Dead-Time Compensation for Wave/String PDEs

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Smith predictorlike designs for compensation of arbitrarily long input delays are commonly available only for finite-dimensional systems. Only very few examples exist, where such compensation has been achieved for partial differential equation (PDE) systems, including our recent result for a parabolic (reaction-diffusion) PDE. In this paper, we address a more challenging wave PDE problem, where the difficulty is amplified by allowing all of this PDE's eigenvalues to be a distance to the right of the imaginary axis. Antidamping (positive feedback) on the uncontrolled boundary induces this dramatic form of instability. We develop a design that compensates an arbitrarily long delay at the input of the boundary control system and achieves exponential stability in closed-loop. We derive explicit formulae for our controller's gain kernel functions. They are related to the open-loop solutions of the antistable wave equation system over the time period of input delay (this simple relationship is the result of the design approach).
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1 Introduction

1.1 Background on Delay Compensation and Predictor Feedback. Despite decades of development, the area of delay systems remains a seemingly endless source of research challenges and continues to be an active area of research, especially in mechanical engineering [1–14].

One problem in control of delay systems that has received continuous attention over the last 5 decades is the problem of systems with input or output delay, or, simply, systems with dead-time. Starting with the Smith predictor [15], ODE systems with dead-time have been studied successfully from numerous angles, leading even to adaptive designs and designs for nonlinear ODEs [16–42].

1.2 Delay Compensation for PDEs. Control of *PDEs* with input delays, or, to be more precise, predictor-based compensation of actuator or sensor delays in control of PDEs, is a new area that is just opening up for research. The motivation partly comes from the classical examples by Datko et al. [43] and Datko [44] who identified some classes of hyperbolic PDE systems, which, although exponentially stabilizable, have zero robustness to inserting a delay in the feedback loop, i.e., arbitrarily small dead-time leads to instability.

In recent work, Guo and Xu [45] presented an observer-based compensator of sensor delay for a wave equation, which Guo and Chang [46], then, followed by an extension to an Euler–Bernoulli beam problem. The wave/string PDE is the subject of continuing advances in boundary control [47–50]. It serves as a benchmark and a stepping stone for the more complex beam equation. While few experiments have been conducted with stringlike systems, several experimental applications of boundary control of beams have been reported [51–54].

In a recent manuscript [55], we initiated an effort for developing predictor feedbacks for PDEs with input delay, via the method of backstepping (see Refs. [56,57,68,69] for other uses of backstepping in boundary control of PDEs). The PDE considered in Ref. [55] is a parabolic (reaction-diffusion) system, which, although open-loop unstable and with a long delay at the input, is not as challenging a problem as wave/hyperbolic equations since this class of PDEs does not exhibit a Datko-type loss of delay robustness margin.

1.3 Contribution of the Present Paper. Our focus in the present paper is on wave equations with input delay, as displayed in Fig. 1. We pursue a problem similar to Ref. [45] but with two differences, one methodological and one in the system being considered. The methodological difference is that we approach the problem with a backstepping-based technique for compensation of the input delay and arrive at a compensator, whose gain functions we derive explicitly, which achieves exponential stabilization in the presence of arbitrarily long actuator delay. The other difference, in the class of systems, is that we consider a wave equation of an unconventional type. The standard wave equation has all of its infinitely many eigenvalues on the imaginary axis. In this paper, we consider a wave equation that contains an antidamping effect on the boundary opposite to the controlled boundary. This antidamping effect results in all of the infinitely many eigenvalues of the uncontrolled system being in the right-half plane, and possibly being arbitrarily far to the right of the imaginary axis.

One of the key tools employed in the design in this paper is a new type of backstepping transformation recently introduced by Smyshlyaev and Krstic [58] for wave equations with boundary antidamping. The rest of the tools are a collection of techniques we recently introduced in Refs. [24,55,59,60].

The difference between the present paper and Ref. [58] is that the presence of the delay requires the construction of an additional backstepping transformation for the delay state, which has to also incorporate the displacement and velocity states of the wave equation. As a result, the construction of the transformation kernels in this paper is much more complex than for the wave equation alone in Ref. [58]. The results of the present paper are of interest even in the absence of antidamping in the wave equation [58]; in that case, one obtains a novel control law for the conventional undamped wave/string system with input delay. The difference between the present paper and Ref. [55] is that the second-order character of the PDE plant makes not only the backstepping transformation of the delay state more complex but it also makes the stability analysis much more involved, requiring, among other things, that the input delay state be quantified not in the L_2 norm (as appropriate when the plant is an ODE) in the H_1 norm (as appropriate when the plant is a parabolic PDE) but in the H_2 norm.

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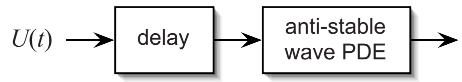

Fig. 1 Antistable wave PDE system with input delay

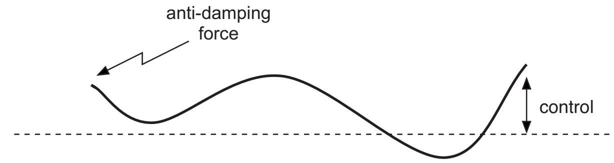

Fig. 3 A diagram of a string with control applied at boundary $x=1$ and an antidamping force acting at the boundary $x=0$

1.4 Physical Motivation for the Wave Equation With Antidamping. The 1D wave equation is a good model for acoustic dynamics in ducts and vibration of strings. The phenomenon of *antidamping* that we include in our study of wave dynamics represents injection of energy in proportion to the velocity field, akin to a damper with a negative damping coefficient. Such a process arises in combustion dynamics, where the pressure field is disturbed in proportion to varying *heat release* rate, which, in turn, is proportional to the rate of change of pressure. Figure 2 shows the classical Rijke tube control experiment [61]. For the sake of our study, we assume that the system is controlled using a loudspeaker, with a pressure sensor (microphone) near the speaker. Both the actuator and the sensor are placed as far as possible from the flame front, for their thermal protection. The process of antidamping is located mostly at the flame front. In a Rijke tube, the flame front may not be at the end of the tube, however, the leaner the fuel/air mixture, the longer it takes for the mixture to ignite, and the further the flame is from the injection point. In the limit, for the leanest mixture for which combustion is sustained, the flame is near the exit of the tube, which corresponds to a situation with boundary antidamping.

An input delay can arise from a computational delay, as depicted in Fig. 2. Alternatively, if the system is controlled not using loudspeaker actuation but using modulation of the fuel injection rate [62–64], a long delay may result both from the delay associated with the servovalve and feed line operation and from the transport process within the combustor from the injection point to the ignition location. The leaner the mixture, the longer the input delay in this configuration.

Another physical example involving antidamping in string dynamics is discussed in Sec. 6.

1.5 Organization of This Paper. This paper is organized as follows. In Sec. 2, we present a control design and state the stability result. In Sec. 3, we derive the gain functions of this control law explicitly. In Sec. 4, we prove exponential stability of the target system resulting from the backstepping construction. In Sec. 5, we establish our main result, through a stability analysis for the system in its original variables. In Sec. 6, we offer some closing comments.

2 Control Design for Antistable Wave PDE With Input Delay

We consider the delay-wave cascade system

$$u_{tt}(x,t) = u_{xx}(x,t), \quad x \in (0,1) \tag{1}$$

$$u_x(0,t) = -qu_t(0,t) \tag{2}$$

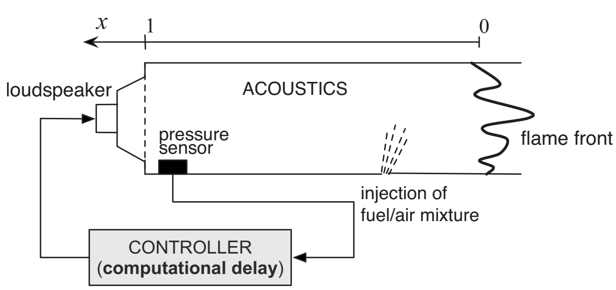

Fig. 2 Control of a thermoacoustic instability in a Rijke tube [61] (a duct-type combustion chamber)

$$u(1,t) = U(t-D) \tag{3}$$

where for $q>0$ and $U(t)\equiv 0$, one obtains an "antistable" wave equation, with all of its infinitely many eigenvalues in the right-half plane and given by Ref. [56], p. 82

$$\sigma_n = \frac{1}{2}\ln\left|\frac{1+q}{1-q}\right| + j\pi\begin{cases} n+\frac{1}{2}, & 0\le q<1 \\ n, & q>1 \end{cases} \tag{4}$$

The structure of the system is such that the destabilizing force acts on the opposite boundary from the input, as represented by the string example in Fig. 3.

