Brief paper

Stochastic source seeking for nonholonomic unicycle

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\begin{abstract}
We apply the recently introduced method of stochastic extremum seeking to navigate a nonholonomic unicycle towards the maximum of an unknown, spatially distributed signal field, using only the measurement of the signal at the vehicle's location but without the measurement of the vehicle's position. Keeping the forward velocity constant and controlling only the angular velocity, we design a stochastic source seeking control law which employs excitation based on filtered white noise, rather than sinusoidal perturbations used in the existing work. We study stability with the help of stochastic averaging theorems that we recently developed for general nonlinear continuous-time systems with stochastic perturbations. We prove local exponential convergence, both almost surely and in probability, to a small neighborhood near the source. We characterize the convergence speed explicitly and provide design guidelines for maximizing it, as well as for minimizing the residual set near the source. We present a detailed simulation study, including a study of the effect of saturation on the steering input.
\end{abstract}

\section{Introduction}

Several source seeking algorithms employing sinusoidal perturbations for nonholonomic vehicles in position-denied environments have been recently proposed. In Zhang, Arnold, Ghods, Siranosian, and Krstic (2007), the angular velocity of the vehicle center is made constant and the control tunes the forward velocity. In Cochran and Krstic (2009), the forward velocity is made constant and the angular velocity is controlled.

In this paper, we investigate a stochastic version of source seeking by navigating the unicycle with the help of a random perturbation, achieving a behavior that mimics the chemotaxis-like motion observed in the bacterium Escherichia coli (E. coli). E. coli is a single celled organism consisting of a cell body with multiple trailing flagella used for propulsion. In the works Berg (2003), and Berg and Brown (1972), it is observed that the bacterium is able to move up chemical gradients towards higher densities of nutrients by switching between alternate behaviors known as “run” and “tumble”. The behavior “run” means that the bacterium moves in essentially a straight line by rotating the flagella counter-clockwise as viewed from behind the cell and the behavior “tumble” means that the bacterium ceases forward motion and spins by turning some flagella in a clockwise direction. It is also observed that the tumble behavior displays apparent random nature, although the net motion of the bacterium is not completely random but is in the direction of higher nutrient concentrations.

Motivated by the chemotactic behavior of E. coli, we consider the problem of stochastic source seeking for a nonholonomic unicycle. The analogy is appropriate since neither the unicycle nor E. coli can exhibit sideways motions, though they can be steered. Our vehicle has no knowledge of its position, nor of the distribution of the signal field. Like E. coli, it can only sense the signal locally.

To find the source, we employ a stochastic extremum seeking approach and provide a stability analysis based on stochastic averaging theorems that we recently developed in Liu and Krstic (in press). With a controller that we design in the paper, the vehicle is driven to approach a small neighborhood of the source in a manner that seems partly random but is convergent in a suitable sense. We present a stability proof for the scheme with a static source and simulation results for both static and moving sources. Convergence is proved both in the ‘almost sure’ sense and ‘in probability’.

It is important to consider the relative merits of the deterministic solution to the source seeking problem in Cochran and Krstic (2009) and the stochastic solution presented here. As expected, the steering inputs in the stochastic approach are less smooth, which is a disadvantage of the stochastic approach from the viewpoint of actuator wear. However, the nearly random motion of the stochastic seeker has its advantage in applications where the seeker itself...
may be pursued by another pursuer. A seeker, which successfully performs the source finding task but with an unpredictable, nearly random trajectory, is a more challenging target, and is hence less vulnerable, than a deterministic seeker.

Motivated by *E. coli* chemotaxis, Mesquita, Hespanha, and Åström (2008) consider a similar problem of seeking the maximum of a scalar signal, using a swarm of autonomous vehicles, and propose a control design which induces the vehicles to perform a biased random walk, with a net motion of the swarm towards the maximum, and achieving higher vehicle densities near the maximum at the end of the search. Besides the difference in the algorithms presented in Mesquita et al. (2008) and in the present paper, different results are proved. Mesquita et al. (2008) guarantee that the probability density function of the positions of the vehicles evolves towards a specified function of the spatial profile of the measured signal, whereas in the present paper we prove convergence (in probability and almost surely), for any single vehicle, to a specific small neighborhood of the source.

Another significant difference is that we establish exponential convergence, and in fact characterize the best achievable value, and the worst-case value, of the exponential convergence rate, as a function of the design parameters. In contrast, in Mesquita et al. (2008) exponential convergence is not shown, nor formally claimed. A considerable difference in performance is also observed in simulations. The algorithm in Mesquita et al. (2008) at best matches the convergence of the deterministic algorithm in Zhang et al. (2007), whereas the present algorithm has superior convergence to that in Zhang et al. (2007) as it does not employ motions that would, in the absence of a gradient, keep a vehicle in place on the average (such as random walk, or the triangle and diamond-shaped gaits in Zhang et al. (2007)), but employs a strategy that keeps the vehicle moving in some average direction even when the gradient is zero, as is the case with the design in Cochran and Krstic (2009). However, it is important to note that the results we prove here are only for signal fields that have circular level sets, whereas in Mesquita et al. (2008) such a restriction is not present.

The present paper adds a new tool to a string of recent successful developments in the area of extremum seeking, both in the deterministic case (Ariyur & Krstic, 2003; Moase, Manzie, & Brear, 2009a,b; Tan, Nešić, & Mareels, 2006) and in the stochastic (discrete time) case (Manzie & Krstic, 2009; Stankovic & Stipanovic, 2009a,b).

The paper is organized as follows. In Section 2 we present the vehicle model and state the problem. In Section 3 we present our stochastic source seeking controller. In Section 4 we prove local exponential convergence for circular level sets, namely, where the signal depends only on the distance from the source and decays quadratically. In Section 5 we calculate the convergence speed, for particular parameter choices for which it is possible to do so explicitly, and characterize the best achievable convergence speed. In Section 6 we present simulations and discussions about dependence on design parameters. In Section 7 we consider signal fields with elliptical level sets.

