this term is small compared to other terms; second, we would like to avoid polynomial functions in a test function because they make the integrals in (35) not computable in closed form. Note also that multiplication of ψ by a constant does not affect the bound (35), therefore we can divide any test function by a constant. As a result, our modified candidate for a test function is

$$\psi_2(x) = \sinh\left(\sigma\left(\frac{x-1}{2}\right)\right) + \frac{K\sinh(\sigma)}{2\cosh\left(\sigma\left(\frac{x-1}{2}\right)\right)}. \quad (40)$$

This function approaches the first eigenfunction as $\sigma \to 0$ (we do not define a metric of closeness of these two functions since our discussion here serves only motivational purposes), however, it does not satisfy the boundary conditions exactly (which is required by Theorem 4). Let us compute

$$\psi_2'(0) - \left(k - \sigma\tanh\left(\frac{\sigma}{2}\right)\right)\psi_2(0) = \frac{\sigma}{\cosh\left(\frac{\sigma}{2}\right)} \quad (41)$$

$$\psi_2'(1) + \left(k - \sigma\tanh\left(\frac{\sigma}{2}\right)\right)\psi_2(1) = \frac{\sigma + k\sinh(\sigma)}{\cosh\left(\frac{\sigma}{2}\right)}. \quad (42)$$

Let us introduce a new candidate $\psi_3(x) = \psi_2(x) - f(x)$, where $f(x)$ should satisfy several conditions. First, it should make the right-hand sides in (41)–(42) zero. Second, it should keep asymptotic properties of ψ_2, in other words, for large σ it should be small compared to ψ_2. Finally, we want to compose $f(x)$ only from hyperbolic functions so that integrals in (35) can be found explicitly. A simple function satisfying all of the above conditions is

$$f(x) = \frac{\sinh(\sigma x)}{\cosh\left(\frac{\sigma}{2}\right)} \quad (43)$$

and our next candidate becomes (after simplification)

$$\psi_3(x) = \frac{K\sinh(\sigma)}{2\cosh\left(\sigma\left(\frac{x-1}{2}\right)\right)} - \tanh\left(\frac{\sigma}{2}\right)\cosh\left(\sigma\left(\frac{x-1}{2}\right)\right).$$

Multiplying the result by a constant factor $1/\sinh(\sigma/2)$, we obtain our final test function $\psi_4 = \psi$

$$\psi(x) = K\frac{\cosh\left(\frac{\sigma}{2}\right)}{\cosh\left(\sigma\left(\frac{x-1}{2}\right)\right)} - \frac{\cosh\left(\sigma\left(\frac{x-1}{2}\right)\right)}{\cosh\left(\frac{\sigma}{2}\right)}. \quad (44)$$

C. Bounds λ_1^* and λ_1^{**}

Substituting (44) into (35) and computing the integrals, we obtain

$$\lambda_1^* = \frac{\frac{\sigma\sinh(\sigma)(K^2-2K-1)+2\sigma^2-\left(2k\cosh^2\left(\frac{\sigma}{2}\right)-\sigma\sinh(\sigma)\right)(K-1)^2}{2\cosh^2\left(\frac{\sigma}{2}\right)}}{\frac{2K\cosh^2\left(\frac{\sigma}{2}\right)(K\sinh(\sigma)-2\sigma)+\sinh(\sigma)+\sigma}{2\sigma\cosh^2\left(\frac{\sigma}{2}\right)}}$$

which, after simplifications, gives the bound (27).

To obtain a simpler (but more conservative) bound, let us first show the following result.

Lemma 5: For any $\sigma \geq 0$ and $k > k_1^*$ the following inequality holds:

$$\sinh(\sigma) + \sigma + K(\cosh(\sigma)+1)(K\sinh(\sigma) - 2\sigma) \geq \frac{K}{4}\sigma^2\cosh\left(\frac{\sigma}{2}\right)(K\sinh(\sigma) - \sigma) \quad (45)$$

where K is given by (29).

Proof: Denote

$$g(\sigma, K) = \sigma + \sinh(\sigma) + K^2\sinh(\sigma)\left(1 + \cosh(\sigma) - \frac{\sigma^2}{4}\cosh\left(\frac{\sigma}{2}\right)\right) - \sigma K\left(2 + 2\cosh(\sigma) - \frac{\sigma^2}{4}\cosh\left(\frac{\sigma}{2}\right)\right). \quad (46)$$

It is easy to check that (45) is equivalent to the condition $g(\sigma, K) \geq 0$.

Using the identity $\cosh(\sigma) = 2\cosh^2(\sigma/2) - 1$, one can rewrite $g(\sigma, K)$ in the following way:

$$\begin{aligned} g(\sigma, K) = {} & 2(K-1)^2\sinh(\sigma)\cosh\left(\frac{\sigma}{2}\right) \\ & + 2(K-1)\cosh\left(\frac{\sigma}{2}\right)(\sinh(\sigma) - \sigma) \\ & + 4(K-1)\cosh^2\left(\frac{\sigma}{2}\right)\left(\sinh\left(\frac{\sigma}{2}\right) - \frac{\sigma}{2}\right) \\ & + 2K\cosh\left(\frac{\sigma}{2}\right)\left(\cosh\left(\frac{\sigma}{2}\right) - 1 - \frac{\sigma^2}{8}\right) \\ & \times (K\sinh(\sigma) - \sigma) \\ & + \sinh(\sigma) - \sigma\cosh(\sigma) + 2\cosh\left(\frac{\sigma}{2}\right) \\ & \times (\sinh(\sigma) - \sigma). \end{aligned} \quad (47)$$