By denoting $v(x,t)\triangleq U(t+x-1-D)$, the delay-wave systems (1)–(3) are alternatively written as transport-wave PDE cascades

$$u_{tt}(x,t) = u_{xx}(x,t), \quad x \in (0,1) \tag{5}$$

$$u_x(0,t) = -qu_t(0,t) \tag{6}$$

$$u(1,t) = v(1,t) \tag{7}$$

$$v_t(x,t) = v_x(x,t), \quad x \in [1,1+D) \tag{8}$$

$$v(1+D,t) = U(t) \tag{9}$$

where $U(t)$ is the overall system input and (u,u_t,v) is the state. Hence, we formulate the control problem as boundary control of a cascade of a transport PDE with a wave equation. For an example of a control problem involving a nonscalar transport PDE with reaction effects and nonlinearities, see Ref. [65].

The spatial variable x, the time variable t, and the delay D are non dimensional. This is achieved by a standard nondimensionalization procedure. Consider a more general wave equation $\check{u}_{\check{t}\check{t}}(\check{x},\check{t})=\sigma^2\check{u}_{\check{x}\check{x}}(\check{x},\check{t})$ on the domain $\check{x}\in(0,L)$, where L is the length of the domain in meters, σ is the wave propagation speed in meters per second, and $\check{t}$ is time in seconds. We introduce a new spatial variable $x=\check{x}/L$ and a new time variable $t=\sigma\check{t}/L$, which are both nondimensional. Then, the new state variable $u(x,t)=\check{u}(Lx,Lt/\sigma)$ satisfies the wave equation $u_{tt}(x,t)=u_{xx}(x,t)$ for $x\in(0,1)$. Similarly, if the boundary input of the wave equation is $\check{u}(L,\check{t})=\check{U}(\check{t}-\check{D})$, the new control is defined as $U(t)=\check{U}(Lt/\sigma)$ and the actuator state is defined as $v(x,t)=\check{U}(L(t+x-1-D)/\sigma)$, where $D=\sigma\check{D}/L$. The quantity $1+D$ in Eqs. (8) and (9) is nondimensional.

We consider the backstepping transformation

$$w(x,t) = u(x,t) - \frac{q(q+c)}{1+qc}u(0,t) + \frac{q+c}{1+qc}\int_0^x u_t(y,t)dy \tag{10}$$

$$z(x,t) = v(x,t) - \int_1^x p(x-y)v(y,t)dy - \theta(x)u(0,t) - \int_0^1 \gamma(x,y)u(y,t)dy - \int_0^1 \rho(x,y)u_t(y,t)dy \tag{11}$$

where Eq. (10) was introduced by Smyshlyaev and Krstic in Ref.

[58], the kernels p,θ,γ,ρ need to be chosen to transform the cascade PDE system into the target system

$$w_{tt}(x,t)=w_{xx}(x,t), \quad x\in(0,1) \tag{12}$$

$$w_x(0,t)=cw_t(0,t) \tag{13}$$

$$w(1,t)=z(1,t) \tag{14}$$

$$z_t(x,t)=z_x(x,t), \quad x\in[1,1+D) \tag{15}$$

$$z(1+D,t)=0 \tag{16}$$

and c is a positive gain, used in the gain kernels of the control law

$$U(t)=\int_1^{1+D}p(1+D-y)v(y,t)dy+\theta(1+D)u(0,t)+\int_0^1\gamma(1+D,y)u(y,t)dy+\int_0^1\rho(1+D,y)u_t(y,t)dy \tag{17}$$

The target system (w,z), which is a transport-wave cascade interconnected through a boundary, is exponentially stable in an appropriate norm, as we shall establish in Sec. 4.

By differentiating Eq. (11) once with respect to t and once with respect to x, equating the two expressions, substituting the PDEs for u and v, and integrating by parts with respect to y, we obtain conditions that the kernels p,θ,γ,ρ need to satisfy in order for the (u,v)-system and the (w,z)-system to be equivalent. The kernel ρ is governed by the PDE

$$\rho_{xx}(x,y)=\rho_{yy}(x,y) \tag{18}$$

$$\rho_y(x,0)=-q\rho_x(x,0) \tag{19}$$

$$\rho(x,1)=0 \tag{20}$$

where $x\in[1,1+D]$ and $y\in(0,1)$. Note that this system has a structure identical to the uncontrolled wave equation plant. Since Eqs. (19) and (20) play the role of boundary conditions, the variable x can be viewed as a timelike variable, even though it plays the role of a spatial variable in the kernel of the transformation Eq. (11). The initial condition of Eq. (18) is

$$\rho(1,y)=-\frac{q+c}{1+qc} \tag{21}$$

$$\rho_x(1,y)=0 \tag{22}$$

After solving for $\rho(x,y)$, the kernels p,θ,γ are obtained as

$$p(s)=-\rho_y(1+s,1), \quad s\in[0,D] \tag{23}$$

$$\theta(x)=-q\rho(x,0) \tag{24}$$

$$\gamma(x,y)=\rho_x(x,y) \tag{25}$$

We present a detailed Lyapunov stability analysis in Secs. 4 and 5, however, we state first a result on closed-loop eigenvalues (for the target system (w,z)).

PROPOSITION 1. *The finite part of the spectrum of systems (12)–(16) is given by*

$$\sigma_n=-\frac{1}{2}\ln\left|\frac{1+c}{1-c}\right|+j\pi\begin{cases}n+\frac{1}{2}, & 0\le c<1\\ n, & c>1\end{cases} \tag{26}$$

where $n\in\mathbb{Z}$.

Proof. Systems (12)–(16) are a cascade connection of the transport PDE $z_t(x,t)=z_x(x,t), z(1+D,t)=0$, whose eigenvalues have real parts equal to negative infinity (see, for example, Ref. [66], Appendix C), and of the boundary-damped wave PDE $w_{tt}(x,t)=w_{xx}(x,t), w_x(0,t)=cw_t(0,t), w(1,t)=0$, whose eigenvalue Eq. (26) was determined in Ref. [56], p. 82.

The inverse backstepping transformation, which is needed in our analysis, is given by

$$u(x,t)=w(x,t)-\frac{c(q+c)}{1+qc}w(0,t)-\frac{q+c}{1+qc}\int_0^x w_t(y,t)dy \tag{27}$$

$$v(x,t)=z(x,t)-\int_1^x\pi(x-y)z(y,t)dy-\eta(x)w(0,t)-\int_0^1\delta(x,y)w(y,t)dy-\int_0^1\mu(x,y)w_t(y,t)dy \tag{28}$$

where the kernels π,η,δ,μ are defined next. The kernel μ is governed by the PDE

$$\mu_{xx}(x,y)=\mu_{yy}(x,y) \tag{29}$$

$$\mu_y(x,0)=c\mu_x(x,0) \tag{30}$$

$$\mu(x,1)=0 \tag{31}$$

where $x\in[1,1+D]$ and $y\in(0,1)$. Again, note that the structure of this system is identical to the target system w. The initial condition of this PDE is

$$\mu(1,y)=\frac{q+c}{1+qc} \tag{32}$$

$$\mu_x(1,y)=0 \tag{33}$$

After solving for $\mu(x,y)$, the kernels π,η,δ are obtained as

$$\pi(s)=-\mu_y(1+s,1), \quad s\in[0,D] \tag{34}$$

$$\eta(x)=c\mu(x,0) \tag{35}$$

$$\delta(x,y)=\mu_x(x,y) \tag{36}$$

3 Explicit Gain Functions

In the standard predictor feedback form, the controller is written as

$$U(t)=-q\rho(1+D,0)u(0,t)+\int_0^1\rho_x(1+D,y)u(y,t)dy+\int_0^1\rho(1+D,y)u_t(y,t)dy-\int_{t-D}^{t}\rho_y(t-\theta,1)U(\theta)d\theta \tag{37}$$

and so, our task is to find the solution $\rho(x,y)$ and its first derivatives with respect to both x and y. The procedure of solving for $\rho(x,y)$ is outlined in Fig. 4.

$$\varpi \xrightarrow{(59)} \vartheta \xrightarrow{(45)} \varsigma \xrightarrow{(43)} \rho$$

Fig. 4 The process of finding the solution to the PDEs (18)–(22) for the gain function $\rho(x,y)$. First, the undamped wave equation with homogeneous boundary conditions, Eqs. (60)–(62) or Eqs. (63)–(65), is solved for ϖ using Lemmas 4 and 5. Second, Lemma 3 yields the solution ϑ to the wave equation with domainwide antidamping λ, Eqs. (47)–(49) or Eqs. (50)–(52). Third, Lemma 2 yields the solution ς to the wave equation with boundary antidamping q, Eqs. (38)–(42). Finally, using Eq. (43), the gain functions ρ,ρ_x,ρ_y, which are needed in the controller Eq. (37), are found in Proposition 6.