### 2. Vehicle model and problem statement

As in Cochran and Krstic (2009), we consider a mobile agent modeled as a unicycle with a sensor mounted at its front end, a distance *R* from the center. Fig. 1 depicts the position, heading, and angular and forward velocities for the center and sensor. The equations of motion for the vehicle center are

\[
\begin{align*}
\dot{x}_c &= v \cos \theta, \\
\dot{y}_c &= v \sin \theta, \\
\dot{\theta} &= u,
\end{align*}
\] (1)

where *v* is the forward velocity of the vehicle and *u* is the angular velocity. The sensor’s orientation is tuned by the extremum seeking control law

\[
\dot{\theta} = a \dot{\eta} + c \xi \sin(\eta) - d_0 \dot{\xi}^2 \sin(\eta),
\] (3)

where \(\dot{\eta} = \frac{1}{v} W[\sigma] \) is the output of the washout filter for the sensor reading *f*, \(\eta = \frac{1}{v} \sqrt{\frac{1}{v}} W[\sigma][\epsilon \in (0, \epsilon_0)]\) is colored noise used as a perturbation in stochastic extremum seeking, and \(V_c, a, c, d_0, \xi, \eta, h > 0\) are design parameters which (along with parameter *R*) influence the performance. The signal \(W(t), t \geq 0\) is a standard Brownian motion defined in a complete probability space \((\Omega, \mathcal{F}, P)\) with the sample space \(\Omega\), the \(\sigma\)-field \(\mathcal{F}\), and the probability measure \(P\).

With the observation that the transfer function from white noise \(\dot{W}\) to \(\dot{\eta}\) is relative degree zero, giving

\[
\dot{\eta} = \frac{g \sqrt{\epsilon}}{\epsilon + 1} \dot{\eta} = \frac{1}{\sqrt{\epsilon}} \frac{g \epsilon + g - \frac{g}{\epsilon}}{\epsilon + 1} [\dot{\eta} + \frac{g}{\sqrt{\epsilon}}] = \frac{g}{\sqrt{\epsilon}} \dot{W} - \frac{1}{\epsilon} \dot{\eta},
\] (4)
the control law is rewritten as
\[
d\theta = -\frac{a}{\varepsilon} \eta dt + (c\xi - d_0\xi^2) \sin(\eta)dt + \frac{ag}{\sqrt{\varepsilon}} dW, \tag{5}
\]
\[
d\eta = \frac{1}{\varepsilon} \eta dt + \frac{g}{\sqrt{\varepsilon}} dW. \tag{6}
\]

Compared with the deterministic case in Cochran and Krstic (2009), where \(\sin(\omega t)\) was used as the probing signal, we use the stochastic signal \(\sin(\eta(t))\) to develop a gradient estimate. It is not essential to choose the sinusoidal nonlinearity \(\sin(\eta)\) in the stochastic design. This choice is primarily made for the ease of deriving the average system in the stability analysis. We can replace \(\sin(\eta)\) with other bounded and odd functions, such as \(ne^{-\theta^2}\), however, the integrals in calculating the expectations in the derivation of the average system become more complicated. In fact, the boundedness of the perturbation (such as \(\sin \eta\) or \(ne^{-\theta^2}\)) is only needed in the analysis, whereas in the simulations, successful convergence is achieved even when \(\sin(\eta)\) is replaced by \(\eta\).

We refer to the term \(-d_0\xi^2 \sin(\eta)\) as the “\(d_0\)-term” or the damping term. This term is not needed in the basic stochastic extremum seeking algorithm for a static map Liu and Krstic (in press). This term is essential for achieving exponential stability in source seeking problems with a vehicle employing constant forward velocity.

### 4. Stability analysis

We assume that the nonlinear map defining the distribution of the signal field is quadratic and takes the form \(J = f(r_c) = f^* - q|r_c - r^*|^2\) where \(r^*\) is the unknown maximizer, \(f^* = f(r^*)\) is the unknown maximum and \(q\) is an unknown positive constant. We define an output error variable \(e = \frac{1}{\sqrt{2\pi}} |J| - f^*\), which allows us to express the signal \(\xi\) after the washout filter, as \(\xi = \frac{1}{\sqrt{2\pi}} |J| = J - f^* - e\), and thus we have \(\dot{\xi} = \dot{h}\xi\).

Stochastic approximation is a good method to find the extremum of a function. However for our source seeking problem, the conditions of convergence analysis of stochastic approximation are hard to verify due to the presence of dynamics and non-holonomic constraints. In this paper, we use our stochastic average theory presented in Appendix to analyze the stability of the closed-loop system.

**Theorem 4.1.** Consider the closed-loop system
\[
dr_c = V_c \dot{e}^\theta dt, \tag{7}
\]
\[
d\theta = -\frac{a}{\varepsilon} \eta dt + (c\xi - d_0\xi^2) \sin(\eta)dt + \frac{ag}{\sqrt{\varepsilon}} dW, \tag{8}
\]
\[
de = \frac{1}{\varepsilon} \eta dt + \frac{g}{\sqrt{\varepsilon}} dW, \tag{9}
\]
\[
\dot{\xi} = (q|\xi - r^*|^2 + e), \tag{10}
\]
\[
\xi = r_c + Re^{\theta}, \tag{11}
\]
\[
dx = \frac{1}{\varepsilon} \eta dt + \frac{g}{\sqrt{\varepsilon}} dW, \tag{12}
\]
where \(c, d_0, h, R, V_c, q, \varepsilon > 0\), and the parameters \(h, V_c, a, g > 0\) are chosen such that
\[
\frac{1}{h} \geq \frac{R}{2V_c} \left(2 - \frac{l_2(2a, g)}{l_1(a, g)}\right), \tag{13}
\]
where \(l_1(a, g) = e^{-\frac{a^2g^2}{4}}, l_2(a, g) = \frac{1}{2} e^{-\frac{(a^2+1)g^2}{4}} - e^{-\frac{a^2g^2}{4}}\).