Since $K \geq 1$, $\sinh(x) \geq x$, $\cosh(x) \geq 1 + x^2/2$ for $x \geq 0$, we get

$$g(\sigma, K) \geq h(\sigma) = \sinh(\sigma) - \sigma\cosh(\sigma) + 2\cosh\left(\frac{\sigma}{2}\right)(\sinh(\sigma) - \sigma). \quad (48)$$

To show that $h(\sigma) \geq 0$, we compute its derivative

$$h'(\sigma) = \sinh\left(\frac{\sigma}{2}\right)(\sinh(\sigma) - \sigma) + 2\sinh(\sigma)\left(\frac{\sinh\left(\frac{\sigma}{2}\right) - \sigma}{2}\right). \quad (49)$$

It's clear that $h'(\sigma) \geq 0$. Since $h(0) = 0$, we have $h(\sigma) \geq 0$ ■

From (27) and (45) we get

$$\lambda_1^* \geq -\frac{16}{K\cosh\left(\frac{\sigma}{2}\right)} \geq -\frac{32}{K}e^{-\sigma/2}. \tag{50}$$

The proof of Theorem 2 is completed.

VI. Conclusion

This note corrects the numerical result in [4, Sec. IV, Fig. 3] which claimed a lack of stabilizability of sharp shock profiles using radiation boundary feedback. We show analytically that any shock profile is stabilizable by radiation feedback with sufficiently high gain, however, the stability margin (the distance of the eigenvalues from the imaginary axis) decays to zero as the shock coefficient grows, irrespective of the gain value.

The vanishing stability margin under radiation feedback amplifies the importance of the backstepping designs in [4], [5]. The backstepping designs achieve arbitrarily fast decay rates, which is established using Lyapunov estimates in [4], [5].

Appendix

Lemma A.1 ([3]): The eigenvalues of the Sturm–Liouville problem

$$\varphi''(x) + (p(x)\varphi(x))' = \mu\varphi(x) \tag{A1}$$
$$\varphi(0) = \varphi(1) = 0 \tag{A2}$$

where $p(x)$ is an arbitrary smooth function, are real and negative.

Theorem A.2: For the eigenvalues $\lambda_n(k)$ of the Sturm–Liouville problem

$$\varphi''(x) + (p(x)\varphi(x))' = \lambda\varphi(x) \tag{A3}$$
$$\varphi_x(0) = k\varphi(0) \tag{A4}$$
$$\varphi_x(1) = -k\varphi(1) \tag{A5}$$

where $p(x)$ is an arbitrary smooth function, the following holds for all $n \in \mathbb{N}$:

(1) $\lambda_n(k)$ is continuously differentiable for all k and

$$\frac{d\lambda_n}{dk} = -\phi_n^2(0) - \phi_n^2(1) \tag{A6}$$

where ϕ_n is the corresponding normalized eigenfunction.

(2) $\lim_{k\to+\infty} |\lambda_n(k) - \mu_n| = 0$, where μ_n are the eigenvalues of the problem (A1)–(A2).

Proof: Statement (1) is a corollary of [2, Theorem 4.2]. Statement (2) follows from [8, Theorem 4.4.3]. ■

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On Almost Sure Stability of Hybrid Stochastic Systems With Mode-Dependent Interval Delays

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Abstract—**This note develops a criterion for almost sure stability of hybrid stochastic systems with mode-dependent interval time delays, which improves an existing result by exploiting the relation between the bounds of the time delays and the generator of the continuous-time Markov chain. The improved result shows that the presence of Markovian switching is quite involved in the stability analysis of delay systems. Numerical examples are given to verify the effectiveness.**

Index Terms—**Almost sure stability, LaSalle-type theorem, Markov chain, stochastic systems, time delays.**

I. Introduction

Since Markov jump systems were firstly introduced in early 1960s (see, e.g., [16] and [23]), hybrid systems driven by continuous-time Markov chains have been widely employed to model many real-life systems where they may experience abrupt changes in system structure and parameters such as $\mathrm{BM/C^3}$ systems, failure prone manufacturing, electric power systems, population dynamics, solar-powered systems, and macroeconomic models of national economy (see [1], [4], [7], [9], [16], [19], [21] and the references therein). Recently, hybrid stochastic delay systems (HSDSs) have received considerable attention (see, e.g., [14], [17] and [21]) since time delays and stochastic perturbation are often encountered in various practical models in many branches of science and engineering. An area of particular interest has been the stability analysis of this class of hybrid systems and its application to automatic control (see [6], [13], [14], [22], [23] and the references therein). The presence of the Markovian switching is quite involved in stability analysis of the hybrid systems (see, e.g., [2], [4], [7], [16]). Even if all the subsystems are stable, the hybrid system may not be stable; on the other hand, the hybrid system may be stable even if all the subsystems are unstable (see, e.g., [2]–[4] and [16]).

The classical stochastic analysis theory studies stability not only in moment sense but also in almost sure sense (see, e.g., [5], [11] and [22]). Among the existing results, [22] studied almost sure stability of HSDSs with the techniques proposed in [11] while most of the others dealt with moment stability. However, the results in [22] require the time delays of all subsystems to be equal to a constant. This may be too restrictive to apply to hybrid systems in many practical situations (see, e.g., Example 4.1). This note extends the results in [22] to hybrid

Manuscript received May 15, 2009; revised August 28, 2009, February 07, 2010, and April 15, 2010; accepted May 04, 2010. Date of publication May 10, 2010; date of current version July 30, 2010. This work was supported by the UK ORSAS and the University of Strathclyde. Recommended by Associate Editor S. A. Reveliotis.

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Digital Object Identifier 10.1109/TAC.2010.2050160