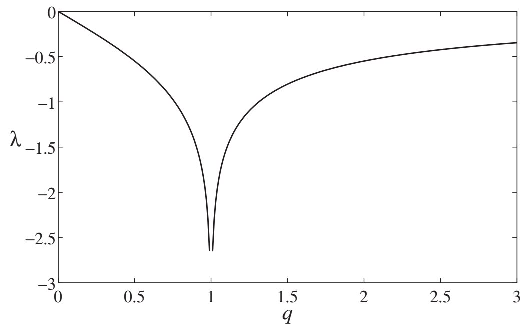

Fig. 5 The graph of the function λ(*q*)

We seek the solution of Eqs. (18)–(22) by seeking the solution of the (space-reversed) PDE system

$$\varsigma_{tt}(x,t)=\varsigma_{xx}(x,t) \tag{38}$$

$$\varsigma(0,t)=0 \tag{39}$$

$$\varsigma_x(1,t)=q\varsigma_t(1,t) \tag{40}$$

with initial conditions

$$\varsigma(x,0)=-\frac{q+c}{1+qc} \tag{41}$$

$$\varsigma_t(x,0)=0 \tag{42}$$

from which we shall obtain

$$\rho(x,y)=\varsigma(1-y,x-1),\quad y\in[0,1],x\geq 1 \tag{43}$$

We present the construction of $\varsigma(x,t)$ through a series of lemmas. Most lemmas that we state are reasonably straightforward to prove, or the proofs can be obtained by direct (albeit possibly lengthy) verification. Hence, we omit most proofs.

The first lemma introduces a backstepping-style transformation, which moves the antidamping effect from the boundary condition $\varsigma_x(1,t)=q\varsigma_t(1,t)$ into the domain, where it can be handled (for the purpose of solving the PDE) more easily.

LEMMA 2. *Consider systems (38)–(40) and the transformations*

$$\vartheta(x,t)=\cosh(\lambda x)\varsigma(x,t)+\int_0^x \sinh(\lambda y)(\varsigma_t(y,t)-\lambda\varsigma(y,t))dy \tag{44}$$

$$\varsigma(x,t)=\cosh(\lambda x)\vartheta(x,t)-\int_0^x \sinh(\lambda y)(\vartheta_t(y,t)+2\lambda\vartheta(y,t))dy \tag{45}$$

where λ *denotes the function*

$$\lambda(q)=\begin{cases}-\tanh^{-1}(q), & q\in[0,1)\\ -\coth^{-1}(q), & q>1\end{cases} \tag{46}$$

which is shown in Fig. 5. For $q\in[0,1)$, *the function* $\varsigma(x,t)$ *satisfies Eqs. (38)–(40) if and only if the function* $\vartheta(x,t)$ *satisfies*

$$\vartheta_{tt}(x,t)+2\lambda\vartheta_t(x,t)+\lambda^2\vartheta(x,t)=\vartheta_{xx}(x,t) \tag{47}$$

$$\vartheta(0,t)=0 \tag{48}$$

$$\vartheta_x(1,t)=0 \tag{49}$$

For $q>1$, *the function* $\varsigma(x,t)$ *satisfies Eqs. (38)–(40) if and only if the function* $\vartheta(x,t)$ *satisfies*

$$\vartheta_{tt}(x,t)+2\lambda\vartheta_t(x,t)+\lambda^2\vartheta(x,t)=\vartheta_{xx}(x,t) \tag{50}$$

$$\vartheta(0,t)=0 \tag{51}$$

$$\vartheta_t(1,t)+\lambda\vartheta(1,t)=0 \tag{52}$$

Proof. By deriving and using the fact that the transformation $u\mapsto w$ yields

$$\vartheta_x(x,t)=\cosh(\lambda x)\varsigma_x(x,t)+\sinh(\lambda x)\varsigma_t(x,t) \tag{53}$$

$$\vartheta_t(x,t)+\lambda\vartheta(x,t)=\sinh(\lambda x)\varsigma_x(x,t)+\cosh(\lambda x)\varsigma_t(x,t) \tag{54}$$

and that the transformation $w\mapsto u$ yields

$$\varsigma_x(x,t)=\cosh(\lambda x)\vartheta_x(x,t)-\sinh(\lambda x)(\vartheta_t(x,t)+\lambda\vartheta(x,t)) \tag{55}$$

$$\varsigma_t(x,t)=-\sinh(\lambda x)\vartheta_x(x,t)+\cosh(\lambda x)(\vartheta_t(x,t)+\lambda\vartheta(x,t)) \tag{56}$$

The reader should note that the relationship between q and a, which is given by Eq. (46), yields

$$\sinh(\lambda x)=\frac{1}{2}\left(\left|\frac{1-q}{1+q}\right|^{x/2}-\left|\frac{1+q}{1-q}\right|^{x/2}\right) \tag{57}$$

$$\cosh(\lambda x)=\frac{1}{2}\left(\left|\frac{1-q}{1+q}\right|^{x/2}+\left|\frac{1+q}{1-q}\right|^{x/2}\right) \tag{58}$$

To make the ς-system easily solvable, we introduce another transformation, given in the next lemma. This lemma completely eliminates the in-domain antidamping but using time-dependent scaling of the state variable.

LEMMA 3. *Let* $\varpi(x,t)=e^{\lambda t}\vartheta(x,t)$, *i.e.,*

$$\vartheta(x,t)=\left|\frac{1+q}{1-q}\right|^{t/2}\varpi(x,t) \tag{59}$$

For $q\in[0,1)$, *the function* $\vartheta(x,t)$ *satisfies Eqs. (47)–(49) if and only if the function* $\varpi(x,t)$ *satisfies*

$$\varpi_{tt}(x,t)=\varpi_{xx}(x,t) \tag{60}$$

$$\varpi(0,t)=0 \tag{61}$$

$$\varpi_x(1,t)=0 \tag{62}$$

For $q>1$, *the function* $\vartheta(x,t)$ *satisfies Eqs. (50)–(52) if and only if the function* $\varpi(x,t)$ *satisfies*

$$\varpi_{tt}(x,t)=\varpi_{xx}(x,t) \tag{63}$$

$$\varpi(0,t)=0 \tag{64}$$

$$\varpi_t(1,t)=0 \tag{65}$$

Furthermore, if the initial conditions of systems (38)–(40) are given by Eqs. (41) and (42), then the initial conditions $\varpi_0(x)\triangleq\varpi(x,0)$ *and* $\varpi_1(x)\triangleq\varpi_t(x,0)$ *for the* ϖ *-system are given by*

$$\varpi_0(x)=-\frac{q+c}{1+qc} \tag{66}$$

$$\varpi_1(x)=0 \tag{67}$$

The ϖ-systems are now readily solvable in explicit form. Their solutions, for arbitrary initial conditions, are stated in the next two lemmas.

LEMMA 4. *For* $q\in[0,1)$, *the solution of systems (60)–(62) with arbitrary initial conditions is*

$$\varpi(x,t) = 2\sum_{n=0}^{\infty} \sin\left((2n+1)\frac{\pi}{2}x\right)\left[\int_0^1 \sin\left((2n+1)\frac{\pi}{2}y\right)\varpi_0(y)dy \cos\left((2n+1)\frac{\pi}{2}t\right) + \frac{2}{(2n+1)\pi}\int_0^1 \sin\left((2n+1)\frac{\pi}{2}y\right)\varpi_1(y)dy \sin\left((2n+1)\frac{\pi}{2}t\right)\right] \quad (68)$$

For $q>1$, the solution of systems (63)–(65) with arbitrary initial conditions is

$$\varpi(x,t) = 2\sum_{n=1}^{\infty} \sin(n\pi x)\left[\int_0^1 \sin(n\pi y)\varpi_0(y)dy \cos(n\pi t) + \frac{1}{n\pi}\int_0^1 \sin(n\pi y)\varpi_1(y)dy \sin(n\pi t)\right] \quad (69)$$

For the initial condition Eqs. (66) and (67), we obtain the explicit solutions $\varpi(x,t)$ as follows.

LEMMA 5. *For $q\in[0,1)$, the solution of systems (60)–(62) with initial conditions given by Eqs. (66) and (67) is*

$$\varpi(x,t) = -\frac{q+c}{1+qc}2\sum_{n=0}^{\infty} \frac{\sin\left((2n+1)\frac{\pi}{2}x\right)\cos\left((2n+1)\frac{\pi}{2}t\right)}{(2n+1)\frac{\pi}{2}} = -\frac{q+c}{1+qc}\sum_{n=0}^{\infty} \frac{\sin\left((2n+1)\frac{\pi}{2}(x+t)\right) - \sin\left((2n+1)\frac{\pi}{2}(t-x)\right)}{(2n+1)\frac{\pi}{2}} \quad (70)$$

For $q>1$, the solution of systems (63)–(65) with initial conditions given by Eqs. (66) and (67) is

$$\varpi(x,t) = -\frac{q+c}{1+qc}2\sum_{m=1}^{\infty} \frac{\sin(2m\pi x)\cos(2m\pi t)}{m\pi} = -\frac{q+c}{1+qc}\sum_{m=1}^{\infty} \frac{\sin(2m\pi(x+t)) - \sin(2m\pi(t-x))}{m\pi} \quad (71)$$

Now, we return to the gain formula (43) and, using Lemmas 2–5, obtain the following explicit expression for the gain kernel $\rho(x,y)$, as well as for the functions $\rho_x(1+D,y)$ and $\rho_y(t-\theta,1)$, which are used in the control law Eq. (37).