(\text{The condition (13) is satisfied for any} \ h > 0 \ \text{and} \ V_c > 0 \ \text{provided} \ g \ \text{is chosen as} \ g = \sqrt{\frac{a}{\varepsilon}} \ \text{and} \ a \ \text{is chosen as} \ 0 < a \ < a^*(\beta) \ \text{if} \ 2 \ln \frac{2\beta - 1}{\beta - 1} \ \text{for any} \ \beta > 0. \ \text{For example, for} \ \beta = 1, a^*(1) \approx 0.24, \ \text{if the initial conditions} \ r_c(0), \ \theta(0), e(0) \ \text{are such that} \ the \ \text{following quantities are sufficiently small},
\]
\[
|\theta(0) - \theta^*(r_c(0)) + \frac{\pi}{2}| < \frac{\pi}{2}, \tag{15}
\]
\[
|\dot{\theta}(0) - \theta^*| = \frac{\pi}{2}, \tag{16}
\]
where
\[
\rho = \sqrt{\frac{V_c l_1(a, g)}{2q_c c R l_2(a, g)}}, \tag{17}
\]
then there exist constants \(C_0, \gamma_0 > 0\) and a function \(T(e) : (0, \gamma_0) \rightarrow \mathbb{N}\) such that for any \(\delta > 0,\)
\[
\liminf\{t \geq 0 : |r_c(t) - r^*| - \rho\}
\]
\[
> C_0 e^{-\gamma_0 t} + \delta, \quad \text{a.s.} \tag{18}
\]
and
\[
\lim P\{|r_c(t) - r^*| - \rho \leq C_0 e^{-\gamma_0 t}+ \delta, \ \forall t \in [0, T(e)]\}
\]
\[
= 1 \tag{19}
\]
with \(\lim_{\varepsilon \rightarrow 0} T(e) = \infty, \text{where the constant} \ C_0 \ \text{is dependent on} \ the \ initial \ condition \ (r_c(0), \ \theta(0), e(0)) \ \text{and on} \ the \ parameters \ a, c, d_0, h, R, V_c, q, \ \text{and the constant} \ \gamma_0 \ \text{is dependent on} \ the \ parameters a, c, d_0, h, R, V_c, q, g. \)

**Proof.** We start by defining the shifted variables
\[
\tilde{r}_c = r_c - r^*, \tag{20}
\]
\[
\tilde{\theta} = \theta - a\eta, \tag{21}
\]
and a map between \(\tilde{r}_c\) and a new quantity \(\tilde{\theta}^*\) given by
\[
\tilde{\theta}^* = \text{arg}(\tilde{r}_c) = \text{arg}(r^* - r_c) \tag{22}
\]
\[
= \begin{cases} 
-\pi - \frac{j}{2} \ln \left(\frac{\tilde{r}_c}{\tilde{r}_c}\right), & \text{if } \tilde{\theta}^* \in \left(\pi, -\frac{\pi}{2}\right), \\
\frac{j}{2} \ln \left(\frac{\tilde{r}_c}{\tilde{r}_c}\right), & \text{if } \tilde{\theta}^* \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\
\pi - \frac{j}{2} \ln \left(\frac{\tilde{r}_c}{\tilde{r}_c}\right), & \text{if } \tilde{\theta}^* \in \left(-\pi, -\frac{\pi}{2}\right),
\end{cases} \tag{23}
\]
where \(\tilde{\theta}^*\) represents the heading angle towards the source located at \(r^*\) when the vehicle is at \(r_c\). Using these definitions, the expression for \(\xi\) is \(\xi = -(q_c(R^2 + |r_c|^2 - 2R|\tilde{r}_c| \cos(\tilde{\theta} - \tilde{\theta}^* + a\eta)) + e)\). Since
\[
d\tilde{\theta} = \dot{\theta} - ad\eta = (c\xi - d_0\xi^2) \sin(\eta)dt, \tag{24}
\]
we obtain the dynamics of the shifted system as
\[
\frac{d\tilde{r}_c}{d\tilde{r}_c} = \frac{dr_c}{dr_c} = V_c e^{(\dot{\theta} + a\eta)}, \tag{25}
\]
\[
\frac{d\tilde{\theta}}{d\tilde{r}_c} = (c\xi - d_0\xi^2) \sin(\eta), \tag{26}
\]
\[
de = -h\dot{q}_c \left(R^2 + |\tilde{r}_c|^2 - 2R|\tilde{r}_c| \cos(\tilde{\theta} - \tilde{\theta}^* + a\eta)\right) - he. \tag{27}
\]
By (12) and the definition of Ito stochastic differential equation, we have
\[ \eta(t) = \eta(0) - \int_{0}^{t} \frac{1}{\sqrt{\sigma}} \eta(s) ds + \int_{0}^{t} \frac{1}{\sqrt{\sigma}} dW(s). \]
Thus it holds that \( \eta(\varepsilon t) = \eta(0) - \int_{0}^{t} \eta(\varepsilon u) du + \int_{0}^{t} \frac{1}{\sqrt{\sigma}} dW(\varepsilon u). \)
Define \( B(t) = \frac{1}{\sqrt{\sigma}} W(\varepsilon t), \) \( \chi(t) = \eta(\varepsilon t). \) Then we have \( d\chi(t) = -\chi(t) dt + \sqrt{\sigma} dB(t), \) where \( B(t) \) is a standard Brownian motion and the process \( \chi(t) \) is an Ornstein–Uhlenbeck (OU) process which is ergodic with
invariant distribution \( \mu(dy) = \frac{1}{\sqrt{\pi \sigma}} e^{-\frac{y^2}{\sigma}} dy. \)

Now we define error variables \( \hat{r}_{c} \) and \( \hat{\theta} \) which represent the distance to the source, and the difference between the vehicle’s heading and the optimal heading, respectively,
\[ \hat{r}_{c} = |\hat{r}_{c}| = |r_{c} - r^{\ast}|, \]
\[ \hat{\theta} = \hat{\theta} - \theta^{\ast}. \]
(28)
(29)