PROPOSITION 6. *The solution of systems (18)–(22) is given by*

$$\rho(x,y) = e^{-\lambda(x-1)}\left[\cosh(\lambda(1-y))\varpi(1-y,x-1) - \int_0^{1-y} \sinh(\lambda s) \times(\varpi_t(s,x-1) + a\varpi(s,x-1))ds\right] \quad (72)$$

where for $q\in[0,1)$, the function $\varpi(\cdot,\cdot)$ is given by Eq. (70) and for $q>1$, the function $\varpi(\cdot,\cdot)$ is given by Eq. (71). Furthermore, the partial derivatives of Eq. (72) are

$$\rho_x(x,y) = -\lambda\rho(x,y) + e^{-\lambda(x-1)}\left[\cosh(\lambda(1-y))\varpi_t(1-y,x-1) - \int_0^{1-y} \sinh(\lambda s)(\varpi_{tt}(s,x-1) + a\varpi_t(s,x-1))ds\right] \quad (73)$$

$$\rho_y(x,y) = e^{-\lambda(x-1)}[-\lambda \sinh(\lambda(1-y))\varpi(1-y,x-1) - \cosh(\lambda(1-y))\varpi_x(1-y,x-1) - \sinh(\lambda(1-y))(\varpi_t(1-y,x-1) + a\varpi(1-y,x-1))] \quad (74)$$

where, for $q\in[0,1)$, the functions $\varpi_x(x,t), \varpi_t(x,t), \varpi_{tt}(x,t)$ are given by

$$\varpi_x(x,t) = -\frac{q+c}{1+qc}2\sum_{n=0}^{\infty} \cos\left((2n+1)\frac{\pi}{2}x\right)\cos\left((2n+1)\frac{\pi}{2}t\right) \quad (75)$$

$$\varpi_t(x,t) = \frac{q+c}{1+qc}2\sum_{n=0}^{\infty} \sin\left((2n+1)\frac{\pi}{2}x\right)\sin\left((2n+1)\frac{\pi}{2}t\right) \quad (76)$$

$$\varpi_{tt}(x,t) = \frac{q+c}{1+qc}2\sum_{n=0}^{\infty} (2n+1)\frac{\pi}{2}\sin\left((2n+1)\frac{\pi}{2}x\right)\cos\left((2n+1)\frac{\pi}{2}t\right) \quad (77)$$

and for $q>1$, the functions $\varpi_x(x,t), \varpi_t(x,t), \varpi_{tt}(x,t)$ are given by

$$\varpi_x(x,t) = -\frac{q+c}{1+qc}2\sum_{m=1}^{\infty} \cos(2m\pi x)\cos(2m\pi t) \quad (78)$$

$$\varpi_t(x,t) = \frac{q+c}{1+qc}2\sum_{m=1}^{\infty} \sin(2m\pi x)\sin(2m\pi t) \quad (79)$$

$$\varpi_{tt}(x,t) = \frac{q+c}{1+qc}2\sum_{m=1}^{\infty} m\pi \sin(2m\pi x)\cos(2m\pi t) \quad (80)$$

4 Stability of the Target System (w,z)

Having completed the derivation of explicit expressions for the control gains in Sec. 3, we now turn our attention to the exponential stability analysis of the target system

$$w_{tt}(x,t) = w_{xx}(x,t), \quad x\in(0,1) \quad (81)$$

$$w_x(0,t) = cw_t(0,t) \quad (82)$$

$$w(1,t) = z(1,t) \quad (83)$$

$$z_t(x,t) = z_x(x,t), \quad x\in[1,1+D) \quad (84)$$

$$z(1+D,t) = 0 \quad (85)$$

The analysis employs two new transformations, whose role is depicted and explained in Fig. 6.

We first denote

$$a(c) = \begin{cases} \tanh^{-1}(c), & c\in[0,1) \\ \coth^{-1}(c), & c>1 \end{cases} \quad (86)$$

and then introduce the transformations

$$\varphi_x(x,t) = \cosh(a(1-x))w_x(x,t) - \sinh(a(1-x))w_t(x,t) \quad (87)$$

$$\varphi_t(x,t) + a\varphi(x,t) = -\sinh(a(1-x))w_x(x,t) + \cosh(a(1-x))w_t(x,t) \quad (88)$$

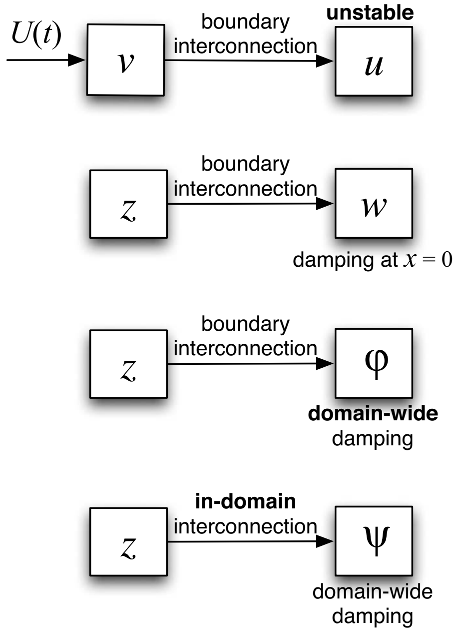

Fig. 6 A sequence of system transformations employed in the analysis. The original plant is (v,u), whereas the backstepping transformation converts the closed-loop system into the autonomous, exponentially stable system (z,w). Due to the boundary interconnection and the boundary damping at $x=0$, the stability of the system (z,w) is hard to analyze. The damping is moved from the boundary to the domain using the transformation $(z,w)\mapsto(z,\varphi)$. Finally, the boundary interconnection, in which an unbounded operator arises, is converted into an easier-to-analyze in-domain interconnection using the transformation $(z,\varphi)\mapsto(z,\psi)$. The transformations between the four representations are indicated in Fig. 7. The stability analysis of the (z,ψ)-system is outlined in Fig. 8. Stability of the (z,ψ)-system is studied in Lemmas 7–10, using the Lyapunov function $\Omega(t)$. Then, stability of the (z,φ)-system is shown in Lemma 11 and Proposition 12.

$$\varphi(1,t)=z(1,t) \tag{89}$$

and

$$w_x(x,t)=\cosh(a(1-x))\varphi_x(x,t)+\sinh(a(1-x))(\varphi_t(x,t)+a\varphi(x,t)) \tag{90}$$

$$w_t(x,t)=\sinh(a(1-x))\varphi_x(x,t)+\cosh(a(1-x))(\varphi_t(x,t)+a\varphi(x,t)) \tag{91}$$

$$w(1,t)=z(1,t) \tag{92}$$

By integrating in x, these transformations are also written as

$$\varphi(x,t)=\cosh(a(1-x))w(x,t)+\int_x^1 \sinh(a(1-y))(w_t(y,t)-aw(y,t))dy \tag{93}$$

$$w(x,t)=\cosh(a(1-x))\varphi(x,t)-\int_x^1 \sinh(a(1-y))(\varphi_t(y,t)+2a\varphi(y,t))dy \tag{94}$$

For $c\in[0,1)$, the transformation converts the w-system into

$$\varphi_{tt}(x,t)+2a\varphi_t(x,t)+a^2\varphi(x,t)=\varphi_{xx}(x,t) \tag{95}$$

$$\varphi_x(0,t)=0 \tag{96}$$

$$(v,u)\xrightarrow{(10),(11)}(z,w)\xrightarrow{(93)}(z,\varphi)\xrightarrow{(101)}(z,\psi)$$

Fig. 7 The transformations among the four system representations in Fig. 6. Only the transformation $(v,u)\mapsto(z,w)$ is of backstepping type, whereas the other two have the role of moving the damping and moving the transport-wave interconnection from a boundary into the domain, which is a form that facilitates the analysis.

$$\varphi(1,t)=z(1,t) \tag{97}$$

For $c>1$, the transformation converts the w-system into

$$\varphi_{tt}(x,t)+2a\varphi_t(x,t)+a^2\varphi(x,t)=\varphi_{xx}(x,t) \tag{98}$$

$$\varphi_t(0,t)+a\varphi(0,t)=0 \tag{99}$$

$$\varphi(1,t)=z(1,t) \tag{100}$$

Even though the z-system is the exponentially stable transport equation $z_t(x,t)=z_x(x,t)$, $z(1,t)=0$, the stability analysis for the φ-system cannot proceed in this form because of $z(1,t)$ entering the φ-system through a boundary condition, which makes the resulting input operator unbounded, and the resulting gain from $z(1,t)$ to $\varphi(x,t)$ (in any suitable norm) unbounded.

We first perform a transformation that shifts $z(1,t)$ into the interior of the domain (0,1). The next lemma introduces this transformation and presents a Lyapunov function for the resulting system (see Fig. 7 for a summary of transformations used in the paper).

LEMMA 7. *Consider the change of variable*

$$\psi(x,t)=\varphi(x,t)-x^2z(1,t) \tag{101}$$

and the resulting system, which, for $c\in[0,1)$, *is*

$$\psi_{tt}(x,t)+2a\psi_t(x,t)+a^2\psi(x,t)=\psi_{xx}(x,t)+g(x,t) \tag{102}$$

$$\psi_x(0,t)=0 \tag{103}$$

$$\psi(1,t)=0 \tag{104}$$

and for $c>1$ *is*

$$\psi_{tt}(x,t)+2a\psi_t(x,t)+a^2\psi(x,t)=\psi_{xx}(x,t)+g(x,t) \tag{105}$$

$$\psi_t(0,t)+a\psi(0,t)=0 \tag{106}$$

$$\psi(1,t)=0 \tag{107}$$

and where

$$g(x,t)=2z(1,t)-x^2(z_{xx}(1,t)+2az_x(1,t)+a^2z(1,t)) \tag{108}$$

Then, for the Lyapunov functional

$$V(t)=\frac{1}{2}\int_0^1((\psi_t(x,t)+a\psi(x,t))^2+\psi_x^2(x,t))dx \tag{109}$$

the following holds:

$$\dot{V}(t)=-2aV(t)+\int_0^1(\psi_t(x,t)+a\psi(x,t))g(x,t)dx \tag{110}$$

for all $t\geq 0$.

Proof. Most of this lemma is obtained by direct verification. The expression

$$g(x,t)=\psi_{tt}(x,t)+2a\psi_t(x,t)+a^2\psi(x,t)-\psi_{xx}(x,t)=\varphi_{tt}(x,t)+2a\varphi_t(x,t)+a^2\varphi(x,t)-\varphi_{xx}(x,t)+2z(1,t)-x^2(z_{xx}(1,t)+2az_x(1,t)+a^2z(1,t))=2z(1,t)-x^2(z_{xx}(1,t)+2az_x(1,t)+a^2z(1,t)) \tag{111}$$

involves $z_x(1,t)$ and $z_{xx}(1,t)$. These quantities are obtained as follows. By setting $x=1$ in $z_t=z_x$, one obtains $z_t(1,t)=z_x(1,t)$. By differentiating $z_t=z_x$ with respect to t, one obtains $z_{tt}=z_{xt}=z_{xx}$. Setting $x=1$ yields $z_{tt}(1,t)=z_{xx}(1,t)$. Substituting $z_t(1,t)=z_x(1,t)$ and $z_{tt}(1,t)=z_{xx}(1,t)$ into Eq. (111), one obtains Eq. (108). The derivative of the Lyapunov function is obtained as

$$\begin{aligned}\dot V(t) = &\int_0^1 ((\psi_t(x,t)+a\psi(x,t))(\psi_{tt}(x,t)+a\psi_t(x,t))\\ &+\psi_x(x,t)\psi_{xt}(x,t))dx = \int_0^1 ((\psi_t(x,t)+a\psi(x,t))(-a\psi_t(x,t)\\ &-a^2\psi(x,t)+\psi_{xx}(x,t)+g(x,t))+\psi_x(x,t)\psi_{xt}(x,t))dx =\\ &-a\int_0^1 (\psi_t(x,t)+a\psi(x,t))^2dx+\int_0^1 ((\psi_t(x,t)+a\psi(x,t))\\ &\times(\psi_{xx}(x,t)+g(x,t))+\psi_x(x,t)\psi_{xt}(x,t))dx = -a\int_0^1 (\psi_t(x,t)\\ &+a\psi(x,t))^2dx+\int_0^1 (-(\psi_{xt}(x,t)+a\psi_x(x,t))\psi_x(x,t)\\ &+\psi_x(x,t)\psi_{xt}(x,t))dx+(\psi_t(x,t)+a\psi(x,t))\psi_x(x,t)|_0^1\\ &+\int_0^1 (\psi_t(x,t)+a\psi(x,t))g(x,t)dx = -2aV(t)+(\psi_t(x,t)\\ &+a\psi(x,t))\psi_x(x,t)|_0^1+\int_0^1 (\psi_t(x,t)+a\psi(x,t))g(x,t)dx\end{aligned} \tag{112}$$

where integration by parts is used in the second to last steps. Equation (110) is obtained by substituting either of the boundary conditions Eq. (103) or Eq. (106) into Eq. (112).

The following lemma is readily verifiable with several applications of Young's inequality.

LEMMA 8. *If functions $V(t)$ and $g(x,t)$ satisfy Eqs. (108)–(110), then they satisfy the following bounds:*

$$\dot V(t) \le -aV(t)+\frac{1}{2a}+\frac{1}{4a}\|g(t)\|^2 \tag{113}$$

$$\|g(t)\|^2 \le 3\left((5+a^2)z^2(1,t)+a^2z_x^2(1,t)+\frac{1}{5}z_{xx}^2(1,t)\right) \tag{114}$$

Next, we turn our attention to the z-systems (84) and (85) and state the following directly verifiable result.

LEMMA 9. *For systems (84) and (85), the following are true:*

$$\frac{d}{dt}\int_1^{1+D} e^{b(x-1)}z^2(x,t)dx = -z(1,t)^2-b\int_1^{1+D} e^{b(x-1)}z^2(x,t)dx \tag{115}$$

$$\frac{d}{dt}\int_1^{1+D} e^{b(x-1)}z_x^2(x,t)dx = -z_x(1,t)^2-b\int_1^{1+D} e^{b(x-1)}z_x^2(x,t)dx \tag{116}$$

$$\frac{d}{dt}\int_1^{1+D} e^{b(x-1)}z_{xx}^2(x,t)dx = -z_{xx}(1,t)^2-b\int_1^{1+D} e^{b(x-1)}z_{xx}^2(x,t)dx \tag{117}$$

for any $b>0$.

With Lemmas 8 and 9, we obtain the following result, as displayed in Fig. 8.

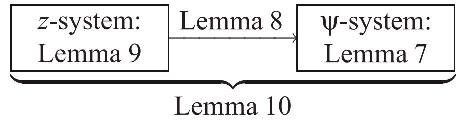

Fig. 8 The outline of the proof of exponential stability of the cascade system (z,ψ). The Lyapunov functional for the autonomous z-system is constructed in Lemma 9. The Lyapunov functional for the ψ-system is constructed in Lemma 7. The input-to-state stability (ISS) of the ψ-system with respect to the z-system is shown in Lemma 8. The Lyapunov stability of the overall (z,ψ)-system is shown in Lemma 10.

LEMMA 10. *For systems (84), (85), and (102)–(108), the following holds:*

$$\Omega(t) \le \Omega_0 e^{-\min\{a,b\}t}, \quad \forall\, t \ge 0 \tag{118}$$

for all $b>0$, where

$$\Omega(t) = V(t)+\frac{3(5+a^2)}{4a}\int_1^{1+D} e^{b(x-1)}(z^2(x,t)+z_x^2(x,t)+z_{xx}^2(x,t))dx \tag{119}$$

Proof. With Lemmas 8 and 9, we obtain

$$\begin{aligned}\dot\Omega(t) \le &-aV(t)+\frac{3}{4a}\left((5+a^2)z^2(1,t)+a^2z_x^2(1,t)+\frac{1}{5}z_{xx}^2(1,t)\right)\\ &-\frac{3(5+a^2)}{4a}(z^2(1,t)+z_x^2(1,t)+z_{xx}^2(1,t))\\ &-\frac{3(5+a^2)}{4a}b\int_1^{1+D} e^{b(x-1)}(z^2(x,t)+z_x^2(x,t)+z_{xx}^2(x,t))dx \le\\ &-aV(t)-\frac{3(5+a^2)}{4a}b\int_1^{1+D} e^{b(x-1)}(z^2(x,t)+z_x^2(x,t)\\ &+z_{xx}^2(x,t))dx\end{aligned} \tag{120}$$

which yields

$$\dot\Omega(t) \le -\min\{a,b\}\Omega(t) \tag{121}$$

and leads to the result of the lemma.

Even though we have obtained exponential stability in the (ψ,z) variables, we have yet to establish exponential stability in the (w,z) variables. We have to consider the chain of transformations

$$(w,z) \mapsto (\varphi,z) \mapsto (\psi,z) \tag{122}$$

as well as their inverses, to establish stability of the (w,z)-system.

The following lemma, which establishes the equivalence between the Lyapunov function $\Omega(t)$ and the appropriate norm of the (w,z)-system, is the key to establishing exponential stability in the (w,z) variables.