Thus we obtain the following dynamics for the error variables
\[ \frac{d\hat{r}_{c}}{dt} = \frac{d\hat{r}_{c}}{dt} - \frac{d\sqrt{\hat{r}_{c}}}{dt} = \frac{1}{2r_{c}} \left( \frac{dr_{c}}{dt} + \hat{r}_{c} \frac{d\hat{r}_{c}}{dt} \right) \]
\[ = -V_{c} \cos(\hat{\theta} + a\chi(t/\varepsilon)) \]
\[ \frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{dt} - \frac{d\hat{\theta}^{*}}{dt} = \frac{d\hat{\theta}}{dt} + \frac{1}{2r_{c}} \left( \frac{dr_{c}}{dt} - \hat{r}_{c} \frac{d\hat{r}_{c}}{dt} \right) \]
\[ = (c - d_{0}\xi) \xi \sin(\chi(t/\varepsilon)) + \frac{V_{c}}{\hat{r}_{c}} \sin(\hat{\theta} + a\chi(t/\varepsilon)). \]
(30)
(31)

We use general stochastic averaging given in Appendix to analyze this error system.

First we calculate the average system of (30)-(31)-(32). Since
\[ \int_{0}^{t} \sin(a\mu) d\mu = \int_{0}^{t} \sin(\mu) \frac{1}{\sqrt{\pi \sigma}} e^{-\frac{\mu^2}{\sigma}} d\mu = 0, \]
\( \int_{0}^{t} \cos(a\mu) d\mu = \int_{0}^{t} \cos(2\mu) \sin(\mu) d\mu = 0, \)
\( \int_{0}^{t} \cos(2\mu) d\mu = \int_{0}^{t} \cos(2\mu) \sin(4\mu) d\mu = 0, \)
and \( \int_{0}^{t} \sin(a\mu) d\mu = \int_{0}^{t} \sin(4\mu) d\mu = 0, \)
they are exponentially stable. Thus by Theorems A.3 and A.4 in Appendix, there exist constants \( C_{0} > 0, \tau_{0} > 0, C_{R} > 0 \) and functions \( T^{(i)}(\varepsilon) : (0, \varepsilon_{0}) \to \mathbb{N}, i = 1, 2, \) such that for any \( \delta > 0, \) and any initial condition \( |A^{(0)}(0)| < r_{0}^{\ast}, \)
\[ \lim_{\varepsilon \to 0} \inf \{ t \geq 0 : |A^{(0)}(t)| > C_{0} |A^{(0)}(0)| e^{-\gamma_{0}^{\ast} t} + \delta \} = \infty, \]
\[ \lim \sup_{\varepsilon \to 0} \inf \{ t \geq 0 : |A^{(0)}(t)| > C_{0} |A^{(0)}(0)| e^{-\gamma_{0}^{\ast} t} + \delta, t \in [0, T^{(i)}(\varepsilon)) \} = 1 \]
with \( \lim_{\varepsilon \to 0} T^{(i)}(\varepsilon) = \infty, \) where \( A^{(0)}(t) = (\tilde{r}(t) - \rho, \tilde{\theta}(t) - \frac{\pi}{2}, e(t) + q_{r}(R^{2} + \rho^{2})), \) and \( A^{(0)}_{2}(t) = (\tilde{r}(t) + \rho, \tilde{\theta}(t) + \frac{\pi}{2}, e(t) + q_{r}(R^{2} + \rho^{2})), \) the results (43), (44), together with the fact \( |\tilde{r}(t) - \tilde{\theta}(t)| < \frac{1}{2}, \)
\[ \tilde{r}(t) - \rho, \tilde{\theta}(t) + \frac{\pi}{2}, e(t) + q_{r}(R^{2} + \rho^{2}) \]
and the definition of \( \tilde{r}_{c}, \) we have
\[ \liminf_{\varepsilon \to 0} \{ t \geq 0 : |r_{c}(t) - r^{\ast} - \rho| > C_{0} e^{-\gamma_{0}^{\ast} t} + \delta \} = \infty, \]
\[ \lim \sup_{\varepsilon \to 0} \{ t \geq 0 : |r_{c}(t) - r^{\ast} - \rho| \leq C_{0} e^{-\gamma_{0}^{\ast} t} + \delta, \forall t \in [0, T^{(i)}(\varepsilon)) \} = 1 \]
with \( \lim_{\varepsilon \to 0} T^{(i)}(\varepsilon) = \infty, \) where \( C_{0}^{(1)} = C_{0}^{(1)}(\tilde{r}(0) - \rho, \tilde{\theta}(0) - \frac{\pi}{2}, e(0) + q_{r}(R^{2} + \rho^{2})), \) and \( C_{0}^{(2)} = C_{0}^{(2)}(\tilde{r}(0) - \rho, \tilde{\theta}(0) + \frac{\pi}{2}, e(0) + q_{r}(R^{2} + \rho^{2})). \) This completes the proof. □
5. Convergence speed

Theorem 4.1 establishes exponential convergence, however, the convergence rate is determined by the complicated cubic polynomial (42), whose roots are hard to find analytically in general. However, for particular parameter choice, they can be found explicitly, as given in the next proposition.

Proposition 5.1. Let the vehicle speed \( V_c \) and the parameter \( h \) of the washout filter be chosen according to the following relation:

\[
V_c = hR. \tag{47}
\]

Then the exponential convergence rate of the source seeking system in Theorem 4.1 is determined by the eigenvalues

\[
\lambda_1 = -h, \tag{48}
\]

\[
\lambda_2 = -\frac{d_0qR^2hR\lambda_1(a,g)l_2(2a,g)}{cl_2(a,g)} \left(1 - \sqrt{1 - \psi}\right), \tag{49}
\]

\[
\lambda_3 = -\frac{d_0qR^2hR\lambda_1(a,g)l_2(2a,g)}{cl_2(a,g)} \left(1 + \sqrt{1 - \psi}\right), \tag{50}
\]

where

\[
\psi = \frac{4c^3l_2^2(a,g)}{d_0qR^2hRl_1(a,g)l_2(2a,g)} > 0, \tag{51}
\]

and the radius of the residual annulus is

\[
\rho = \sqrt{\frac{hR\lambda_1(a,g)}{2qcl_2(a,g)}}. \tag{52}
\]