LEMMA 11. *Consider the transformation Eqs. (93), (94), and (101), along with the energy function $\Omega(t)$ defined in Eq. (119). The following holds:*

$$\alpha_1\Xi(t) \le \Omega(t) \le \alpha_2\Xi(t) \tag{123}$$

where

$$\begin{aligned}\Xi(t) = &\int_0^1 (w_x^2(x,t)+w_t^2(x,t))dx+\int_1^{1+D} (z^2(x,t)+z_x^2(x,t)\\ &+z_{xx}^2(x,t))dx\end{aligned} \tag{124}$$

and

$$\alpha_1 = \min\left\{\frac{3(5+a^2)}{4a},\quad \frac{1}{8\cosh(2a)\max\{1,2a\}}\right\} \tag{125}$$

$$\alpha_2 = \max\left\{2\cosh(2a), \quad \frac{8}{5}(1+a^2) + \frac{3(5+a^2)}{4a}e^{bD}\right\} \quad (126)$$

Proof. To save on notation, in this proof, we use the symbol $\|\cdot\|$ to mean both $\|\cdot\|_{L_2[0,1]}$ and $\|\cdot\|_{L_2[1,1+D]}$. We first consider the transformations $w \mapsto \varphi$ and $\varphi \mapsto w$. Squaring up Eq. (87), we get

$$w_x^2(x,t) \le 2\cosh^2(a(1-x))\varphi_x^2(x,t) + 2\sinh^2(a(1-x))(\varphi_t(x,t) + a\varphi(x,t))^2 \quad (127)$$

Doing the same with Eqs. (88), (90), and (91), integrating from 0 to 1, and majorizing $\cosh^2(a(1-x))$ and $\sinh^2(a(1-x))$ over (0,1) under the integrals, we get

$$\|w_x(t)\|^2 + \|w_t(t)\|^2 \le 2(\cosh^2(a) + \sinh^2(a))(\|\varphi_x(t)\|^2 + \|\varphi_t(t) + a\varphi(t)\|^2) \quad (128)$$

and

$$\|\varphi_x(t)\|^2 + \|\varphi_t(t) + a\varphi(t)\|^2 \le 2(\cosh^2(a) + \sinh^2(a))(\|w_x(t)\|^2 + \|w_t(t)\|^2) \quad (129)$$

From Eq. (101), we get

$$\varphi_t(x,t) + a\varphi(x,t) = \psi_t(x,t) + a\psi(x,t) + x^2(z_x(1,t) + az(1,t)) \quad (130)$$

$$\varphi_x(x,t) = \psi_x(x,t) + 2xz(1,t) \quad (131)$$

where we have used the fact that $z_t(1,t) = z_x(1,t)$. Taking the L_2 norm of both sides of both equations, we obtain

$$\|\varphi_t(t) + a\varphi(t)\|^2 \le 2\|\psi_t(t) + a\psi(t)\|^2 + \frac{2}{5}(z_x(1,t) + az(1,t))^2 \quad (132)$$

$$\|\varphi_x(t)\|^2 \le 2\|\psi_x(t)\|^2 + \frac{2}{3}z^2(1,t) \quad (133)$$

as well as

$$\|\psi_t(t) + a\psi(t)\|^2 \le 2\|\varphi_t(t) + a\varphi(t)\|^2 + \frac{2}{5}(z_x(1,t) + az(1,t))^2 \quad (134)$$

$$\|\psi_x(t)\|^2 \le 2\|\varphi_x(t)\|^2 + \frac{2}{3}z^2(1,t) \quad (135)$$

Using the fact that $z(1,t) \equiv 0$ and $z_x(1,t) \equiv 0$, where the latter follows from the fact that $z_t(1,t) \equiv 0$, with Agmon's inequality, we get that

$$\|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2 \le 2(\|\psi_t(t) + a\psi(t)\|^2 + \|\psi_x(t)\|^2) + \frac{16}{5}\|z_{xx}(t)\|^2 + 4\left(\frac{4}{5}a^2 + \frac{2}{3}\right)\|z_x(t)\|^2 \quad (136)$$

$$\|\psi_t(t) + a\psi(t)\|^2 + \|\psi_x(t)\|^2 \le 2(\|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2) + \frac{16}{5}\|z_{xx}(t)\|^2 + 4\left(\frac{4}{5}a^2 + \frac{2}{3}\right)\|z_x(t)\|^2 \quad (137)$$

With further majorizations, we achieve simplifications of expressions

$$\|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2 \le 4V(t) + \frac{16}{5}(1+a^2)(\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2) \quad (138)$$

$$V(t) \le \|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2 + \frac{8}{5}(1+a^2)(\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2) \quad (139)$$

Now, we first focus on Eq. (138)

$$\|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2 \le 4\left(V(t) + \frac{4}{5}(1+a^2)(\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2)\right) \le 4\left(V(t) + \frac{4}{5}(1+a^2)\int_1^{1+D} e^{b(x-1)}(z^2(x,t) + z_x^2(x,t) + z_{xx}^2(x,t))dx\right) \quad (140)$$

Invoking Eq. (119), we get

$$\|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2 \le 4\max\left\{1, \frac{\frac{4}{5}(1+a^2)}{\frac{3(5+a^2)}{4a}}\right\}\Omega(t) \quad (141)$$

With a few steps of majorization, it is easy to see that

$$\frac{\frac{4}{5}(1+a^2)}{\frac{3(5+a^2)}{4a}} \le \frac{16a}{15} < 2a \quad (142)$$

Hence

$$\|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2 \le 4\max\{1, 2a\}\Omega(t) \quad (143)$$

Recalling Eq. (128), we get

$$\|w_x(t)\|^2 + \|w_t(t)\|^2 \le 8(\cosh^2(a) + \sinh^2(a))\max\{1,2a\}\Omega(t) = 8\cosh(2a)\max\{1,2a\}\Omega(t) \quad (144)$$

where we have used the fact that $\cosh^2(a) + \sinh^2(a) = \cosh(2a)$. Furthermore, from Eq. (119), we get

$$\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2 \le \frac{4a}{3(5+a^2)}\Omega(t) \quad (145)$$

With Eqs. (144) and (145), we obtain the left side of inequality (123) with α_1 given by Eq. (125). Now, we turn our attention to Eq. (139) and to proving the right-hand-side of inequality (123). From Eqs. (129) and (139), we get

$$V(t) \le 2(\cosh^2(a) + \sinh^2(a))(\|w_x(t)\|^2 + \|w_t(t)\|^2) + \frac{8}{5}(1+a^2) \times (\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2) = 2\cosh(2a)(\|w_x(t)\|^2 + \|w_t(t)\|^2) + \frac{8}{5}(1+a^2)(\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2) \quad (146)$$

Then, with Eqs. (119) and (146), we obtain

$$\Omega(t) \le 2\cosh(2a)(\|w_x(t)\|^2 + \|w_t(t)\|^2) + \frac{8}{5}(1+a^2)(\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2) + \frac{3(5+a^2)}{4a}\int_1^{1+D} e^{b(x-1)}(z^2(x,t) + z_x^2(x,t) + z_{xx}^2(x,t))dx \le 2\cosh(2a)(\|w_x(t)\|^2 + \|w_t(t)\|^2)$$

$$+\left(\frac{8}{5}(1+a^2)+\frac{3(5+a^2)}{4a}e^{bD}\right)(\|z(t)\|^2+\|z_x(t)\|^2+\|z_{xx}(t)\|^2) \tag{147}$$

This completes the proof of the right side of inequality (123) with α_2 given by Eq. (126).

With Lemmas 10 and 11, we prove the following result on exponential stability of the target system (w,z).

PROPOSITION 12. *For systems (81)–(85), the following inequality holds for the norm Eq. (124) for all* $b>0$:

$$\Xi(t)\le\frac{\alpha_2}{\alpha_1}\Xi_0 e^{-\min\{a,b\}t},\quad \forall t\ge 0 \tag{148}$$

5 Stability in the Original Plant Variables (u,v)

We now return to the backstepping transformations $(u,v)\mapsto(w,z)$ and $(w,z)\mapsto(u,v)$ in Sec. 2. After the substitution of the gain kernels expressed in terms of $\rho(x,y)$ and $\mu(x,y)$, the backstepping transformation is written as

$$w(x,t)=u(x,t)-\frac{q(q+c)}{1+qc}u(0,t)+\frac{q+c}{1+qc}\int_0^x u_t(y,t)dy \tag{149}$$

$$z(x,t)=v(x,t)+\int_1^x \rho_y(1+x-y,1)v(y,t)dy+q\rho(x,0)u(0,t)-\int_0^1\rho_x(x,y)u(y,t)dy-\int_0^1\rho(x,y)u_t(y,t)dy \tag{150}$$

and the inverse backstepping transformation is written as

$$u(x,t)=w(x,t)-\frac{c(q+c)}{1+qc}w(0,t)-\frac{q+c}{1+qc}\int_0^x w_t(y,t)dy \tag{151}$$

$$v(x,t)=z(x,t)+\int_1^x\mu_y(1+x-y,1)z(y,t)dy-c\mu(x,0)w(0,t)-\int_0^1\mu_x(x,y)w(y,t)dy-\int_0^1\mu(x,y)w_t(y,t)dy \tag{152}$$

Since the stability result in Proposition 12 is given in terms of

$$\Xi(t)=\|w_x(t)\|^2+\|w_t(t)\|^2+\|z(t)\|^2+\|z_x(t)\|^2+\|z_{xx}(t)\|^2 \tag{153}$$

and we want to establish stability in terms of

$$\Upsilon(t)=\|u_x(t)\|^2+\|u_t(t)\|^2+\|v(t)\|^2+\|v_x(t)\|^2+\|v_{xx}(t)\|^2 \tag{154}$$

we need to derive the expressions for all of the five normed quantities appearing in $\Xi(t)$ and all of the five normed quantities appearing in $\Upsilon(t)$.

The normed quantities appearing in $\Xi(t)$ are given by

$$w_x(x,t)=u_x(x,t)+\frac{q+c}{1+qc}u_t(x,t) \tag{155}$$

$$w_t(x,t)=\frac{q+c}{1+qc}zu_x(x,t)+u_t(x,t) \tag{156}$$

$$z(x,t)=v(x,t)+\int_1^x \rho_y(1+x-y,1)v(y,t)dy+q\rho(x,0)u(0,t)-\int_0^1\rho_x(x,y)u(y,t)dy-\int_0^1\rho(x,y)u_t(y,t)dy \tag{157}$$

$$z_x(x,t)=v_x(x,t)+\rho_y(1,1)v(x,t)+\int_1^x\rho_{xy}(1+x-y,1)v(y,t)dy+q\rho_x(x,0)u(0,t)-\int_0^1\rho_{xx}(x,y)u(y,t)dy-\int_0^1\rho_x(x,y)u_t(y,t)dy \tag{158}$$

$$z_{xx}(x,t)=v_{xx}(x,t)+\rho_y(1,1)v_x(x,t)+\rho_{xy}(1,1)v(x,t)+\int_1^x\rho_{xxy}(1+x-y,1)v(y,t)dy+q\rho_{xx}(x,0)u(0,t)-\int_0^1\rho_{xxx}(x,y)u(y,t)dy-\int_0^1\rho_{xx}(x,y)u_t(y,t)dy \tag{159}$$

The normed quantities appearing in $\Upsilon(t)$ are given by

$$u_x(x,t)=w_x(x,t)-\frac{q+c}{1+qc}w_t(x,t) \tag{160}$$

$$u_t(x,t)=-\frac{q+c}{1+qc}w_x(x,t)+w_t(x,t) \tag{161}$$

$$v(x,t)=z(x,t)+\int_1^x\mu_y(1+x-y,1)z(y,t)dy-c\mu(x,0)w(0,t)-\int_0^1\mu_x(x,y)w(y,t)dy-\int_0^1\mu(x,y)w_t(y,t)dy \tag{162}$$

$$v_x(x,t)=z_x(x,t)+\mu_y(1,1)z(x,t)+\int_1^x\mu_{xy}(1+x-y,1)z(y,t)dy-c\mu_x(x,0)w(0,t)-\int_0^1\mu_{xx}(x,y)w(y,t)dy-\int_0^1\mu_x(x,y)w_t(y,t)dy \tag{163}$$

$$v_{xx}(x,t)=z_{xx}(x,t)+\mu_y(1,1)z_x(x,t)+\mu_{xy}(1,1)z(x,t)+\int_1^x\mu_{xxy}(1+x-y,1)z(y,t)dy-c\mu_{xx}(x,0)w(0,t)-\int_0^1\mu_{xxx}(x,y)w(y,t)dy-\int_0^1\mu_{xx}(x,y)w_t(y,t)dy \tag{164}$$

In establishing a relation between $\Upsilon(t)$ and $\Xi(t)$, first we establish a relation between $\|u_x(t)\|^2+\|u_t(t)\|^2$ and $\|w_x(t)\|^2+\|w_t(t)\|^2$.

LEMMA 13. *Consider the transformation Eqs. (149) and (151), along with the norm Eqs. (153) and (154). The following are true:*

$$\|u_x(t)\|^2+\|u_t(t)\|^2\le 2(1+\eta^2)(\|w_x(t)\|^2+\|w_t(t)\|^2)\le 2(1+\eta^2)\Xi(t) \tag{165}$$

$$\|w_x(t)\|^2+\|w_t(t)\|^2 \le 2(1+\eta^2)(\|u_x(t)\|^2+\|u_t(t)\|^2) \le 2(1+\eta^2)\Upsilon(t) \quad (166)$$

where

$$\eta = \frac{q+c}{1+qc} \quad (167)$$

Next, we focus on relating $\|v(t)\|^2+\|v_x(t)\|^2+\|v_{xx}(t)\|^2$ to $\Xi(t)$ and $\|z(t)\|^2+\|z_x(t)\|^2+\|v_{xx}(t)\|^2$ to $\Upsilon(t)$.

LEMMA 14. *For the transformation Eq. (152) along with the norm Eq. (153), the following are true:*

$$\|v(t)\|^2 \le \gamma_0 \Xi(t) \quad (168)$$

$$\|v_x(t)\|^2 \le \gamma_1 \Xi(t) \quad (169)$$

$$\|v_{xx}(t)\|^2 \le \gamma_2 \Xi(t) \quad (170)$$

where

$$\gamma_0 = 5\max\left\{1 + D\int_1^{1+D}\mu_y^2(x,1)dx, 4\left(c\int_1^{1+D}\mu^2(x,0)dx + \int_1^{1+D}\int_0^1 \mu_x^2(x,y)dydx\right), \int_1^{1+D}\int_0^1 \mu^2(x,y)dydx\right\} \quad (171)$$

$$\gamma_1 = 6\max\left\{1, \mu_y^2(1,1) + D\int_1^{1+D}\mu_{xy}^2(x,1)dx, 4\left(c\int_1^{1+D}\mu_x^2(x,0)dx + \int_1^{1+D}\int_0^1 \mu_{xx}^2(x,y)dydx\right), \int_1^{1+D}\int_0^1 \mu_x^2(x,y)dydx\right\} \quad (172)$$

$$\gamma_2 = 7\max\left\{1, \mu_y^2(1,1), \mu_{xy}^2(1,1) + D\int_1^{1+D}\mu_{xxy}^2(x,1)dx, 4\left(c\int_1^{1+D}\mu_{xx}^2(x,0)dx + \int_1^{1+D}\int_0^1 \mu_{xxx}^2(x,y)dydx\right), \int_1^{1+D}\int_0^1 \mu_{xx}^2(x,y)dydx\right\} \quad (173)$$

For the transformation Eq. (150) along with the norm Eq. (154), the following are true:

$$\|z(t)\|^2 \le \delta_0 \Upsilon(t) \quad (174)$$

$$\|z_x(t)\|^2 \le \delta_1 \Upsilon(t) \quad (175)$$

$$\|z_{xx}(t)\|^2 \le \delta_2 \Upsilon(t) \quad (176)$$

where

$$\delta_0 = 5\max\left\{1 + D\int_1^{1+D}\rho_y^2(x,1)dx, 4\left(q\int_1^{1+D}\rho^2(x,0)dx + \int_1^{1+D}\int_0^1 \rho_x^2(x,y)dydx\right), \int_1^{1+D}\int_0^1 \rho^2(x,y)dydx\right\} \quad (177)$$

$$\delta_1 = 6\max\left\{1, \rho_y^2(1,1) + D\int_1^{1+D}\rho_{xy}^2(x,1)dx, 4\left(q\int_1^{1+D}\rho_x^2(x,0)dx + \int_1^{1+D}\int_0^1 \rho_{xx}^2(x,y)dydx\right), \int_1^{1+D}\int_0^1 \rho_x^2(x,y)dydx\right\} \quad (178)$$

$$\delta_2 = 7\max\left\{1, \rho_y^2(1,1), \rho_{xy}^2(1,1) + D\int_1^{1+D}\rho_{xxy}^2(x,1)dx, 4\left(q\int_1^{1+D}\rho_{xx}^2(x,0)dx + \int_1^{1+D}\int_0^1 \rho_{xxx}^2(x,y)dydx\right), \int_1^{1+D}\int_0^1 \rho_{xx}^2(x,y)dydx\right\} \quad (179)$$