Proof. With \( V_c = hR \), the stability condition (13) becomes

\[
0 > -\frac{l_2(2a,g)}{2l_1(a,g)l_2(2a,g)}, \tag{53}
\]

which is satisfied for all parameters \( a, g, h, R > 0 \). Thus the characteristic polynomial (42) has all three roots with negative real parts. Let

\[
H \triangleq 4d_0\rho^2q^2R^2l_2(2a,g), \tag{54}
\]

\[
M \triangleq \frac{2\sqrt{c}l_2^2(a,g)}{\rho^2}, \tag{55}
\]

\[
Q \triangleq 4d_0\rho^2q^2R^2[2Vc\lambda_1(a,g)l_2(2a,g) + hR(l_2(2a,g) - 2l_1(a,g)l_2(a,g))]. \tag{56}
\]

Then we write the characteristic polynomial compactly as

\[
\lambda^2 + (h + H)\lambda^2 + (M + Q)\lambda + hM = 0. \tag{57}
\]

Denote by \( \lambda_i, i = 1, 2, 3 \), the roots of the polynomial (57). Then by the relation between the roots and the coefficients in the polynomial, we have

\[
h + H = -\lambda_1 - \lambda_2 - \lambda_3, \tag{58}
\]

\[
hM = -\lambda_1\lambda_2\lambda_3, \tag{59}
\]

\[
M + Q = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1. \tag{60}
\]

At this point one can just verify (48)–(50) by direct substitution into (58)–(60); however, we explain how we have arrived at (48)–(50). Let \( \lambda_1 = -h \). We shall show that this choice satisfies (58)–(60) by also finding \( \lambda_2 \) and \( \lambda_3 \) which satisfy (58)–(60). With \( \lambda_1 = -h \), (58)–(60) become

\[
H = -\lambda_2 - \lambda_3, \tag{61}
\]

\[
M = \lambda_2\lambda_3, \tag{62}
\]

\[
Q = hH. \tag{63}
\]

With substitution of \( V_c = hR \) into (56), we immediately see that (63) is verified. From (61) and (62) we see that we only need to solve the quadratic equation \( \lambda^2 + h\lambda + M = 0 \). Applying the formula for the roots of a quadratic equation, we arrive at

\[
\lambda_2 = -\frac{d_0\rho^2q^2R^2l_2(2a,g)}{cl_2(a,g)} + \frac{1}{\rho} \sqrt{4d_0\rho^2q^2R^2l_2(2a,g) - 2Vc^2l_2^2(a,g)} \tag{64}
\]

\[
\lambda_3 = -\frac{d_0\rho^2q^2R^2l_2(2a,g)}{cl_2(a,g)} - \frac{1}{\rho} \sqrt{4d_0\rho^2q^2R^2l_2(2a,g) - 2Vc^2l_2^2(a,g)} \tag{65}
\]

which, with some simplifications, gives (48)–(51). \( \square \)

Of the three eigenvalues in Proposition 5.1 one is real and can be placed arbitrarily far to the left by choosing \( h \) large, whereas the other two can either be real or conjugate complex. The optimal choice is where the eigenvalues \( \lambda_2 \) and \( \lambda_3 \) are equal, because otherwise, either one or both of these eigenvalues are closer to the imaginary axis than when \( \lambda_2 = \lambda_3 \). Unfortunately, this optimal eigenvalue placement cannot be achieved by intent, since the design parameters would have to depend on the unknown \( q_r \), however, in the next corollary we state this result in order to note what the best achievable convergence speed is.

Corollary 5.1. Let \( V_c = hR \) and let the damping parameter be chosen as

\[
d_0 = \frac{1}{\sqrt{q_rRhl_2(2a,g)}} \frac{2}{\sqrt{c^3l_2^2(a,g)}} l_1(a,g). \tag{66}
\]

Then the exponential convergence rate of the source seeking system in Theorem 4.1 is determined by the eigenvalues

\[
\lambda_1 = -h, \tag{67}
\]

\[
\lambda_2 = \lambda_3 = -\frac{2R}{\sqrt{q_rhl_2(2a,g)}} l_2(a,g), \tag{68}
\]

whereas the residual annulus is as in (52).

From Corollary 5.1 we note that the optimizing damping coefficient \( d_0 \) grows, whereas the convergence rate \( \lambda_2 = \lambda_3 \) decays, with a decrease of the parameter \( q_r \), namely, with the flattening of the extremum, as should be expected. Not surprisingly, the residual annulus (52) also grows with the flattening of the extremum. The convergence speed grows, whereas the annulus size shrinks, with the tuning gain \( c \).

Proposition 5.2. For a fixed \( a \), the optimal convergence speed (68) has a non-monotonic dependence on the noise intensity \( g \), with the maximal convergence speed achieved for

\[
g^* = \frac{1}{a} \ln \frac{2a^2 + 1 + 2a}{2a^2 + 1 - 2a}. \tag{69}
\]
Proof. By considering (68) and maximizing \( I_1(a, g)I_2(a, g) = \frac{1}{2}e^{\frac{x^2}{2}} - e^{-\frac{1}{2}2^x - e^{-\frac{1}{2}2^x}} \) with respect to \( g^2 \). \( \square \)

The non-monotonic dependence of the convergence speed on the noise intensity \( g \) is intuitive. If the noise is low, the gradient exploration is insufficient and the tuning process is ineffective. Too much noise, and the perturbation takes the trajectories too far from the average trajectory, slowing the approach to the annulus.