Proof. We only prove inequality (170). All of the other inequalities are easier to prove. Starting from Eq. (164), we get

$$\|v_{xx}(t)\|^2 \le 7\Bigg(\|z_{xx}(t)\|^2 + \mu_y^2(1,1)\|z_x(t)\|^2 + \mu_{xy}^2(1,1)\|z(t)\|^2 + D\int_1^{1+D}\mu_{xxy}^2(x,1)dx\|z(t)\|^2 + c\int_1^{1+D}\mu_{xx}^2(x,0)dx w^2(0,t) + \int_1^{1+D}\int_0^1 \mu_{xxx}^2(x,y)dydx\|w(t)\|^2 + \int_1^{1+D}\int_0^1 \mu_{xx}^2(x,y)dydx\|w_t(t)\|^2\Bigg) \quad (180)$$

where we have used the fact that

$$\int_1^{1+D}\int_0^{x-1}\mu_{xxy}^2(1+s,1)dsdx = \int_1^{1+D}(1+D-x)\mu_{xxy}^2(x,1)dx \le D\int_1^{1+D}\mu_{xxy}^2(x,1)dx \quad (181)$$

Employing Agmon's inequality, we get

$$\|v_{xx}(t)\|^2 \le 7\Bigg[\|z_{xx}(t)\|^2 + \mu_y^2(1,1)\|z_x(t)\|^2 + \left(\mu_{xy}^2(1,1) + D\int_1^{1+D}\mu_{xxy}^2(x,1)dx\right)\|z(t)\|^2 + 4\left(c\int_1^{1+D}\mu_{xx}^2(x,0)dx + \int_1^{1+D}\int_0^1 \mu_{xxx}^2(x,y)dydx\right)\|w(t)\|^2 + \int_1^{1+D}\int_0^1 \mu_{xx}^2(x,y)dydx\|w_t(t)\|^2\Bigg] \quad (182)$$

from which Eq. (170) follows with Eq. (173).

With Lemmas 13 and 14, we get the following relation between $\Upsilon(t)$ and $\Xi(t)$.

LEMMA 15. *The following relation holds between the norm Eqs. (153) and (154):*

$$\alpha_3 \Upsilon(t) \le \Xi(t) \le \alpha_4 \Upsilon(t) \quad (183)$$

where

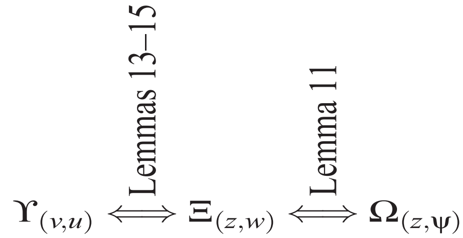

Fig. 9 The norms used in the stability analysis of the closed-loop system. Their equivalence is established in the lemmas indicated in the figure.

$$\alpha_3 = 1/[2(1+\eta^2)+\gamma_0+\gamma_1+\gamma_3] \tag{184}$$

$$\alpha_4 = 2(1+\eta^2)+\delta_0+\delta_1+\delta_3 \tag{185}$$

Finally, we obtain our main result on exponential stability of the (u,v)-system with the help of the lemmas, as indicated in Fig. 9.

THEOREM 16. *Consider the closed-loop system consisting of the plant Eqs. (5)–(9) and the control law Eq. (37). The following holds for all* $b>0$*:*

$$\Upsilon(t) \leq \frac{\alpha_2\alpha_4}{\alpha_1\alpha_3}\Upsilon_0 e^{-\min\{a,b\}t}, \quad \forall\, t \geq 0 \tag{186}$$

where the system norm $\Upsilon(t)$ *is defined in Eq. (154) and a is defined in Eq. (86).*

Since the coefficient b is arbitrary, we can choose it as $b=a$. Then, we obtain the following corollary.

COROLLARY 17. *Consider the closed-loop system consisting of the plant Eqs. (5)–(9) and the control law Eq. (37). The following holds:*

$$\Upsilon(t) \leq \frac{\alpha_2\alpha_4}{\alpha_1\alpha_3}\Upsilon_0 \left|\frac{1-c}{1+c}\right|^{t/2}, \quad \forall\, t \geq 0 \tag{187}$$

It should be noted that all of the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ depend on $a(c)$. In addition, α_3 and α_4 depend on q. Finally, it is important to observe that α_2, as well as α_3 and α_4, are nondecreasing functions of D.

6 Conclusions

The result of this paper offers an alternative approach to addressing the problem of input delays in control of wave PDEs, brought up by the Datko challenge. Stabilization in the presence of arbitrarily long delay is achieved for a wave equation plant with antidamping, which has all of its infinitely many open-loop eigenvalues arbitrarily far to the right of the imaginary axis.

Some interesting open problems arise from the considerations in this paper. First, one would want to consider the problem of robustness to small errors in D that is employed in the predictor feedback. While we were successful in establishing robustness to delay error for an ODE with predictor feedback [24], we are not very hopeful that such a robustness result would hold for a wave PDE plant.

Second, the foremost problem would be to consider adaptive stabilization of the antistable wave equation with unknown delay, namely, of the systems

$$u_{tt}(x,t) = u_{xx}(x,t), \quad x \in (0,1) \tag{188}$$

$$u_x(0,t) = -qu_t(0,t) \tag{189}$$

$$u(1,t) = v(1,t) \tag{190}$$

$$Dv_t(x,t) = v_x(x,t), \quad x \in [1,2) \tag{191}$$

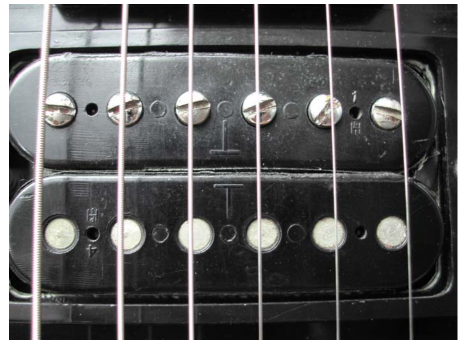

Fig. 10 A pickup on an electric guitar. Based on Faraday's law of induction, it converts the string velocity into voltage. Connected into a high-gain amplifier, this system results in (domainwide) antidamping, which manifests itself as a swell in volume, up to a saturation of the amplifier, which guitarists refer to, simply, as feedback.

$$v(2,t) = U(t) \tag{192}$$

where D and q are unknown. An adaptive design for this system would indirectly address the Datko [44] question but in a more challenging setting, where D is not small but it is large and has a large uncertainty and where the wave equation plant is not neutrally stable but antistable. The q-adaptive problem for the antistable wave equation with $D=0$ was solved in Ref. [60]. The delay-adaptive control problem for ODE plants was recently solved in Ref. [67].

It is worth mentioning that, even though we have considered only a state feedback case in this paper, there is no problem to extend the results to output feedback. For example, the observers (Ref. [58], Eqs. (110)–(112))

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) - \frac{q+\tilde{c}}{1+q\tilde{c}}[u_{xt}(1,t) - \hat{u}_{xt}(1,t)] \tag{193}$$

$$\hat{u}_x(0,t) = -q\hat{u}_t(0,t) - \frac{q(q+\tilde{c})}{1+q\tilde{c}}[u_x(1,t) - \hat{u}_x(1,t)] \tag{194}$$

$$\hat{u}(1,t) = U(t-D) \tag{195}$$

where $\tilde{c}>0$ is an observer gain, can be combined with the predictor feedback law

$$U(t) = -q\rho(1+D,0)\hat{u}(0,t) + \int_0^1 \rho_x(1+D,y)\hat{u}(y,t)dy + \int_0^1 \rho(1+D,y)\hat{u}_t(y,t)dy - \int_{t-D}^t \rho_y(t-\theta,1)U(\theta)d\theta \tag{196}$$

to achieve output feedback stabilization using the measurement $u_x(1,t)$, which is collocated with the control $u(1,t)=U(t-D)$, as in Fig. 2. The proof of this fact is more complicated than in the present paper due to the need to incorporate the observer error system in the analysis. Furthermore, a predictor feedback can be designed for the architecture, where the control and measurement are switched, namely, where $u_x(1,t)=U(t-D)$ is a Neumann input and $u(1,t)$ is a Dirichlet output, as in Ref. [58], Eqs. (63)–(65) and (76).

Another example of a wave system with antidamping is that of electrically amplified stringed instruments (for example, electric guitar). Such instruments employ an electromagnetic pickup, shown in Fig. 10, where the voltage on the terminals of the pickup is proportional, according to Faraday's law of induction, to the velocity of the string above it. The pickup's voltage is then amplified using an electric amplifier. The loudspeaker of a high-gain amplifier, when played at high volume, is capable of producing an acoustic excitation of such intensity that its force acts to mechanically excite the string. This is a positive feedback loop, where the string velocity is converted to voltage, multiplied by high-gain, and then applied back as a force on the string. The phenomenon is not the same as our antidamping at the boundary but it appears in the form $u_{tt}(x,t)=u_{xx}(x,t)+gu_t(p,t)$, where g is the gain and $p \in (0,1)$ is the location of the pickup along the length of the string. The instability described here manifests itself as a loud, sustaining, tone, even when the string is not being "plucked," although, when used in a control manner, it can be employed musically (leading to "swells" of sound that the musician can induce on chosen notes). Among electric guitarists, this phenomenon is referred to, simply, as *feedback* [68,69].

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