**Proposition 5.3.** For \( a \geq 1/2 \) the annulus radius \( \rho \) defined in (52) is a decreasing function of noise intensity \( g \). For \( a \in (0, 1/2) \) the radius \( \rho \) has a non-monotonic dependence on \( g \), with the minimal \( \rho \) achieved for

\[
\rho^o = \sqrt{\frac{1}{a} \ln \frac{1 + 2a}{1 - 2a}}. \tag{70}
\]

**Proof.** By considering (52) and minimizing \( \frac{I_2(a, g)}{I_1(a, g)} = \frac{2^a e^{\frac{1}{2}x}}{2^{2a} - e^{-a2x}} \) with respect to \( g^2 \). \( \square \)

Since we want to operate with a relatively small perturbation parameter \( a \), the annulus-minimizing value of \( g \) in (70) is of interest. Both very large and very low intensity of perturbation noise result in a large annulus, whereas a medium range of \( g \) is optimal. It is worth comparing the optimizing \( g \) for convergence speed in (69) with the optimizing \( g \) for the annulus in (70). For small \( a \) they are similar, which is very fortunate.

### 6. Simulations, dependence on design parameters, effect of constraints of the angular velocity, and design alternatives

#### 6.1. Basic simulations

Without loss of generality, we let the unknown location of the source be at the origin \( r^* = (0, 0) \). We pick the design parameters as \( V_c = 0.1, c = 10000, d_0 = 10, a = 0.1, h = 1, g = 1, \varepsilon = 0.01, R = 0.1 \) and take the parameters of the map as \( f^* = 0, q_* = 1.5 \). The simulation results are given in Fig. 3. We observe that the trajectories of the vehicle center go to a small neighborhood of the source and the vehicle motion involves a random perturbation component, instead of a sinusoidal perturbation employed in the deterministic case Cochran and Krstic (2009). In the simulations we use band-limited white noise to approximate the white noise.

The stochastic source seeking approach can also be used for pursuit of non-stationary sources. For the case where the source is performing a “figure eight” motion, unknown to the pursuing vehicle, the simulation result is shown in Fig. 4.

#### 6.2. Dependence of annulus radius \( \rho \) on parameters

From (17), we see the radius \( \rho \) of the attractive annulus is dependent on the model parameters \( q_* \), \( R \) and design parameters \( V_c \), \( c \), \( a \), \( g \), and that it can be made as small as desired. Hence, by (18) and (19), by making \( \rho \) as small as desired, the vehicle can converge as closely to the source as desired.

The dependence of \( \rho \) on the noise intensity is characterized by Proposition 5.3. Fig. 5 show some of this dependence. For a fixed small \( a = 0.1 \), the radius for \( g = 2 \) is \( \rho = 0.021 \), which is smaller than the radius \( \rho = 0.029 \) for \( g = 1 \).

![Figure 3](image1.png)

**Fig. 3.** (a) The trajectory of the vehicle center for the case of source with circular level sets. The trajectory converges to an annulus; (b) A zoomed in section of the vehicle trajectory, displaying the vehicle motion more clearly. For both simulations: \( V_c = 0.1, c = 10000, d_0 = 10, a = 0.1, g = 1, \varepsilon = 0.01, R = 0.1, f^* = 0, h = 1, q_* = 1.5 \). The source is at \( r^* = (0, 0) \).

![Figure 4](image2.png)

**Fig. 4.** Vehicle following a moving source with circular level sets. The simulation parameters are \( V_c = 0.1, c = 10000, d_0 = 10, a = 0.1, g = 1, \varepsilon = 0.01, R = 0.1, f^* = 0, h = 1, q_* = 1.5 \). The source moves according to \( x_w(t) = 0.5 \sin(0.13t), y_w(t) = 0.5 \sin(0.26t) \).

#### 6.3. Effect of constraints of the angular velocity

A physical vehicle always has a steering constraint, namely, a limit on the angular velocity \( \dot{\theta} \). This type of a unicycle model is commonly referred to as the Dubins vehicle. Fig. 6 depicts the trajectories of the vehicle center when the angular velocity is restricted to a symmetric interval, \([-u_{\text{max}}, +u_{\text{max}}]\), for several values of \( u_{\text{max}} \). We observe that, for \( u_{\text{max}} \) as small as 20, our control law successfully steers the vehicle to the annulus, and keeps the vehicle near the source, see Fig. 6(a). In addition, the vehicle moves more smoothly for smaller \( u_{\text{max}} \), see Fig. 6(b). However, if the actuator constraint \( u_{\text{max}} \) is too small, for example, \( u_{\text{max}} = 10 \), the algorithm cannot keep the vehicle very near the source, as observed in Fig. 7.
are the same, typically \( \sin(\omega t) \). Looking at the probing equation

\[ \dot{a} \cos(\theta) \eta - \frac{ag^2}{2\xi} \sin(\eta) + c\xi \sin(\eta) - d_0 \xi^2 \sin(\eta) \]

and thus (8) in the closed-loop system changes to

\[ d\theta = \left[ -\frac{a}{E} \cos(\theta) \eta - \frac{ag^2}{2E} \sin(\eta) \right] dt \]

\[ + (c\xi - d_0 \xi^2) \sin(\eta) dt + \frac{ag}{\sqrt{E}} \cos(\eta) dW, \]

where the additional term \(-\frac{ag^2}{2E} \sin(\eta)\) results from the Ito formula. Consequently, the two terms \( \cos(\theta + a\chi(t/\xi)) \) and \( \sin(\theta + a\chi(t/\xi)) \) in the error system (30)–(31)–(32) should be replaced by \( \cos(\theta + a\sin(\chi(t/\xi))) \) and \( \sin(\theta + a\sin(\chi(t/\xi))) \), respectively. It is hard to obtain the corresponding analytical average error system because we need to calculate two integrals: \( \int_{-\infty}^{+\infty} \cos(a\sin(y)) \sin(y) e^{-\frac{t^2}{2}} dy \) and \( \int_{-\infty}^{+\infty} \sin(a\sin(y)) \sin(y) e^{-\frac{t^2}{2}} dy \) and it is hard to obtain the analytical results though we can obtain numerical results. Fig. 8 depicts the trajectory of the vehicle center when the control law (71) is used. From the simulation, there is no noticeable difference relative to the trajectory in Fig. 3(a).
that depend on the distance from the source only. In this section, 

\[\text{(2)}\]

Fig. 9. The trajectory of the vehicle center under the control law (73). The simulation parameters are \(V_c = 0.1, c = 10000, d_0 = 10, a = 0.1, g = 1, \varepsilon = 0.01, R = 0.1, f^* = 0, h = 1, \varphi = 1.5\). The source is at \(r^* = (0, 0)\). Then

Now we analyze the radius of the annulus for three alternative perturbation signals. Let \(V_c = 0.1, c = 10000, d_0 = 10, a = 0.1, g = 1, \varepsilon = 0.01, R = 0.1, f^* = 0, \varphi = 1.5\). 

(1) For the probing signal \(\eta\) in (21) and demodulation signal \(\sin(\eta)\) in (24), we obtain the radius of the annulus as \(\rho^1 = 0.0293\).

(2) If we use \(\sin(\eta)\) to replace \(\eta\) as the probing signal in (21), the expressions \(I_1(a, g)\) and \(I_2(a, g)\) are replaced by \(I'_1(a, g)\) and \(I'_2(a, g)\), where \(I'_1(a, g) \triangleq \int_{g}^{\infty} \cos(a \sin(y)) \mu(dy)\) and \(I'_2(a, g) \triangleq \int_{g}^{\infty} \sin(a \sin(y)) \mu(dy)\),

\[
\sin(y) \mu(dy) = \int_{g}^{\infty} \sin(a \sin(y)) \sin(y) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy.
\]

By calculating the integrals numerically, we obtain \(I'_1(0.1, 1) = 0.9984\) and \(I'_2(0.1, 1) = 0.0316\). Thus, we get the radius of the annulus as \(\rho^1 = \left(\frac{V_c}{2\pi a \rho} \right)^\frac{1}{2} = 0.0325\), which is a little larger than \(\rho^1\).

(3) If we use the bounded function \(\eta \exp(-\eta^2)\) to replace both \(\eta\) as the probing signal in (21) and \(\sin(\eta)\) as the demodulation signal in (24), by numerical calculation we obtain

\[
\int_{g}^{\infty} \cos(0.1y \exp(-y^2)) \mu(dy) = \int_{g}^{\infty} \cos(0.1y \exp(-y^2)) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy = 0.9995, \int_{g}^{\infty} \sin(0.1y \exp(-y^2)) \exp(-\eta^2) \mu(dy) = \int_{g}^{\infty} \sin(0.1y \exp(-y^2)) \exp(-\eta^2) \mu(dy) = 0.0096.
\]

Therefore, from the point of view of the annulus radius, our choice \(\eta\) as the probing signal in (21) and \(\sin(\eta)\) as the demodulation signal in (24), achieves the best performance, in addition to facilitating the analysis.

If OU process \(\eta(t), t \geq 0\) is used not only as the probing signal, but also as the demodulation signal in (24), the extremum seeking control law (3) is replaced by

\[
\dot{\varphi} = a\varphi + (c\xi - d_0\xi^2)\eta . \tag{73}
\]

With \(\sin(\eta)\) replaced by \(\eta\) as a demodulation signal, where the latter signal is not uniformly bounded, the local Lipschitz condition (Assumption A.1 in Appendix) is not satisfied uniformly in the perturbation process for the resulting closed-loop system. For this reason, we cannot use the general stochastic averaging theorem to analyze stability. However from simulation results given by Fig. 9, we observe that the vehicle achieves convergence to a an annulus near the source under the control law (73).

7. System behavior for elliptical level sets

Our analysis is limited to circular level sets, namely, to fields that depend on the distance from the source only. In this section,

\[\text{(73)}\]

we present simulation results for elliptical level sets. Without loss of generality, we assume the source is at \(r^* = (0, 0)\), and the signal distribution in space is given (at the sensor location) by

\[
\begin{align*}
J = f(r_0) & = f^* - q_i |r_i|^2 - q_p (r_i^2 + r_p^2) \\
& = f^* - (q_i + 2q_p) \lambda^2 - (q_p - 2q_p) \eta_2^2 \\
& = f^* - q_i |r_c + Re^{j\theta}|^2 - q_p (|r_c + Re^{j\theta}|^2 + (r_c + Re^{-j\theta})^2) . \tag{74}
\end{align*}
\]

Fig. 10 depicts the trajectory of the vehicle center for a signal field with elliptical level sets. The vehicle reaches a small neighborhood of the source, however, the average motion is not circular revolution around the source, nor elliptical revolution, but a motion bias to one of the flatter sides of the ellipse. More than one such attractor exists. It depends on the initial condition and on the noise sequence which of the average attractors the trajectory will converge to.

Fig. 11 depicts the trajectories of the vehicle center with different \(d_0\) values in the control law. From Fig. 11(a), we see that for larger \(d_0\) the vehicle undergoes a “roundabout” behavior and then moves into a small neighborhood of the source. This is different than the situation for circular level sets, with either stochastic or deterministic source seeking algorithms. However, from Fig. 11(b), we observe a difference relative to the results obtained for elliptical level sets in the deterministic case in Cochran and Krsic (2009). The value of \(d_0\) does not affect the shape and size of the system attractors—the motion near the source is limited to an elliptical shape.

8. Concluding remarks

We have proposed and analyzed an algorithm for stochastic source seeking, employing, in a suitable way, colored noise perturbations, instead of periodic deterministic perturbations. We have adapted the general stochastic averaging theory for nonlinear continuous-time systems with stochastic perturbation to establish exponential convergence, both almost surely and in probability, to an annulus shaped region around the source. For particular values of design parameters we have calculated the convergence rate explicitly. If the cutoff frequency of the washout filter is chosen as high, then the dominant dependence of the best achievable convergence speed, given in (88), shows an increasing dependence on all of the relevant parameters—the vehicle length \(R\), the tuning gain \(c\), and the peak sharpness coefficient \(q_i\). However, both the convergence speed and the annulus size show non-monotonic dependence on the noise intensity \(g\), for which we provide optimizing values.
The results of this paper would not be difficult to extend to 3D source seeking, as in Cochran, Ghods, Siranosian, and Krstic (2009a), for underwater vehicle applications, or even to source seeking for fish models, as in Cochran, Kanso, Kelly, Xiong, and Krstic (2009b).

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Appendix. General stochastic averaging

Consider the system
\[ \frac{dX_t^\epsilon}{dt} = a(X_t^\epsilon, Y_t^\epsilon), \quad X_0^\epsilon = x, \tag{A.1} \]
where \( X_t^\epsilon \in \mathbb{R}^n \), \( Y_t \in \mathbb{R}^m \) is a time homogeneous continuous Markov process defined on a complete probability space \((\Omega, \mathcal{F}, P)\), where \( \Omega \) is the sample space, \( \mathcal{F} \) is the \( \sigma \)-field, and \( P \) is the probability measure. The initial condition \( X_0^\epsilon = x \) is deterministic, \( \epsilon \) is a small parameter in \((0, \epsilon_0)\) with fixed \( \epsilon_0 > 0 \). Let \( D \subset \mathbb{R}^n \) be a domain (open connected set) of \( \mathbb{R}^n \) and \( S_Y \) be the living space of the perturbation process \((Y_t, t \geq 0)\). Notice that \( S_Y \) may be a proper (e.g., compact) subset of \( \mathbb{R}^m \).

Consider the following two assumptions:

**Assumption A.1.** The vector field \( a(x, y) \) is a continuous function of \((x, y)\), and for any \( x \in D \), it is a bounded function of \( y \). Further it satisfies the locally Lipschitz condition in \( x \in D \) uniformly in \( y \in S_Y \), i.e., for any compact subset \( D_0 \subset D \), there is a constant \( k_{00} \) such that for all \( x', x'' \in D_0 \) and all \( y \in S_Y \), \( |a(x', y) - a(x'', y)| \leq k_{00} |x' - x''| \).

**Assumption A.2.** The perturbation process \((Y_t, t \geq 0)\) is ergodic with invariant distribution \( \mu \).

Under **Assumption A.2**, we obtain the average system of system (A.1) as follows:
\[ \frac{d\bar{X}_t}{dt} = \bar{a}(\bar{X}_t), \quad \bar{X}_0 = x \tag{A.2} \]
where
\[ \bar{a}(x) = \int_{S_Y} a(x, y) \mu(dy). \tag{A.3} \]

For the case \( D = \mathbb{R}^n \), under **Assumptions A.1** and A.2, and the assumption of the existence of the solution, we have proved general stochastic averaging theorems in Liu and Krstic (in press). The existence assumption is formulated in Liu and Krstic (in press) as follows: for any \( x \in \mathbb{R}^n \) and the perturbation process \((Y_t, t \geq 0)\), system (A.1) has a unique (almost surely) continuous solution on \([0, \infty)\). When the domain \( D \) is a proper subset of \( \mathbb{R}^n \), this condition is a strong restriction on system (A.1), because it is hard to restrict the solution of a stochastic system within such a domain. In this paper, we can eliminate this condition.

Before presenting the main results, we give two lemmas. To this end, for any point \( x \in D \), we define by \( d(x', \partial D) \) the distance between \( x' \) and the boundary \( \partial D \) of the domain \( D \), i.e., \( d(x', \partial D) = \inf |x' - y| : y \in \partial D \). By convention \( d(x', \partial D) = \infty \). Since \( D \) is a domain, for any \( x \in D \), we have that \( d(x', \partial D) > 0 \). If \( A \) is a subset of \( D \), we define by \( d(A, \partial D) \) the distance between \( A \) and \( \partial D \) as follows:
\[ d(A, \partial D) = \inf_{x \in A} d(x, \partial D) = \inf_{y \in \partial D} |x - y| : x \in A, y \in \partial D. \]

Throughout this part of the paper, we assume that \( x \in D \), where \( x \) is the initial value of system (A.1). System (A.1) is a stochastic ordinary differential equation (stochastic ODE), and its solution can be defined for each sample path of the perturbation process \((Y_{1/\epsilon} : t \geq 0)\). If system (A.1) satisfies **Assumption A.1**, then for any compact subset \( D_0 \subset D \) and the constant \( k_{00} \) stated in **Assumption A.1**, it holds that for any \( \omega \in \Omega \), any \( t \geq 0 \), any \( \epsilon \in (0, \epsilon_0) \), and all \( x', x'' \in \Omega \), \( |a(x', Y_{1/\epsilon}(\omega)) - a(x'', Y_{1/\epsilon}(\omega))| \leq k_{00} |x' - x''| \). Thus by the theorem on local existence and uniqueness of solutions of nonlinear systems (see, e.g., Theorem 3.1 of Khalil (2002)), for any \( \epsilon \in (0, \epsilon_0) \) and any \( \omega \in \Omega \), system (A.1) has a unique solution \( X_t^\epsilon(\omega) \) with the life time \( l_\epsilon(\omega) > 0 \), where \( l_\epsilon(\omega) = \sup \{ t \geq 0 : X_t^\epsilon(\omega) \in \partial D \} \). For \( t > l_\epsilon(\omega) \), we define \( X_t^\epsilon(\omega) = X_t^\epsilon(\omega) \), i.e., as soon as the solution reaches the boundary of the domain \( D \), we fix it and maintain it at that constant value thereafter.

**Lemma A.1.** Consider system (A.1) under **Assumptions A.1** and A.2. If \( d(|\bar{X}_t|, t \geq 0), \partial D) > 0 \), then for any \( T > 0 \), we have that
\[ \lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} |X_t^\epsilon - \bar{X}_t| = 0, \text{ a.s.} \tag{A.4} \]

**Proof.** Fix \( T > 0 \) and define \( A_T = \{|X_t| : 0 \leq t \leq T\} \). Then by the assumption that \( d(|\bar{X}_t|, t \geq 0), \partial D) > 0 \), we have that
Lemma A.2. Consider system (A.1) under Assumptions A.1 and A.2. If
\[ d((\bar{x}, t \geq 0), \partial D) > 0, \]
then for any \( \delta > 0 \), we have \( \lim_{\varepsilon \to 0} \inf_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t| = +\infty, \) a.s.


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